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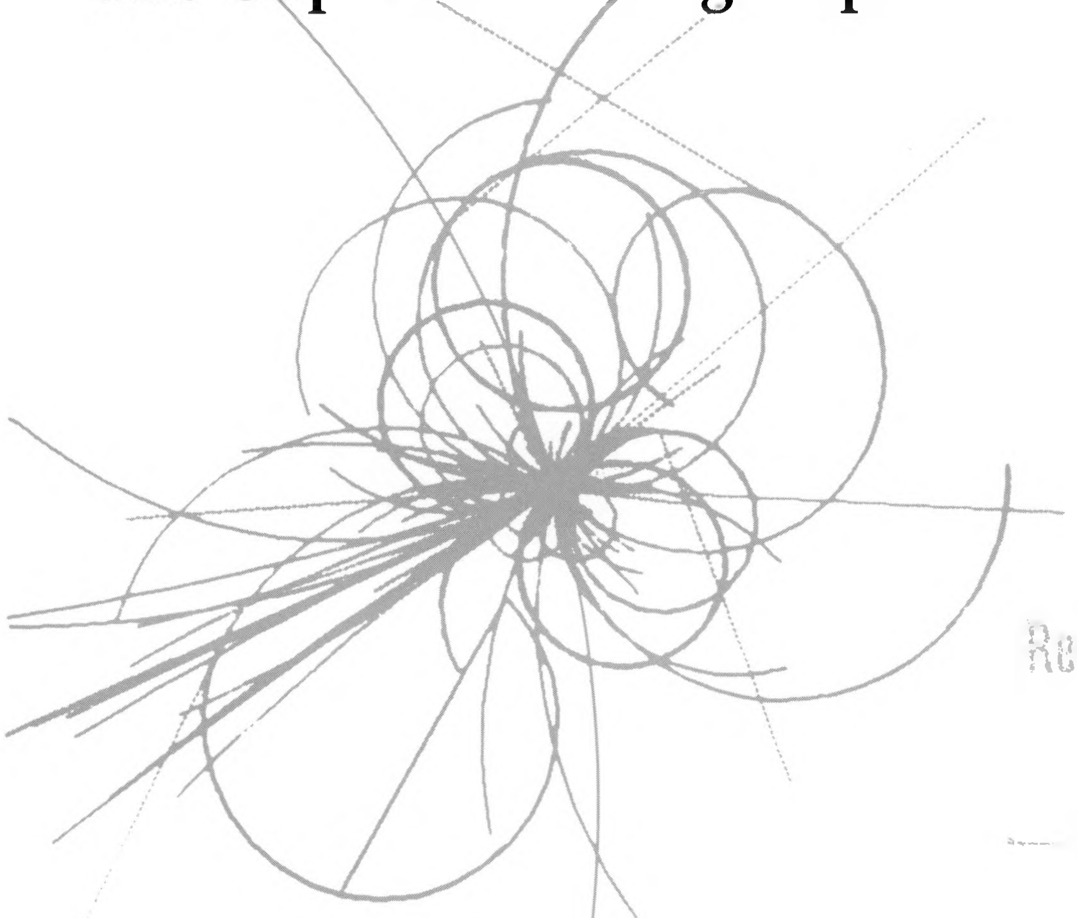

On Multiple Siberian Snakes

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I. INTRODUCTION

The acceleration of polarized proton beams has been done in a number of synchrotrons such as at ANL, Saclay, BNL, and KEK. The theory and technique have been well established up to the energy of a few tens of GeV; however, the same technique cannot be applied to an energy range of TeV as in the SSC. For such a machine we need Siberian snakes. As for this clever scheme, however, nothing is known experimentally and even theoretical knowledge is not enough to make reliable predictions for the SSC. It is generally thought that the single and double Siberian snake schemes are insufficient and that we need many pairs of double snakes. The point of controversy now is how many pairs are necessary and sufficient.

A most drastic depolarization during acceleration can occur when the spin-orbit resonance condition

$$\nu = m_0 + m_1\nu_x + m_2\nu_y + m_3\nu_s \tag{1.1}$$

is satisfied. Here, ν is the spin precession tune, ν_x, ν_y, ν_s are the horizontal betatron, vertical betatron and synchrotron tunes and m's are integers. In planar rings, since the spin tune is given by

$$\nu = \gamma a = E/0.52335\text{GeV} \tag{1.2}$$

to a good approximation, where γ is the beam energy in units of the rest mass and a is the coefficient of the anomalous magnetic moment, the spin tune varies over a large range during acceleration, which makes resonances inevitable. The Siberian snake is an idea to keep the spin tune at some value, usually 1/2, away from any resonance. However, in very high energy rings, perturbations such as machine imperfections and betatron oscillations may change the spin tune considerably so as to cause a resonance. Also, even if the spin tune does not satisfy Eq.(1.1), we may have non-resonant depolarization during acceleration.

In this paper we shall discuss this problem according to the following steps:

(a) The first question is whether or not Eq.(1.1) is satisfied at some energy. If yes, we have a resonance crossing and have to follow the standard procedure: compute the resonance strength (Fourier harmonic), put it into the Froissart-Stora formula, and, if the resonance is neither strong enough nor weak enough, consider the cures. In some cases further investigation is necessary for the possible overlapping resonances. Obviously, however, we prefer the answer "no" for this first question, because in very high energy rings we usually have a tremendous number of resonances which cannot be cured in practice. Thus, the first question imposes a condition

$$|\delta\nu| = |\nu - \frac{1}{2}| \ll \frac{1}{2}\,. \tag{1.3}$$

(b) When the answer of the first question is "no," we next consider the spin direction. We express the perturbed spin motion at each fixed energy using action-angle variables; *i.e.*, decompose the spin vector $\mathbf{s}$ into its component along a vector denoted by $\mathbf{n}$ and the precession around it. The projection of $\mathbf{s}$ onto $\mathbf{n}$ is the action variable J. The vector $\mathbf{n}$ deviates from its unperturbed value $\mathbf{n}_0$ due to perturbation. Then the second question is whether or not $|\mathbf{n} - \mathbf{n}_0| \ll 1$ is satisfied at the final energy. If not, there is no hope of polarization at that energy. Also, we ask whether it is satisfied during acceleration.

(c) If not at some energy, it does not immediately mean depolarization. The spin might come back to $\mathbf{n}_0$ after passing this region of energy. In this case we have to consider the speed of acceleration. Since J is an adiabatic invariant, the polarization will be restored if the acceleration is slow enough.

(d) The final question is the possible resonance overlap, as in the case of resonance crossing. Especially when a very slow acceleration is required in (c), the contribution of another hamonic can be significant. But this question is beyond the scope of the present paper.

In Sec.2 we shall develop basic formulae using perturbation theory. In later sections we discuss the above stated points (a) to (c) for betatron oscillations and machine imperfections.

II. PERTURBATION EXPANSION

In this section we shall apply the well-known, old-fashioned perturbation theory to find the spin tune and the spin action-angle variables for a quasi-periodic system (*i.e.*, in the absense of acceleration). There are several mothods for this purpose. The present method is not a smart one but seems to be the most familiar. The spin tune will be given up to the second order of perturbation and the action-angle variables up to the first order. The resulting formulae have been given by Derbenev and Kondratenko.[1]

Let us start from the equation of motion

$$\frac{d\mathbf{s}}{d\theta} = (\mathbf{\Omega}_0(\theta) + \delta\mathbf{\Omega}(\theta)) \times \mathbf{s}, \tag{2.1}$$

where θ is the generalized machine azimuth of the ring and the components of the spin vector $\mathbf{s}$ are measured in the system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, $\mathbf{e}_x$ being radial and $\mathbf{e}_z$ being longitudinal unit vector. We assume the unperturbed spin angular velocity vector $\mathbf{\Omega}_0$ is a periodic function of θ with period 2π. The perturbation $\delta\mathbf{\Omega}$ may contain different frequencies like the betatron tune.

The unperturbed equation

$$\frac{d\mathbf{s}}{d\theta} = \mathbf{\Omega}_0(\theta) \times \mathbf{s} \tag{2.2}$$

has three orthonormal solutions $\mathbf{n}_{01}, \mathbf{n}_{02}, \mathbf{n}_{03}$. The third vector $\mathbf{n}_{03}$, which we often denote as $\mathbf{n}_0$, is periodic in θ and $\mathbf{k}_0 \equiv \mathbf{n}_{01} + i\mathbf{n}_{02}$ has the quasi-periodicity

$$\mathbf{k}_0(\theta + 2\pi) = e^{-2\pi i \nu_0} \mathbf{k}_0(\theta), \tag{2.3}$$

where ν_0 is the unperturbed spin tune. We introduce a complex periodic vector

$$\tilde{\mathbf{k}}_0(\theta) = e^{i\nu_0\theta} \mathbf{k}_0(\theta). \tag{2.4}$$

Then the general solution to the unperturbed equation can be written as

$$\mathbf{s} = J_0 \mathbf{n}_0(\theta) + \sqrt{1 - J_0^2}\, \mathcal{R}e[e^{-i\psi_0} \tilde{\mathbf{k}}_0(\theta)]. \tag{2.5}$$

Here, J_0 and ψ_0 are the action-angle variables of the unperturbed system described by the Hamiltonian $\nu_0 J_0$.

The purpose of this section is to write the spin vector in the same form as Eq.(2.5) in the presense of perturbation, which can either be machine imperfections or betatron oscillations. Since the spin tunes of particles in a bunch have some spread normally, one can assume the distribution of the angle variable is uniform except in transient cases such as during resonance crossing. Therefore, the second term in Eq.(2.5) vanishes by average over particles and the degree of polarization is just the average of J. (If $\mathbf{n}_0$ depends on particles as in the case of betatron oscillation which we shall see later, one also has to average $\mathbf{n}_0$.)

Now, consider the perturbed Hamiltonian

$$H(J_0, \psi_0, \theta) = \nu_0 J_0 + \delta\mathbf{\Omega}(\theta) \cdot \mathbf{s}(J_0, \psi_0, \theta), \tag{2.6}$$

with s given in Eq. (2.5). A generating function $W(J, \psi_0, \theta)$ causes the canonical transformation

$$J_0 = \frac{\partial W}{\partial \psi_0} \tag{2.7}$$

$$\psi = \frac{\partial W}{\partial J} \tag{2.8}$$

$$H_{new} = \nu_0 \frac{\partial W}{\partial \psi_0} + \delta\mathbf{\Omega} \cdot \mathbf{n}_0 \frac{\partial W}{\partial \psi_0} + \sqrt{1 - (\frac{\partial W}{\partial \psi_0})^2}\, \mathcal{R}e(e^{-i\psi_0} \tilde{\mathbf{k}}_0 \cdot \delta\mathbf{\Omega}) + \frac{\partial W}{\partial \theta}. \tag{2.9}$$

Expanding W according to the order in $\delta\mathbf{\Omega}$ as

$$W = \psi_0 J + w_1 + w_2 + ..., \tag{2.10}$$

we get, to the second order,

$$H_{new} = \nu_0 J + \nu_0 \frac{\partial w_1}{\partial \psi_0} + J\delta\mathbf{\Omega}\cdot\mathbf{n}_0 + \sqrt{1-J^2}\,\mathcal{R}e(e^{-i\psi_0}\tilde{\mathbf{k}}_0\cdot\delta\mathbf{\Omega}) + \frac{\partial w_1}{\partial\theta}$$
$$+\nu_0\frac{\partial w_2}{\partial\psi_0} + \frac{\partial w_1}{\partial\psi_0}\delta\mathbf{\Omega}\cdot\mathbf{n}_0 - \frac{J}{\sqrt{1-J^2}}\frac{\partial w_1}{\partial\psi_0}\,\mathcal{R}e(e^{-i\psi_0}\tilde{\mathbf{k}}_0\cdot\delta\mathbf{\Omega}) + \frac{\partial w_2}{\partial\theta}. \quad (2.11)$$

The first order Hamiltonian is

$$H_{new,1} = J\left\langle \mathbf{n}_0\cdot\delta\mathbf{\Omega}\right\rangle, \quad (2.12)$$

where $\langle\ \rangle$ is the long term average; i.e.,

$$\langle f(\theta)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f(\theta)\,d\theta. \quad (2.13)$$

When $\delta\mathbf{\Omega}$ contains the orbital oscillation, we can treat them as another degree of freedom, by introducing orbital action-angle variables (I,φ). Then, the average would be

$$\langle f(\theta,\varphi)\rangle = \frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\varphi f(\theta,\varphi) \quad (2.14)$$

assuming f is periodic in θ and φ. However, since the influence of spin on the orbit is extremely small, we may treat the orbital motion as an external force, *i.e.*, treat φ as an explicit function of θ. Then, the average takes the form (2.13).

The first order generator satisfies

$$\nu_0\frac{\partial w_1}{\partial\psi_0} + \frac{\partial w_1}{\partial\theta} = -J(\mathbf{n}_0\cdot\delta\mathbf{\Omega} - \langle\mathbf{n}_0\cdot\delta\mathbf{\Omega}\rangle) - \sqrt{1-J^2}\,\mathcal{R}e(e^{-i\psi_0}\tilde{\mathbf{k}}_0\cdot\delta\mathbf{\Omega}). \quad (2.15)$$

The solution which has vanishing long term average is

$$w_1 = -J\int_{-\infty}^{\theta}d\theta(\mathbf{n}_0\cdot\delta\mathbf{\Omega} - \langle\mathbf{n}_0\cdot\delta\mathbf{\Omega}\rangle) - \sqrt{1-J^2}\,\mathcal{R}e\left[e^{-i\psi_0}e^{i\nu_0\theta}\int_{-\infty}^{\theta}e^{-i\nu_0\theta}\tilde{\mathbf{k}}_0\cdot\delta\mathbf{\Omega}d\theta\right]. \quad (2.16)$$

We shall often encounter integrals from minus infinity. They are to be understood as the following. If the integrand is expanded into Fourier series, then

$$\int_{-\infty}^{\theta}e^{i\kappa\theta}d\theta = \frac{e^{i\kappa\theta}}{i\kappa} \quad (2.17)$$

term by term. (If the integrand contains a zero-harmonic, which can happen at exact resonances, the integral cannot be defined.) This recipe can also be written by introducing an infinitesimal damping factor, as

$$\int_{-\infty}^{\theta}f(\theta)d\theta \stackrel{\text{def}}{=} \lim_{\epsilon\to 0+}\int_{-\infty}^{\theta}e^{\epsilon\theta}f(\theta)d\theta. \quad (2.18)$$

If the integrand has the periodicity $f(\theta + 2\pi) = e^{i\alpha} f(\theta)$, then one can reduce the integral to finite interval;

$$\begin{aligned}\int_{-\infty}^{\theta} f(\theta)d\theta &= \lim_{\epsilon\to 0+} \sum_{n=0}^{\infty} \int_{\theta-2(n+1)\pi}^{\theta-2n\pi} e^{\epsilon\theta} f(\theta)d\theta \\ &= \lim_{\epsilon\to 0+} \sum e^{-(\epsilon+i\alpha)(n+1)} \int_{\theta}^{\theta+2\pi} e^{\epsilon\theta} f(\theta)d\theta \\ &= \frac{1}{e^{i\alpha}-1} \int_{\theta}^{\theta+2\pi} f(\theta)d\theta. \qquad (2.19)\end{aligned}$$

Thus the new action-angle variables are found to be

$$\begin{aligned}J &= J_0 - \frac{\partial w_1}{\partial \psi_0} \\ &= J_0 + \sqrt{1-J^2}\, \mathcal{I}m \left[e^{-i\psi_0} e^{i\nu_0\theta} \int_{-\infty}^{\theta} e^{-i\nu_0\theta} \tilde{\mathbf{k}}_0 \cdot \delta\mathbf{\Omega} d\theta \right] \qquad (2.20)\end{aligned}$$

$$\begin{aligned}\psi &= \psi_0 + \frac{\partial w_1}{\partial J} \\ &= \psi_0 - \int_{-\infty}^{\theta} d\theta (\mathbf{n}_0 \cdot \delta\mathbf{\Omega} - \langle \mathbf{n}_0 \cdot \delta\mathbf{\Omega} \rangle) + \frac{J}{\sqrt{1-J^2}} \mathcal{R}e \left[e^{-i\psi_0} e^{i\nu_0\theta} \int_{-\infty}^{\theta} e^{-i\nu_0\theta} \tilde{\mathbf{k}}_0 \cdot \delta\mathbf{\Omega} d\theta \right] \\ & \qquad (2.21)\end{aligned}$$

up to the first order of perturbation.

