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INCLUDING RESONANCES IN THE MULTIPERIPHERAL MODEL*

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Abstract

A simple generalization of the multiperipheral model (MPM) and the Mueller-Regge Model (MRM) is given which has improved phenomenological capabilities by explicitly incorporating resonance phenomena, and still is simple enough to be an important theoretical laboratory.

The model is discussed both with and without charge. In addition, the one channel, two channel, three channel and N channel cases are explicitly treated. Particular attention is paid to the constraints of charge conservation and positivity in the MRM. The recently proven equivalence between the MRM and MPM is extended to this model, and is used extensively.

I. INTRODUCTION

For many years the Chew-Pignotti multiperipheral model[1] (CPM) has been an important testing ground for various theoretical ideas. This is in spite of the fact that it is fundamentally at odds with the data.[2] Recently it has been proven that the N-channel multiperipheral model is totally equivalent to the N-channel Mueller-Regge model, and that one can derive either model from the other quite simply.[3]

We want to point out a simple generalization of the CPM and the equivalent Mueller Regge model which: 1) provides a generalization of CPM which has improved phenomenological capabilities, 2) still is simple enough to be an important theoretical laboratory, 3) adds additional phenomena that are expected on physical grounds.

This model has been known in gas dynamics for years as the δ-function interaction, but in high energy physics might justifiably be called a resonance model.[4] In the Veneziano model[5] resonances are dual to an infinite set of Regge trajectories. In this model the δ-function interaction, while not identical to either real resonance or Regge exchanges, gives effects similar to low mass resonances or to low lying Regge trajectories.

From the CPM point of view what this means is that one selects a subset of the infinite set of trajectories and daughters to be treated in the normal way and approximates the remaining trajectories by resonances (which we take here to be δ-functions.

In section II we discuss the simple scalar model. Section II contains most of the physics implications, without the complications of the more detailed models which are discussed in Sections III and IV. In Section III, we generalize the problem to N channels with k channels containing normal Regge poles and N-k channels containing δ -functions. In Section IV, a specific three channel example with charge is worked out in great detail. Whereas in Section II we start the discussion from the multiperipheral model, deriving the equivalent Mueller model in the end, in Section IV we start by contrast with the Mueller Model, and in the end find the equivalent multiperipheral model. This is instructive to the demonstration of the complete equivalence of the Mueller and multiperipheral models.

II. PHYSICAL IMPLICATIONS

A. One Channel Model

Consider the one dimensional one-channel CPM model, written in terms of the rapidity variables where $Y = \ln (s/M^2)$, and where the longitudinal momentum of the i^{th} produced particle is $p^i_{||} = \kappa_i \sinh y_i$ and $\kappa_i = (m_i^2 + \langle p_T^2 \rangle)^{\frac{1}{2}}$. Then the prong cross section is given by[6]

$$\sigma_{n+2}(Y) = g e^{-Y} \int_0^Y dy_1 \int_{y_1}^Y dy_2 \int_{y_2}^Y dy_3 \ldots \int_{y_{n-1}}^Y dy_n \tag{2.1}$$

$$K(y_1)\, K(y_2 - y_1) \ldots\ldots K(y_n - y_{n-1}) K(Y - y_n)$$

where $K(z) = g e^{z\beta}$ and $\beta = 2\,\alpha_{in} - 1$. This equation implies that

$$\sigma_n(Y) = e^{-Y} \int_0^Y dy\, K(Y-y)\, e^y\, \sigma_{n-1}(y). \tag{2.2}$$

Both of these integrals are diagonalized by Laplace transforms to

$$A_n(J) = g(K(J))^{n+1} \tag{2.3}$$

$$A_n(J) = K(J)\, A_{n-1}(J) \tag{2.4}$$

where $A_n(J)$ is the Laplace transform of $e^Y \sigma_{n+2}(Y)$

$$A_n(J) = \int_0^\infty dY\, e^{-(J-1)Y} \sigma_{n+2}(Y), \tag{2.5}$$

and K(J) is the Laplace transform of K(z),

$$K(J) = \frac{g}{J-\beta} \quad . \tag{2.6}$$

From summing either (2.3) or (2.4) over n one finds that[1]

$$\sigma_{tot}(Y) = g^2 e^{(\alpha_0 - 1)\, Y} \tag{2.7}$$

where $\alpha_0 = 2\alpha_{in} - 1 + g$. Another important result is that[1]

$$< n > = g\, Y \tag{2.8}$$

where $< n >$ is the average number of produced particles of all types. Experimentally it appears that $\frac{d< n >}{dY}$ is greater than 2.0.[7] Assuming that $\alpha_0 = 1$ for a constant total cross section this implies $\alpha_{in} < 0$. On the other hand it is generally believed that the set of trajectories at $J = 1/2$ must be important in the inelastic process. Since the above α_{in} represents an average position of important trajectories there must be in addition to the trajectory at 1/2 a range of J from 1/2 to -2 or -3 contributing.

Another problem of the CPM is that it predicts a Poisson multiplicity distribution for produced particles of all types. The experimental distribution, even neglecting diffraction effects, is broader. Again this is what one would expect if a range of trajectories contributed. The lower trajectories would contribute mainly to high multiplicities while the higher trajectories would contribute mainly to lower multiplicities. This agrees with the observation[7] that the low multiplicity prong cross sections are dropping approximately like $s^{-0.8}$, more like a trajectory at 1/2 than one below zero.

After this discussion it is obvious that a simple model which corrects the above problems must in some way take into account the effect of lower trajectories. To illustrate this point let us consider the problem where we add one lower trajectory. Although in general this would require more than one channel, we will simplify to the one channel case here. The kernel for such a model is

$$K(J) = \frac{g_1}{J-\beta_1} + \frac{g_2}{J-\beta_2} \tag{2.9}$$

with $\beta_1 \approx 0$ and $\beta_2 < 0$, (maybe $\beta_2 \approx -2$). A considerable simplification results by approximating the second term by a constant. Then

$$K(J) = \frac{g}{J-\beta} + \lambda \ . \tag{2.10}$$

This can be obtained from (2.9) by letting $g_2 = -\beta_2 \lambda$ and letting β_2 go to $-\infty$. Poles very low in the J plane contribute at the lowest subenergies and physically account for the <u>excess</u> of the resonance contribution not included in the first poles.[1] The (inverse) transform of eq. (2.10) is

$$K(x) = g e^{\beta x} + \lambda \delta (x) \tag{2.11}$$

showing that the total contribution of the λ-term comes at zero rapidity difference. Its presence in the kernel means that in the differential cross section, two particles will have a probability λ of having the same rapidity. Since it can act repeatedly 3 particles have a probability λ^2 of having the same rapidity, 4 particles have probability λ^3, etc. Obviously λ must be less than one.

We consider the sets of particles with identical rapidities to be idealized versions of real resonances. The probability of a "resonance" decaying into m particles is λ^{m-1}.

We of course do not expect groups of particles with identical rapidities, although the model as it is written implies that. One can imagine this as a resonance with zero Q-value or a resonance which decays purely transversely. In a more realistic treatment, one would include a decay distribution factor for each resonance. If we omit these decay distribution factors this model will still yield correct results for integrated quantities but for differential distributions it will be in error.

We now investigate the properties of a model which has a kernel of the form (2.10), (there is no particular reason but convenience for using the same g and λ in the end couplings)

$$A(z,J) = g z \sum_{n=0}^{\infty} (zK(J))^{n+1} = \frac{z^2 gK}{1-zK}$$

$$= \frac{z^2 g^2 + z^2 g \lambda (J-\beta)}{(J-\beta)(1-z\lambda) - zg} \qquad (2.12)$$

where we have included the "fugacity" parameter z.[8] The imaginary forward amplitude is obtained by setting $z = 1$; thus,

$$A(J) = g\left(\frac{G}{J-\alpha} + \Lambda\right)$$

where

$$G = \frac{g}{(1-\lambda)^2}$$

$$\alpha = \beta + \frac{g}{1-\lambda} \qquad (2.13)$$

$$\Lambda = \frac{\lambda}{1-\lambda} .$$

We again see that λ must be less than one. Notice an important feature of this kernel; it generates an amplitude that has the same form of the original kernel, which infact has a number of obvious theoretical advantages and implications.

