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Exact Magnetic Diffusion Solutions for Magnetohydrodynamic Code Verification

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In this paper, the authors present several new exact analytic space and time dependent solutions to the problem of magnetic diffusion in R-Z geometry. These problems serve to verify several different elements of an MHD implementation: magnetic diffusion, external circuit time integration, current and voltage energy sources, spatially dependent conductivities, and ohmic heating. The exact solutions are shown in comparison with 2D simulation results from the Ares code. (U)

Introduction

With the continued growth of interest in pulsed power there is a corresponding growth in the need for simulation codes to accurately predict experimental results. And magnetohydrodynamics (MHD) is an important area of physics in some of these experiments. For a simulation code to be truly useful tool, experimenters and designers must have high level of confidence in the code's accuracy. And one very important step in building said confidence is verification – the determination that the simulation code is correctly and accurately solving the intended equations by direct comparison with known solutions. In this paper we derive a number of new exact solutions which may prove useful for just this purpose. The problems are all pure magnetic diffusion – there is no hydrodynamic motion involved. Also, all of the problems are limited to R-Z cylindrical symmetry. The resulting solutions are then explicitly dependent on the time *t* and radius *r*.

The Equations

To begin we first derive the magnetic diffusion equation for the cylindrical geometry and symmetry in which we are interested. We adopt the approximation that the fields are slowly changing with respect to the speed of light and hence drop the time derivative in Ampere's law. This is sometimes referred to as the MHD approximation. Then we have

Ampere's law:

$$\vec{\nabla} x \vec{B} = 4\pi \vec{j} \qquad 1.$$

Faraday's Law:

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} x \vec{E} = 0 \qquad 2.$$

and Ohm's Law:

$$\vec{j} = \sigma \vec{E} \qquad 3.$$

Combining these three equations we find that

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} x \left(\frac{1}{4\pi\sigma} \vec{\nabla} x \vec{B}\right) \qquad 4.$$

We wish to apply this equation to the geometry as shown in Fig 1. This is a cylindrically symmetric conductor (wires and hollow cylinders) of outer radius R_w and length Z. As part of the computational problem, the conductor is surrounded by a cylindrical region of vacuum which extends out to radius R_B. For the symmetry we are interested in, the magnetic field points solely in the angular direction and is only a function of radius and time. Then Eq. 4 reduces

$$\frac{\partial (rB)}{\partial t} - r\frac{\partial}{\partial r}\left(\frac{1}{4r\pi\sigma}\frac{\partial (rB)}{\partial r}\right) = 0 \qquad 5.$$

This is the magnetic diffusion equation for which we will seek solutions. Our diffusion unknown is

$I(r,t) = rB(r,t)$. The current density provides Ohmic heating via the material temperature equation

$$\rho C_v \frac{\partial T}{\partial t} = \frac{1}{\sigma} j^2 \qquad 6.$$

T is the material temperature, ρ is the mass density and C_v is the specific heat.

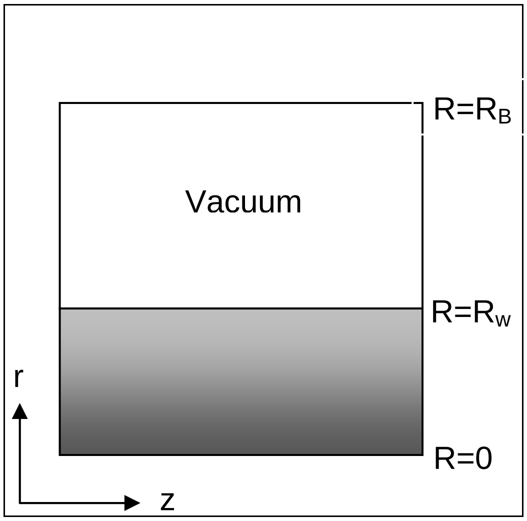

Figure 1

Boundary Conditions

In all problems considered here, we have that the boundary condition at the center of the conductor is $B(r=0,t) = 0$ (or in the case of a hollow vacuum filled cylinder, zero at the inner surface of the conductor). The boundary conditions on the ends of the cylinder are such that the normal magnetic flux is zero. This eliminates the need to consider edge effects and preserves the symmetry of the solution. The boundary condition at the outer surface will depend on whether we drive the problem with a current source or a voltage source and the verification solutions will be divided into two groups based on this difference.
With a current source, the value of the total current is part of the problem specification. If the total current is $S(t)$, then we know that the value of the magnetic field everywhere exterior to the conductor is easily computed from the integral form of Ampere's law