We can define a new orthonormal basis $(\mathcal{R}e\, \tilde{\mathbf{k}}, \mathcal{I}m\, \tilde{\mathbf{k}}, \mathbf{n})$ so that the spin component seen in this frame is $(\sqrt{1-J^2}\cos\psi,\ \sqrt{1-J^2}\sin\psi,\ J)$. Comparing Eq.(2.5) and

$$\mathbf{s} = J\mathbf{n} + \sqrt{1-J^2}\, \mathcal{R}e(e^{-i\psi}\tilde{\mathbf{k}}), \qquad (2.22)$$

we get, to the first order of $\delta\mathbf{\Omega}$,

$$\begin{aligned}\mathbf{n} &= \mathbf{n}_0 + \mathcal{I}m[\tilde{\mathbf{k}}_0^* e^{i\nu_0\theta} \int_{-\infty}^{\theta} e^{-i\nu_0\theta} \tilde{\mathbf{k}}_0 \cdot \delta\mathbf{\Omega} d\theta] \\ &= \mathbf{n}_0 + \mathcal{I}m[\mathbf{k}_0^* \int_{-\infty}^{\theta} \mathbf{k}_0 \cdot \delta\mathbf{\Omega} d\theta] \qquad (2.23)\end{aligned}$$

$$\tilde{\mathbf{k}} = \tilde{\mathbf{k}}_0 - i\tilde{\mathbf{k}}_0 \int_{-\infty}^{\theta} (\mathbf{n}_0 \cdot \delta\mathbf{\Omega} - \langle \mathbf{n}_0 \cdot \delta\mathbf{\Omega} \rangle) d\theta + i\mathbf{n}_0 e^{i\nu_0\theta} \int_{-\infty}^{\theta} e^{-i\nu_0\theta} \tilde{\mathbf{k}}_0 \cdot \delta\mathbf{\Omega} d\theta. \qquad (2.24)$$

Now, let us proceed to the second order, for which we shall derive the spin tune only. From Eq.(2.11) we get

$$H_{new,2} = -\frac{J}{\sqrt{1-J^2}} \left\langle \frac{\partial w_1}{\partial \psi_0} \mathcal{R}e(e^{-i\psi_0} \tilde{\mathbf{k}}_0 \cdot \delta\mathbf{\Omega}) \right\rangle, \qquad (2.25)$$

since the term $\partial w_1/\partial\psi_0 \mathbf{n}_0 \cdot \delta\mathbf{\Omega}$ vanishes by the average over ψ_0. Using Eq.(2.16) we have

$$\begin{aligned} H_{new,2} &= J\left\langle \mathcal{I}m[e^{-i\psi_0}e^{i\nu_0\theta}\int_{-\infty}^{\theta} e^{-i\nu_0\theta}\tilde{\mathbf{k}}_0\cdot\delta\mathbf{\Omega}d\theta]\ \mathcal{R}e(e^{i\psi_0}\tilde{\mathbf{k}}_0^*\cdot\delta\mathbf{\Omega})\right\rangle \\ &= \frac{1}{2}J\,\mathcal{I}m\left\langle \tilde{\mathbf{k}}_0^*\cdot\delta\mathbf{\Omega}e^{i\nu_0\theta}\int_{-\infty}^{\theta} e^{-i\nu_0\theta}\tilde{\mathbf{k}}_0\cdot\delta\mathbf{\Omega}d\theta\right\rangle. \end{aligned} \tag{2.26}$$

Thus, combining Eq.(2.12) and (2.26), we finally get the spin tune up to the second order in $\delta\mathbf{\Omega}$;

$$\nu = \nu_0 + \Delta\nu_1 + \Delta\nu_2 \tag{2.27}$$

$$\Delta\nu_1 = \langle \mathbf{n}_0\cdot\delta\mathbf{\Omega}\rangle \tag{2.28}$$

$$\begin{aligned} \Delta\nu_2 &= \frac{1}{2}\mathcal{I}m\left\langle \tilde{\mathbf{k}}_0^*\cdot\delta\mathbf{\Omega}e^{i\nu_0\theta}\int_{-\infty}^{\theta} e^{-i\nu_0\theta}\tilde{\mathbf{k}}_0\cdot\delta\mathbf{\Omega}d\theta\right\rangle \\ &= \frac{1}{2}\mathcal{I}m\left\langle \mathbf{k}_0^*\cdot\delta\mathbf{\Omega}\int_{-\infty}^{\theta}\mathbf{k}_0\cdot\delta\mathbf{\Omega}d\theta\right\rangle. \end{aligned} \tag{2.29}$$

These formulae are entirely general but in the following we discuss the effect of transverse displacement only, for which we have

$$\delta\mathbf{\Omega} = R(\gamma a+1)(x''\mathbf{e}_y - y''\mathbf{e}_x). \tag{2.30}$$

Here, a prime denotes the derivative w.r.t. the design orbit length s and R is the average machine radius, which appears because our independent variable is not s but θ. Furthermore, we shall discuss only the case where $\mathbf{n}_0$ is vertical (up or down) almost everywhere. (We do not discuss single Siberian snakes.) With these assumptions we find that only a horizontal displacement contributes to the tune shift in first order and only the vertical in second order;

$$\Delta\nu_1 = R(\gamma a+1)\langle \mathbf{n}_0\cdot\mathbf{e}_y\, x''\rangle \tag{2.31}$$

$$\Delta\nu_2 = \frac{1}{2}R^2(\gamma a+1)^2\,\mathcal{I}m\left\langle \mathbf{k}_0^*\cdot\mathbf{e}_x y''\int_{-\infty}^{\theta}\mathbf{k}_0\cdot\mathbf{e}_x y'' d\theta\right\rangle. \tag{2.32}$$

In passing, let us give explicit formulae of the vectors $\mathbf{n}_0$ and $\mathbf{k}_0$ for our model ring. If the ring is planar and the bends are distributed uniformly, they are given by

$$\mathbf{n}_0(\theta) = -\mathbf{e}_y \tag{2.33}$$

$$\mathbf{k}_0(\theta) = e^{-i\nu_0\theta}(\mathbf{e}_x + i\mathbf{e}_z) \qquad \text{with} \qquad \nu_0 = \gamma a. \tag{2.34}$$

The sign of $\mathbf{n}_0$ is so chosen that the spin tune is $+\gamma a$ for our right-handed basis. When we discuss Siberian snakes, we shall take the following model of the ring. The ring has M sections, each of which consists of a first-kind snake, an arc, a second-kind snake and another arc. The two arcs in a section must have the same bending angles. We shall call this unit section a "snake period." The number of snake periods, M, must be odd in order to have $\nu_0 = 1/2$. We do not take into account the details of the snake structures and assume they simply rotate the spin by 180 degrees. We denote the locations of the first kind snakes by $\theta_0(=0), \theta_2, \theta_4, ..., \theta_{2M-2}$ and those of the second-kind snakes by $\theta_1, \theta_3, ..., \theta_{2M-1}$. The vector $\mathbf{n}_0$ is always vertical and given by

$$\mathbf{n}_0(\theta) = (-1)^j \mathbf{e}_y \qquad \text{for} \qquad \theta_{j-1} < \theta < \theta_j. \tag{2.35}$$

If all the sections are identical, as we shall assume in most cases, θ_j is given by

$$\theta_j = \frac{j\pi}{M} \qquad j = 0, 1, ...2M-1. \tag{2.36}$$

and the bending angle of one snake period is θ_2. In the first snake period, the vector $\mathbf{k}_0$ is

$$\begin{aligned} \mathbf{k}_0(\theta) &= e^{-i\gamma a\theta}(\mathbf{e}_x + i\mathbf{e}_z) && (0 < \theta < \theta_1) \\ &= e^{+i\gamma a(\theta-\theta_2)}(\mathbf{e}_x - i\mathbf{e}_z) && (\theta_1 < \theta < \theta_2) \end{aligned} \tag{2.37}$$

and the periodicity

$$\mathbf{k}_0(\theta + \theta_2) = -\mathbf{k}_0(\theta) \tag{2.38}$$

gives the values for $\theta > \theta_2$.

III. MACHINE IMPERFECTION

TUNE SHIFT

In this section we discuss the depolarization due to machine imperfections. First, consider the spin tune shift.

The first-order shift only comes from the horizontal closed-orbit distortion

$$\Delta\nu_1 = R(\gamma a + 1)\frac{1}{2\pi}\int_0^{2\pi} (\mathbf{n}_0 \cdot \mathbf{e}_y) x''_{\text{cod}} d\theta. \tag{3.1}$$

In planar rings this is exactly zero since $\mathbf{n}_0 \cdot \mathbf{e}_y = 1$ and x_{cod} is in fact closed.

In the rings equipped with Siberian snakes, the first-order shift is

$$\Delta\nu_1 = \frac{\gamma a + 1}{2\pi} \sum_{j=0}^{2M-1} (-1)^j \left[x'_{\text{cod}}(\theta_{j+1}-) - x'_{\text{cod}}(\theta_j+)\right], \tag{3.2}$$

where the sign following θ indicates just before or after a snake. In contrast to the case of planar rings, this is not necessarily zero. If $x_{\rm cod}$ is random, we have statistically

$$\Delta\nu_{1,\rm rms} = \frac{\gamma a + 1}{2\pi}\sqrt{4M}x'_{\rm cod,rms}. \tag{3.3}$$

At 20 TeV, $x'_{\rm cod,rms}$ of one microradian gives $\Delta\nu_1 \sim 0.01\sqrt{M}$. This is not important unless M is of the order of 100. Also, even if $\Delta\nu_1$ is large, we can correct the orbit to eliminate it, because the correction is independent of energy.

Next, let us estimate the second order shift. The expression Eq.(2.32) of $\Delta\nu_2$ can be rewritten, using Eq.(2.19) and (2.3), as

$$\Delta\nu_2 = \frac{1}{2}R^2(\gamma a + 1)^2 \mathcal{Im}\left[\frac{1}{e^{-2\pi i\nu_0} - 1}\frac{1}{2\pi}\int_0^{2\pi} d\theta h^*(\theta)y''_{\rm cod}\int_\theta^{\theta+2\pi} d\theta' h(\theta')y''_{\rm cod}\right], \tag{3.4}$$

where

$$h(\theta) = \mathbf{k}_0 \cdot \mathbf{e}_x(\theta). \tag{3.5}$$

This integral shows very different features between the cases with and without closed-orbit correction. For simplicity let us assume that all M sections are identical with respect not only to the snake scheme but also to the orbital optics.

If we do not correct the orbit, $y_{\rm cod}$ is dominated by the Fourier harmonic close to $\pm\nu_y + kM$, k being an integer. Within a half snake period, between first and second kind snakes, $h(\theta)$ oscillates with the phase $\pm\gamma a\theta$. Therefore, $\Delta\nu_2$ can become very large near the beam energy corresponding to $\gamma a = \pm\nu_y + kM$. It is not important at other energies. This phenomenon may be called "local resonance." It does not cause infinity in the expression (3.4), for the effect does not accumulate over revolutions but only within a half snake period. However, if the tune shift due to this local resonance is large, it may cause a real resonance Eq.(1.1). In this circumstance the multiple Siberian snake breaks the accumulation before it becomes large and cancels it with that in the following snake periods.

On the other hand, if the orbit is well corrected, the spectrum of $y_{\rm cod}$ is white and there is almost no correlation of $y_{\rm cod}$ between different snake periods unless, for example, every FODO cell is equipped with snakes. There is no special value of energy which causes local resonances. The tune shift can be equally important or unimportant at every energy apart from the factor $\gamma a + 1$ in Eq.(3.4).

Let us estimate (3.4) when one Fourier harmonic κ, an integer close to $\pm\nu_y + kM$, dominates;

$$y''_{\rm cod} = Y_\kappa e^{i\kappa\theta} + Y_\kappa^* e^{-i\kappa\theta} \tag{3.6}$$

The phase of the integrand is either $(\kappa + \gamma a)\theta$ or $(\kappa - \gamma a)\theta$. When $\gamma a = \kappa$, we can ignore the former. For one pair of snakes $M = 1$, the integral is

$$
\begin{aligned}
& \int_0^{2\pi} d\theta h^* y''_{\rm cod} \int_\theta^{\theta+2\pi} d\theta' h y''_{\rm cod} \\
= & \int_0^{\pi} y''_{\rm cod} e^{+i\kappa\theta} d\theta \left[\int_\theta^{\pi} e^{-i\kappa\theta'} + \int_\pi^{2\pi} e^{i\kappa(\theta'-2\pi)} - \int_{2\pi}^{\theta+2\pi} e^{-i\kappa(\theta'-2\pi)} \right] y''_{\rm cod} d\theta' \\
& + \int_\pi^{2\pi} y''_{\rm cod} e^{i\kappa(2\pi-\theta)} d\theta \left[\int_\theta^{2\pi} e^{i\kappa(\theta'-2\pi)} - \int_{2\pi}^{3\pi} e^{-i\kappa(\theta'-2\pi)} - \int_{3\pi}^{\theta+3\pi} e^{i\kappa(\theta'-4\pi)} \right] y''_{\rm cod} d\theta' \\
= & \int_0^{\pi} Y^*_\kappa \left[(\pi - \theta) Y_\kappa + \pi Y^*_\kappa - \theta Y_\kappa \right] + \int_\pi^{2\pi} Y_\kappa \left[(2\pi - \theta) Y^*_\kappa - \pi Y_\kappa - (\theta - \pi) Y^*_\kappa \right] \\
= & \pi^2 (Y^{*2}_\kappa - Y^2_\kappa).
\end{aligned}
$$

Thus we get

$$\Delta\nu_2 = +\frac{\pi}{4} R^2 (\gamma a + 1)^2 \,\mathcal{I}m\, Y^2_\kappa. \tag{3.7}$$

This can be expressed by a more familiar quantity, the resonance strength often used for planar rings;

$$\epsilon_\kappa = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{k}_0 \cdot \delta\mathbf{\Omega} d\theta = -\frac{R(\gamma a + 1)}{2\pi} \int_0^{2\pi} e^{-i\gamma a\theta} y''_{\rm cod} d\theta \tag{3.8}$$

where, if $\gamma a = \kappa$, the integrand is periodic. We take Y_κ term only and get

$$\epsilon_\kappa = -R(\gamma a + 1) Y_\kappa. \tag{3.9}$$

Using ϵ_κ we can write Eq.(3.7) as

$$\Delta\nu_2 = \frac{\pi}{4} \,\mathcal{I}m\, \epsilon^2_\kappa. \tag{3.10}$$

One can easily derive the tune shift for multiple snake case in the same manner with the result

$$\Delta\nu_2 = \frac{\pi}{4M} \,\mathcal{I}m\, \epsilon^2_\kappa \tag{3.11}$$

which agrees with the small ϵ_κ limit of the formula given by R.D.Ruth.[2] Thus we find the multiple Siberian snake can reduce the spin tune shift as long as one single Fourier harmonic dominates in $y_{\rm cod}$.