To obtain the generating function for this model we find the pole in (2. 12) as a function of the fugacity:

$$a(z) = \beta + \frac{zg}{1-z\lambda} . \qquad (2.14)$$

The asymptotic forms of the moments of the multiplicity distribution are

$$f_1 = \langle n \rangle = \left[\frac{\partial a}{\partial z} \right]_{z=1} Y = \frac{gY}{(1-\lambda)^2} \qquad (2.15\text{ a})$$

$$f_2 = \left[\frac{\partial^2 a}{\partial z^2} \right]_{z=1} Y = \frac{2g\lambda Y}{(1-\lambda)^3} \qquad (2.15\text{ b})$$

and in general

$$f_n = \frac{n!\, g\, \lambda^{n-1}}{(1-\lambda)^{n+1}} Y, \; n \geqslant 1 . \qquad (2.15\text{ c})$$

To aid ones understanding of this model, we note that dividing numerator and denominator of (2. 12) by $(1-z\lambda)$ gives

$$A(z, J) = zg \frac{\bar{K}(z, J) + \frac{z\lambda}{1-z\lambda}}{1 - \bar{K}(z, J)} = zg \left[\bar{K}(z, J) + \frac{z\lambda}{1-z\lambda} \right] \sum_{n=0}^{\infty} \left[\bar{K}(z, J) \right]^n \quad (2.16)$$

where

$$\bar{K}(z, J) = \frac{\left(\frac{zg}{1-z\lambda}\right)}{J-\beta} . \qquad (2.17)$$

Hence $\overline{K}(z, J)$ is the kernel to produce one resonance. Since it is not polynominal in z the resonance decays into an arbitrary number of particles. By introducing a fugacity for the resonance, ζ, where

$$\zeta = \frac{z(1-\lambda)}{1-z\lambda} , \tag{2.18}$$

we can write

$$\overline{K}(z, J) = \frac{\zeta \frac{g}{1-\lambda}}{J-\beta} . \tag{2.19}$$

This shows that the coupling constant for a resonance is $\frac{g}{1-\lambda}$. We also can find the average number of particles per resonance

$$\langle n \rangle_r = \left. \frac{\partial \zeta}{\partial z} \right|_{z=1} = \frac{1}{1-\lambda} . \tag{2.20}$$

The equation (2.19) implies also that all the results of the Chew Pignotti model hold, except now for the resonances, not the particles. For instance, the multiplicity distribution for the number of resonances, n_r, is Poisson with

$$\langle n_r \rangle = \frac{g}{1-\lambda} Y . \tag{2.21}$$

Since the decay probability of the resonances are independent we have the relationship that the average number of particles is the average number per resonance times the average number of resonances.

$$\langle n \rangle = \langle n \rangle_r \langle n_r \rangle = \frac{gY}{(1-\lambda)^2} \tag{2.22}$$

To obtain the complete generating function we must also have the residue of the pole in $A(z, J)$ as a function of z:

$$r(z) = \frac{g^2 z^2}{(1-\lambda z)^2} = \left(\frac{\zeta g}{1-\lambda}\right)^2 \tag{2.23}$$

Upon transforming back to energy (or Y) we have

$$\sigma(z, Y) = r(z)\, e^{(\alpha(z)-1)\, Y} = \frac{g^2 z^2}{(1-\lambda z)^2} \exp \left\{ (\beta-1)\, Y + \frac{g z Y}{1-\lambda z} \right\} \tag{2.24}$$

which can be evaluated explicitly in terms of Laguerre polynomials $L_n(x)$ through the identities

$$\sum_{n=0}^{\infty} L_n^{(1)}(x)\, z^n = (1-z)^{-2} \exp\left(\frac{xz}{z-1}\right) \tag{2.25 a}$$

$$L_n^{(1)}(x) = \frac{n+1}{x} \left[L_n(x) - L_{n+1}(x) \right] \tag{2.25 b}$$

Writing $\sigma(z, Y) = \sum_{n=0}^{\infty} \sigma_{n+2}(Y)\, z^{n+2}$, we find for the exclusive cross sections

$$\sigma_{n+2}(Y) = g^2 e^{(\beta-1)\, Y}\, \frac{(n+1)\,\lambda^{n+1}}{gY} \left\{ L_{n+1}\left(-\frac{gY}{\lambda}\right) - L_n\left(-\frac{gY}{\lambda}\right) \right\} . \tag{2.26}$$

To obtain the parameters for the corresponding Mueller-Regge Model we note that

$$A(z, J) = zg \sum_{n=0}^{\infty} \left[z \left(\frac{g}{J-\beta} + \lambda \right) \right]^{n+1} \tag{2.27 a}$$

$$= z^2 g\, \frac{K(J)}{1-K(J)} \sum_{n=0}^{\infty} \left[\frac{(z-1)\, K(J)}{1-K(J)} \right]^n \tag{2.27 b}$$

$$= \left(\frac{z}{z-1} \right)^2 \frac{g}{G}\, (z-1)\, G \sum_{n=0}^{\infty} \left[(z-1) \left(\frac{G}{J-\alpha} + \Lambda \right) \right]^{n+1} \tag{2.27 c}$$

This means that except for the factor $\frac{g}{G} \left[z/(z-1) \right]^2$, the generating function transforms into itself under the substitution

$$\begin{aligned} z &\to z-1 \\ g &\to G \\ \beta &\to \alpha \\ \lambda &\to \Lambda \ . \end{aligned} \qquad (2.28)$$

[This is the transformation of the multiperipheral model into the Mueller model in the simple one channel case. We will see more complicated examples of this later.] We, therefore, know immediately that the inclusive cross sections are given by

$$\sigma_{tot}\, \rho_n(Y) = gGe^{(\alpha-1)\,Y}\ \frac{(n+1)\,\Lambda^{n+1}}{GY}\ \{L_{n+1}\left(-\frac{GY}{\Lambda}\right) - L_n\left(-\frac{GY}{\Lambda}\right)\} \quad (2.29)$$

where $A(z, J) = z^2\ \Sigma_n (z-1)^n\ \sigma_{tot}\, \rho_n(J)$ defines the inclusive cross sections.

B. Two Channel Generalization

In more general problems we treat each exchanged object as a separate channel. This is, of course, important when including charge and when considering more general coupling schemes. We should point out that the previous problem could have been treated as a two-channel problem with the propagator matrix F (using the notation of Ref. 3) given by

$$F = \begin{pmatrix} \frac{1}{J-\beta} & 0 \\ 0 & 1 \end{pmatrix} \qquad (2.30)$$

and the coupling matrix

$$G = \begin{pmatrix} g & \sqrt{g\lambda} \\ \sqrt{g\lambda} & \lambda \end{pmatrix} \ . \qquad (2.31)$$

As an example of this multi-channel approach let us consider the previous problem within the more general framework. Now

$$F = \begin{pmatrix} \frac{1}{J-\beta} & 0 \\ 0 & 1 \end{pmatrix} ; \quad G = \begin{pmatrix} g & h \\ h & \lambda \end{pmatrix} \tag{2.32}$$

and an end coupling vector $D = \begin{pmatrix} d \\ 0 \end{pmatrix}$ which means that we assume (for convenience) there are no resonances on the ends. Then

$$A(z, J) = \sum_{n=0}^{\infty} zD^T F (zGF)^n Dz = z^2 D^T F (I-zGF)^{-1} D. \tag{2.33}$$

The position of the pole of A is determined from

$$\mathrm{Det}\,(I-zGF) = 0 \quad . \tag{2.34}$$

This yields for the pole position

$$\alpha = \beta + \frac{gz(1-\lambda z) + h^2 z^2}{1-\lambda z} \quad . \tag{2.35}$$

Notice again that if $h^2 = \lambda g$ this reduces to the previous result. Here h^2 represents the coupling of two-particle resonances, and $h^2\lambda^n$ represents the coupling for $(2+n)$-particle resonances; g is the coupling for single particle states. Thus with this slight generalization we have the possibility of having a) only resonances, $g = 0$; b) only stable particles and 2-particle resonances, $\lambda = 0$; or c) only 2 particle resonances, $g = \lambda = 0$.

With only a little work we can show that the entire imaginary amplitude is

$$A(J, z) = \frac{d^2 z^2}{J-\alpha(z)} \quad . \tag{2.36}$$

This transforms into

$$\sigma(z, Y) = d^2 z^2 \; e^{(\alpha(z)-1)Y}$$

$$= d^2 z^2 \exp\left\{ (\beta-1)Y + \left(g - \frac{h^2}{\lambda}\right) zY + \frac{h^2 Y}{\lambda}\left(\frac{z}{1-\lambda z}\right)\right\} \quad (2.37)$$

for the generating function of the cross sections, which has the same form as the generating function proposed by Frazer, Peccei, Pinsky, and Tan (FPPT).[9] Following them (their eqs. (22-24) we may conclude that this leads to

$$\sigma_{n+2}(Y) = d^2 \; e^{(\beta-1)Y} \sum_{j=0}^{n} \frac{\eta^j Y^j}{j!} \lambda^{n-j} \left[L_{n-j}\left(-\frac{h^2 Y}{\lambda^2}\right) - L_{n-j-1}\left(-\frac{h^2 Y}{\lambda^2}\right)\right] \quad (2.38)$$

where $\eta = \left(g - \frac{h^2}{\lambda}\right)$. For $\eta = 0$, (2.38) differs from (2.26) due to the neglect in (2.33) of resonances coupling at the ends. Note also that we define $L_{-1}(x) \equiv 0$.