$$B(r,t)_{exterior} = \frac{2S(t)}{r} \qquad 7.$$

This then provides the boundary condition on the surface

$$I(R_w,t) = 2S(t) \qquad 8.$$

Equation 8 is always true regardless of the type of source, but only for current sources does it alone specify the boundary condition on the outer surface of the conductor. For problems where we have some external circuit specification providing a voltage source, we don't a priori know $S(t)$ and so the boundary condition at the surface of the conductor is more complicated. To derive it we apply Faraday's law of induction to the exterior vacuum region. Faraday's law applied to the exterior vacuum region states

$$\frac{-d}{dt}\int_{exterior} \vec{B} \bullet d\vec{A} = \oint_{exterior} \vec{E} \bullet d\vec{l} \qquad 9.$$

where the line integral in Eq. 9 is taken in the counter clockwise direction. From Eq. 7 we know that

$$\int_{exterior} \vec{B} \bullet d\vec{A} = Z\int_{R_w}^{R_B} \frac{2S(t)}{r} dr = 2Z\ln(\frac{R_B}{R_w})S(t) \qquad 10.$$

And from the symmetry we know that the electric field only points in the z direction so that

$$\oint_{vacuum} \vec{E} \bullet d\vec{l} = ZE(r=R_w,t) - ZE(r=R_B,t) \qquad 11.$$

At the surface of the conductor we can get the electric field from $I(r,t)$ as

$$\vec{E}(R_w,t) = \left(\frac{1}{4\pi\sigma}\vec{\nabla}x\vec{B}\right)_{r=R_w} \qquad 12.$$

which in our geometry reduces to

$$E(R_w,t) = \frac{1}{4\pi\sigma R_w}\left(\frac{\partial}{\partial r}I(r,t)\right)_{r=R_w} \qquad 13.$$

What about the electric field at $r=R_B$? This is just the voltage applied to our problem from the external circuit divided by the length of the load

$$E(R_B,t) = \frac{V_{applied}}{Z} \qquad 14.$$

Putting together Eqs. 9-13 we have

$$Z\ln(\frac{R_B}{R_w})\frac{d}{dt}I(R_w,t) + \frac{Z}{4\pi\sigma R_w}\left(\frac{\partial}{\partial r}I(r,t)\right)_{r=R_w} = V_{applied} \qquad 15.$$

So, Eq. 8 provides our exterior surface boundary condition when the problem is driven by a current source, and Eq. 15 when it is driven by a voltage source.

Solution Method

In each case the solution method is generally the same: take the Laplace transform in the time variable of the magnetic equation Eq. 5, and the relevant boundary conditions, solve the resulting ordinary differential equation in the radius variable, then perform the inverse Laplace transform to obtain the complete magnetic field solution. Then, the current density is derived from the Magnetic field B via Eq. 1, Ampere's law. For voltage driven problems, the total current is derived from the magnetic field via Eq. 7. If the heat capacity and electrical conductivity are a constant, the Ohmic heating can then be computed by time integrating Eq. 6, resulting in an exact solution for the material temperature as well.

Current Source Solutions

Solution 1: Consider a solid cylindrical conductor of radius R_w, driven by a current source

$$S(t) = S_0 (b/t)^{5/2} (2b/t - 3) e^{-b/t} \qquad 16.$$

where b is a constant. Use a spatially dependent conductivity given by

$$\sigma(r) = \frac{\sigma_0}{r^2} \qquad 17.$$

Define $D = 4\pi\sigma_0$. Then the magnetic field and current density are given by Equations 18 and 19.

