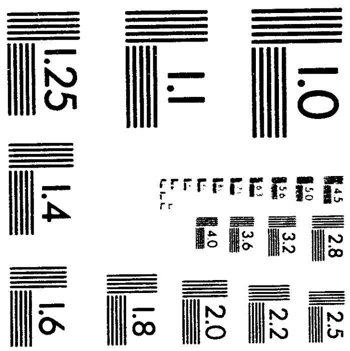
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Quantum Groups, Non-Commutative Differential Geometry and Applications*

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Abstract

The topic of this thesis is the development of a versatile and geometrically motivated differential calculus on non-commutative or quantum spaces, providing powerful but easy-to-use mathematical tools for applications in physics and related sciences. A generalization of unitary time evolution is proposed and studied for a simple 2-level system, leading to non-conservation of microscopic entropy, a phenomenon new to quantum mechanics. A Cartan calculus that combines functions, forms, Lie derivatives and inner derivations along general vector fields into one big algebra is constructed for quantum groups and then extended to quantum planes. The construction of a tangent bundle on a quantum group manifold and an BRST type approach to quantum group gauge theory are given as further examples of applications.

The material is organized in two parts: Part I studies vector fields on quantum groups, emphasizing Hopf algebraic structures, but also introducing a 'quantum geometric' construction. Using a generalized semi-direct product construction we combine the dual Hopf algebras $\mathcal{A}$ of functions and $\mathcal{U}$ of left-invariant vector fields into one fully bicovariant algebra of differential operators. The pure braid group is introduced as the commutant of $\Delta(\mathcal{U})$. It provides invariant maps $\mathcal{A} \rightarrow \mathcal{U}$ and thereby bicovariant vector fields, casimirs and metrics. This construction allows the translation of undeformed matrix expressions into their less obvious quantum algebraic counter parts. We study this in detail for quasitriangular Hopf algebras, giving the determinant and orthogonality relation for the 'reflection' matrix. Part II considers the additional structures of differential forms and finitely generated quantum Lie algebras — it is devoted to the construction of the Cartan calculus, based on an undeformed Cartan identity. We attempt a classification of various types of quantum Lie algebras and present a fairly general example for their construction, utilizing pure braid methods, proving orthogonality of the adjoint representation and giving a (Killing) metric and the quadratic casimir. A reformulation of the Cartan calculus as a braided algebra and its extension to quantum planes, directly and induced from the group calculus, are provided.

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It is a pleasure to thank my advisor, Professor Bruno Zumino, who introduced me to the topics of this thesis, for many things — his support and care, for sharing his amazing intuition and being patient with his young student's ignorance — and especially for creating a pleasant, cooperative and productive environment.

In this context I also would like to thank my colleagues Chryss Chryssomalakos and Paul Watts for cooperation rather than competition, for constructive criticism, ongoing discussions and generally for a fun time. The same goes to my fellow graduate students and friends at LBL, on campus, at Caltech and in Heidelberg. Thanks also to the professors, postdocs and staff at LBL and on campus, and especially to Sandy Ewing, Betty Moura, Luanne Neumann, Anne Takizawa and Donna Sakima. We had many interesting conversations, not only about physics, and their help in bureaucratic matters made life at Berkeley easier and quite enjoyable.

Special thanks to Professor Nicolai Yu. Reshetikhin for always having time to help me with my many questions and for sharing his elegant approach to Quantum Groups in general and Ribbon Tangles in particular and to Professor Stanley Mandelstam for helpful discussion, for being on my committee and for reading my thesis.

Helpful conversations about aspects of the subject matter of this thesis with Paolo Aschieri, Scott Hotes, John Madore, Peter Neu, Marc Rosso, Michael Schlieker, Alexander Sevrin, Leehwa Yeh and participants of the XXII-th DGM conference are gratefully acknowledged. I would also like to thank Professor W. J. Freeman, Dirk Graudenz, Andreas Herz and David MacKay for initiating me into the interesting field of neural systems and Professor L. Falicov for introducing me to condensed matter theory. I am indebted to Professor Jörn Knoll for early teaching and guidance and to Dr. Heister and Renker GmbH. for their support and faith in my abilities as a student.

This thesis would not have been possible without the ongoing support and encouragement of my parents Hans and Marlis Schupp. Finally, no thanks could be enough for Claudia Herold, for her love, encouragement, welcome distraction and her patience with the INS, physics and me.