Next, consider the case with orbit correction. Since $h(\theta + 2\pi) = -h(\theta)$, we can rewrite Eq.(3.4) as

$$\Delta\nu_2 = -\frac{R^2(\gamma a + 1)^2}{8\pi} \,\mathcal{I}m \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' y''_{\rm cod}(\theta) y''_{\rm cod}(\theta') h^*(\theta) h(\theta') {\rm sgn}(\theta' - \theta) \tag{3.12}$$

where $\text{sgn}(x) = -1, 0, 1$ for $x < 0, = 0, > 0$ respectively. Let us approximate $y''_{\text{cod}} R d\theta$ by thin lens kick angle ϕ_n, n being the sequential number of kicks. They can be due to either errors or correctors. Then Eq.(3.12) can be written as

$$\Delta\nu_2 = -\frac{(\gamma a + 1)^2}{8\pi} \mathcal{I}m \sum_{m,n=0}^{N_k - 1} h_m^* h_n \phi_m \phi_n \text{sgn}(n - m), \tag{3.13}$$

where N_k is the total number of kicks and h_n is the value of h at n-th kick.

Now, estimate the expectation of $\Delta\nu_2$ after orbit correction. Since the correlation of ϕ_m with ϕ_n is usually very small if $|m - n|$ is more than a few, we may ignore the correlation if the m-th and n-th kicks are in different half snake periods. When they are in the same half snake period, the spin phase function in Eq.(3.13) is

$$h_m^* h_n = e^{\pm i\gamma a \Delta\theta_{mn}} \qquad \text{in the same half snake period} \tag{3.14}$$

where $\Delta\theta_{mn}$ is the distance between m-th and n-th kicks in machine azimuth and the upper (lower) sign is to be taken when the kicks are downstream of first (second) kind snake. Now, the expectation of $\Delta\nu_2$ becomes

$$\begin{aligned} \langle \Delta\nu_2 \rangle &= -\frac{(\gamma a + 1)^2}{8\pi} \sum_{m,n=0}^{N_k - 1} \pm \sin(\gamma a |\Delta\theta_{mn}|) \langle \phi_m \phi_n \rangle \\ &= 0, \end{aligned} \tag{3.15}$$

because the plus terms and minus terms cancel each other within each snake period. Thus we find that the expectation of $\Delta\nu_2$ vanishes.

The next step is to estimate its standard deviation. For this we further simplify our model, assuming that all kicks are equally spaced so that $\Delta\theta_{mn} = 2\pi(m - n)/N_k$. We introduce the correlation function of ϕ;

$$\langle \phi_m \phi_n \rangle = C_{|m-n|} \phi_{\text{rms}}^2. \tag{3.16}$$

From Eq.(3.13) we get

$$\begin{aligned} \langle \Delta\nu_2^2 \rangle = \frac{(\gamma a + 1)^4}{(8\pi)^2} \sum_{m,n} \sum_{m',n'} (\mathcal{I}m\, h_m^* h_n)(\mathcal{I}m\, h_{m'}^* h_{n'}) \\ \langle \phi_m \phi_n \phi_{m'} \phi_{n'} \rangle \text{sgn}(n - m) \text{sgn}(n' - m'). \end{aligned} \tag{3.17}$$

Ignoring the four-point correlation, we decompose the product of ϕ's into three terms, $\langle \phi_m \phi_n \rangle \langle \phi_{m'} \phi_{n'} \rangle$, $\langle \phi_m \phi_{m'} \rangle \langle \phi_n \phi_{n'} \rangle$ and $\langle \phi_m \phi_{n'} \rangle \langle \phi_{m'} \phi_n \rangle$. The first one vanishes because it gives $\langle \Delta\nu_2 \rangle^2$. The third term gives the same contribution as the second. Thus, we have

$$\begin{aligned}\langle \Delta\nu_2^2 \rangle &= \frac{(\gamma a+1)^4}{(8\pi)^2} 2 \sum_{m,n} \sum_{m',n'} (\mathcal{I}m\, h_m^* h_n)(\mathcal{I}m\, h_{m'}^* h_{n'}) \\ &\qquad \phi_{\rm rms}^4 C_{|m-m'|} C_{|n-n'|} {\rm sgn}(n-m) {\rm sgn}(n'-m') \\ &= 2 \left[\frac{(\gamma a+1)^2 \phi_{\rm rms}^2}{8\pi} \right]^2 \sum_{m,n} \sum_{j,j'} \sin\left[\frac{2\pi\gamma a}{N_k} |m-n| \right] \\ &\qquad \sin\left[\frac{2\pi\gamma a}{N_k} |m-n+j-j'| \right] C_{|j|} C_{|j'|},\end{aligned}$$

and, approximating $\sin^2$ by 1/2 and sin cos by 0, obtain

$$\Delta\nu_{2,rms} = \frac{(\gamma a+1)^2 \phi_{\rm rms}^2 N_k}{8\pi} \left| 1 + 2 \sum_{j=1}^{\infty} \cos\left(\frac{2\pi\gamma a}{N_k} j \right) C_{|j|} \right|. \tag{3.18}$$

The factor $|C_1|$ is usually about 0.5 or less and the rest are small. So, we estimate the factor in the asolute sign to be about 2 at maximum. Thus, we have

$$\Delta\nu_{2,rms} = \frac{(\gamma a+1)^2 \phi_{\rm rms}^2 N_k}{4\pi} \tag{3.19}$$

This formula is derived for a somewhat oversimplified model but is expected to apply to more general cases apart from the factor 2 due to the residual correlation. In general the factor $\phi_{\rm rms}^2 N_k$ should be understood as the sum of ϕ_n^2.

The contributions to $\phi_{\rm rms}^2 N_k$ mainly come from three kinds of kicks, namely, the off-centered orbit in quads, vertical kick $\phi_{\rm B}$ due to the roll of bends (rotation around longitudinal axis) and the kick $\phi_{\rm C}$ by the correctors. The strength of each corrector $\phi_{\rm C}$ is a sum $\phi_{\rm CB} + \phi_{\rm CQ}$, where $\phi_{\rm CQ}$ and $\phi_{\rm CB}$ are the required strengths to correct the effects of the misalignment of quads and the roll of bends, respectively. We may assume that $\phi_{\rm CQ}$ and $\phi_{\rm CB}$ have no correlation.

Usually, the correctors and monitors are placed close to quads and the origin of the monitors are adjusted to the center of quads. Therefore, we have

$$N_k \phi_{\rm rms}^2 \sim N_{\rm Q} \left\langle (q y_{\rm mon} + \phi_{\rm CQ} + \phi_{\rm CB})^2 \right\rangle + N_{\rm B} \left\langle \phi_{\rm B}^2 \right\rangle \tag{3.20}$$

where $y_{\rm mon} = y_{\rm cod} - y_{\rm mis}$ is the residual closed orbit distortion with respect to the center of quads, $y_{\rm mis}$ is the misalignment of quads, q is the inverse focal length of the quads and $N_{\rm Q}$ and $N_{\rm B}$ are the lengths of a quad and a bend. The three terms in the parenthesis have only small correlation to each other. As is shown in appendix A, we statistically have

$$N_{\rm C}\left\langle\phi_{\rm CQ}^2\right\rangle \sim N_{\rm Q}\left\langle(qy_{\rm mis})^2\right\rangle$$
$$N_{\rm C}\left\langle\phi_{\rm CB}^2\right\rangle \sim N_{\rm B}\left\langle\phi_{\rm B}^2\right\rangle. \tag{3.21}$$

Also, note that $\phi_{\rm B} = 2\pi\theta_{\rm B}/N_{\rm B}$, $\theta_{\rm B}$ being the roll angle of bends. Therefore, $\Delta\nu_{2,rms}$ can be estimated as

$$\Delta\nu_{2,rms} \lesssim \frac{(\gamma a+1)^2}{4\pi}\left[N_{\rm Q}q^2y_{\rm mon,rms}^2 + N_{\rm Q}q^2y_{\rm mis,rms}^2 + n\theta_{\rm B,rms}^2\cdot 4\pi^2/N_{\rm B}\right] \tag{3.22}$$
$$n=2 \qquad (\gamma a/N_{\rm B}\ll 1) \qquad =1 \qquad (\gamma a/N_{\rm B}\gg 1)$$

The factor n comes from the following consideration. We have estimated the spin kick angle due to the roll of the bends by $\gamma a+1$ times the orbit kick angle $\phi_{\rm B}$. However, when $2\pi\gamma a/N_{\rm B}$, the precession angle in one bending magnet, is very large, the spin kick due to the roll is much smaller than this estimation. Therefore, in such a case the last term of Eq.(3.20) can be ignored. But the term of the same form coming from $\phi_{\rm CB}$ through Eq.(3.21) still exists, giving rise to $n=1$. Otherwise, both terms, the direct term and $\phi_{\rm CB}$ term, contribute and give n=2.

When the roll angle of a bend is not uniform due to fabrication errors, we should cut the bend into short pieces and treat each piece independently by using a larger number for $N_{\rm B}$.

Let us estimate the tolerance for the SSC. The upper limit of the tune shift that does not cause resonance is not obvious, especially because we don't know how high an order of resonance can be significant, but here we assume up to $\Delta\nu_{\rm max}$ =0.15 is tolerated and take three standard deviations as a safety factor because, as is seen in Fig.2(b) to be described later, $\Delta\nu$ is a very rapidly varying function of energy and the condition of small $\Delta\nu$ has to be satisfied at all the energies. Typical SSC parameters [$N_{\rm Q}$(number of quads in the arc)=700, q=0.01/m, $N_{\rm B}$=3800, γa =38000 for 20 TeV] and the condition $\Delta\nu_{\rm rms}\lesssim$0.05 lead to the tolerance of quadrupole misalignment $y_{\rm mis,rms}\lesssim$ 80 microns if the errors come from quadupole misalignment only. The tolerance of closed orbit after correction is also $y_{\rm mon,rms}\lesssim$ 80 microns. The tolerance of roll of bends is $\theta_{\rm B,rms}\lesssim$200 microns if the errors come from bends only. ($n=1$ is used since $\gamma a/N_{\rm B}=10$.) Each dipole magnet is about 16m long, which is too long to consider to be uniform. If we assume four 4 meter long independent pieces, then the tolerance becomes $\theta_{\rm B,rms}\lesssim$400 microns. Bear in mind that these tolerances are inversely proportional to the beam energy.

Figures 1 to 3 show an example of computer simulation. The adopted model is as follows. The ring consists of N_c=330 identical FODO cells, each of which is 200m long and consists of a pair of thin lens quads of equal strength and two bends. Monitors are attached to all the quads and the correctors are located at the center of one of the bends in each cell (one per cell). (This point is different from the above theory.) The corrector strengths are determined by least square method. The Siberian snakes are inserted equidistantly. Since $N_c = 330 = 2 \times 3 \times 5 \times 11$, the possible numbers of snake pairs are $M = 1, 3, 5, 11, 15, 33, 55$ and 165. Adopted parameters are:

quadrupole misalignment $y_{\rm mis} = \pm$ 100 microns (uniform)
monitor error = $\pm$ 50 microns (uniform)
tune $\nu_y = 53.213$ ($\sim$ 58 degrees per cell)
roll of bends not included

Figure 1(a) shows the spectrum of $y''_{\rm cod}$ before correction. Clearly, it is dominated by the harmonics, $330 - \nu_y$, ν_y, $165 + \nu_y$ and $165 - \nu_y$. The last two come from the alternating sign of the quads. The r.m.s. cod is 1.2 mm. After correction it becomes 25 microns and the spectrum becomes as shown in Fig.1(b). No particular structure is seen.

The tune shift $\Delta\nu = \nu - 0.5$ for one pair of snakes $M = 1$ before correction is plotted in Fig.2(a) as a function of γa for $37620 = 114 \times N_c < \gamma a < 38940 = 118 \times N_c$. (In this figure the misalignment is reduced by factor 10, since otherwise $\Delta\nu$ is too large.) One finds sharp spikes due to local resonance. If we have M pairs of snakes, the spikes become M times lower and M times wider.

Figure 2(b) is the tune shift after correction. No resonance-like structure is seen but instead one finds a suppression of $\Delta\nu$ near $\gamma a = 116 \times N_c$. This is due to our oversimplified model for simulation. Since the location of every kick (by quads or by correctors) is a multiple of $2\pi/(4N_c)$ in our model, the precession angle between successive kicks is a multiple of 360 degrees when γa is a multiple of $4 \times N_c$. There is no net preccession. This effect can be explained, to some extent, by the cosine term in Eq.(3.18), where C_1 is usually negative. This effect will not be seen clearly in real machines due to various asymmetries.

In the case of one pair snakes dnu is a rapidly varying function of γa, as in this figure. If we have M pairs, it is less rapid ($\Delta\nu/d\gamma a \sim 1/M$), but $\Delta\nu_{\rm rms}$ is still the same.

SPIN PRECESSION AXIS

The next problem is the deviation of the precession axis $\mathbf{n}$ from $\mathbf{n}_0$; $\delta\mathbf{n} = \mathbf{n} - \mathbf{n}_0$. A large value of $|\delta\mathbf{n}|$ in some narrow energy region during acceleration does not immediately mean depolarization, because the spin direction might be restored after passing this region. It can at least be said, however, that a large $|\delta\mathbf{n}|$ at the energy for experiments is fatal. One should also note that, if $|\delta\mathbf{n}|$ is large, our perturbation approach fails.

From Eqs.(2.22), (2.30) and (2.19) we have, for $\nu_0 = 1/2$,

$$\delta\mathbf{n}(\theta) = \mathcal{Im}\, \mathbf{k}_0^* F(\theta) \tag{3.23}$$

with

$$F(\theta) = \frac{1}{2} R(\gamma a + 1) \int_\theta^{\theta+2\pi} h(\theta) y''_{\rm cod}(\theta) d\theta. \tag{3.24}$$

Since $|\delta\mathbf{n}|^2 = |F|^2$, we have only to discuss the scalar function $F(\theta)$. First, let us estimate it in the case without orbit correction. When one Fourier harmonic dominates one can easily integrate (3.24), ignoring the rapidly oscillating terms. At the locations of snakes θ_j defined in Sec.II we have

$$F(\theta_j) = \frac{\pi}{M} R(\gamma a{+}1)\, |Y_\kappa| \cos(\arg Y_\kappa + \frac{j\pi}{2} + (j{+}1)\frac{\kappa\pi}{M}) \times \left\{ \begin{array}{ll} e^{-i\kappa\pi/M} & j{=}\text{even} \\ e^{i\pi/2} & j{=}\text{odd} \end{array} \right\}. \tag{3.25}$$

For $\theta_j < \theta < \theta_{j+1}$, $F(\theta)$ is on the line segment from $F(\theta_j)$ to $F(\theta_{j+1})$ in the complex plane. Therefore, the maximum of $|F(\theta)|$ never exceeds, and in fact is very close to, the value given by replacing cos in Eq.(3.25) by 1. Thus, we get

$$|\delta\mathbf{n}|_{\rm max} = \frac{\pi}{M} R(\gamma a + 1)\, |Y_\kappa| = \frac{\pi}{M}\, |\epsilon_\kappa|\,. \tag{3.26}$$

Comparing this expression with $\Delta\nu_2$ in Eq.(3.11) we find that, in the case when ϵ_κ is large so that we need large M, $|\delta\mathbf{n}|$ is always much smaller than unity if $\Delta\nu_2$ is.