$$B(r,t) = \frac{4S_0}{r}\left(\frac{b}{t}\right)^{5/2}\left(1+\frac{1}{2}\sqrt{\frac{D}{b}}\ln\left(\frac{R_w}{r}\right)\right)\left(\frac{b}{t}\left(1+\frac{1}{2}\sqrt{\frac{D}{b}}\ln\left(\frac{R_w}{r}\right)\right)^2-\frac{3}{2}\right)e^{-\frac{b}{t}\left(1+\frac{1}{2}\sqrt{\frac{D}{b}}\ln\left(\frac{R_w}{r}\right)\right)^2} \qquad 18.$$

$$j(r,t) = \frac{S_0\sqrt{D}b^2}{16\pi r^2 t^{4.5}}\left(\begin{array}{l} 12t\left(t-4b-\sqrt{D}\ln\left(\frac{R_w}{r}\right)\left(4\sqrt{b}+\sqrt{D}\ln\left(\frac{R_w}{r}\right)\right)\right)+ \\ \left(2\sqrt{b}+\sqrt{D}\ln\left(\frac{R_w}{r}\right)\right)^4 \end{array}\right)e^{-\frac{b}{t}\left(1+\frac{1}{2}\sqrt{\frac{D}{b}}\ln\left(\frac{R_w}{r}\right)\right)^2} \qquad 19.$$

For a specific example, set S_0=1.2, b=0.5, R_w=1.0 cm, and σ_0=1.0 cm/milliOhm. In Figures 2 and 3 we show the solutions for the magnetic field and current densities at $r = 0.5*R_w$ (curves A and C) and $r=0.9*R_w$ (curves B and D). The red curves represent the exact solution and the blue curves are the Ares result.

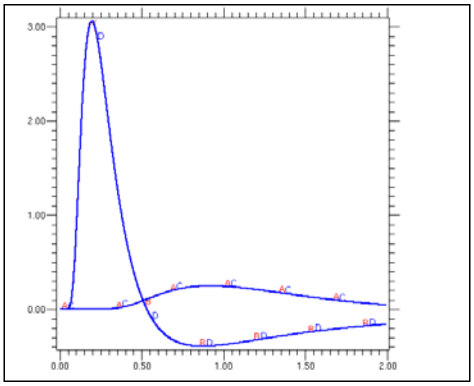

Figure 2: Magnetic Field, MegaGauss versus μs for solution 1

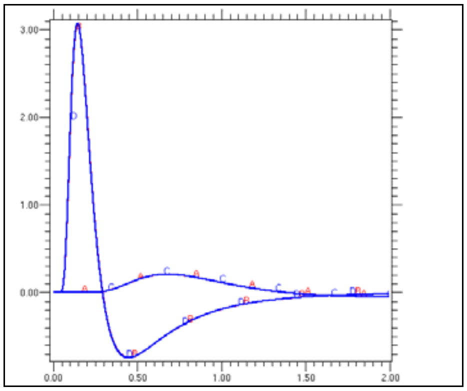

Figure 3: Current Density, DegaMegaAmps/cc versus μs for solution 1

Solution 2: Consider a solid cylindrical conductor of radius R_w, driven by constant current $S(t) = S_0$ and possessing constant electrical conductivity σ and specific heat C_v. Now we can also solve for the Ohmic heating. Define $D = 4\pi\sigma$ and $b = R_w\sqrt{D}$. Let y_n be the n[th] nonzero roots of $J_1(y)=0$. Then the magnetic field, current density and material temperature are given by Eqs. 20, 21 and 22.

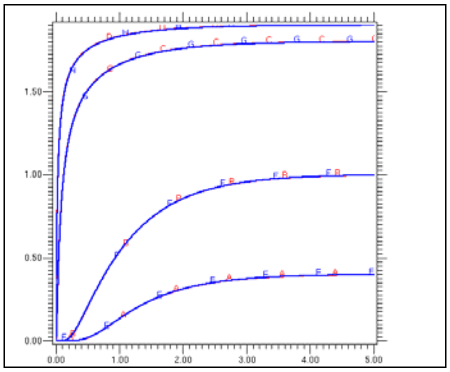

Figure 4: Magnetic Field, MegaGauss versus μs for solution 2

$$B(r,t) = \frac{2rS_0}{R_w^2}\left[1 + 2\sum_{n=1}^{\infty}\frac{R_w}{ry_n}\frac{J_1\left(\frac{r}{R_w}y_n\right)}{J_0(y_n)}e^{-t\frac{y_n^2}{b^2}}\right] \qquad 20.$$

$$j(r,t) = \frac{S_0}{\pi R_w^2}\left[1 + \sum_{n=1}^{\infty}\frac{J_0\left(\frac{r}{R_w}y_n\right)}{J_0(y_n)}e^{-t\frac{y_n^2}{b^2}}\right] \qquad 21.$$

S_0 = 1.0 DecaMegaAmps, initial temperature T_0 = 1.0e-3 KeV. The results are shown in Figures 4, 5 and 6. The red curves represent the exact solution and the blue curves are the Ares result.
The edits are taken at r=0.2cm (curves A and E), r=0.5cm (curves B and F), r=0.9cm (curves C and G), and r=0.95cm (curves D and H).