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Contents

Introduction

The topic of this thesis is non-commutative geometry in general and the development of powerful and easy to use differential calculi on quantum spaces and some examples of their application in particular. I will try to give an as geometric picture as possible while including all necessary mathematical tools. The emphasis will be on the formation of concepts (Begriffsbildung).

In classical differential geometry we have a choice between two dual and equivalent descriptions: we can either work with points on a manifold $\mathcal{M}$ or with the algebra $C(\mathcal{M})$ of functions on $\mathcal{M}$. Non-commutative geometry is based on the idea that the algebra $C(\mathcal{M})$ need not be commutative. Such a space is called a quantum space — in analogy to the quantization of the commutative algebra of functions on phase-space that yields the non-commutative operator algebra of quantum mechanics. More general, a non-commutative algebra, viewed as if it was a function algebra on a (possibly non-existing) topological space, is called a quantum or pseudo space. One could call it a "theory of shadows" — shadows of classical concepts and objects.

The poor understanding of physics at very short distances indicates that the small scale structure of space-time might not be adequately described by classical continuum geometry. At the Planck scale one expects that the notion of classical geometry has to be generalized to incorporate quantum effects. No convincing alternative is presently known, but several possibilities have been proposed; one of them is the introduction into physics of non-commutative geometry. Such new physical theories would allow, roughly speaking, the necessary fuzziness for a successful description of the space-time "foam" expected at tiny distances. See for instance the interesting gedanken experiment [1] concerning generalized uncertainty relations.

This certainly was one of the motivations behind the work on quantum deformations of the Lorentz and Poincare groups [3, 4, 5] and of Minkowski space in terms of a parameter q and of course behind Connes program [2] of non-commutative geometry, but there are also many other possible applications of non-commutative calculi in physics like generalized symmetries (*e.g.* quantum group gauge theory) and stochastics (master equations, random walks, ...), to mention a few. Continuous deformations of symmetry groups in physical theories have historically been proven to be rather successful in enlarging the class of phenomena that these theories describe well; one of the most famous examples is special relativity. For this reason it would be very interesting in elementary particle physics to study deformations of semi-simple Lie groups. Unfortunately these groups allow only trivial deformations as long as one stays within the category of Lie groups, hence giving another motivation for the study of the less rigid quantum groups.

Such generalizations of physical theories might have welcome and also unexpected side effects: One of them is the possibility that some q-deformed quantum field theories might be naturally finite. This is expected if the deformation parameter has dimensions of length, in analogy to amplitudes in string theory which were proven to be finite to all orders by S. Mandelstam [6]. Even if q turns out not to be a physical parameter, such a theory might still be interesting as a new way to regularize infinities [7, 8], using q-identities, known from the study of q-functions, which were first introduced in the context of combinatorics nearly a century ago. Here we should also mention a quick and easy approach, due to [9], to lattice gauge theory based on a minimal non-commutative calculus. In chapter 5 we will show at the example of a simple toy model that modified time evolution equations, that could be motivated from deformed space time symmetries, lead to non-conservation of entropy. This might be of interest in connection with the black hole evaporation paradox. Connes [10] and Connes & Lott [11] consider a minimal generalization of classical gauge theory and study a Kaluza-Klein theory with a 2-point internal space and use non-commutative geometric methods to define metric properties; note that it is also possible with these methods to gauge *discrete* spaces. This lead to a new approach to the standard model. Fröhlich and collaborators [12] introduced gravity in this context. As an example of new symmetries in "old" theories we would like to mention the work of the Hamburg Group of Mack and collaborators [13]: They showed that the internal symmetries of (low-dimensional) quantum field theories with braid group statistics form a larger class than groups and were able to motivate from basic axioms of such field theories that elements of weak quasitriangular quasi Hopf algebras with $*$-structures should act as symmetry operators in the Hilbert space of physical states. Particle physics *phenomenology* from q-deformed Poincare algebra is for example considered in [14], where evidence of q-deformed space time is sought in the observed spectrum of $\rho - a, \omega - f, K^0$ mesons and remarkably good agreement of theory and experiment, similar to, if not better, than Regge pole theory is found.