It is interesting to point out the error in FPPT that gave them the above generating function without any δ-function interaction. The point is that in the calculation of three body and higher correlations, FPPT neglect a particular contribution which is zero if the meson Regge trajectory is replaced by a δ-function. For example, the contribution to the 3 body correlation that was neglected by FPPT is shown in figure 1, where the graphs are to be integrated over $y_1 < y_2 < y_3$. When the meson trajectory is replaced by a δ-function in figure 1 the contribution is zero. Thus neglecting these terms as in FPPT is equivalent to the δ-function model.[13]

III. N-CHANNEL FORMALISM

In this section we consider the case of N-channels, k of which have poles at finite J and N-k of which have δ -function interactions. We neglect charge, and by using the generating function approach obtain the transformation which relates the Mueller Regge (MRM) and multiperipheral (MPM) models. This is a generalization of previous work[3] which discussed the equivalence when only poles were involved. One might expect that since the resonance interaction can be treated as the limit of a normal pole going to infinity that the result would be a trivial extension of previous work. This turns out not to be the case. What was previously an orthogonal similarity transformation relating the two models turns out now to be of the type.

$$\Gamma = S^T GS \tag{3.1}$$

with $S^T \neq S^{-1}$ so that the class properties are reduced. Because of these complexities we proceed carefully, even repeating previous parts for completeness. We start out in the multiperipheral framework, working in the context of the N-channel, one dimensional model.[6] We have

$$\sigma_{n+2}(s) = \frac{1}{s} A_n(s) \tag{3.2}$$

which has as its Laplace transform

$$A_n(J) = D^T F(J) (GF(J))^n D \tag{3.3}$$

where F and G are the NxN propagator and coupling matrices respectively and D is an N-dimensional vector. We form the generating function

$$Q(z, J) = z^2 \sum_n z^n A_n(J)$$

$$= z^2 D^T [F^{-1} - zG]^{-1} D \quad . \tag{3.4}$$

We are considering that k of the N channels have normal Regge singularities, here approximated by poles, and that N-k of them are δ-function interactions. Thus the diagonal "propagator matrix", F(J), has an inverse of the form

$$F^{-1}(J) = \begin{pmatrix} J-\ell_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & J-\ell_k & & & & \\ & & & -\ell_{k+1} & & & \\ & 0 & & & \ddots & & \\ & & & & & -\ell_N \end{pmatrix} \tag{3.5}$$

We block out G into Regge and resonance parts

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \tag{3.6}$$

where G_{11} is a kxk matrix, G_{12} is a kx (N-k) matrix, etc. Similarly D is put in block form.

To transform from the multiperipheral to the Mueller Regge picture, we use the identity

$$[F^{-1} - zG]^{-1} = [F^{-1} - G]^{-1} \sum_{n=0}^{\infty} [(z-1) G (F^{-1} - G)^{-1}]^n \tag{3.7}$$

so that the generating function can be written

$$Q(z, J) = \sum_n z^2 D^T (F^{-1} - G)^{-1} [(z-1) G (F^{-1} - G)^{-1}]^n D \tag{3.8}$$

To complete the transformation we need to diagonize F^{-1} -G. Defining the projection matrix that projects onto the Regge channels,

$$I_k = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{3.9}$$

we write $F^{-1}(J) = J I_k - L$ where $L = \delta_{ij} \ell_i$ for $i, j = 1, \ldots, N$. We remark that ℓ_{k+1} to ℓ_N are not independent parameters, but could be included in the couplings. For definitness, we can always define these parameters to be -1. The "output" or inclusive poles will be solution to the equation

$$\mathrm{Det}\,[F^{-1}(J) - G] = \mathrm{Det}\,[J I_k - L - G] = 0 \tag{3.10}$$

and hence there will be k of them. Thus in the Mueller picture there will also be k Regge poles and (N-k) δ-function channels. We can choose to have the poles first so that the diagonalized form of $F^{-1}(J) - G$ which we call Φ^{-1}, will have the form

$$\Phi^{-1}(J) = J I_k - \Lambda \tag{3.11}$$

with Λ diagonal. <u>We want to make this diagonalization with a J independent matrix</u>, S such that

$$S^T [J I_k - L - G] S = J I_k - \Lambda \quad . \tag{3.12}$$

For S to be J independent it must satisfy both

$$S^T[L + G]\, S = \Lambda \tag{3.13 a}$$

$$S^T I_k S = I_k \quad . \tag{3.13 b}$$

We construct S by writing it and the matrix $L+G \equiv \mathcal{L}$ in the block form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{3.14}$$

$$L + G = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} . \tag{3.15}$$

Since L and G are symmetric we have $\mathcal{L}_{11} = \mathcal{L}_{11}{}^T$, $\mathcal{L}_{22} = \mathcal{L}_{22}{}^T$ and $\mathcal{L}_{12} = \mathcal{L}_{21}{}^T$. Then eq. (3.13 b) yields

$$S_{11}{}^T S_{11} = I \tag{3.16 a}$$

$$S_{11}^T\, S_{12} = S_{12}{}^T S_{11} = S_{12}^T S_{12} = 0 \tag{3.16 b}$$

which implies S_{12} is zero and S_{11} is orthogonal. Incorporating this with (3.13 a) we find, since Λ is diagonal, the equations

$$S_{22}^T \mathcal{L}_{21} S_{11} + S_{22}^T \mathcal{L}_{22} S_{21} = 0 \tag{3.17 a}$$

$$S_{11}^T \mathcal{L}_{12} S_{22} + S_{21}^T \mathcal{L}_{22} S_{22} = 0 \tag{3.17 b}$$

$$S_{22}^T \mathcal{L}_{22} S_{22} = \Lambda_{22} \tag{3.17 c}$$

$$S_{11}^T \mathcal{L}_{11} S_{11} + S_{11}^T \mathcal{L}_{12} S_{21} + S_{21}^T \mathcal{L}_{21} S_{11} + S_{21}^T \mathcal{L}_{22} S_{21} = \Lambda_{11} \quad . \tag{3.17 d}$$

We satisfy the first two of these equations by requiring

$$S_{11}^T \mathcal{L}_{12} + S_{21}^T \mathcal{L}_{22} = 0$$

or

$$S_{21}^T = - S_{11}^T \mathcal{L}_{12} \mathcal{L}_{22}^{-1} \quad . \tag{3.18}$$

Substituting this equation for S_{21} into (3. 17 d) yields

$$S_{11}^T (\mathcal{L}_{11} - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}) S_{11} = \Lambda_{11} \tag{3.19}$$

where Λ_{11} is diagonal. Since S_{11} is orthogonal we may transpose to get the eigenvalue equation

$$(\mathcal{L}_{11} - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}) (S_{11})_i = \lambda_i (S_{11})_i \tag{3.20}$$

showing that S_{11} is the matrix whose i^{th} column is the i^{th} eigenvector of the above equation. Notice that the eigenvalue equation

$$\mathrm{Det}[\lambda I - (\mathcal{L}_{11} - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{12}^{T})] = 0 \tag{3.21}$$

is the same as the more normal form (3. 10),

$$\mathrm{Det}\,(\lambda I_k - (L + G)) = 0 \quad ,$$

even though the first is the determinant of a kxk matrix while the last is the determinant of an NxN matrix.[10]

Having determined the eigenvalues and eigenfunctions (3. 20), and hence S_{11}, we use eq. (3. 18) to determine S_{21}. We only have S_{22} remaining undetermined. Equation (3. 17 c)

$$S_{22}^T \mathcal{L}_{22} S_{22} = \Lambda_{22}$$

constrains S_{22} but contains a large amount of freedom. We can however, choose S_{22} to be orthogonal. We will discuss another choice in appendix A as well as the question of finding the inverse transformation to (3.12). With S_{22} orthogonal, (3.17 c) gives the eigenvalue equation

$$\mathcal{L}_{22}(S_{22})_i = \lambda_i (S_{22})_i \tag{3.22}$$

where S_{22} is the matrix of the eigenfunctions.