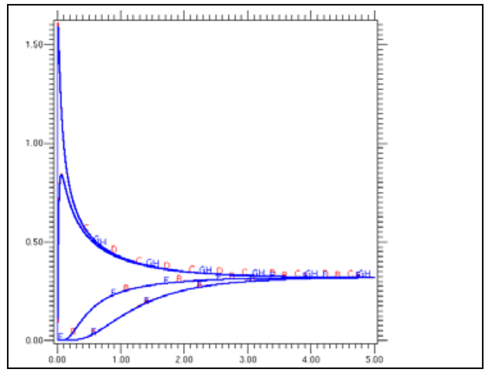

Figure 5: current density in DecaMegaAmps/cc versus μs for solution 2

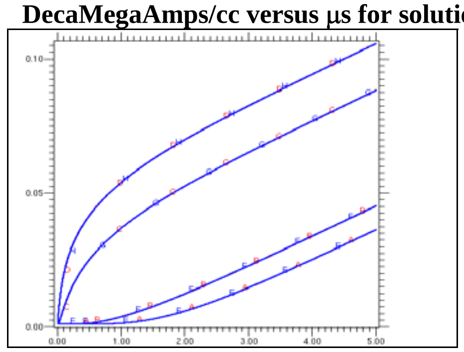

Figure 6: Material temperature in KeV versus μs for solution 2

$$T(r,t) = T_0 + \frac{S_0^2}{\pi^2 R_w^4 \sigma\rho C_v}\left[\begin{array}{l} t + 2\sum_{n=1}^{\infty}\left(\frac{b^2}{y_n^2}\frac{J_0\left(\frac{r}{R_w}y_n\right)}{J_0(y_n)}\left(1 - e^{-t\frac{y_n^2}{b^2}}\right)\right) \\ + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{b^2}{\left(y_n^2 + y_m^2\right)}\frac{J_0\left(\frac{r}{R_w}y_n\right)J_0\left(\frac{r}{R_w}y_m\right)}{J_0(y_n)J_0(y_m)}\left(1 - e^{-t\frac{\left(y_n^2+y_m^2\right)}{b^2}}\right)\right)\end{array}\right] \qquad 22.$$

As a specific example, use radius R_w = 1.0 cm, σ = 1/milliohm/cm, specific heat C_v = 1.0 Eu/gm/KeV, density ρ = 8.93 gm/cc,

Voltage Source Solutions

Solution 3: Consider a solid cylindrical conductor of radius R_w, driven by a voltage source representing the external circuit shown in Figure 7, where V is a constant voltage, L is a constant inductor, and R is a constant resistor.

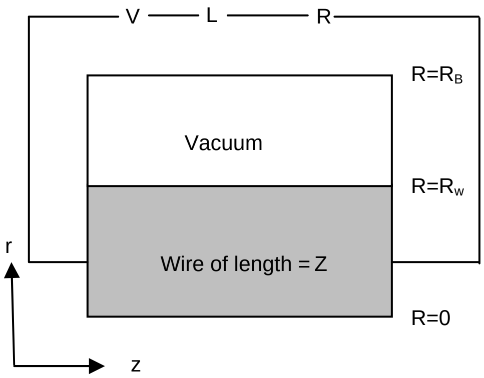

Figure 7

Again use a spatially dependent conductivity given by $\sigma(r) = \sigma_0 / r^2$. Define $D = 4\pi\sigma_0$, $\beta = \ln(R_B/R_w) + L/(2Z)$. Require that $R = Z/(2D\beta)$. Then the magnetic field, current density and total current are given by Eqns. 23, 24 and 25.