The theory of non-commutative spaces is quite old, going back to early work of Kac [15], Taksaki [16] and Schwarz & Enock [17]. Recently, the interest got revived by the discovery of non-trivial examples. Quantum groups, which are a content rich example of quantum spaces, arise naturally in several different branches of physics and mathematics: in the context of integrable models, quantum inverse scattering method, Yang-Baxter-equations and their solutions, the so called R-matrices, Knizhnik-Zamolodchikov equations, rational conformal field theory and in the theory of knot and ribbon invariants. Concerning knot theory we should in particular mention the discovery of the Jones polynomial [18] and its generalizations, which were then reconstructed from quantum R-matrices in the work of Reshetikhin & Turaev [19] and later related to the topological Chern-Simons action by Witten [20]. It was pointed out by Drinfeld that these examples find an adequate description in the language of Hopf algebras.

There are at least three major approaches to the construction of quantum deformations of Lie groups: Drinfeld and Jimbo introduce a deformation parameter on the Lie algebra level and provided us with consistent deformations for all semi-simple Lie groups. The St Petersburg Group impose q-dependent commutation relations in terms of numerical R-matrices among the matrix elements of a matrix representation. Manin finally identifies quantum groups with endomorphisms of quantum planes.

A large part of this thesis is devoted to the study of differential calculi on quantum groups rather than quantum planes (these will be considered in the second part of this thesis). This path was in part taken because quantum groups have more structure than quantum planes and hence provide more guidance in the search for the correct axioms. Apart from this purely practical reason, the importance of differential geometry in the theory of (quantum) Lie groups and vice versa should, however, not be underestimated. Lie groups make their appearance in differential geometry, *e.g.* in principal and associated fiber bundles and in the infinite graded Lie algebra of the Cartan generators $(\boldsymbol{\mathcal{L}}, \boldsymbol{i}, \mathbf{d})$. Differential geometry on group manifolds on the other hand gives rise to the concepts of tangent Lie algebra and infinitesimal representation — and infinitesimal group generators, like *e.g.* the angular momentum operator play obviously a very important role in physics. Covariant differential calculi on quantum *groups* were first introduced by S. Woronowicz [21]; differential calculi on linear quantum *planes* were constructed by J. Wess & B. Zumino [22]. Since then much effort [23, 24, 25, 26, 27] has been devoted to the construction of differential geometry on quantum groups. Most approaches are unfortunately rather specific: many papers deal with the subject by considering the quantum group in question as defined by its R-matrix, and others limit themselves to particular cases. In this thesis we will develop a more abstract formulation which depends primarily on the underlying Hopf algebraic structure of a quantum group; it will therefore be a generalization of many previously obtained results, and the task of constructing specific examples of differential calculi is greatly simplified. We have to stop short of giving a "cook book recipe", however, because of case specific problems in the identification of finite bases of generators.

The thesis is divided into two parts: Part I studies vector fields on quantum groups; an algebraic and a geometric construction of a bicovariant quantum algebra of differential operators is given. Here we are mainly interested in the underlying Hopf algebra and bicovariance considerations, introducing the pure braid group and the canonical element in this context. Part II introduces additional structure in form of a Cartan calculus of differential forms, Lie derivatives and inner derivations; it is devoted to differential calculi on quantum groups and quantum planes and examples of their application.

Part I

Bicovariant Quantum Algebras

Chapter 1

Quantum Algebras and Quantum Groups

1.1 Introduction

There are two dual approaches to the quantization of Lie groups. Drinfeld [28] and Jimbo [29] have given quantum deformations of all simple Lie algebra in terms of a numerical parameter q. For the case of $\mathrm{SL}_q(2)$ one has for instance

$$[H, X_\pm] = \pm 2X_\pm, \qquad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{1.1}$$

and consistent rules for taking tensor product representations, given in terms of co-products, that we will come back to later. The second approach is due to the Russian school of Faddeev, Reshetikhin and Takhtadzhyan. Consider again $\mathrm{SL}_q(2)$ which can be defined in terms of a two by two matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.2}$$

its fundamental representation. But instead of behaving like C-numbers, the group parameters a, b, c, d now obey non-trivial commutation relations

$$\begin{array}{ccc} ab = qba, & ac = qca, & bc = cb, \\ bd = qdb, & cd = qdc, & ad - da = \lambda bc \end{array} \tag{1.3}$$

where $\lambda = (q - q^{-1})$, and

$$\det_q(T) = ad - qbc = 1. \tag{1.4}$$

The remarkable property of such quantum matrices is that, given two identical but mutually commuting copies of these matrices, their matrix product is again a quantum matrix who's elements satisfy the same commutation relations, as given above. Later we will express this property in terms of the coproduct of T, which is an algebra homomorphism.

In the following we will give a more formal introduction to quantum groups.