With S determined, we now proceed to construct the equivalent Mueller Regge theory. From equation (3.8) we see that the generating function can be written as (note, however, appendix B)

$$Q(z, J) = z^2 \sum_{n=0}^{\infty} (z-1)^n P_n(J) \tag{3.23}$$

where

$$P_n(J) = \Delta^T \Phi(J) (\Gamma \Phi(J))^n \Delta \quad . \tag{3.24}$$

Because of the form of (3.12), Γ and Δ are given by

$$\Gamma = S^T G S \tag{3.25}$$

$$\Delta = S^T D \quad . \tag{3.26}$$

This gives us all the components of the MRM, and therefore the full model.

From the multiperipheral point of view, this completes our discussion of the N channel model with k-poles. We have demonstrated the equivalence as well as constructing the generating function. When one starts from the Mueller picture, the model must be supplemented by positivity (and charge conservation when included) requirements. We discuss the formulation of these constraints briefly now. This formulation requires the inverse of the transformation that we used above to obtain MRM from MPM and is discussed in appendix A.

We construct G from Γ using the transformation S^{-1} (q. v. Eqs. A9 - A12). The elements of G must be positive. The most straight forward way to enforce these conditions is simply to calculate G using the known transformation and then explicitly enforce it. It is also possible to impose some of the conditions without calculating G, using only the class properties of S^{-1}. Preparing for the inclusion of charge, we add now an extra subscript to G and Γ, writing G^i, and Γ^i respectively. The transformation (A7) gives for G^i,

$$G^i = (S^{-1})^T \Gamma^i S^{-1}$$

$$= \begin{pmatrix} S_{11} & (S^{-1})^T_{21} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} \Gamma^i_{11} & \Gamma^i_{12} \\ \Gamma^i_{21} & \Gamma^i_{22} \end{pmatrix} \begin{pmatrix} S^T_{11} & 0 \\ (S^{-1})_{21} & S^T_{22} \end{pmatrix} \tag{3.27}$$

$$= \begin{pmatrix} \begin{array}{l} S_{11}\Gamma^i_{11}S^T_{11} + S_{11}\Gamma^i_{12}(S^{-1})_{21} + \\ (S^{-1})^T_{21}\Gamma^i_{21}S^T_{11} + (S^{-1})^T_{21}\Gamma^i_{22}(S^{-1})_{21} \end{array} & S_{11}\Gamma^i_{12}S^T_{22} + (S^{-1})^T_{21}\Gamma^i_{22}S^T_{22} \\ S_{22}\Gamma^i_{21}S^T_{11} + S_{22}\Gamma^i_{22}(S^{-1})_{21} & S_{22}\Gamma^i_{22}S^T_{22} \end{pmatrix}$$

From this matrix we obtain the submatrices G_{11} and G_{22}:

$$G_{22} = S_{22}\ \{\Gamma^i_{22}\}\ S^T_{22} \tag{3.28 a}$$

$$G_{11} = S_{11}\{\Gamma^i_{11} - \Gamma^i_{12}(\Lambda-\Gamma)^{-1}_{22}(\Lambda-\Gamma)^T_{12} - (\Lambda-\Gamma)_{12}(\Lambda-\Gamma)^{-1}_{22}\Gamma^i_{21} \tag{3.28 b}$$

$$+(\Lambda-\Gamma)_{12}(\Lambda-\Gamma)^{-1}_{22}\Gamma^i_{22}(\Lambda-\Gamma)^{-1}_{22}(\Lambda-\Gamma)^T_{12}\}\, S^T_{11}\ .$$

Since S_{11} and S_{22} are orthogonal transformations, the positivity of the elements of G_{11} and G_{22} imply the following conditions on the bracketed terms:

$$\mathrm{Tr}\,(\{\ \}^n) \geq 0 \tag{3.29}$$

$$\text{for } n = 1 \text{ to } N\ .$$

There are also the non-class constraints (i.e. those not related by similarity transformation) that the elements of

$$G^i_{21} = S_{22}\left(\Gamma^i_{21} - \Gamma^i_{22}(\Lambda-\Gamma)^{-1}_{22}(\Lambda-\Gamma)^T_{12}\right) S^T_{11}$$

$$= S_{22}\left(\Gamma^i_{21} + \Gamma^i_{22}(\Lambda-\Gamma)^{-1}_{22}\Gamma^T_{12}\right) S^T_{11} \tag{3.30 a}$$

and

$$G^i_{12} = S_{11}\left(\Gamma^i_{12} + \Gamma_{12}(\Lambda-\Gamma)^{-1}_{22}\Gamma^i_{22}\right) S^T_{22} \tag{3.30 b}$$

are positive. The last conditions imply

$$G^i_{21}G^i_{12} = S_{22}\{[\Gamma^i_{21} + \Gamma^i_{22}(\Lambda-\Gamma)^{-1}_{22}\Gamma^T_{12}]\ [\Gamma^i_{12} + \Gamma_{12}(\Lambda-\Gamma)^{-1}_{22}\Gamma^i_{22}]\}S^T_{22} \tag{3.31 a}$$

and

$$G^i_{12}G^i_{21} = S_{11}\{[\Gamma^i_{12} + \Gamma_{12}(\Lambda-\Gamma)^{-1}_{22}\Gamma^i_{22}]\ [\Gamma^i_{21} + \Gamma^i_{22}(\Lambda-\Gamma)^{-1}_{22}\Gamma^T_{12}]\}S^T_{11} \tag{3.31 b}$$

have positive elements, leading again to class constraints (3.29) on the bracketed quantities in (3.31a) and (3.31b).

IV. 3 CHANNEL MODEL WITH CHARGE

In this section we explicitly calculate a charge conserving 3 channel model starting from the Mueller picture. We have one isospin zero Regge pole and two δ-functions, one with a mixture of isospin zero and two, and one with isospin one. The isospin of the δ-function is the isospin of the Regge exchange that the δ-function replaces. The propagator matrix is

$$\Phi(J)^{-1} = \begin{pmatrix} J-\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{pmatrix} = JI_1 - \Lambda \tag{4.1}$$

where I_1 is the matrix with only 1 in the (11) element, and λ_2 is the $I = 0$ δ-function. The coupling matrices to the various charge states are

$$\Gamma^+ = \begin{pmatrix} g_0 & a_1 & a_2 \\ a_1 & d_1 & d_2 \\ -a_2 & -d_2 & d_3 \end{pmatrix} ; \; \Gamma^- = (\Gamma^+)^T ; \; \Gamma^o = \begin{pmatrix} \epsilon g_0 & c & 0 \\ c & f & 0 \\ 0 & 0 & h \end{pmatrix} . \tag{4.2}$$

The forms of the coupling matrices are determined by the quantum numbers of the exchange. The fact we only allow G parity positive to be exchanged, assumes the dominance of pions. The full matrix, following Ref. 3, is

$$\Gamma = \Gamma^+ + \Gamma^- + \Gamma^o = \begin{pmatrix} g_0(2+\epsilon) & 2a_1+c & 0 \\ 2a_1+c & 2d_1+f & 0 \\ 0 & 0 & 2d_3+h \end{pmatrix} \tag{4.3}$$

where we will denote the elements of Γ by γ_{ij} in what follows. To find the MPM parameters, we must find an S^{-1} satisfying

$$L = (S^{-1})^T (\Lambda-\Gamma) S^{-1} , \tag{4.4}$$

where L is the diagonal matrix $\ell_i \delta_{ij}$. We use this equation to construct S^{-1} by the same method discussed in the previous section. Then

$$\Lambda - \Gamma \equiv \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 - \gamma_{11} & -\gamma_{12} & 0 \\ -\gamma_{12} & \lambda_2 - \gamma_{22} & 0 \\ 0 & 0 & \lambda_3 - \gamma_{33} \end{pmatrix}, \tag{4.5}$$

and

$$S^{-1} \equiv \begin{pmatrix} \bar{S}_{11} & \bar{S}_{12} \\ \bar{S}_{21} & \bar{S}_{22} \end{pmatrix} . \tag{4.6}$$