$$B(r,t) = \frac{VD}{rZ}\left[\begin{array}{l} 4\beta\, erfc\left(\frac{1}{2}\sqrt{\frac{D}{t}}\ln\left(\frac{R_w}{r}\right)\right) - 4\sqrt{\frac{t}{\pi D}}\, e^{-\frac{D}{4t}\ln\left(\frac{R_w}{r}\right)^2} + \\ 2\left(\frac{t}{\beta D} + \ln\left(\frac{R_w}{r}\right) - 2\beta\right) erfc\left(\frac{1}{2\beta}\sqrt{\frac{t}{D}} + \frac{1}{2}\sqrt{\frac{D}{t}}\ln\left(\frac{R_w}{r}\right)\right) e^{\frac{1}{2\beta}\ln\left(\frac{R_w}{r}\right) + \frac{t}{4D\beta^2}} \end{array}\right] \qquad 23.$$

$$j(r,t) = \frac{VD}{4\pi r^2 Z}\left[\begin{array}{l} 2\sqrt{\frac{D}{\pi t}}\left(2\beta - \ln\left(\frac{R_w}{r}\right)\right) e^{-\frac{D}{4t}\ln\left(\frac{R_w}{r}\right)^2} + \\ \left(\begin{array}{l} 2\sqrt{\frac{D}{\pi t}}\left(\frac{t}{\beta D} + \ln\left(\frac{R_w}{r}\right) - 2\beta\right) e^{-\left(\frac{1}{2\beta}\sqrt{\frac{t}{D}} + \frac{1}{2}\sqrt{\frac{D}{t}}\ln\left(\frac{R_w}{r}\right)\right)^2} \\ -\frac{1}{\beta}\left(\frac{t}{\beta D} + \ln\left(\frac{R_w}{r}\right)\right) erfc\left(\frac{1}{2\beta}\sqrt{\frac{t}{D}} + \frac{1}{2}\sqrt{\frac{D}{t}}\ln\left(\frac{R_w}{r}\right)\right) \end{array}\right) e^{\frac{1}{2\beta}\ln\left(\frac{R_w}{r}\right) + \frac{t}{4D\beta^2}} \end{array}\right] \qquad 24.$$

$$S(t) = \frac{VD}{2Z}\left[4\beta - 4\sqrt{\frac{t}{\pi D}} + 2\left(\frac{t}{\beta D} - 2\beta\right) erfc\left(\frac{1}{2\beta}\sqrt{\frac{t}{D}}\right) e^{\frac{t}{4D\beta^2}}\right] \qquad 25.$$

As a specific example, set V=10.0 decakilovolts, σ_0=1.0 cm/milliOhm, R_w=1.3cm, R_B=1.5cm. The inductance and resistance are set to a specific value which satisfies the requirements of the problem but still allows them to have the same value. This is

$$L = R = Z\left(\sqrt{\ln(R_B/R_W)^2 + 1/D} - \ln(R_B/R_W)\right) \qquad 26.$$

In figures 8, 9 , and 10 we plot the exact solutions and the Ares simulations results. The edits are taken at r=0.2cm (curves A and E), r=0.5cm (curves B and F), r=0.9cm (curves C and G), and r=0.95cm (curves D and H). The red curves represent the exact solution and the blue curves are the Ares result.