1.1.1 Quasitriangular Hopf Algebras

A Hopf algebra $\mathcal{A}$ is an algebra $(\mathcal{A}\cdot, +, k)$ over a field k, equipped with a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, an antipode $S : \mathcal{A} \to \mathcal{A}$, and a counit $\epsilon : \mathcal{A} \to k$, satisfying

$$
\begin{aligned}
(\Delta \otimes id)\Delta(a) &= (id \otimes \Delta)\Delta(a), \qquad \text{(coassociativity)}, && (1.5)\\
\cdot(\epsilon \otimes id)\Delta(a) &= \cdot(id \otimes \epsilon)\Delta(a) = a, \qquad \text{(counit)}, && (1.6)\\
\cdot(S \otimes id)\Delta(a) &= \cdot(id \otimes S)\Delta(a) = 1\epsilon(a), \qquad \text{(coinverse)}, && (1.7)
\end{aligned}
$$

for all $a \in \mathcal{A}$. These axioms are dual to the axioms of an algebra. There are also a number of consistency conditions between the algebra and the coalgebra structure,

$$
\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b), && (1.8)\\
\epsilon(ab) &= \epsilon(a)\epsilon(b), && (1.9)\\
S(ab) &= S(b)S(a), \qquad \text{(antihomomorphism)}, && (1.10)\\
\Delta(S(a)) &= \tau(S \otimes S)\Delta(a), \qquad \text{with} \quad \tau(a \otimes b) \equiv b \otimes a, && (1.11)\\
\epsilon(S(a)) &= \epsilon(a), \qquad \text{and} && (1.12)\\
\Delta(1) &= 1 \otimes 1, \qquad S(1) = 1, \qquad \epsilon(1) = 1_k, && (1.13)
\end{aligned}
$$

for all $a, b \in \mathcal{A}$. We will often use Sweedler's [30] notation for the coproduct:

$$\Delta(a) \equiv a_{(1)} \otimes a_{(2)} \qquad \textit{(summation is understood).} \qquad (1.14)$$

Note that a Hopf algebra is in general non-cocommutative, i.e. $\tau \circ \Delta \neq \Delta$.

A quasitriangular Hopf algebra $\mathcal{U}$ [28] is a Hopf algebra with a *universal* $\mathcal{R} \in \mathcal{U}\hat{\otimes}\mathcal{U}$ that keeps the non-cocommutativity under control,

$$\tau(\Delta(a)) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \qquad (1.15)$$

and satisfies,

$$
\begin{aligned}
(\Delta \otimes id)\mathcal{R} &= \mathcal{R}^{13}\mathcal{R}^{23}, \quad \text{and}\\
(id \otimes \Delta)\mathcal{R} &= \mathcal{R}^{13}\mathcal{R}^{12}, && (1.16)
\end{aligned}
$$

where *upper* indices denote the position of the components of $\mathcal{R}$ in the tensor product *algebra* $\mathcal{U}\hat{\otimes}\mathcal{U}\hat{\otimes}\mathcal{U}$: if $\mathcal{R} \equiv \alpha_i \otimes \beta_i$ *(summation is understood),* then *e.g.* $\mathcal{R}^{13} \equiv \alpha_i \otimes 1 \otimes \beta_i$. Equation (1.16) states that $\mathcal{R}$ generates an algebra map $\langle \mathcal{R}, . \otimes id \rangle$: $\mathcal{U}^* \to \mathcal{U}$ and an antialgebra map $\langle \mathcal{R}, id \otimes . \rangle$: $\mathcal{U}^* \to \mathcal{U}$.* The following equalities are consequences of the axioms:

$$
\begin{aligned}
\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} &= \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}, \quad \text{(quantum Yang-Baxter equation)}, && (1.17) \\
(S \otimes id)\mathcal{R} &= \mathcal{R}^{-1}, && (1.18) \\
(id \otimes S)\mathcal{R}^{-1} &= \mathcal{R}, \quad \text{and} && (1.19) \\
(\epsilon \otimes id)\mathcal{R} &= (id \otimes \epsilon)\mathcal{R} = 1. && (1.20)
\end{aligned}
$$