In addition to (4.4), S^{-1} must also satisfy

$$(S^{-1})^T \, I_k \, S^{-1} = I_k \tag{4.7}$$

which implies

$$\bar{S}_{12} = 0 \text{ and } \bar{S}_{11}^T = \bar{S}_{11}^{-1} . \tag{4.8}$$

Since $\bar{S}_{11}$ is a 1×1 matrix, we find immediately that

$$\bar{S}_{11} = 1 . \tag{4.9}$$

By the procedure discussed in section III (see also eq. A10) we find

$$\begin{aligned} \bar{S}_{21} &= -\mathcal{L}_{22}^{-1} \, \mathcal{L}_{21} \, \bar{S}_{11} \\ &= -\begin{pmatrix} \frac{1}{\lambda_2 - \gamma_{22}} & 0 \\ 0 & \frac{1}{\lambda_3 - \gamma_{33}} \end{pmatrix} \begin{pmatrix} -\gamma_{12} \\ 0 \end{pmatrix}, \end{aligned} \tag{4.10}$$

therefore

$$\bar{S}_{21} = \begin{pmatrix} \eta \\ 0 \end{pmatrix} \tag{4.11}$$

where

$$\eta = \frac{\gamma_{12}}{\lambda_2 - \gamma_{22}} \quad . \tag{4. 12}$$

The equation that determines $\bar{S}_{22}$ (A9) is

$$\bar{S}_{22}^T \begin{pmatrix} \lambda_2 - \gamma_{22} & 0 \\ 0 & \lambda_3 - \gamma_{33} \end{pmatrix} \bar{S}_{22} = \begin{pmatrix} \ell_2 & 0 \\ 0 & \ell_3 \end{pmatrix} \quad . \tag{4. 13}$$

Since $\bar{S}_{22}$ is an orthogonal transformation and $\ell_2 = \ell_3$ by charge symmetry, then

$$\lambda_2 - \gamma_{22} = \lambda_3 - \gamma_{33} = \ell_2 = \ell_3 \equiv \lambda - \gamma \equiv \ell \quad . \tag{4. 14}$$

Therefore, $\bar{S}_{22}$ has the following form in terms of the transformation angle θ:

$$\bar{S}_{22} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad . \tag{4. 15}$$

In summary, the full transformation is

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \eta & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{4. 16}$$

and the position of the "input" Regge cut is (Eq. A11)

$$\ell_1 = \lambda_1 - \gamma_{11} - \gamma_{12}\,\eta \quad . \tag{4. 17}$$

Now let us use S^{-1} to calculate the coupling in the MPM from our couplings in the MRM. The form that the coupling must have in the MPM will enforce charge conservation and positivity:

$$G^+ = \begin{pmatrix} 0 & 0 & B \\ A & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (S^{-1})^T \begin{pmatrix} g_0 & a_1 & a_2 \\ a_1 & d_1 & d_2 \\ -a_2 & -d_2 & d_3 \end{pmatrix} S^{-1} \quad (4.18)$$

$$= \begin{pmatrix} g_0 + 2a_1\,\eta + d_1\eta^2 & C(a_1 + \eta d_1) + S(a_2 + \eta d_2) & C(a_2 + \eta d_2) - S(a_1 + \eta d_1) \\ C(a_1 + d_1\eta) - S(a_2 + d_2\eta) & C^2 d_1 + S^2 d_3 & d_2 - CSd_1 + CSd_3 \\ -S(a_1 + d_1\eta) - C(a_2 + d_2\eta) & -d_2 - CSd_1 + CSd_3 & C^2 d_3 + S^2 d_1 \end{pmatrix}$$

where $C = \cos\theta$ and $S = \sin\theta$. The solution to (4.18) is

$$d_1 = d_2 = d_3 = 0 \quad (4.19\text{ a})$$

$$g_0 + 2a_1\,\eta = 0 \,. \quad (4.19\text{ b})$$

$$\cos^2\theta = \sin^2\theta \quad (4.19\text{ c})$$

$$a_1 \cos\theta = -a_2 \sin\theta \quad (4.19\text{ d})$$

$$A = a_1 \cos\theta - a_2 \sin\theta > 0 \quad (4.19\text{ e})$$

$$B = a_2 \cos\theta - a_1 \sin\theta > 0 \,. \quad (4.19\text{ f})$$

Let us consider this solution in more detail. First, (4.19 c) implies

$$\cos\theta = \pm \sin\theta$$

and hence using (4.19 d),

$$a_1 = \pm a_2 \,. \quad (4.20)$$

Since $A + B = (a_1 + a_2)(\cos\theta - \sin\theta)$ is strictly positive, $\cos\theta$ and $\sin\theta$ must have opposite signs; a_1 and a_2 must have the same signs; $\cos\theta$ and a_1 must have the same signs. Therefore we have two possibilities:

$$\begin{array}{lcl} \cos\theta = +1/\sqrt{2} & & \cos\theta = -1/\sqrt{2} \\ \sin\theta = -1/\sqrt{2} & \text{or} & \sin\theta = 1/\sqrt{2} \\ a_1 = a_2 > 0 & & a_1 = a_2 < 0 \end{array} \tag{4.21}$$

Experimentally a_1 and a_2 are the terms that govern the approach to scaling and therefore should experimentally be negative to have inclusive distributions scale from below.[11] This completely determines S^{-1}:

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \eta & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & +1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \tag{4.22}$$

$$A = B = -a_1\sqrt{2} \quad . \tag{4.23}$$

Now let us consider the neutral couplings

$$G_0 = \begin{pmatrix} E & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} = \begin{pmatrix} 1 & \eta & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \tag{4.24}$$

$$\times \begin{pmatrix} \epsilon g_0 & c & 0 \\ c & f & 0 \\ 0 & 0 & h \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \eta & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & +1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} .$$

Solving this as we did above we find

$$\begin{aligned} f &= h \\ D &= f \\ c &= -\eta f \\ E &= \epsilon g_0 - \eta^2 f \quad . \end{aligned} \tag{4.25}$$

We now assemble our results. Roughly, the constraints arise as follows:

1) Constraints on the Mueller parameters due to charge conservation

$$a_1 = a_2;\ d_1 = d_2 = d_3 = 0;\ f = h\ ;$$

$$c = \frac{2a_1 f}{(-\lambda)}\ ;\quad g_o = \frac{4a_1{}^2}{(-\lambda)}\ ; \tag{4.26}$$

$$\lambda_2 = \lambda_3 = \lambda\ .$$

2) Relations between Mueller parameters and Multiperipheral parameters, and constraints of positivity

$$A = B = -a_1\sqrt{2} > 0 \tag{4.27}$$

$$\ell_2 = \ell_3 = \ell = \lambda - f$$

$$\ell_1 = \lambda_1 - g_0\ (1 + \epsilon - f/(-\lambda))$$

$$f = D > 0$$

$$E = g_0\ (\epsilon - f/(-\lambda)) > 0\ .$$

The end couplings are determined similarly by means of

$$(S^{-1})^T \Delta^+ = D^+\ . \tag{4.28}$$

Therefore

$$\begin{pmatrix} 1 & \eta & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & d_1 \\ d_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.29}$$

(where d_1 and d_2 should not be confused with the parameters appearing in (4.2) which we found to be zero). The solution is

$$\delta_{22} = \delta_{23} = \delta_{32} = \delta_{33} = \delta_{12} = 0$$

$$\delta_{13} = d_1 > 0, \; \delta_{11} = -\delta_{21}\eta = \eta d_2/\sqrt{2}$$

$$\delta_{31} = d_2/\sqrt{2} > 0, \; \delta_{21} = -\delta_{31} = -d_2/\sqrt{2}\;, \tag{4.30}$$

and therefore,

$$\Delta^{+} = \begin{pmatrix} +d_2\eta/\sqrt{2} & 0 & d_1 \\ -d_2/\sqrt{2} & 0 & 0 \\ d_2/\sqrt{2} & 0 & 0 \end{pmatrix}. \tag{4.31}$$

We solve for Δ^- and Δ^o in the same manner, and obtain for $\Delta\,(xyz) \equiv x\Delta^+ + y\Delta^- + z\Delta^o$:

$$\Delta = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\,zd_{11} + \eta\, d_2(x+y) & \sqrt{2}\,yd_1 + z\eta\, d_{22} & \sqrt{2}\,xd_1 + z\,\eta d_{22} \\ -d_2\,(x+y) & -zd_{22} & -zd_{22} \\ d_2\,(x-y) & zd_{22} & -zd_{22} \end{pmatrix} \tag{4.32}$$

The generating function is (see Ref. 3)

$$Q\,(xyz\,J) = \bar{\Delta}^T \frac{1}{\Phi^{-1}(J) - [\,\Gamma^+(x-1) + \Gamma^-(y-1) + \Gamma^o(z-1)]}\Delta \tag{4.33}$$

where $\bar{\Delta}^T = \Delta^T\,(x\leftrightarrow y)$. Each element of the matrix Q is the generating function for a particular initial state labeled by its charge. The rows and columns of the matrix correspond to charge 0 + - respectively; to find the generating function for initial charges a, b one takes the a $\bar{b}$ element. This is a result of the conventions that in the multiperipheral graph for $\sigma_n(Y)$ particle b is flowing out rather than in.