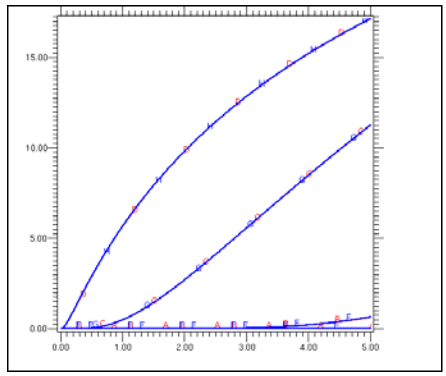

Figure 8: Magnetic field in megaGauss versus μs for solution 3

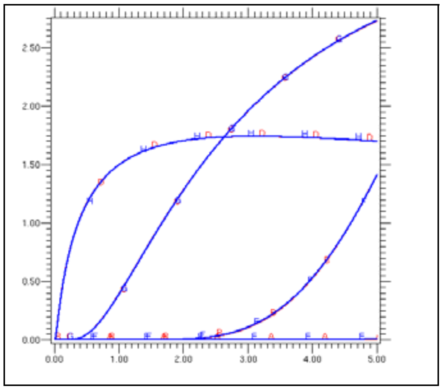

Figure 9: current density in decaMegaAmps/cc verus μs for solution 3

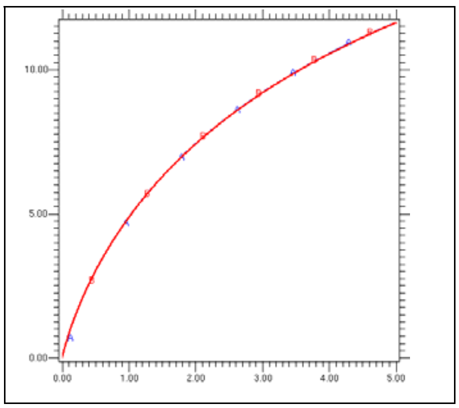

Figure 10: Total current in decaMegaAmps versus μs for solution 3

Solution 4: For this solution we use the same circuit driven picture as in Figure 7 but let the conductivity now be a constant, σ=*const*. Define $D = 4\pi\sigma$, $\gamma = R/2$, $b = R_w\sqrt{D}$, $\beta = L/2 + Z\ln(R_B/R_w)$, and $\delta = Z/(DR_w)$. Let x_n be the nth root of

$$R_w(\gamma - \beta x)J_1(b\sqrt{x}) + \delta b\sqrt{x}J_0(b\sqrt{x}) = 0 \quad 27.$$

Now define $y_n = b\sqrt{x_n}$ and

$$G_n = \frac{2\delta V}{2\delta R_w\gamma + \delta^2 y_n^2 + R_w^2(\gamma - \beta x_n)^2} \quad 28.$$

With these definitions, the magnetic field, current density and total current are given by equations 29, 30, and 31.

$$B(r,t) = \frac{Vr}{R_w(R_w\gamma + 2\delta)} - \sum_{n=1}^{\infty} G_n e^{-x_n t}\frac{J_1(\frac{r}{R_w}y_n)}{J_1(y_n)} \quad 29.$$

$$j(r,t) = \frac{1}{4\pi R_w}\left[\frac{2V}{(R_w\gamma + 2\delta)} - \sum_{n=1}^{\infty} G_n y_n e^{-x_n t}\frac{J_0(\frac{r}{R_w}y_n)}{J_1(y_n)}\right] \quad 30.$$

$$S(t) = \frac{R_w}{2}\left[\frac{V}{(R_w\gamma + 2\delta)} - \sum_{n=1}^{\infty} G_n e^{-x_n t}\right] \quad 31.$$

As a specific example, use conductor radius R_w = 1.3 cm, outer radius of the vacuum R_B = 1.5 cm, conductivity σ = 0.1/milliOhm/cm, conductor length Z = 2 cm, inductance L = 0.13 nanoHenries, resistance R_s= 0.13 milliOhms, voltage V= 10.0 decaKiloVolts. Figures 11 and 12 are plots of the magnetic field and current density solutions at three different radii: r=0.1R_w (curves A and D), r=0.5R_w (curves B and E), r=0.9R_w (curves C and F). Figure 13 is the total current as a function of time (in μs). The red curves represent the exact solution and the blue curves are the Ares result.