Now, consider the case with orbit correction. We can rewrite (3.24) as a sum over kicks;

$$F(\theta) = -\frac{\gamma a + 1}{2} \sum_n h_n \phi_n, \tag{3.27}$$

where the summation is over the kicks between θ and $\theta + 2\pi$. Its square average is

$$\begin{aligned} \left\langle |F(\theta)|^2 \right\rangle &= \left(\frac{\gamma a + 1}{2}\right)^2 \sum_{m,n} h_m^* h_n \langle \phi_m \phi_n \rangle \\ &= \left(\frac{\gamma a + 1}{2} \phi_{\rm rms}\right)^2 \sum_{n,j} h_{n+j}^* h_n C_{|j|}. \end{aligned} \tag{3.28}$$

With the same assumption as before we use Eq.(3.15) and obtain

$$\left\langle |F|^2 \right\rangle = \left(\frac{\gamma a + 1}{2} \phi_{\rm rms}\right)^2 \sum_n \left[1 + 2 \sum_{j=1}^{\infty} \cos\left(\frac{2\pi\gamma a}{N_k} j\right) C_j \right]. \tag{3.29}$$

By estimating the quantity in the square brackets to be about 2 as before, we get

$$|\delta \mathbf{n}|_{\text{rms}} \sim \frac{1}{\sqrt{2}}(\gamma a + 1)\sqrt{N_k}\phi_{\text{rms}} \sim \sqrt{2\pi \Delta\nu_{2,\text{rms}}}. \tag{3.30}$$

The last line comes from Eq.(3.19). Thus, we find, also in the corrected orbit case, that if $\Delta\nu$ is small, so is $\delta\mathbf{n}$. The depolarization due to the tilt of $\mathbf{n}$ is $|\delta\mathbf{n}|^2/2 \sim \pi\Delta\nu_{2,\text{rms}}$.

IV. BETATRON OSCILLATION

In this section we discuss the depolarization due to betatron oscillation. The effects resemble those of the uncorrected c.o.d. in that they show the local resonance when the sum of or the difference between the spin precession tune per snake period $\gamma a/M$ and the betatron tune ν_y/M becomes an integer.

TUNE SHIFT

Obviously, the first order correction $\Delta\nu_1$ vanishes by the average over θ. Let us consider $\Delta\nu_2$. We can decompose y''_β into two parts;

$$y''_\beta(\theta) = Y^{(+)}(\theta) + Y^{(-)}(\theta) \tag{4.1}$$

so that $Y^{(-)}$ is the complex conjugate of $Y^{(+)}$ and $Y^{(\pm)}$ has the quasi-periodicity

$$Y^{(\pm)}(\theta + 2\pi) = e^{\pm 2\pi i \nu_y} Y^{(\pm)}(\theta) \tag{4.2}$$

where ν_y is the vertical betatron tune. Then, using the quasi-periodicity of $\mathbf{k}_0$, Eq.(2.3), we get

$$\int_{-\infty}^{\theta} h(\theta) y''_\beta d\theta = \sum_{\pm} \frac{1}{e^{2\pi i(\pm\nu_y - \nu_0)} - 1} \int_{\theta}^{\theta+2\pi} h(\theta) Y^{(\pm)} d\theta. \tag{4.3}$$

By the average over θ the terms $Y^{(+)}Y^{(+)}$ and $Y^{(-)}Y^{(-)}$ vanish. The remaining terms, $Y^{(\mp)}Y^{(\pm)}$, are periodic in θ. Thus, we obtain

$$\Delta\nu_2 = \frac{1}{2}\sum_{\pm} \mathcal{Im}\left[\frac{R^2(\gamma a + 1)^2}{e^{2\pi i(\pm\nu_y - \nu_0)} - 1}\frac{1}{2\pi}\int_0^{2\pi} d\theta Y^{(\mp)} h^*(\theta) \int_{\theta}^{\theta+2\pi} Y^{(\pm)} h(\theta') d\theta'\right]. \tag{4.4}$$

As an example, if the ring is planar, i.e., $h = e^{-i\gamma a\theta}$, and consists of N_c identical FODO cells with length L_c, this formula gives

$$\Delta\nu_2 = \frac{WN_c(\gamma a + 1)^2}{\pi L_c} \frac{\sin(\mu_{cy}/2)\sin(\mu_c/2)}{\cos(\mu_{cy}/2)} \frac{\cos(\mu_c/2) - \cos^2(\mu_{cy}/2)}{\sin^2(\mu_c/2) - \sin^2(\mu_{cy}/2)}. \tag{4.5}$$

Here, W is the Courant-Snyder invariant of the betatron oscillation, $\mu_{cy} = 2\pi\nu_y/N_c$ is the phase advance per cell and $\mu_c = 2\pi\gamma a/N_c$. The focusing and defocusing quads (thin lens) are assumed to have equal strength and the focusing effect of bends is ignored. Near a resonance $\nu_0 = \pm\nu_y + k$, k being an integer, the integrand of the r.h.s. of Eq.(4.3) is nearly periodic. It determines the resonance strength

$$\epsilon_\kappa = R(\gamma a + 1)\frac{1}{2\pi}\int_0^{2\pi}\left[Y^{(\pm)}\mathbf{k}_0 \cdot \mathbf{e}_x\right]_{\nu_0=\kappa=\pm\nu_y+k} d\theta. \tag{4.6}$$

The machine configuration stated above gives

$$|\epsilon_\kappa| = \frac{\gamma a + 1}{\pi} N_c \sqrt{\frac{W}{L_c}} \sin(\mu_{cy}/2) \frac{\sqrt{1 + \sin(\mu_{cy}/2)} + \sqrt{1 - \sin(\mu_{cy}/2)}}{\sin \mu_{cy}} \tag{4.7}$$

As an example, let us take a machine with N_c=330, L_c=200m and μ_{cy}=60 degrees. We get the resonance strength

$$|\epsilon_\kappa| = 2.1\sqrt{\frac{E}{20\text{TeV}}}\sqrt{\frac{\gamma W}{10^{-6}\text{rad.m}}}, \tag{4.8}$$

where γW is the normalized Courant-Snyder invariant. This is about the same as the one estimated by E. D. Courant *et al* [3] for a more realistic model of the SSC.

The integral in Eq.(4.4) can be expressed by this parameter near the resonance. Then, we have

$$\Delta\nu_2 = \frac{1}{2}\frac{|\epsilon_k|^2}{\nu_0 \mp \nu_y - k}. \tag{4.9}$$

Since we are using a perturbation expansion, these formulae do not apply in the close vicinity of resonances.

The formula (4.5) was checked by a computer simulation for a planar ring by tracking a spin at fixed energy and by using Fourier transformation to find the spin tune. The agreement was excellent.

For rings equipped with Siberian Snakes, $\Delta\nu_2$ is zero if the following conditions are satisfied.

(1) unperturbed tune ν_0 is 1/2.

(2) betatron optic has the mirror symmetry so that $Y^{(\pm)}(-\theta) = Y^{(\mp)}(\theta)$.

(3) with respect to this symmetry point, $\mathbf{k}_0 \cdot \mathbf{e}_x = h$ is symmetric, i.e., $h(-\theta) = h(\theta)$, which is satisfied in our model ring. (Note that h cannot be an odd function since $|h| = 1$.)

Let us prove this. Using (1) and the quasi-periodicity of $Y^{(\pm)}$ and h, one can change the integration range $(\theta, \theta + 2\pi)$ in Eq.(4.4) to $(0, 2\pi)$ as

$$\Delta\nu_2 = \frac{R^2(\gamma a+1)^2}{8\pi \cos \pi\nu_y} \mathcal{I}m \sum_{\pm} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \left(Y^{(\mp)}h^*\right)_\theta \left(Y^{(\pm)}h\right)_{\theta'} \mathrm{sgn}(\theta-\theta') e^{\pm\pi i\nu_y \mathrm{sgn}(\theta-\theta')}. \tag{4.10}$$

If one replaces θ and θ' with $2\pi - \theta$ and $2\pi - \theta'$, respectively, the integrand becomes

$$\left(Y^{(\mp)}h^*\right)_{2\pi-\theta} \left(Y^{(\pm)}h\right)_{2\pi-\theta'} \mathrm{sgn}(\theta'-\theta) e^{\pm\pi i\nu_y \mathrm{sgn}(\theta'-\theta)}.$$

Using

$$Y^{(\pm)}(2\pi-\theta) = e^{\pm 2\pi i\nu_y} Y^{(\pm)}(-\theta) = e^{\pm 2\pi i\nu_y} Y^{(\mp)}(\theta)$$

and $h(2\pi-\theta) = -h(-\theta) = -h(\theta)$, one can rewrite the integrand as

$$\left(Y^{(\pm)}h^*\right)_\theta \left(Y^{(\mp)}h\right)_{\theta'} \mathrm{sgn}(\theta'-\theta) e^{\pm\pi i\nu_y \mathrm{sgn}(\theta'-\theta)}.$$

By exchanging $\pm$ and $\mp$, one finds the same expression as Eq.(4.10) but the overall sign is changed. Thus, we have $\Delta\nu_2 = 0$.

This fact was confirmed by a computer simulation for the above simple model, using Fourier transformation to find the spin tune. An interesting fact is that $\Delta\nu = 0$ seems to hold exactly not only to the second order of perturbation. Even when the amplitude of betatron oscillation is so large that spectrum lines of very high orders are seen, all the lines can be identified either as integer$\times\nu_y$ or $1/2 -$ integer $\times\ \nu_y$. The proof of vanishing $\Delta\nu$ to arbitrary orders may not be easy but we can conclude that the spin tune shift due to the betatron oscillation is very small. Therefore, there is no real resonance.

SPIN PRECESSION AXIS

Next, consider the tilt of the precession axis $\delta\mathbf{n} = \mathbf{n} - \mathbf{n}_0$ due to the betatron oscillation. By reducing the integration in Eq.(2.23) to a finite interval, we get

$$\delta\mathbf{n}(\theta) = \mathcal{I}m\, \mathbf{k}_0^* F(\theta) \tag{4.11}$$

with

$$F(\theta) = -\sum_{\pm} \frac{R(\gamma a+1)}{e^{2\pi i(\pm\nu_y - \nu_0)} - 1} \int_\theta^{\theta+2\pi} Y^{(\pm)}(\theta) h(\theta) d\theta. \tag{4.12}$$

$|\delta\mathbf{n}| = |F|$ gives the magnitude of the tilt. For a planar machine near a resonance, the function F is closely related to the resonance strength defined in Eq.(4.6) as

$$|F| = \left|\frac{\pi\epsilon_K}{\sin\pi(\nu_0 \pm \nu_y)}\right| \qquad \text{(planar ring).} \tag{4.13}$$

For rings with Siberian snakes, F is usually small except at the local resonance, where the precession due to the betatron oscillation accumulate within a half snake period. If the ring consists of N_c identical FODO cells, the local resonance occurs when $(\gamma a \pm \nu_y)/N_c$ is an integer. If γa is far from local resonances, the typical value of F is

$$|F| \sim (\gamma a + 1) q \sqrt{W \beta_y} \tag{4.14}$$

which is the precession in one quadrupole magnet. Here, q is the inverse focal length, W the Courant-Snyder invariant. An absolutely necessary condition to have high polarization is that this precession angle be much smaller than one. But this does not impose a severe condition in practice.

At a local resonance $\gamma a \sim \pm\nu_y + kM \equiv \kappa$, we can estimate F by the same method which we have used in deriving Eq.(3.25) for the uncorrected c.o.d. By the approximation

$$y''_\beta \sim Y_\kappa e^{i\kappa\theta} + Y^*_\kappa e^{-i\kappa\theta} \tag{4.15}$$

and by ignoring rapidly oscillating terms, we obtain

$$F(\theta_j) = \frac{\pi}{M} \frac{|\epsilon_\kappa|}{\cos(\kappa\pi/M)} \cos(\arg\epsilon_\kappa + \frac{j\pi}{2} + j\frac{\kappa\pi}{M}) \times \left\{ \begin{array}{ll} e^{-i\kappa\pi/M} & j\text{=even} \\ e^{i\pi/2} & j\text{=odd} \end{array} \right\}, \tag{4.16}$$

$$(\epsilon_\kappa = R(\gamma a + 1) Y_\kappa)$$

which is quite similar to Eq.(3.25) except for the extra denominator $\cos(\kappa\pi/M) = \pm\cos(\nu_y\pi/M)$. The physical meaning of this factor is that, if the betatron tune in one snake period, ν_y/M, is nearly one half, the effect of the next period adds up because the snake causes another one half rotation. The factor can never be zero unless the fractional part of the betatron tune is 1/2, which means a real spin-orbit resonance. Nevertheless, it can still be very small for some combinations of ν_y and M. Therefore, the improvement of polarization by increasing the number of snakes is not monotonic. (In particular, this fact must be kept in mind for computer simulation.) However, by carefully choosing ν_y and M, it is possible to make $\cos(\kappa\pi/M)$ close to unity. Thus, if

$$\left| \frac{\pi\epsilon_\kappa}{M} \right| \ll 1 \tag{4.17}$$

is satisfied, the depolarization due to the tilt of axis is small even at the local resonance.

However, this is not a necessary condition to have high polarization. Even if the l.h.s. of (4.17) is large, the polarization might be restored after crossing the local resonance. We shall derive a more relaxed condition later.

LOCAL RESONANCE CROSSING

So far, we have discussed the effects of the betatron oscillation at fixed energy. If there is no acceleration, the spin motion can be written in the form (2.22) with constant J. When the vector $\mathbf{n}$ is close to $\mathbf{n}_0$, i.e., $|\delta\mathbf{n}| \ll 1$, the spin can remain almost vertical in the presense of the betatron oscillation if it is initially.

J is no longer a constant of motion, however, when the beam is accelerated. We shall now discuss the change of J during the acceleration, in particular, during local resonance crossing. Basically, since J is an adiabatic invariant, we may expect that, if the acceleration is slow enough, J is almost conserved.