At this point we believe it instructive to illustrate some of the above results and conventions by means of an explicit example. We treat the case of an initial state of two positive particles, for which the cross sections and inclusive rates will be the "23" matrix element of the appropriate terms in (4. 33). The generating function is also given by the "23" element, and after some algebra, is found to be (the following calculation can be done easily in the MPM formalism)

$$Q_{++}(x, y, z \mid J) = \frac{x^2 \left(d_1 + \dfrac{\sqrt{\dfrac{g_0}{2}}\;\dfrac{d_{22} z}{\sqrt{-\lambda}}}{1-(z-1)\,(f/(-\lambda))} \right)^2}{J-\lambda_1 - g_0 \left\{ \epsilon\,(z-1) + \dfrac{(xy-1) + (f/-\lambda)^2\,(z-1)^2}{1-(z-1)\,(f/-\lambda)} \right\}} \tag{4.34}$$

We can also calculate the inclusive cross sections directly, by setting x=y=z=1 in Δ, and by using (3. 24). For example, the ++ cross section (in the j-plane) is

$$P^{o}_{(++)} = \left(\bar{\Delta}^T \Phi \quad \Delta \right)_{23} \tag{4.35}$$

$$= \frac{1}{2} \left(\sqrt{2}\, d_1 + \eta\, d_{22},\ - d_{22},\ d_{22}\right) \begin{pmatrix} \frac{1}{J-\lambda} & & \\ & \frac{1}{-\lambda} & \\ & & \frac{1}{-\lambda} \end{pmatrix} \begin{pmatrix} \sqrt{2}\, d_1 + \eta\, d_{22} \\ -d_{22} \\ -d_{22} \end{pmatrix}$$

$$= \frac{\left(d_1 + \sqrt{\dfrac{g_0}{2}}\;\dfrac{d_{22}}{\sqrt{(-\lambda)}} \right)^2}{J-\lambda_1}$$

As a cross check $Q\,(111 \mid J) = P^{o}_{(++)}$ and of course (4. 35) and (4. 34) agree.

The generating function for the (++) cross sections is obtained from (4. 34) by performing the inverse Laplace transform. Normalizing the generating function to unity when x=y=z=1, we find

$$I(x, y, z \mid Y) = x^2 \left(d_1 + \frac{\sqrt{\frac{g_0}{2}} \frac{d_{22} z}{\sqrt{-\lambda}}}{1-(z-1)\,(f/(-\lambda))} \right)^2 \tag{4.36}$$

$$\frac{1}{\left(d_1 + \sqrt{\frac{g_0}{2}} \frac{d_{22}}{\sqrt{(-\lambda)}} \right)^2} \exp \left[g_0 Y \left(\epsilon \;(z-1) + \frac{(xy-1) + (f/(-\lambda))^2 (z-1)^2}{1-(z-1)\,(f/(-\lambda))} \right) \right] .$$

As with the simple models of Sec. II we again see a number of interesting properties. The function

$$\frac{1}{1-(z-1)\,(f/(-\lambda))} \tag{4.37}$$

indicates the presence of the resonances, which decay into any number of neutrals with the production probability for n being $(f/-\lambda)^n$. As mentioned earlier, λ is not an independent parameter but rather always occurs in conjunction with the δ-function couplings f and d_{22}. Because of the simple dependence of the generating function on x and y we see that our resonances can decay into any number of neutrals but only one +-pair.[11] This represents a possible short coming of the model from a phenomenological point of view. We remark that this model, taken as a crude approximation to more realistic MPM's, is in rather good agreement with the integrated values of the correlation parameters at present energies[(12)]. For example, note that if we only look at the negatives (i.e. $x = z = 1$), then

$$I\,(1, y, 1 \mid Y) = e^{g_0 Y\,(y-1)} , \tag{4.38}$$

which is of course a Poisson distribution for σ_{n-}. Also the fact that the generating function has no Y^2 term indicates that

$$f_2^- = 0 \quad , \tag{4.39}$$

a result that is difficult to obtain from a multiperipheral model of the normal type. These results certainly agree with the trend of present data. Similarly the average number of negatives is

$$\langle n_- \rangle = g_0 Y \quad . \tag{4.40}$$

To obtain an estimate of g_o, recall that the position of the input cut is

$$\ell_1 = 2\alpha_{in} - 1 = \lambda_1 - g_0 (1 + \epsilon - f/(-\lambda)) \quad . \tag{4.41}$$

It is generally expected that $\alpha_{in} \simeq 1/2$ and $\lambda_1 \simeq 1$. Therefore

$$g_0 \simeq \frac{1}{1 + \epsilon - f/(-\lambda)} \quad ; \tag{4.42}$$

from other phenomenological considerations[11] $\epsilon \simeq 1$ and $f/(-\lambda) \cong 1/2$ implying $g_0 \simeq 2/3$ which is quite reasonable, considering the crudeness of the model. One should note that since resonances cannot be produced among m charged particles for $m > 2$, the specific model considered in this section in principle suffers from some of the same criticism as the original CPM. Namely, if the data were to favor a large value of $g_0 \gtrsim 4/3$, then α_{in} would have to be negative. Nevertheless, present data favor $d\langle n \rangle/dY \sim 2\text{-}3$, so that $g_o \sim 2/3\text{-}1$ and the model is in no trouble.

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APPENDIX A. Inverse Transformation

We want to consider the question of the transformations preserving unity of the lower diagonal elements of F(J) in the transformation to Φ (J). This will be related to the questions of inverses so we take this topic up first. Temporarily we assume that neither the lower diagonal elements of F or Φ are unity but only that the transformation which relates them is orthogonal. That is, using the results of Sec. III,

$$S_{22}^T \ (L+G)_{22} S_{22} = \Lambda_{22} \tag{A1}$$

$$S_{22}^T = S_{22}^{-1} \tag{A2}$$

$$S_{21} = -(L+G)_{22}^{-1} (L+G)_{12}^T S_{11} \tag{A3}$$

$$S_{11}^T[(L+G)_{11} - (L+G)_{12} (L+G)_{22}^{-1} (L+G)_{12}^T] S_{11} = \Lambda_{11} \tag{A4}$$

$$S_{11}^T = S_{11}^{-1} \tag{A5}$$

$$S_{12} = 0 \tag{A6}$$

These are the equations determining S if one starts with the MPM. Then of course

$$\Gamma = S^T G S \tag{A7}$$

and Λ determined in the previous equations determines a MRM. Starting from the MRM one can also determine the transformation to the MPM, namely S^{-1}. Since the equation for S^{-1} analogous to (3. 13 a) is

$$L = (S^{-1})^T (\Lambda - \Gamma) S^{-1} \ , \tag{A8}$$

the equations one would get starting from the MRM are

$$(S^{-1})_{22}{}^{T}\,(\Lambda-\Gamma)_{22}\,(S^{-1})_{22} = L_{22} \tag{A9}$$

$$(S^{-1})_{21} = -\,(\Lambda-\Gamma)_{22}{}^{-1}\,(\Lambda-\Gamma)_{12}{}^{T}\,(S^{-1})_{11} \tag{A10}$$

$$(S^{-1})_{11}^{T}\left[\,(\Lambda-\Gamma)_{11} - (\Lambda-\Gamma)_{12}\,(\Lambda-\Gamma)_{22}^{-1}\,(\Lambda-\Gamma)_{12}^{T}\right]\,(S^{-1})_{11} = L_{11}\;. \tag{A11}$$

In considering whether these are consistent with (A1-A6) we note that since S has the form

$$S = \begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}$$

then

$$S^{-1} = \begin{pmatrix} S_{11}^{-1} & 0 \\ (S^{-1})_{21} & S_{22}^{-1} \end{pmatrix} \tag{A12}$$

where $(S^{-1})_{21} = -S_{22}^{-1} S_{21} S_{11}^{-1}$. Equations (A1) and (A9) are obviously consistent. After writing out the submatrices of eq. (A7), making frequent use of eq. (A3) one can show that eqs. (A9) - (A12) are consistent with eqs. (A1) - (A6).