An example of a quasitriangular Hopf algebra that is of particular interest here is the deformed universal enveloping algebra $U_q\mathbf{g}$ of a Lie algebra g. Dual to $U_q\mathbf{g}$ is the Hopf algebra of "functions on the quantum group" Fun(G_q) ; in fact, $U_q\mathbf{g}$ and Fun(G_q) are *dually paired.* We call two Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ dually paired if there exists a non-degenerate inner product $<\,,\,>: \mathcal{U} \otimes \mathcal{A} \to k$, such that:

$$
\begin{aligned}
< xy, a > &= < x \otimes y, \Delta(a) > \equiv < x, a_{(1)} >< y, a_{(2)} >, && (1.21) \\
< x, ab > &= < \Delta(x), a \otimes b > \equiv < x_{(1)}, a >< x_{(2)}, b >, && (1.22) \\
< S(x), a > &= < x, S(a) >, && (1.23) \\
< x, 1 > &= \epsilon(x), \qquad \text{and} \qquad < 1, a > = \epsilon(a), && (1.24)
\end{aligned}
$$

for all $x, y \in \mathcal{U}$ and $a, b \in \mathcal{A}$. In the following we will assume that $\mathcal{U}$ (quasitriangular) and $\mathcal{A}$ are dually paired Hopf algebras, always keeping $U_q\mathbf{g}$ and Fun(G_q) as concrete realizations in mind.

In the next subsection we will sketch how to obtain Fun(G_q) as a matrix representation of $U_q\mathbf{g}$.

1.1.2 Dual Quantum Groups

We cannot speak about a quantum group G_q directly, just as "phase space" loses its meaning in quantum mechanics, but in the spirit of geometry on non-commuting spaces the (deformed) functions on the quantum group Fun(G_q) still make sense. This can be made concrete, if we write Fun(G_q) as a pseudo matrix group [31], generated by the elements of an $N \times N$ matrix $A \equiv (A^i{}_j)_{i,j=1\ldots N} \in M_N(\text{Fun}(G_q))^\dagger$. We require

*Notation: "." denotes an argument to be inserted and "id" is the identity map, *e.g.* $\langle \mathcal{R}, id \otimes f \rangle \equiv \alpha_i \langle \beta_i, f \rangle$; $\mathcal{R} \equiv \alpha_i \otimes \beta_i \in \mathcal{U}\hat{\otimes}\mathcal{U}$, $f \in \mathcal{U}^*$.

†We are automatically dealing with $GL_q(N)$ unless there are explicit or implicit restrictions on the matrix elements of A.

that $\rho^i{}_j \equiv < . , A^i{}_j >$ be a matrix representation of $U_q\mathbf{g}$, i.e.

$$\begin{array}{l} \rho^i{}_j : U_q\mathbf{g} \to k, \\ \rho^i{}_j(xy) = \sum_k \rho^i{}_k(x)\rho^k{}_j(y), \qquad \text{for } \forall x, y \in U_q\mathbf{g}, \end{array} \tag{1.25}$$

just like in the classical case[‡]. The universal $\mathcal{R} \in U_q\mathbf{g}\hat{\otimes}U_q\mathbf{g}$ coincides in this representation with the numerical R-matrix:

$$< \mathcal{R}, A^i{}_k \otimes A^j{}_l >= R^{ij}{}_{kl}. \tag{1.26}$$

It immediately follows from (1.21) and (1.25) that the coproduct of A is given by matrix multiplication [31, 23],

$$\Delta A = A\dot{\otimes}A, \qquad \text{i.e. } \Delta(A^i{}_j) = A^i{}_k \otimes A^k{}_j. \tag{1.27}$$

Equations (1.15), (1.22), and (1.25) imply [28, 23],

$$\begin{array}{rcl} < x, A^j{}_s A^i{}_r > & = & < \Delta x, A^j{}_s \otimes A^i{}_r > \\ & = & < \tau \circ \Delta x, A^i{}_r \otimes A^j{}_s > \\ & = & < \mathcal{R}(\Delta x)\mathcal{R}^{-1}, A^i{}_r \otimes A^j{}_s > \\ & = & R^{ij}{}_{kl} < \Delta x, A^k{}_m \otimes A^l{}_n > (R^{-1})^{mn}{}_{rs} \\ & = & < x, R^{ij}{}_{kl}A^k{}_m A^l{}_n (R^{-1})^{mn}{}_{rs} >, \end{array} \tag{1.28}$$

i.e. the matrix elements of A satisfy the following commutation relations,

$$R^{ij}{}_{kl}A^k{}_m A^l{}_n = A^j{}_s A^i{}_r R^{rs}{}_{mn}, \tag{1.29}$$

which can be written more compactly in tensor product notation as:

$$R_{12}A_1A_2 \;=\; A_2A_1R_{12}; \tag{1.30}$$

$$R_{12} = (\rho_1 \otimes \rho_2)(\mathcal{R}) \;\equiv\; < \mathcal{R}, A_1 \otimes A_2 > . \tag{1.31}$$

Lower numerical indices shall denote here the position of the respective matrices in the tensor product of *representation spaces (modules)*. The contragredient representation [32] $\rho^{-1} =< . , SA >$ gives the antipode of $\mathrm{Fun}(G_q)$ in matrix form: $S(A^i{}_j) = (A^{-1})^i{}_j$. The counit is: $\epsilon(A^i{}_j) =< 1, A^i{}_j >= \delta^i{}_j$.