First, let us consider the case when the condition (4.17) is satisfied so that our perturbation expansion is valid. As is shown in Appendix B, the depolarization due to acceleration can be written as

$$\Delta P = \frac{1}{2}\left|\int_{-\infty}^{\infty} d\theta \mathbf{k}_0 \cdot \frac{\partial \delta\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}\right|^2 \tag{4.18}$$

if the acceleration is slow enough. Using (4.11) and the fact $\mathbf{k}_0 \cdot \mathbf{k}_0 = 0$ and $\mathbf{k}_0 \cdot \mathbf{k}_0^* = 2$, the integrand can be rewritten as

$$\mathbf{k}_0 \cdot \frac{\partial \delta\mathbf{n}}{\partial\gamma} = \mathbf{k}_0 \frac{\partial}{\partial\gamma} \mathcal{I}m\, \mathbf{k}_0^* F = \frac{1}{2i}\mathbf{k}_0 \frac{\partial}{\partial\gamma}(\mathbf{k}_0^* F) = -i\left(\frac{\partial F}{\partial\gamma} + \mathbf{k}_0 \cdot \frac{\partial \mathbf{k}_0^*}{\partial\gamma}F\right). \tag{4.19}$$

For our model ring, the vector $\mathbf{k}_0$ in Eq.(2.37) can be expanded in a Fourier series, owing to the periodicity (2.38), in the form

$$\mathbf{k}_0(\theta) = \sum_{n=-\infty}^{\infty} \hat{\mathbf{k}}_n e^{iM(n+1/2)\theta}. \tag{4.20}$$

The coefficient is found to be

$$\hat{\mathbf{k}}_n = \frac{M}{2\pi}\sum_{\pm}(\pm i\mathbf{e}_x - \mathbf{e}_z)\frac{e^{-i\pi[\gamma a \pm M(n+1/2)]/M} - 1}{\gamma a \pm M(n+\frac{1}{2})}, \tag{4.21}$$

which is analytic w.r.t. γa.

Putting all together we find that the integral in Eq.(4.21) can be written as

$$\sum_n \int_{-\infty}^{\infty} d\theta f_{n,\pm}(\gamma a) e^{i(\pm\nu_y + M(n+1/2))\theta}, \tag{4.22}$$

where the function $f_{n,\pm}$ can be written in the form

$$(\text{rational functionof } \gamma a)e^{-i\pi\gamma a/M} + (\text{rational function of } \gamma a)$$

and is analytic in the entire complex γa plane, i.e., the singularities of these rational functions cancel each other. Let us assume a constant acceleration $\gamma a = (\gamma a)_0 + \alpha\theta/2\pi$, where α is the increment of γa per revolution. For each term of Eq.(4.22) we close the integration path by a large upper (lower) hemicircle in the complex θ-plane if $\pm\nu_y + M(n+\frac{1}{2})$ is positive (negative). Then, if

$$\left|\frac{\alpha}{2M}\right| < \left|\pm\nu_y + M(n+\frac{1}{2})\right|, \tag{4.23}$$

the contribution of the hemicircle vanishes and, since the integrand is free from singularities, the integral vanishes. If (4.23) holds for every n, or equivalently,

$$\alpha < 2M^2\delta\nu_y \tag{4.24}$$

where $\delta\nu_y$ is the distance between ν_y/M and the nearest half odd integer, the depolarization will be negligible.

This condition is in practice well satisfied even if $M = 1$, unless ν_y/M is very close to a half integer. Thus, as long as the perturbation is small so that (4.17) is satisfied, there is no depolarization due to acceleration.

However, this conclusion was derived within the framework of the perturbation theory. We need a non-perturbative theory when (4.17) does not hold, in particular near non-linear resonances.

Now, let us try a non-perturbative approach with a help of computer simulation, for which we use spinor formalism. The same approach has already been developed by S. Y. Lee and S. Tepikian.[4] The equation of motion of the spin component $s_j = \mathbf{s}\cdot\mathbf{n}_{0j}$ is given by

$$\frac{ds_j}{d\theta} = \sum_{kl}\epsilon_{jkl}(\delta\mathbf{\Omega}\cdot\mathbf{n}_{0k})s_l, \tag{4.25}$$

ϵ_{jkl} being the completely anti-symmetric tensor. This is equivalent to the equation for a two-component spinor Ψ

$$\frac{d\Psi}{d\theta} = -\frac{i}{2}\sum_j(\delta\mathbf{\Omega}\cdot\mathbf{n}_{0j})\sigma_j\Psi \tag{4.26}$$

with

$$s_j = \Psi^*\sigma_j\Psi, \tag{4.27}$$

σ_j being the Pauli matrices. Near a local resonance we may approximate

$$\delta\mathbf{\Omega} = -R(\gamma a+1)y''\mathbf{e}_x = (\epsilon_\kappa e^{i\kappa\theta} + \epsilon_\kappa^* e^{-i\kappa\theta})\mathbf{e}_x. \tag{4.28}$$

Using the explicit expression of $\mathbf{k}_0 = \mathbf{n}_{01} + i\mathbf{n}_{02}$ in Eq.(2.37) we obtain the spin transfer matrix for one snake period;

$$\Psi(\theta_2) = e^{-\frac{i}{2}(-\Delta\sigma_3 + \epsilon_\kappa e^{i\kappa\theta_2}\sigma_+ + \text{h.c.})\theta_1} e^{-\frac{i}{2}(\Delta\sigma_3 + \epsilon_\kappa^*\sigma_+ + \text{h.c.})\theta_1}\Psi(0) \tag{4.29}$$

Here, $\Delta = \gamma a - \kappa$, $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ and h.c. denotes the Hermitian conjugate of σ_+ term. For the next snake period one has to use $\epsilon_\kappa e^{i\kappa\theta_2}$ instead of ϵ_κ to take into account the betatron phase advance.

In the above expression the basis $(\mathcal{R}e\,\mathbf{k}_0, \mathcal{I}m\,\mathbf{k}_0, \mathbf{n}_0)$ is used. One has to multiply it by the rotation around the third axis by 180 degrees, $i\sigma_3$, for using the periodic basis $(\mathcal{R}e\,\tilde{\mathbf{k}}_0,\ \mathcal{I}m\,\tilde{\mathbf{k}}_0,\ \mathbf{n}_0)$.

A computer simulation was done using this transfer matrix. Initially the spin points to the third axis and Δ is a large negative value (typically -20). Every revolution Δ advances by α and becomes a large positive value (typically 20) finally. Several particles are tracked with different initial betatron phase but with the same amplitude and the final polarization is obtained by averaging them. There are four parameters, namely the number of snake pairs M, the betatron tune ν_y, the resonance strength $|\epsilon_\kappa|$ and the acceleration rate α, but they scales w.r.t. M so that the independent parameters are ν_y/M, $|\epsilon_\kappa|/M$ and α/M^2. Thus, simulations for $M = 1$ suffice.

Figure 4 shows the result of the simulation schematically in $(\nu_y,\ |\epsilon_\kappa|)$ plane for fixed acceleration rate α. The shaded area is the depolarizing region. One finds that around $|\epsilon_\kappa| = 1$ the beam is totally depolarized regardless of the tune. Larger ϵ_κ is even better but it cannot be accepted in practice because the resonance strengths are different from resonace to resonance in real rings. The significance of the point $|\epsilon_\kappa| = 1$ has been pointed out by J. Buon.[5]

Figure 5 shows the final polarization as a function of the tune for seven values of $|\epsilon_\kappa|$, 0.4, 0.5, 0.6, 0.8, 1.0, 1.5 and 2.0. All the resonances are identified as $\nu_y/M = 1/2 - \text{integer}/(2l+1)$, or equivalently

$$\text{spin tune per snake period} = \frac{1}{2} = n \pm (2l+1)\frac{\nu_y}{M}, \tag{4.30}$$

n and l being integers. Figures 6(a) and 6(b) shows the relation between the final polarization and the acceleration rate. In Fig.6(a) $\nu_y = 0.213$ and $|\epsilon_\kappa| = 0.2$. The vertical line is the onset of adiabaticity breaking predicted by the perturbation theory Eq.(4.24). The slight depolarization below it is presumably caused by stepwise acceleration in the simulation. In Fig.6(b) a resonating value of tune $\nu_y = 1/6$ is chosen, where the perturbation theory fails in spite of small value of $|\epsilon_\kappa| = 0.1$. One sees that a slower acceleration is even worse. The depolarization is proportional to $1/\alpha^2$. Physically speaking, if the system is just on resonance, any slow perturbation cannot be slow enough to be adiabatic, since the unperturbed system has an infinitely slow oscillation component. The phenomena found by J. Buon[5] was exactly this situation since he used $\nu_y = 0.3$ which is just on the 5-th order ($l = 2$) resonance.

In Figs.7(a) and 7(b) the final polarization is plotted as a function of the resonance strength. The parameters are $\nu_y = 1/6$ [Fig.7(a)], 0.180 [Fig.7(b)] and $\alpha = 0.02$.

One finds from Fig.5 that if the tune is carefully chosen, a relatively large value of $|\epsilon_\kappa|$, say 0.5, does not cause significant depolarization. Thus, we conclude, from the simulation, that if

$$\frac{|\epsilon_\kappa|}{M} \lesssim 0.5 \tag{4.31}$$

is satisfied the depolarization will be small. One may adopt a little larger value for the r.h.s. at some more increased risk but it must never exceed 1. For the SSC, using (4.8), we have $|\epsilon_\kappa| \sim 5$ for the normalized Courant-Snyder invariant 6π mm.mrad, which corresponds to 95 percent emittance. Thus, we need $M \gtrsim 11$ pairs of snakes to avoid the depolarization due to betatron oscillation.

V. SUMMARY

We have discussed the depolarization in a large, high-energy proton ring equipped with multiple Siberian snakes. Our main concern is the effects of machine imperfections after orbit correction and those of the betatron oscillation. They were studied separately and turned out to be very different from each other. The betatron oscillation causes strong resonances at the beam energy corresponding to $\gamma a = kM \pm \nu_y$, which can be cured by multiple Siberian snakes whereas the imperfection effects are equally important at every energy and cannot be reduced by increasing the number of snakes.

For the imperfection problem we employed the perturbative approach and found that the spin tune shift is given by Eq.(3.22), which leads to the tolerance of the misalignment of quads, the roll of bends and the orbit correction. For the SSC, we have 80μm, 200μrad and 80μm (r.m.s.), respectively. If the spin tune shift is small, the tilt of the precession axis is small too, and the acceleration rate is not very important.

For the betatron oscillation we derived the required number of snakes, mainly using computer simulations. The spin tune shift turned out to be unimportant. The results are Eq.(4.30) which shows the betatron tune to be avoided and Eq.(4.31) which gives the required number of snake pairs M, once the resonance strength is given. We got $M \gtrsim 11$ for the SSC.

There are still many problems to be discussed. For example, when evaluating the tolerances, we put the criterion $\Delta\nu_{\rm rms} \lesssim 0.05$ somewhat arbitrarily, thinking that the beam will be depolarized if the spin tune becomes an integer (or maybe even integer$\pm\nu_y$) even at several standard deviation. But what happens in such a case is nontrivial.

Another problem left open is the betatron oscillation in the presense of corrected closed-orbit distortion. We can now estimate the required number of snakes and the tolerances if we consider the betatron oscillation and the imperfection separately. Now the question is whether or not we get more severe numbers when two effects interfere.

Also, we have to study design of snakes which we treated here just as points.

For the SSC, the obtained tolerances seem to be quite marginal. Thus, the next thing to do is to refine our coarse estimation of tolerances and to consider the possible cures.

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APPENDIX A

In this appendix is discussed the statistical relation between the r.m.s. error kicks and the corrector strengths

$$N_C \phi^2_{C,rms} \sim N_{err} \phi^2_{err,rms} \tag{A.1}$$

which is used in Sec.3. Here, N_C is the number of correctors, $\phi_{C,rms}$ the r.m.s. kick angle of correctors and N_{err} and $\phi_{err,rms}$ are those of error kicks.

We use the following simplified model of a ring. There are many sources of error kicks so that they are almost continuously distributed over the ring. The smooth approximation of the beta function is valid and the correctors are located equidistantly. We determine the corrector strengths using the least squares method.

Let $\xi(\theta)$ be the curvature due to error field at θ. The closed orbit at θ due to this error is written as

$$y_{err}(\theta) = \int_0^{2\pi} M(\theta - \theta')\xi(\theta')Rd\theta', \tag{A.2}$$

where R is the average machine radius, and the response function $M(\theta)$, which is periodic in θ, is given by

$$\begin{aligned} M(\theta) &= \frac{R/\nu}{\sin \pi\nu} \cos(\nu\pi - \nu\theta) \qquad (0 < \theta < 2\pi) \\ &= \frac{R}{2\pi} \sum_{m=-\infty}^{\infty} \frac{1}{\nu^2 - m^2} e^{im\theta}. \end{aligned} \tag{A.3}$$

Let η_α $(\alpha = 0, 1, ..N_C)$ be the kick angle of the corrector located at $\theta = \theta_\alpha = 2\pi\alpha$. The closed orbit due to the correctors only is

$$y_{cor}(\theta) = \sum_{\alpha=0}^{N_C - 1} M(\theta - \theta_\alpha)\eta_\alpha. \tag{A.4}$$

Using the Fourier transform, we can write the normal equation to minimize $\int_0^{2\pi}(y_{err} + y_{cor})^2 d\theta$ in the form

$$\sum_{\beta=0}^{N_C - 1} A_{\alpha\beta}\eta_\beta = -\int_0^{2\pi} Rd\theta \left[\sum_{m=-\infty}^{\infty} \frac{e^{im(\theta_\alpha - \theta)}}{(\nu^2 - m^2)^2} \right] \xi(\theta) \tag{A.5}$$

where the matrix A is given by

$$A_{\alpha\beta} = \sum_{m=-\infty}^{\infty} \frac{e^{im(\theta_\alpha - \theta_\beta)}}{(\nu^2 - m^2)^2}. \tag{A.6}$$

One can easily see that the inverse of A can be written as

$$A^{-1}_{\alpha\beta} = \frac{1}{N_C^2} \sum_{n=0}^{N_C-1} \frac{1}{c_n} e^{in(\theta_\alpha - \theta_\beta)} \tag{A.7}$$

where

$$c_n = \sum_{k=-\infty}^{\infty} \frac{1}{[\nu_2 - (n + kN_C)^2]^2}. \tag{A.8}$$

Thus, the least square solution is

$$\eta_\alpha = -\frac{R}{N_C} \sum_{n=0}^{N_C-1} \sum_{k=-\infty}^{\infty} \int_0^{2\pi} d\theta \frac{1}{c_n} \frac{e^{in\theta_\alpha - i(n+kN_C)\theta}}{[\nu^2 - (n + kN_C)^2]^2}. \tag{A.9}$$

Now, let us estimate the square sum of η_α statistically, assuming that the error field $\xi(\theta)$ at different θ has no correlation;

$$\langle \xi(\theta)\xi(\theta') \rangle = C_\xi \delta(\theta - \theta') \tag{A.10}$$

where δ is the Dirac delta function. It can be shown that, if the number of error sources is finite but large, the coefficient C_ξ can be related to the r.m.s. error kick angle as

$$2\pi R^2 C_\xi = N_{\text{err}} \phi^2_{\text{err,rms}}. \tag{A.11}$$

After some manipulations using these expressions, we obtain

$$N_C \phi^2_{\text{C,rms}} \equiv \sum_{\alpha=0}^{N_C-1} \langle \eta_\alpha^2 \rangle = K(\nu, N_C) N_{\text{err}} \phi^2_{\text{err,rms}} \tag{A.12}$$

with

$$K(\nu, N_C) = \frac{1}{N_C} \sum_{n=0}^{N_C-1} \frac{\sum_{k=-\infty}^{\infty} 1/[\nu^2 - (n + kN_C)^2]^4}{\left(\sum_{k=-\infty}^{\infty} 1/[\nu^2 - (n + kN_C)^2]^2\right)^2}, \tag{A.13}$$

which does not have either zero or infinity as a function of ν since the singularities of the numerator are canceled by those in the denominator. $K(\nu, N_C)$ is actually a function of ν/N_C only, when N_C is large, say > 10, which is well satisfied except for a very small machine. It is plotted in Fig.8. Since, in practice, the ratio of tune to the number of correctors, ν/N_C, is smaller than 0.3, we conclude

$$N_C \phi^2_{\text{C,rms}} \sim 0.9 N_{\text{err}} \phi^2_{\text{err,rms}} \tag{A.14}$$

as seen from the figure.