Having shown for transformations with S_{22} orthogonal that the inverse transformation (MRM → MPM) is the inverse of the transformation (MPM → MRM), we proceed to examine the transformations where we give up the orthogonality of S_{22} in order to preserve a simple form for the propagator matrices. Suppose F has for its "22" component the identity matrix. We denote by S_{2} the orthogonal transformation from F to Φ, with $F_{22} = I$ and with Φ_{22} determined from the eigenvalues, i.e.

$$-\Phi_{22}^{-1} \equiv \Lambda_{22} = \begin{pmatrix} \lambda_{k+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}. \tag{A13}$$

We denote by $\mathcal{S}$ the transformation which makes Φ_{22} the identity. Consider the matrix, assuming $\lambda_j < 0$ for $j = k+1, \ldots, n$,

$$\Lambda_{22}^{\frac{1}{2}} = \begin{pmatrix} \sqrt{-\lambda_{k+1}} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \cdot \\ & & \sqrt{-\lambda_n} \end{pmatrix} \tag{A14}$$

Then obviously $\mathcal{S}_{22} = S_{22} \Lambda_{22}^{-\frac{1}{2}}$. If not all λ_i's are negative then $\Lambda_{22}^{\frac{1}{2}}$ has to be modified accordingly, and one obtains for Λ_{22} a diagonal matrix with elements +1 or -1. Call this latter type matrix $I_{\pm}$.

One might also wish to start the Mueller Regge theory with $\Lambda_{22} = I_{\pm}$ and use an orthogonal transformation to get to the multiperipheral model which, of course, would not have $L_{22} = -I$ but rather $L_{22} = \ell_i \delta_{ij}$. For $i > k$, ℓ_i is determined from the eigenvalue equation,

$$(\Lambda_{22} - \Gamma_{22})\,(T_{22})_i = \ell_i (T_{22})_i \;.$$

where T_{22} is an orthogonal matrix. By constructing

$$L_{22}^{\frac{1}{2}} = \begin{pmatrix} \sqrt{-\ell_{k+1}} & & 0 \\ & \cdot & \\ 0 & & \sqrt{-\ell_n} \end{pmatrix} \tag{A15}$$

we then obtain a transformation converting Φ into F with $F_{22} = I$. Note that the ℓ_i must be negative if they are to be interpreted as the result of low lying trajectories, due to the positivity of the coupling matrix. Writing

$$L_2^{\frac{1}{2}} \equiv \begin{pmatrix} I_k & 0 \\ 0 & L_{22}^{\frac{1}{2}} \end{pmatrix} \tag{A16}$$

we have

$$\tilde{G} = T^T \Gamma T$$

$$G = L_2^{-\frac{1}{2}} \tilde{G} L_2^{-\frac{1}{2}} = \mathcal{J}^T \Gamma \mathcal{J} \qquad (A17)$$

where

$$\mathcal{J} = T L_2^{-\frac{1}{2}} . \qquad (A18)$$

The inverse transformations, assuming now both $\Lambda_{22} = I_{\pm}$ and $L_{22} = -I$, are

$$\tilde{\Gamma} = S^T G S \qquad (A19)$$

$$\Gamma = \Lambda_2^{-\frac{1}{2}} \tilde{\Gamma} \Lambda_2^{-\frac{1}{2}} = \mathcal{S}^T G \mathcal{S}$$

where analogously we have defined

$$\Lambda_2^{\frac{1}{2}} = \begin{pmatrix} I_k & 0 \\ 0 & \Lambda_{22}^{\frac{1}{2}} \end{pmatrix} \qquad (A20)$$

$$\mathcal{S} = S \Lambda_2^{-\frac{1}{2}} \qquad (A21)$$

The connection between $\mathcal{S}$ and $\mathcal{J}$ is

$$\mathcal{J} = \mathcal{S}^{-1} \qquad (A22)$$

or equivalently

$$T = \Lambda_2^{\frac{1}{2}} S^{-1} L_2^{\frac{1}{2}} \qquad (A23)$$

Schematically we have the picture in Fig. 2.

APPENDIX B. End Couplings in the MRM

In Ref. 3 the general equivalence between the Mueller Regge (one dimensional) model and the multiperipheral (one dimensional) model is displayed. However, the "leading particles" were never treated in a truly inclusive manner. In their eqs. (5) and (6)

$$Q(z,J) = \sum_{n=0}^{\infty} z^{n+2} Q_n(J) \tag{B1}$$

$$= z^2 \sum_{n=0}^{\infty} (z-1)^n P_n(J) \tag{B2}$$

where $Q_n(J)$ is the Laplace transform of s times the exclusive cross section for producing n particles (n+2 particles in the final state), one sees that $P_n(J)$ are the (transforms of the) integrals of the n-particle inclusive cross sections only over the pionization particles and not over the leading particles. The same assumption is made in the present paper in Section II.

One way to state this problem is to note that in the final prescription given in Ref. 3 for calculating the contribution of $P_n(J)$ to $Q(z,J)$, the end couplings Δ are multiplied by z. Actually for only one kind of particle this is not so bad, but for charged particles it is worse since $\Delta(x,y,z) = x\Delta^+ + y\Delta^- + z\Delta^o$. There should be therefore, a prescription in the Mueller Regge theory for calculating the totally inclusive distributions, summed over all final particles, leading or not. We shall give such a prescription here. First we write $\Delta(x,y,z)$ as

$$\Delta(x,y,z) = \Delta + (x-1)\Delta^+ + (y-1)\Delta^- + (z-1)\Delta^o \tag{B3}$$

where $\Delta \equiv \Delta^+ + \Delta^- + \Delta^o$. In Mueller Regge diagrams, Δ is the conventional end coupling, Fig. 3a, and Δ^+, Δ^- or Δ^o is the "dragonfly vertex," Fig. 3b, where a "leading" particle of charge +, -, or o is emitted respectively. Then to calculate the average multiplicity of type i, we sum the set of diagrams in Fig. 4; correspondingly the integrated two particle inclusive cross section ($\langle n(n-1)\rangle_i$) is given in Fig. 5. Analogous to these drawings are the equations in a true Mueller Regge theory (entirely inclusive instead of mixed exclusive-inclusive) which lead to

$$Q(x, y, z|J) = \sum_{n_+ n_- n_o} P_{n_+ n_- n_o}(J) \quad (x-1)^{n_+} (y-1)^{n_-} (z-1)^{n_o} \tag{B4}$$

where $P_{n_+ n_- n_o}(J)$ includes all diagrams, with the proper number of positive, negative or neutral particles, leading or otherwise.

Footnotes and References

1. G. F. Chew and A. Pignotti, Phys. Rev. 276 2112 (1968).
2. The most apparent problem of the CPM is its prediction of a Poisson distribution for particle multiplicities. Data show important correlations requiring substantial modification of the simple theory. For a guide to literature on current data and models for multiplicities see D. Sivers and G. H. Thomas, Argonne report ANL/HEP 7328 (1973).
3. S. S. Pinsky, D. R. Snider and G. H. Thomas, Argonne Report ANL/HEP 7331 (1973).
4. David K. Campbell and S. J. Chang, Institute for Advanced Study report entitled "Multiplicity Distributions in Multiperipheral Models with Isospin," (c. May 1973).
5. For a review of dual resonance models see for example E. L. Berger, in Phenomenology in Particle Physics 1971, proceedings of the conference held at Cal Tech 1971, edited by C. B. Chiu, G. C. Fox, and A. J. G. Hey (Cal Tech, Pasadena 1971), p. 83.
6. C. E. DeTar, Phys. Rev. D3, 128 (1971).
7. See e.g. T. Ferbel, Rochester report COO-3-65-41, to be published in Annals of the New York Academy of Sciences.
8. e.g. A. H. Mueller, Phys. Rev. D4, 150 (1970); B. R. Webber, Nucl. Phys. B43, 541 (1972).
9. W. R. Frazer, R. D. Peccei, S. S. Pinsky, and C.-I. Tan, Phys. Rev. D7, 2647 (1973).

10. F. R. Gantmacher, The Theory of Matrices (Chelsea Publishing Company, New York 1960).

11. This follows since the MPM propagator has no neutral δ-function. This implies further that the model as constructed does not conserve isotopic spin.

12. For MRM analysis of present data, see S. S. Pinsky and G. H. Thomas, Argonne report ANL/HEP 7345 (1973).

13. The identical diagrams were also neglected in the following: S. Pinsky, "Exchange Degeneracy as a Consequence in the Mueller-Regge Model," University of California at Riverside preprint (unpublished); S. Pinsky, "Constraints of Charge Conservation on the Mueller-Regge Model," University of California at Riverside preprint (unpublished).