Higher (tensor product) representations can be constructed from A:
A_1A_2, $A_1A_2A_3$, ..., $A_1A_2\cdots A_m$. We find numerical $\boldsymbol{R}$-matrices [33] for any pair of

‡The quintessence of this construction is that the coalgebra of $\mathrm{Fun}(G_q)$ is undeformed i.e. we keep the familiar matrix group expressions of the classical theory.

such representations:

$$\begin{aligned}
\boldsymbol{R}_{\underbrace{(1',2',\ldots,n')}_{I},\underbrace{(1,2,\ldots,m)}_{II}} &\equiv \; <\mathcal{R}, A_{1'}A_{2'}\cdots A_{n'}\otimes A_1A_2\cdots A_m> \\
&= \begin{array}{llll} R_{1'm} & \cdot R_{1'(m-1)} & \cdot \ldots \cdot & R_{1'1} \\ \cdot R_{2'm} & \cdot R_{2'(m-1)} & \cdot \ldots \cdot & R_{2'1} \\ \vdots & \vdots & & \vdots \\ \cdot R_{n'm} & \cdot R_{n'(m-1)} & \cdot \ldots \cdot & R_{n'1} \end{array}
\end{aligned} \tag{1.32}$$

Let $\boldsymbol{A}_I \equiv A_{1'}A_{2'}\cdots A_{n'}$ and $\boldsymbol{A}_{II} \equiv A_1A_2\cdots A_m$, then:

$$\boldsymbol{R}_{I,II}\,\boldsymbol{A}_I\boldsymbol{A}_{II} = \boldsymbol{A}_{II}\boldsymbol{A}_I\boldsymbol{R}_{I,II}\,. \tag{1.33}$$

$\boldsymbol{R}_{I,II}$ is the "partition function" of exactly solvable models. We will need it in section 3.1.1.

We can also write $U_q\mathbf{g}$ in matrix form [23, 32] by taking representations ϱ — *e.g.* $\varrho = <\,.\,,\boldsymbol{A}>$ — of $\mathcal{R}$ in its first or second tensor product space,

$$L^+_\varrho \;\equiv\; (id\otimes\varrho)(\mathcal{R}), \qquad L^+ \equiv\, <\mathcal{R}^{21}, A\otimes id>, \tag{1.34}$$

$$SL^-_\varrho \;\equiv\; (\varrho\otimes id)(\mathcal{R}), \qquad SL^- \equiv\, <\mathcal{R}, A\otimes id>, \tag{1.35}$$

$$L^-_\varrho \;\equiv\; (\varrho\otimes id)(\mathcal{R}^{-1}), \qquad L^- \equiv\, <\mathcal{R}, SA\otimes id>\,. \tag{1.36}$$

The commutation relations for all these matrices follow directly from the quantum Yang-Baxter equation, *e.g.*

$$\begin{aligned} 0 &= \; <\mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12} - \mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23}\,,\; id\otimes A_1\otimes A_2> \\ &= \; R_{12}L_2^+L_1^+ - L_1^+L_2^+R_{12}\,, \end{aligned} \tag{1.37}$$

where upper "algebra" indices should not be confused with lower "matrix" indices. Similarly one finds:

$$R_{12}L_2^-L_1^- = L_1^-L_2^-R_{12}, \tag{1.38}$$

$$R_{12}L_2^+L_1^- = L_1^-L_2^+R_{12}. \tag{1.39}$$

1.2 Quantized Algebra of Differential Operators

Here we would like to show how two dually paired Hopf algebras can be combined using a Hopf algebra analog of a semi-direct product construction. We obtain an algebra of differential operators consisting of elements of $U_q\mathbf{g}$ with function coefficients from $\mathrm{Fun}(G_q)$. Both the inner product with and the action on elements of $\mathrm{Fun}(G_q)$ by elements of $U_q\mathbf{g}$ will be encoded in the product of the new combined algebra. Using this construction we can avoid having to work with convolution products and similar abstract and sometimes clumsy constructions. In fact we will be able to extend the R-matrix approach of [23] so that all (Hopf algebra) relations can be written in terms of simple commutation relations of operator-valued matrices; see for example [24].