APPENDIX B

We shall derive a general formula for the depolarization due to an adiabatic crossing of a resonance which applies to local resonances as well. Suppose that the system has a slowly varying parameter γ, which is not necessarily the beam energy so that the vectors $\mathbf{n}$ and $\mathbf{k}$ are functions of θ and γ. We assume that $\partial\mathbf{n}/\partial\gamma$ can be large only in some region of γ. We also assume that the depolarization is small.

Since $\mathbf{n}(\theta,\gamma)$ is a solution to the BMT equation (2.1) for constant γ, the equation of motion of J for changing γ is

$$\frac{dJ}{d\theta} = \frac{d}{d\theta}\mathbf{s}\cdot\mathbf{n}(\theta,\gamma) = \mathbf{s}\cdot\frac{\partial\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta} = \sqrt{1-J^2}\,\mathcal{R}e\left[e^{-i\psi}\tilde{\mathbf{k}}\cdot\frac{\partial\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}\right] \tag{B.1}$$

where we have used $\mathbf{n}\cdot\partial\mathbf{n}/\partial\gamma = 0$. Similarly, the equation for ψ reads

$$\frac{\partial\psi}{\partial\theta} = \nu - \frac{J}{\sqrt{1-J^2}}\,\mathcal{I}m\left[e^{-i\psi}\tilde{\mathbf{k}}\cdot\frac{\partial\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}\right]. \tag{B.2}$$

We have to solve these equations carefully, because they are singular at $J=1$. We can deduce from these equations

$$\frac{d}{d\theta}\sqrt{1-J^2}e^{i(\psi-\varphi(\theta))} = -J\,e^{-i\varphi(\theta)}\tilde{\mathbf{k}}\cdot\frac{\partial\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}, \tag{B.3}$$

where

$$\varphi(\theta) = \int_0^\theta \nu d\theta. \tag{B.4}$$

With the initial condition $J=1$ ($\theta\to-\infty$), we can approximately solve Eq.(B.3) as

$$\sqrt{1-J^2}e^{i\psi} = -e^{i\varphi(\theta)}\int_{-\infty}^{\theta} d\theta e^{-i\varphi(\theta)}\tilde{\mathbf{k}}\cdot\frac{\partial\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}, \tag{B.5}$$

assuming the change of J is small. Then the equation for J becomes

$$\frac{dJ}{d\theta} = -\,\mathcal{R}e\left\{e^{-i\varphi(\theta)}\tilde{\mathbf{k}}\cdot\frac{\partial\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}\int_{-\infty}^{\theta} d\theta'\left[e^{i\varphi(\theta)}\tilde{\mathbf{k}}^*\cdot\frac{\partial\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}\right]_{\theta'}\right\} \tag{B.6}$$

from which we get the depolarization in the form

$$\Delta P = \Delta J = \frac{1}{2}\left|\int_{-\infty}^{\infty} d\theta e^{-i\varphi(\theta)}\tilde{\mathbf{k}}\cdot\frac{\partial\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}\right|^2. \tag{B.7}$$

This can be applied, for example, to the well-known adiabatic spin-flip problem. In the present context of perturbation approach, we may replace ν by ν_0 and $\tilde{\mathbf{k}}$ by $\tilde{\mathbf{k}}_0$ so that $e^{-i\varphi}\tilde{\mathbf{k}} = \mathbf{k}_0$. Since $\partial\mathbf{n}_0/\partial\gamma = 0$, we obtain

$$\Delta P = \frac{1}{2}\left|\int_{-\infty}^{\infty} d\theta\mathbf{k}_0\cdot\frac{\partial\delta\mathbf{n}}{\partial\gamma}\frac{d\gamma}{d\theta}\right|^2. \tag{B.8}$$

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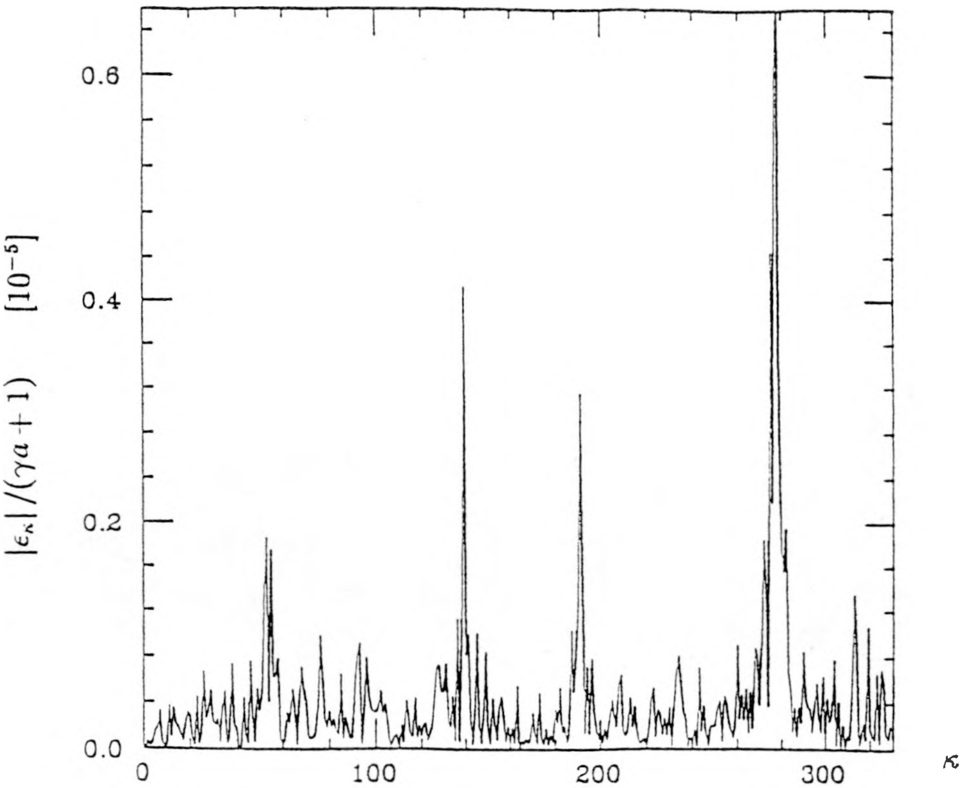


Fig.1a. Fourier spectrum of $y''_{\rm cod}$ without orbit correction.

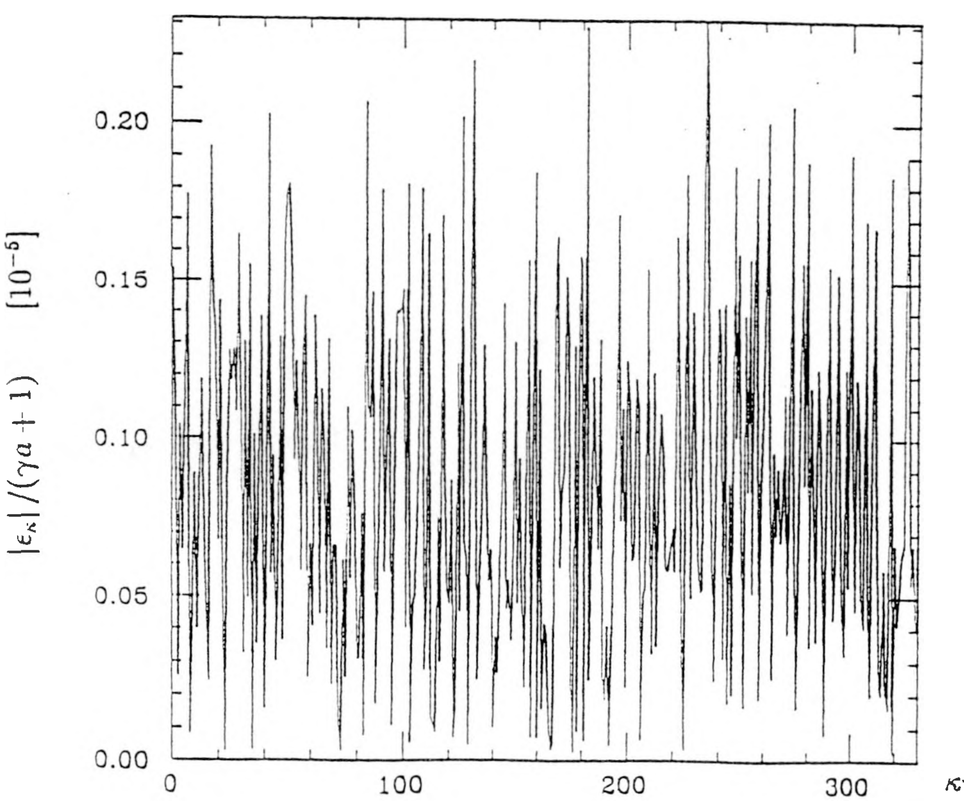


Fig.1b. Fourier spectrum of $y''_{\rm cod}$ with orbit correction.

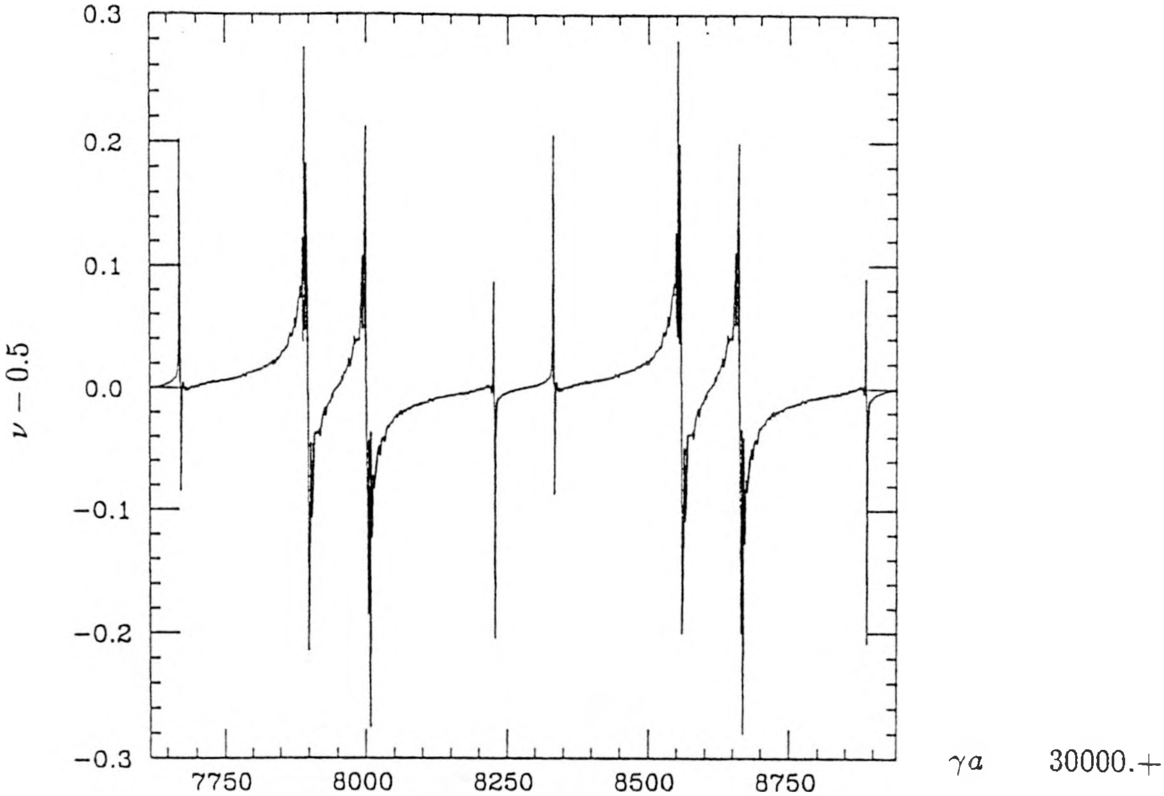


Fig.2a. Spin tune shift due to imperfection without correction. Quadrupole misalignment $\pm 10\mu$m (uniform).

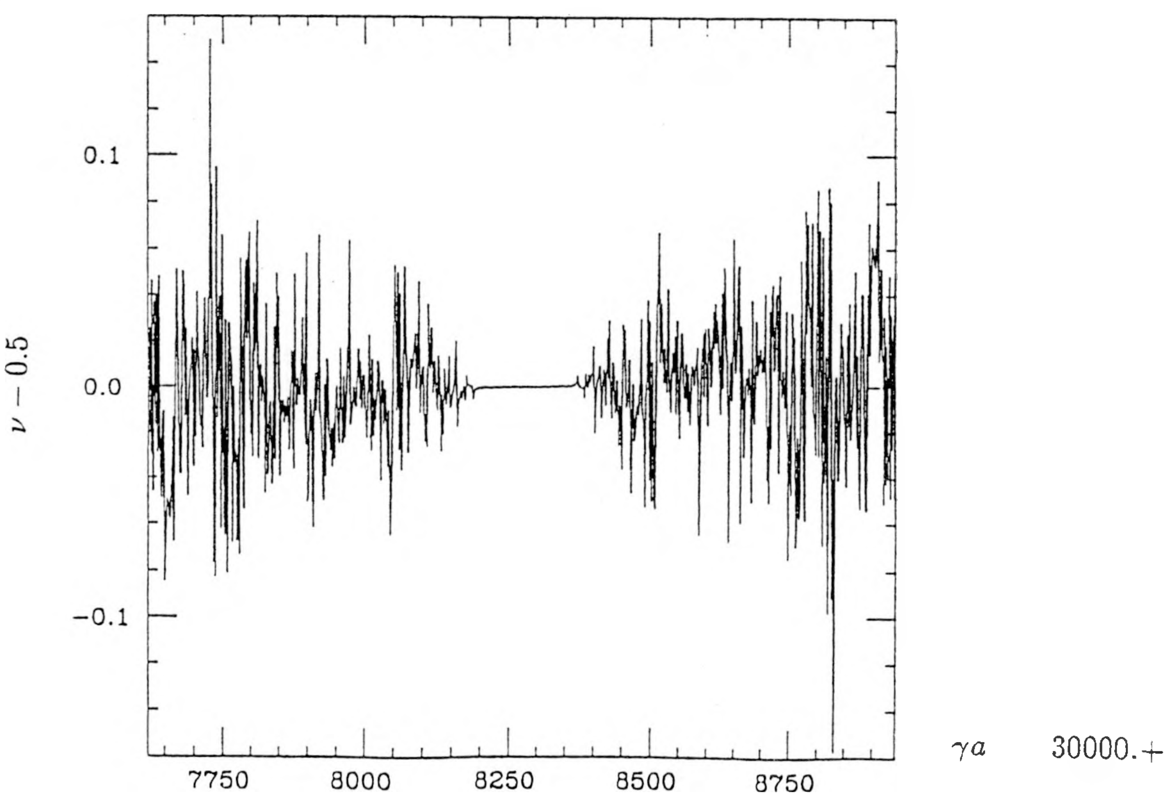


Fig.2b. Spin tune shift due to imperfection with correction. Quadrupole misalignment $\pm 100\mu$m (uniform) and residual $y_{\rm cod} = 25\mu$m.