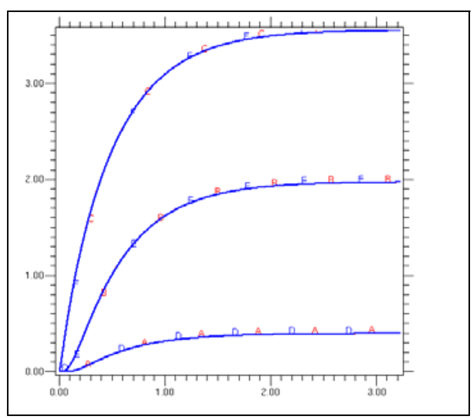

Figure 11: Magnetic field in megaGauss versus μs for solution 4

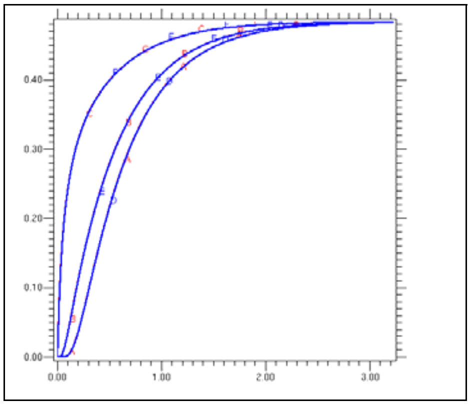

Figure 12: current density in decaMegaAmps/cc verus μs for solution 4

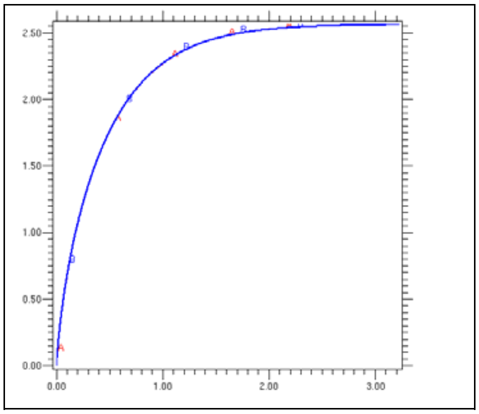

Figure 13: Total current in decaMegaAmps versus μs for solution 4

Solution 5: For this solution we use the same problem setup as in solution 4 but now add a constant capacitor C in series with the other circuit elements. Again use $D = 4\pi\sigma$, $\gamma = R/2$, $b = R_w\sqrt{D}$, $\beta = L/2 + Z\ln(R_B/R_w)$, and $\delta = Z/(DR_w)$. But let us also define $\alpha = 1/(2C)$. Proceeding as in solution 4, let x_n be the nth root of

$$R_w(\gamma - \beta x - \frac{\alpha}{x})J_1(b\sqrt{x}) + \delta b\sqrt{x}J_0(b\sqrt{x}) = 0 \quad 32.$$

Define $y_n = b\sqrt{x_n}$ and

$$G_n = \frac{\delta(\frac{Q_0}{C} - V)}{2\delta R_w(\gamma - \frac{2\alpha}{x_n}) + \delta^2 y_n^2 + R_w^2(\gamma - \beta x_n - \frac{\alpha}{x_n})^2} \quad 33.$$

With these definitions, the magnetic field, current density and total current is given by equations 34, 35, and 36.

$$B(r,t) = 2\sum_{n=1}^{\infty} G_n e^{-x_n t}\frac{J_1(\frac{r}{R_w}y_n)}{J_1(y_n)} \quad 34.$$

$$j(r,t) = \frac{1}{2\pi R_w}\sum_{n=1}^{\infty} G_n y_n e^{-x_n t}\frac{J_0(\frac{r}{R_w}y_n)}{J_1(y_n)} \quad 35.$$

$$J(t) = R_w\sum_{n=1}^{\infty} G_n e^{-x_n t} \quad 36.$$

Solution 5 has two different modes of operation - the critically damped and oscillatory damped modes. Which mode one sees depends on the nature of the roots of equation 32. If all of the roots are real then you get the critically damped mode. If any of the roots are complex, then you get the damped oscillatory behavior.

For a critically damped example, use the following parameters: R_w = 1.3 cm, R_B = 1.5 cm, σ = 0.1/milliOhm/cm, Z = 2 cm, L = 10.0 nanoHenries, R= 1.0 milliOhms, voltage V= 10.0 decaKiloVolts, C=5.0 milliFarads. For these parameters all of the roots x_n are real and positive.

Figures 14 and 15 are plots of the magnetic field and current density solutions at three different radii: r=0.1R_w (curves A and D), r=0.5R_w (curves B and E), r=0.9R_w (curves C and F). Figure 16 is the total current as a function of time (in μs). The red curves represent the exact solution and the blue curves are the Ares result.

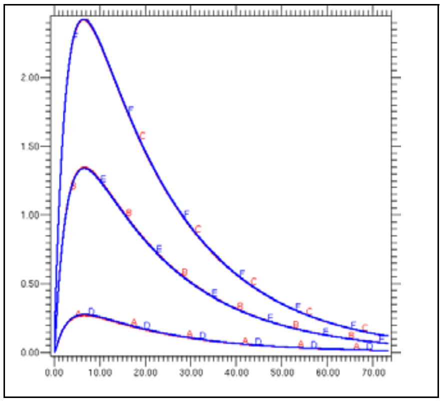

Figure 14: Magnetic field in megaGauss versus μs for solution 5, critically damped