1.2.1 Actions and Coactions

Actions. A *left action* of an algebra A on a vector space V is a bilinear map,

$$\triangleright : A \otimes V \rightarrow V : x \otimes v \mapsto x \triangleright v, \tag{1.40}$$

such that:

$$(xy) \triangleright v = x \triangleright (y \triangleright v), \qquad 1 \triangleright v = v. \tag{1.41}$$

V is called a left A-module. In the case of the left action of a Hopf algebra H on an algebra A' we can in addition ask that this action preserve the algebra structure of A', i.e. $x \triangleright (ab) = (x_{(1)} \triangleright a)\,(x_{(2)} \triangleright b)^{*}$ and $x \triangleright 1 = 1\,\epsilon(x)$, for all $x \in H, a, b \in A'$. A' is then called a left H-module algebra. Right actions and modules are defined in complete analogy. A left action of an algebra on a (finite dimensional) vector space induces a right action of the same algebra on the dual vector space and vice versa, via pullback. Of particular interest to us is the left action of $\mathcal{U}$ on $\mathcal{A}$ induced by the right multiplication in $\mathcal{U}$:

$$\begin{aligned} &< y, x \triangleright a > := < yx, a > = < y \otimes x, \Delta a > = < y, a_{(1)} < x, a_{(2)} >>, \\ &\Rightarrow \quad x \triangleright a = a_{(1)} < x, a_{(2)} >, \qquad \text{for } \forall\, x, y \in \mathcal{U},\ a \in \mathcal{A}, \end{aligned} \tag{1.42}$$

where again $\Delta a \equiv a_{(1)} \otimes a_{(2)}$. This action of $\mathcal{U}$ on $\mathcal{A}$ respects the algebra structure of $\mathcal{A}$, as can easily be checked. The action of $\mathcal{U}$ on itself given by right or left multiplication does *not* respect the algebra structure of $\mathcal{U}$; see however (1.63) as an example of an algebra-respecting "inner" action.

Coaction. In the same sense as comultiplication is the dual operation to multiplication, *right* or *left coactions* are dual to left or right actions respectively. One therefore defines a right coaction of a coalgebra C on a vector space V to be a linear map,

$$\Delta_C : V \rightarrow V \otimes C : v \mapsto \Delta_C(v) \equiv v^{(1)} \otimes v^{(2)'}, \tag{1.43}$$

such that,

$$(\Delta_C \otimes id)\Delta_C = (id \otimes \Delta)\Delta_C, \qquad (id \otimes \epsilon)\Delta_C = id. \tag{1.44}$$

---

* $x\triangleright$ is called a *generalized derivation*.

Following [33] we have introduced here a notation for the coaction that resembles Sweedler's notation (1.14) of the coproduct. The prime on the second factor marks a right coaction. If we are dealing with the right coaction of a Hopf algebra H on an algebra A, we say that the coaction respects the algebra structure and A is a right H-comodule algebra, if $\Delta_H(a \cdot b) = \Delta_H(a) \cdot \Delta_H(b)$ and $\Delta_H(1) = 1 \otimes 1$, for all $a, b \in A$. In the case of a coaction on a Hopf algebra, there might be additional compatability relations between its coproduct and antipode and the coaction.

Duality of Actions and Coactions. If the coalgebra C is dual to an algebra A in the sense of (1.21), then a *right* coaction of C on V will induce a *left* action of A on V and vice versa, via

$$x \triangleright v = v^{(1)} < x, v^{(2)'} >, \qquad (\textit{general}), \tag{1.45}$$

for all $x \in A$, $v \in V$. Applying this general formula to the specific case of our dually paired Hopf algebras $\mathcal{U}$ and $\mathcal{A}$, we see that the right coaction $\Delta_{\mathcal{A}}$ of $\mathcal{A}$ on itself, corresponding to the left action of $\mathcal{U}$ on $\mathcal{A}$, as given by (1.42), is just the coproduct Δ in $\mathcal{A}$, i.e. we pick:

$$\Delta_{\mathcal{A}}(a) \equiv a^{(1)} \otimes a^{(2)'} = a_{(1)} \otimes a_{(2)}, \qquad \text{for } \forall a \in \mathcal{A}. \tag{1.46}$$