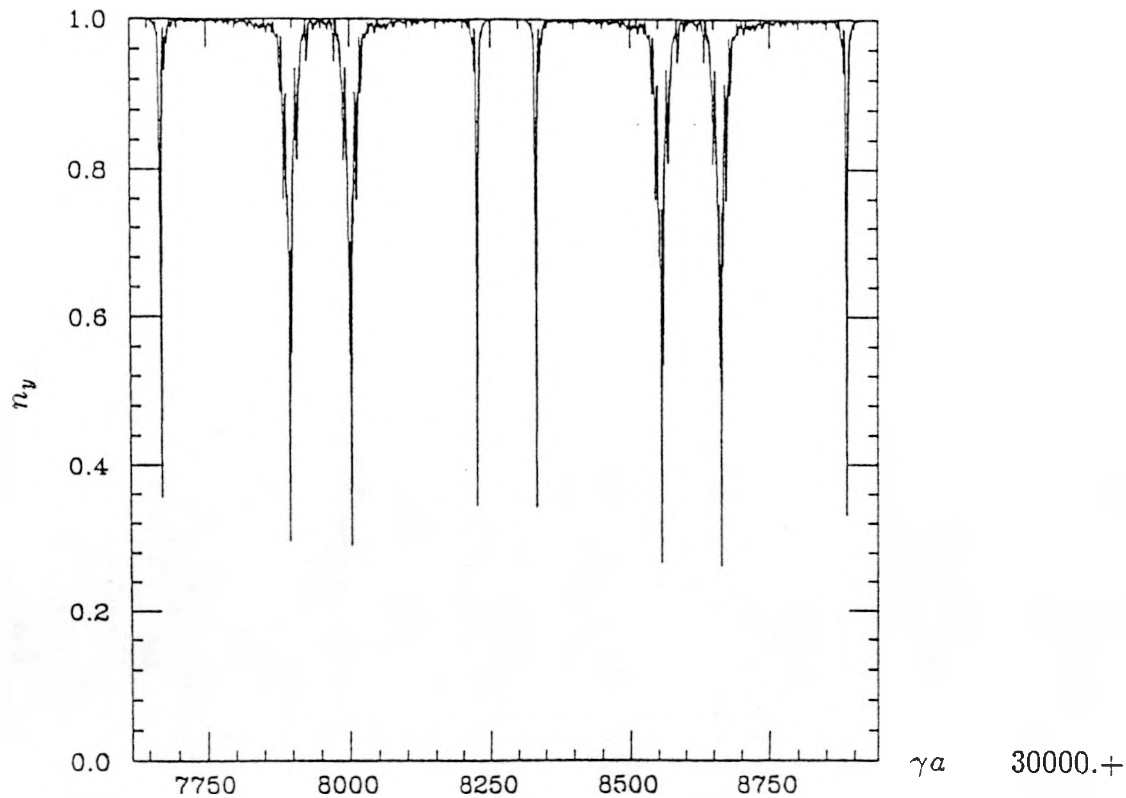


Fig.3a Vertical component of precession axis n_y without orbit correction.

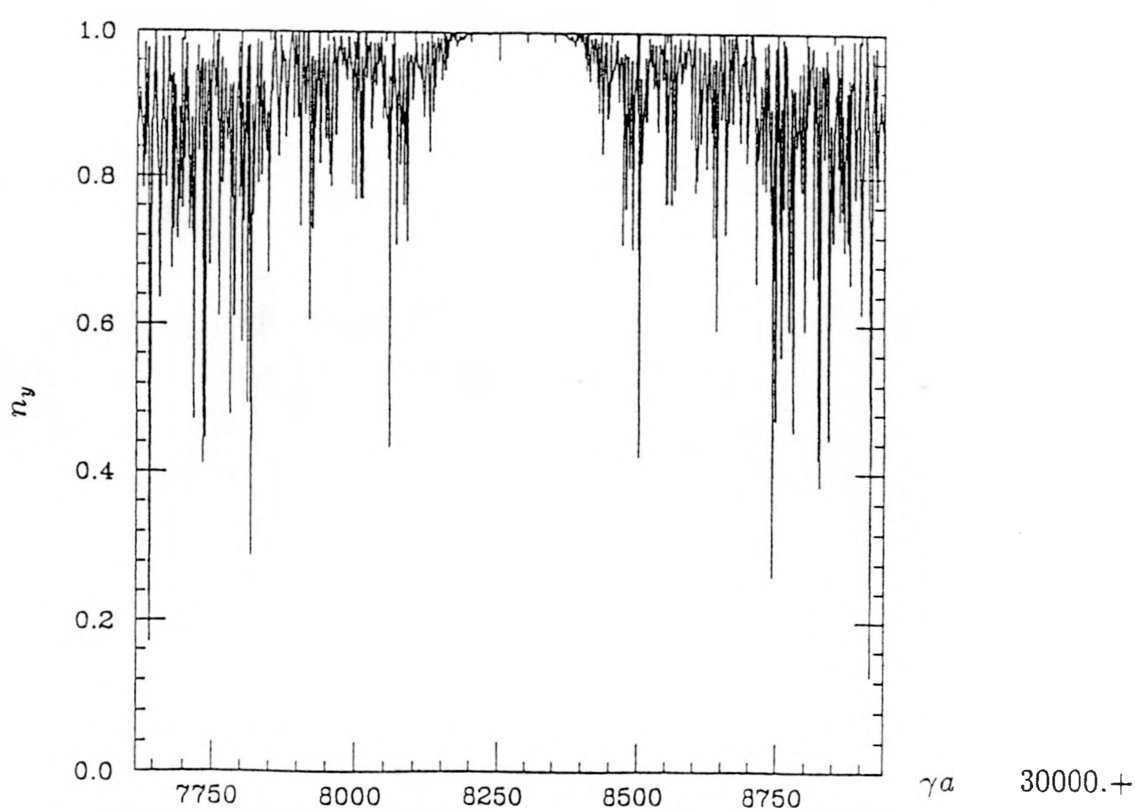


Fig.3b Vertical component of precession axis n_y with orbit correction.

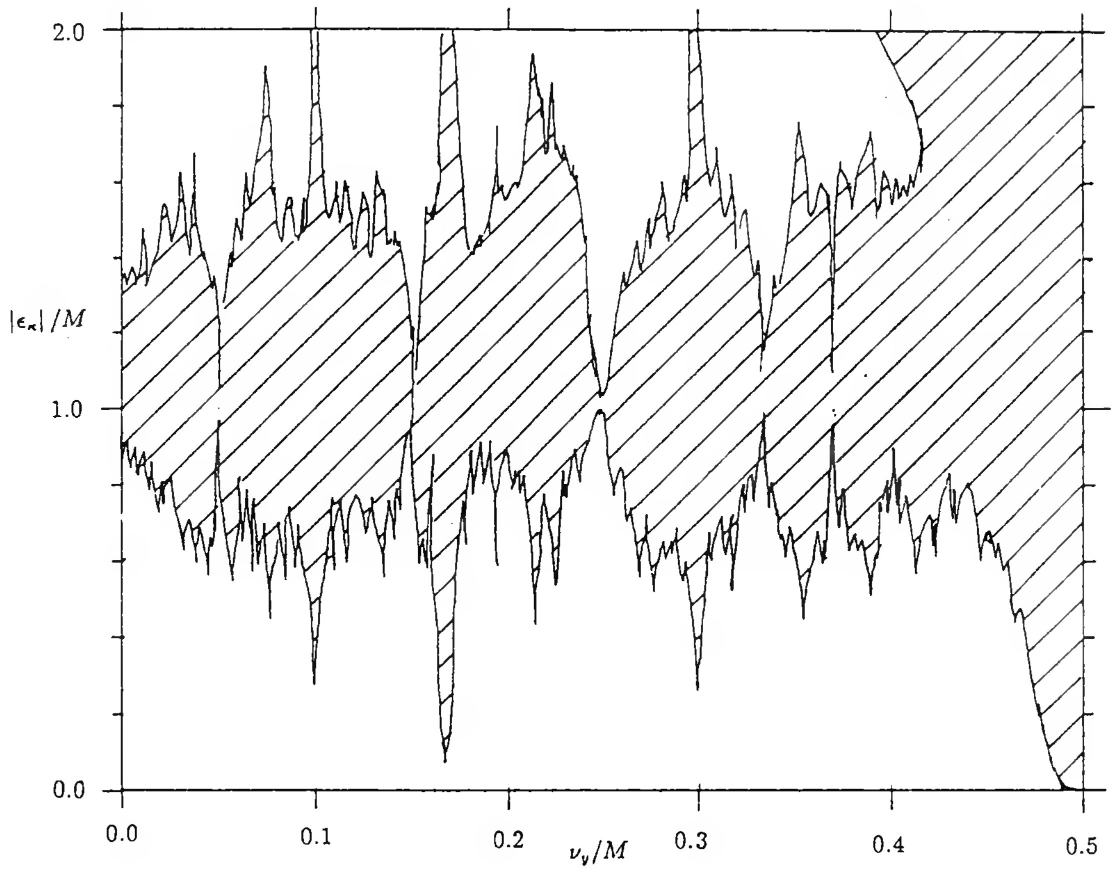


Fig.4. Schematic plot of depolarizing region due to betatron resonance.

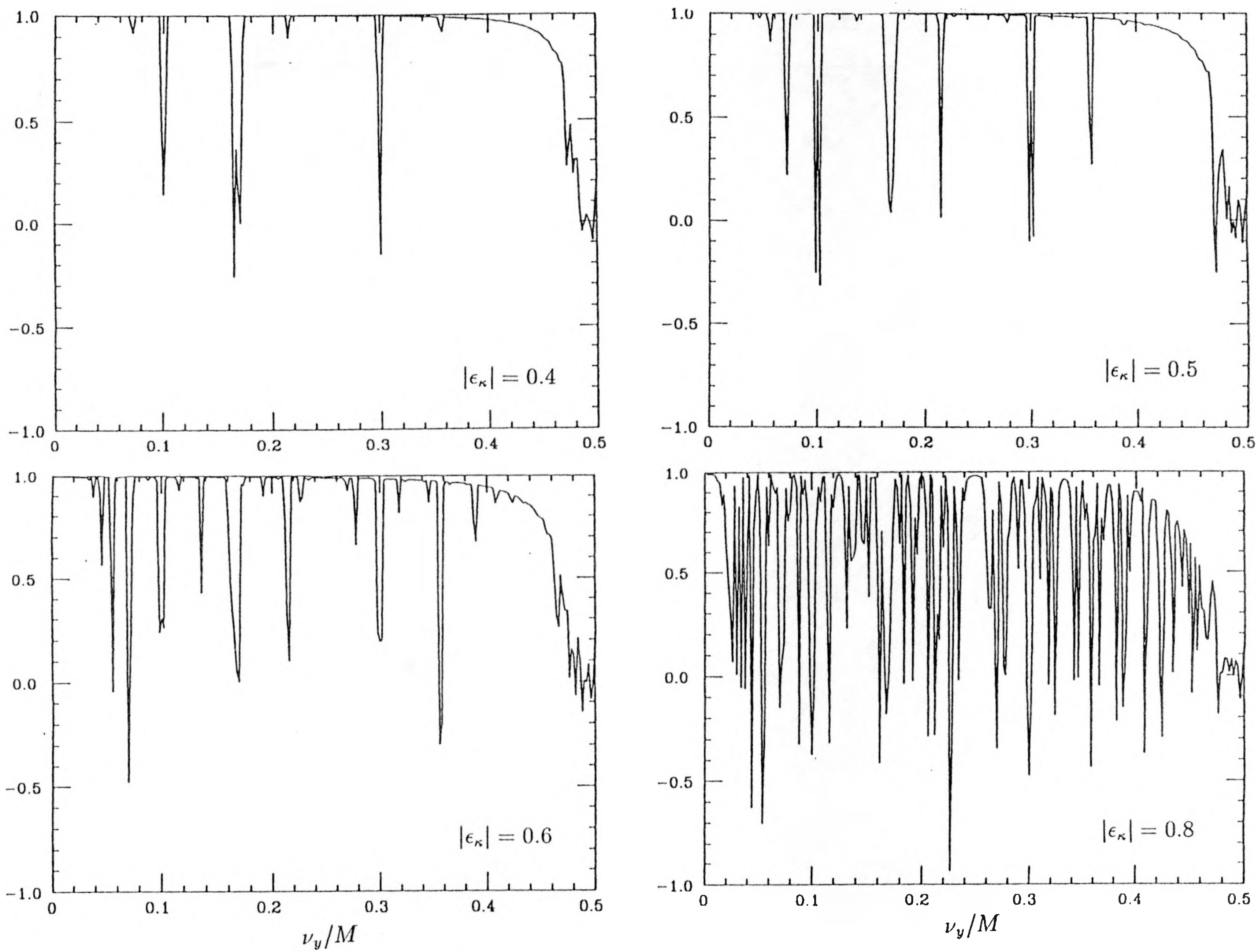
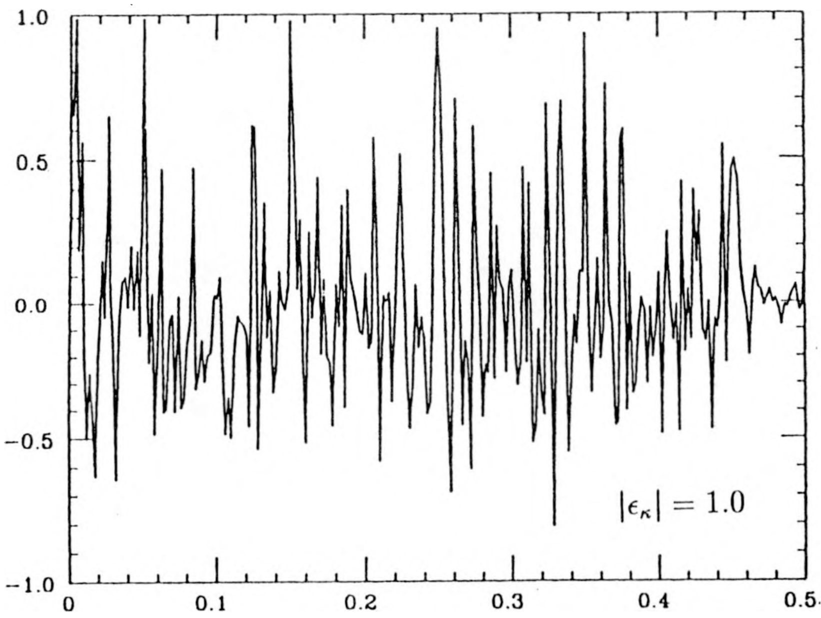
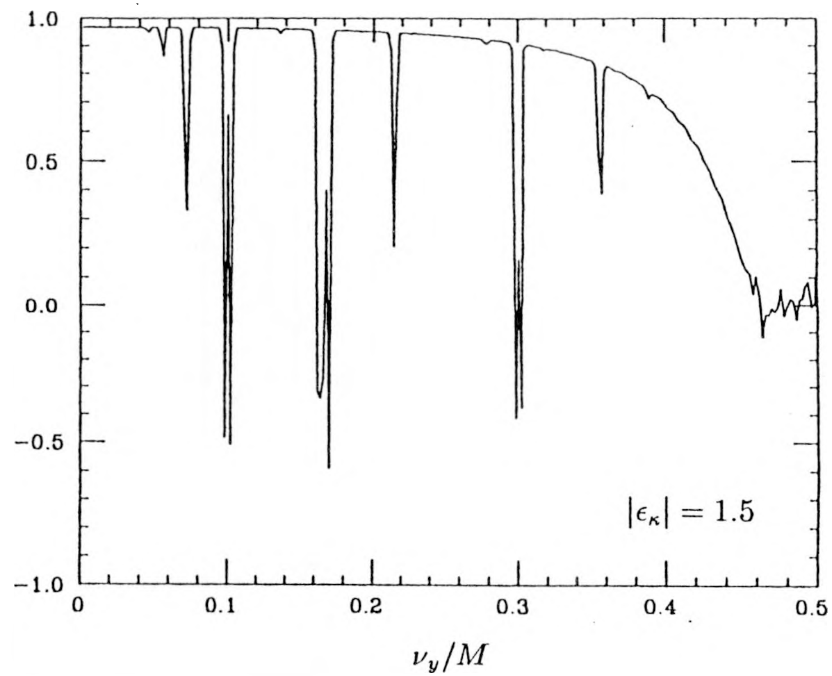


Fig.5. Polarization after passing betatron resonance as a function of the tune for various values of $|\epsilon_\kappa|$.