Figure Captions

1. One of the graphs left out of FPPT. This contribution to C_3 vanishes in the δ-function model.

2. Schematic connection between MRM and MPM in the N channel model containing δ-function interactions. "Orthogonal" refers to the '22' components of matrices.

3. MRM end couplings: a) represents the conventional end coupling Δ in Appendix B; b) gives the end coupling Δ^{i}.

4. Diagrams contributing to the average multiplicity as discussed in Appendix B.

5. Diagrams contributing to the second moment $\langle n(n-1)\rangle$ as discussed in Appendix B.

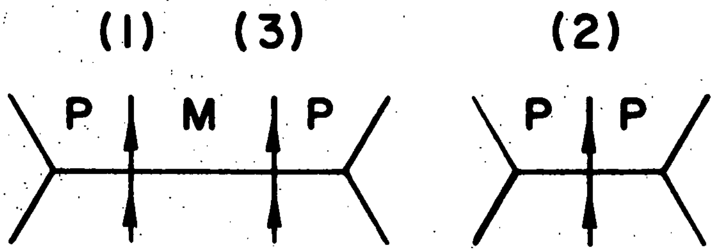
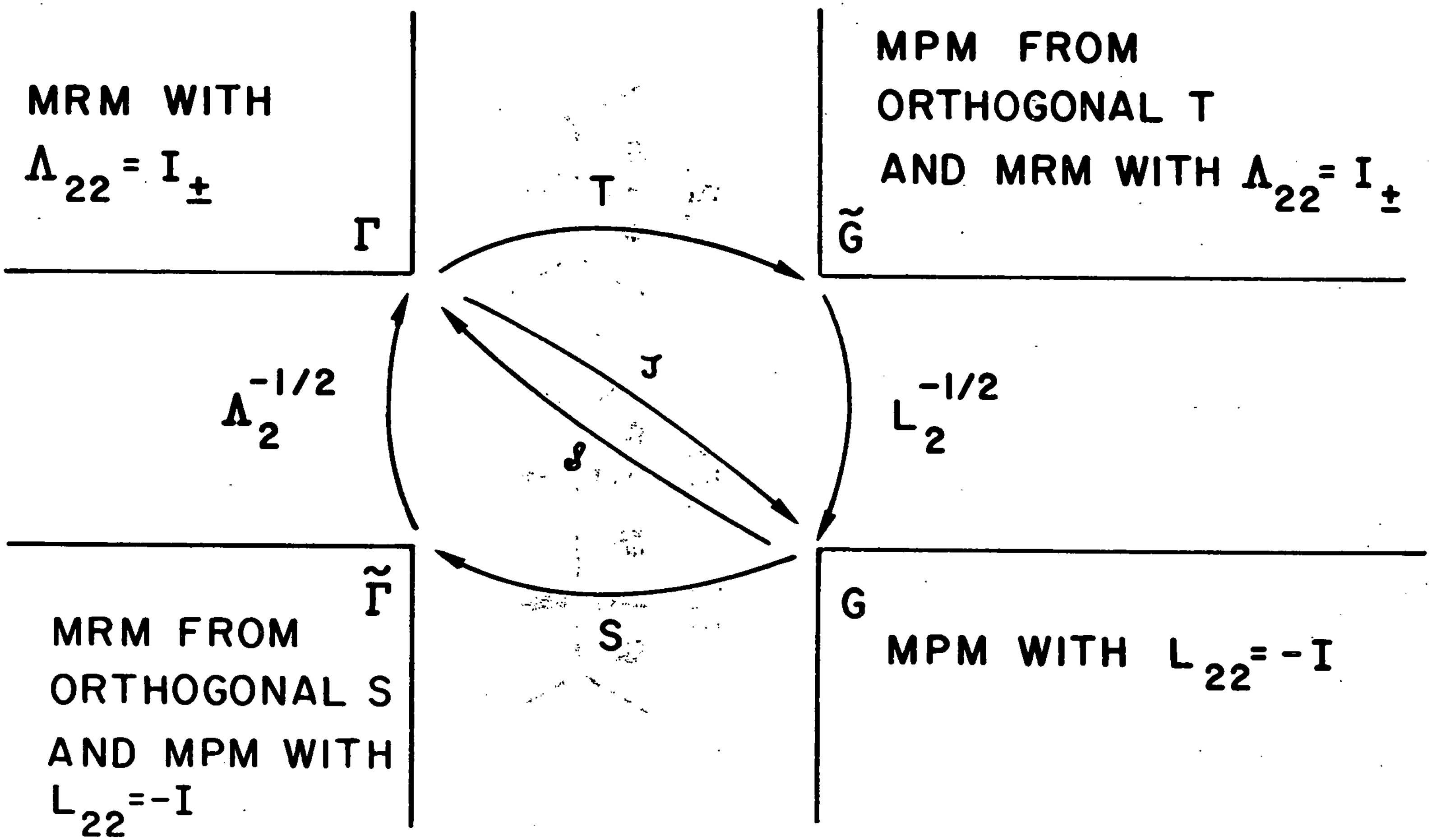
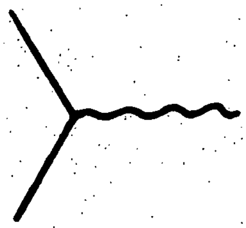


Figure 1

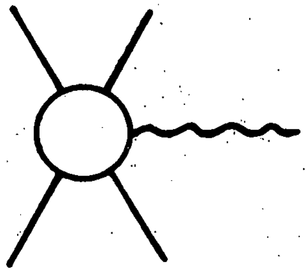
Figure 2



($\tilde{\Gamma}$ is called Γ in the text)

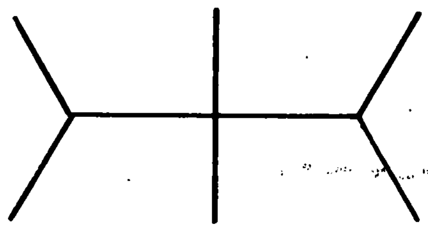


3a

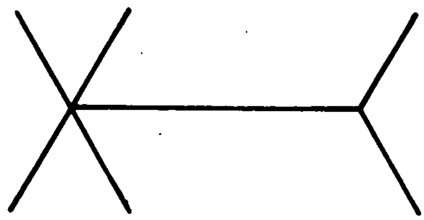


3b

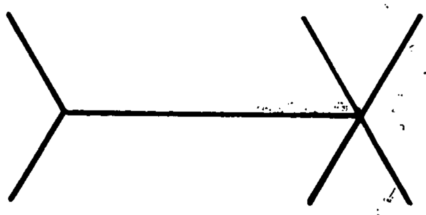
Figure 3



$$= \Delta \Phi \Gamma^{i} \Phi \Delta$$

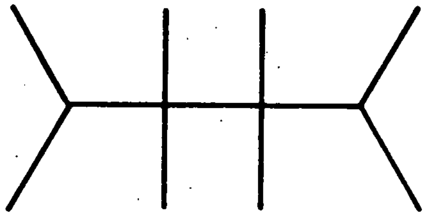


$$= \Delta^{i} \Phi \Delta$$

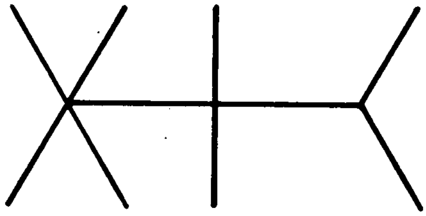


$$= \Delta \Phi \Delta^{i}$$

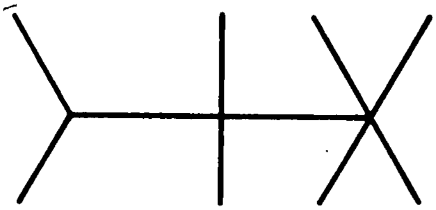
Figure 4



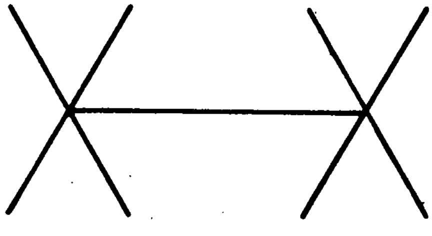
$$= \Delta \Phi \Gamma^{i} \Phi \Gamma^{j} \Phi \Delta$$



$$= \Delta^{i} \Phi \Gamma^{j} \Phi \Delta$$



$$= \Delta \Phi \Gamma^{i} \Phi \Delta^{j}$$



$$= \Delta^{i} \Phi \Delta^{j}$$

Figure 5