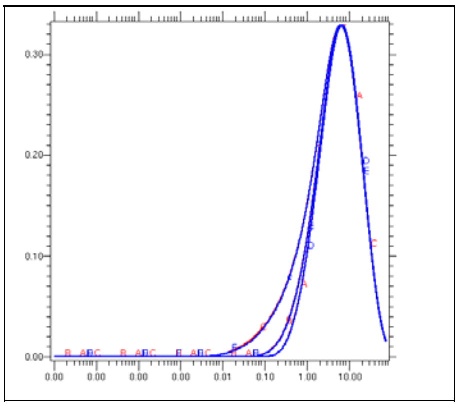

Figure 15: current density in decaMegaAmps/cc verus μs for solution 5, critically damped

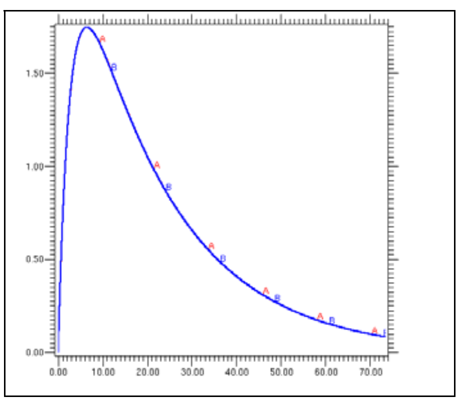

Figure 16: Total current in decaMegaAmps versus μs for solution 5, critically damped

To get an oscillatory damped example, just change the value of the capacitor to C=0.1 milliFarads. Now the first two roots of equation 32 are complex (they are the complex conjugates x_1= 0.20889+0.90466i and x_2= 0.20889-0.9046i) and all the other roots are real and positive definite. Again we plot the magnetic field and current density solutions at three different radii: r=0.1R_w (curves A and D), r=0.5R_w (curves B and E), r=0.9R_w (curves C and F). Figure 19 is the total current as a function of time. Again, the red curves represent the exact solution and the blue curves are the Ares result.

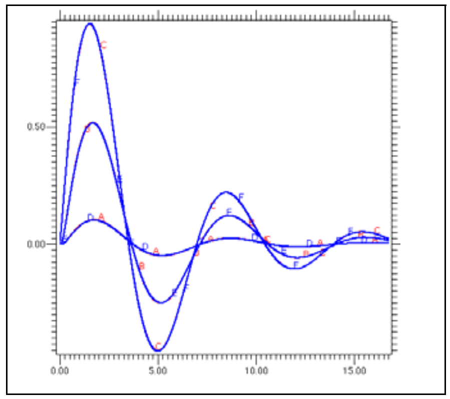

Figure 17: Magnetic field in megaGauss versus μs for solution 5, oscillatory

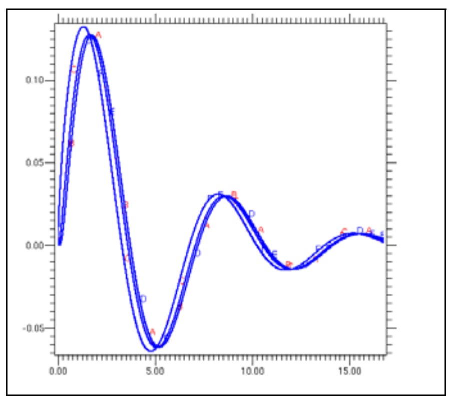

Figure 18: current density in decaMegaAmps/cc verus μs for solution 5, oscillatory

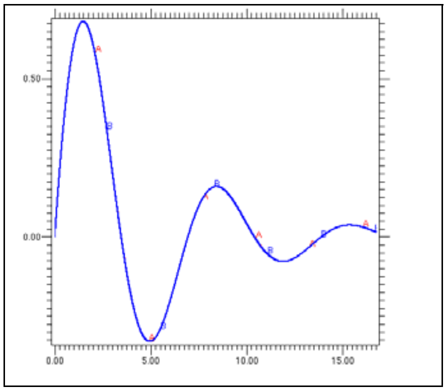

Figure 19: Total current in decaMegaAmps versus μs for solution 5, oscillatory

Conclusions

These verification problems have proven useful to the Ares team for building confidence in our MHD implementations. Notice that while all of the Ares numerical solutions provided were run in a 2D r-z mode, the solutions are equally valid run in 3D. These problems were in fact also used to verify the 3D mhd implementation in Ares, as well as the 2D. And while it may seem that these problems are very similar to each other, they each combine different elements of difficulty in implementation and/or solution. Together they give a good base of coverage for basic magnetic diffusion (with Ohmic heating) verification. Hopefully other code groups will find these problems useful as well.

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