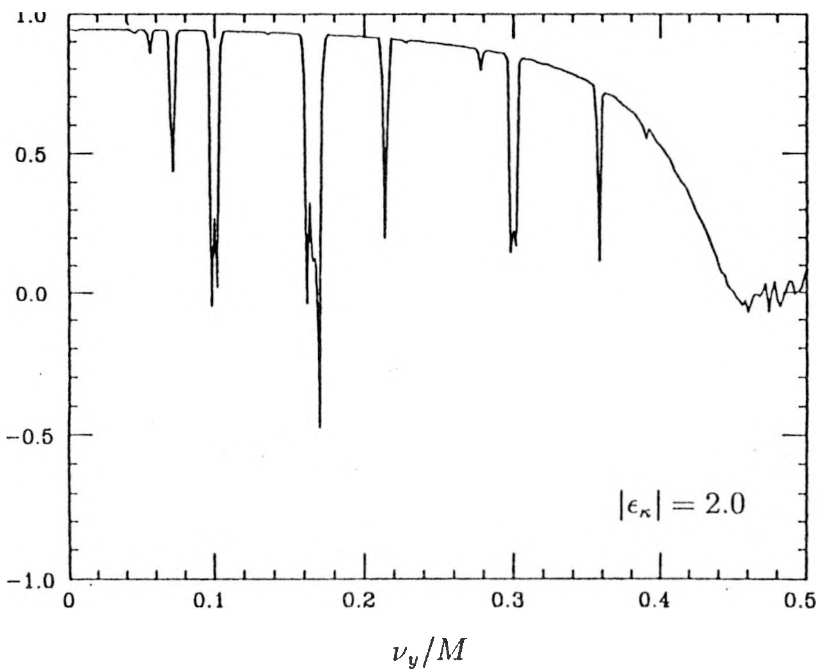


Fig.5 (continued.)

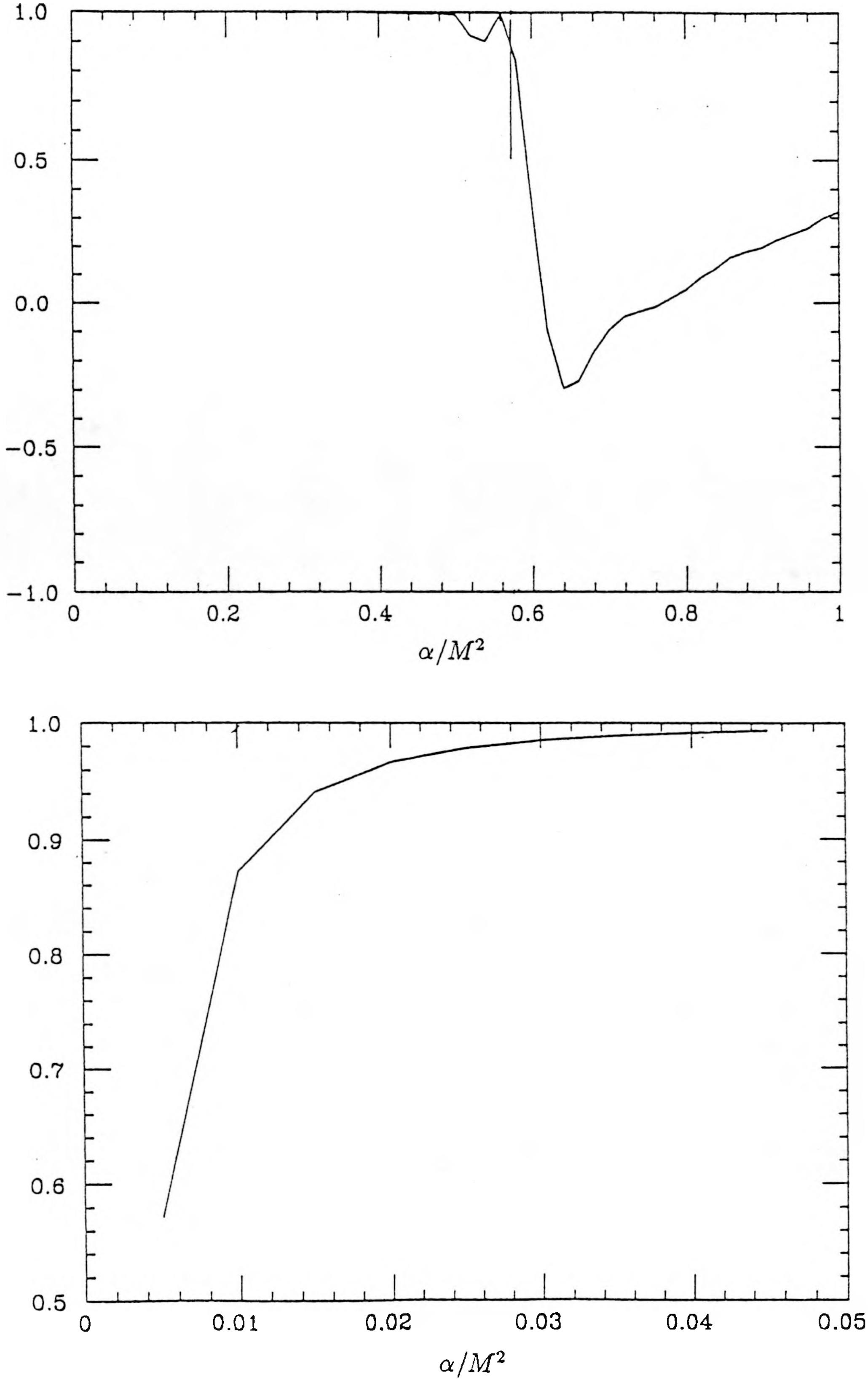


Fig.6. Final polarization as a function of the acceleration rate α/M^2

for (a) $|\epsilon_\kappa|/M = 0.2$, $\nu_y/M = 0.213$ and (b) $|\epsilon_\kappa|/M = 0.1$, $\nu_y/M = 1/6$.

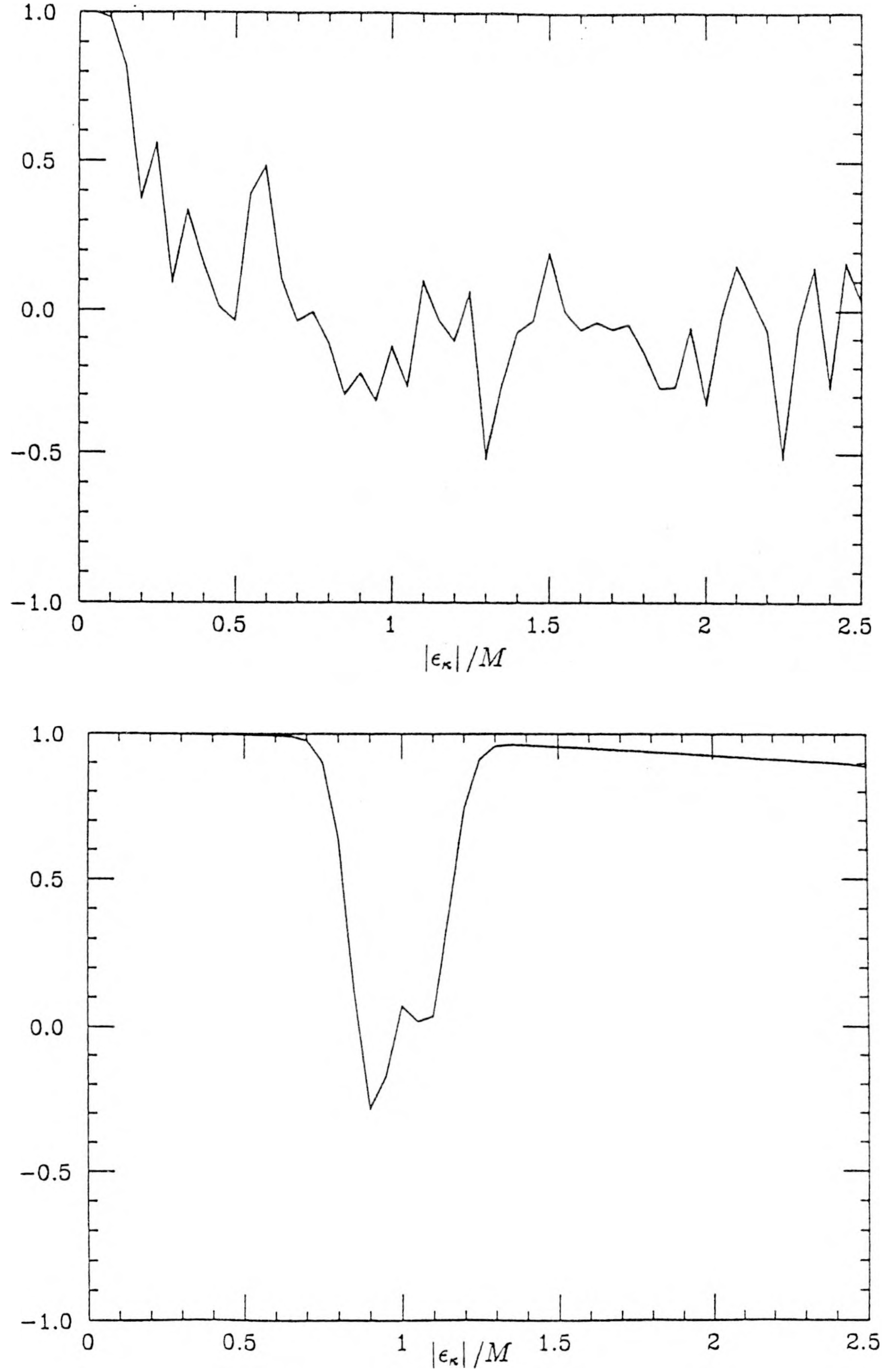


Fig.7. Final polarization as a function of resonance strength.

for (a) $\nu_y/M = 1/6$ and (b) $\nu_y/M = 0.180$.

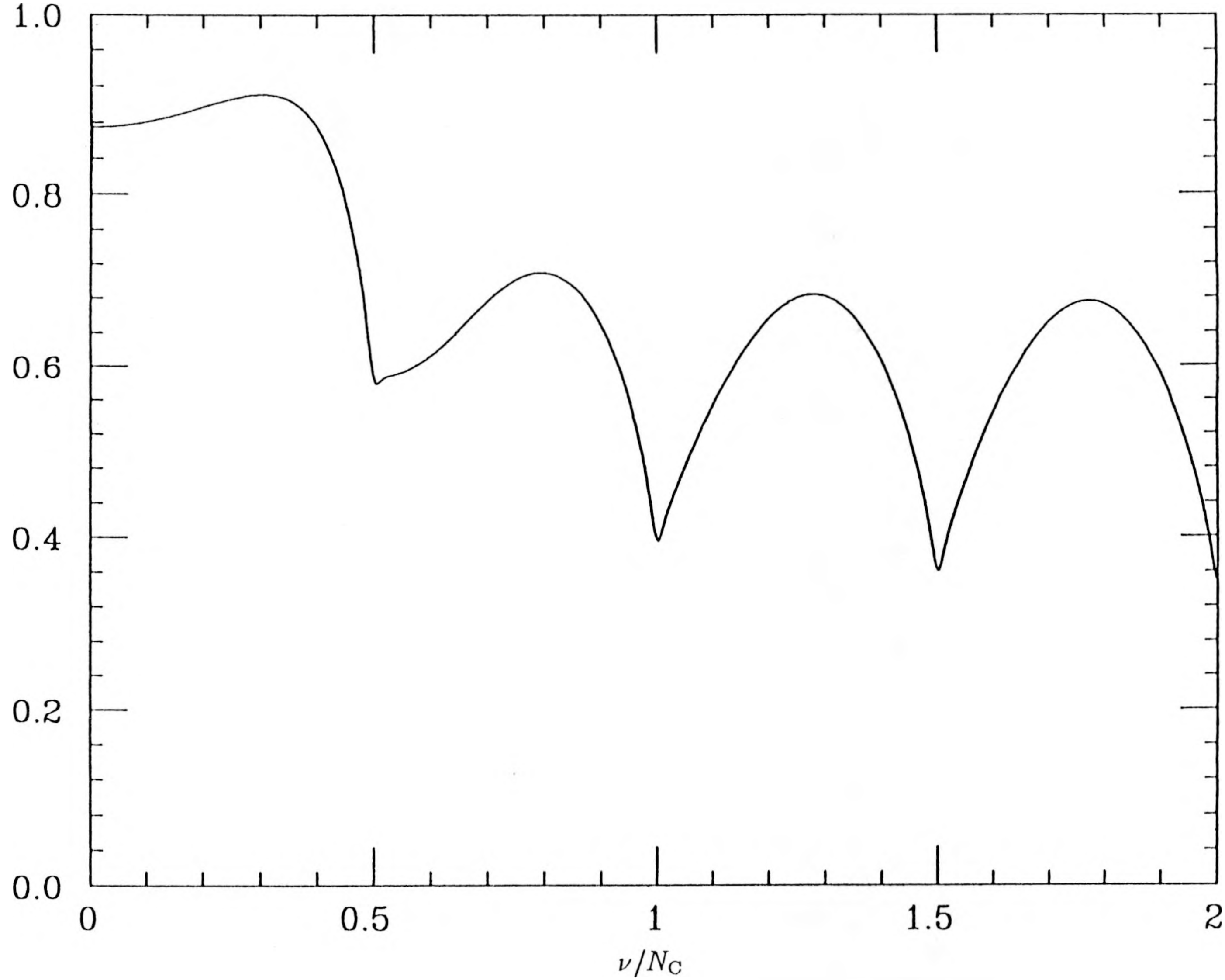


Fig.8 Function $K(\nu, N_C)$ defined in appendix A for $N_C = 50$.