To get an intuitive picture we may think of the left action (1.42) as being a generalized *specific left translation* generated by a left invariant "tangent vector" $x \in \mathcal{U}$ of the quantum group. The coaction $\Delta_{\mathcal{A}}$ is then the generalization of an *unspecified translation*. If we supply for instance a vector $x \in \mathcal{U}$ as transformation parameter, we recover the generalized specific transformation (1.42); if we use $1 \in \mathcal{U}$, i.e. evaluate at the "identity of the quantum group", we get the identity transformation; but the quantum analog to a classical finite translation through left or right multiplication by a *specific* group element does not exist. In section 4.2 we will give a much more detailed and geometric discussion of these matter.

Quantum Matrix Formulation. The dual quantum group in its matrix form stays very close to the classical formulation and we want to use it to illustrate some of the above equations. For the matrix $A \in M_N(\text{Fun}(G_q))$ and $x \in U_q\mathbf{g}$ we find,

$$\begin{aligned} &\text{Fun}(G_q) \to \text{Fun}(G_q) \otimes \text{Fun}(G_q): \\ &\Delta_{\mathcal{A}} A = AA', \qquad \text{(right coaction)}, \end{aligned} \tag{1.47}$$

$$\begin{aligned} &\text{Fun}(G_q) \to \text{Fun}(G_q) \otimes \text{Fun}(G_q): \\ &{}_{\mathcal{A}}\Delta A = A'A, \qquad \text{(left coaction)}, \end{aligned} \tag{1.48}$$

$$
\begin{aligned}
&U_q\mathbf{g} \otimes \mathrm{Fun}(G_q) \rightarrow \mathrm{Fun}(G_q) : \\
&x \triangleright A = A < x, A >, \qquad \text{(left action)},
\end{aligned}
\tag{1.49}
$$

where matrix multiplication is implied. Following common custom we have used a prime to distinguish copies of the matrix A in different tensor product spaces. We see that in complete analogy to the classical theory of Lie algebras, we first evaluate $x \in U_q\mathbf{g}$, interpreted as a left invariant vector field, on $A \in M_n(\mathrm{Fun}(G_q))$ at the "identity of G_q", giving a numerical matrix $< x, A > \in M_n(k)$, and then shift the result by left matrix multiplication with A to an unspecified "point" on the quantum group. Unlike a Lie group, a quantum group is not a manifold in the classical sense and we hence cannot talk about its elements, except for the identity (which is also the counit of $\mathrm{Fun}(G_q)$). For $L^+ \in M_N(U_q\mathbf{g})$ equation (1.49) becomes,

$$
L_2^+ \triangleright A_1 \quad = \quad A_1 < L_2^+, A_1 > \quad = \quad A_1 R_{12}, \tag{1.50}
$$

and similarly for $L^- \in M_N(U_q\mathbf{g})$:

$$
L_2^- \triangleright A_1 \quad = \quad A_1 < L_2^-, A_1 > \quad = \quad A_1 R_{21}^{-1}. \tag{1.51}
$$

1.2.2 Commutation Relations

The left action of $x \in \mathcal{U}$ on products in $\mathcal{A}$, say bf, is given via the coproduct in $\mathcal{U}$,

$$
\begin{aligned}
x \triangleright bf &= (bf)_{(1)} < x, (bf)_{(2)} > \\
&= b_{(1)} f_{(1)} < \Delta(x), b_{(2)} \otimes f_{(2)} > \\
&= \cdot \Delta x \triangleright (b \otimes f) = b_{(1)} < x_{(1)}, b_{(2)} > \ x_{(2)} \triangleright f.
\end{aligned}
\tag{1.52}
$$

Dropping the "$\triangleright$" we can write this for arbitrary functions f in the form of commutation relations,

$$
x\, b = \Delta x \triangleright (b \otimes id) = b_{(1)} < x_{(1)}, b_{(2)} > \ x_{(2)}. \tag{1.53}
$$

This commutation relation provides $\mathcal{A} \otimes \mathcal{U}$ with an algebra structure via the *cross product*,

$$
\begin{aligned}
&\cdot : (\mathcal{A} \otimes \mathcal{U}) \otimes (\mathcal{A} \otimes \mathcal{U}) \rightarrow \mathcal{A} \otimes \mathcal{U} : \\
&ax \otimes by \mapsto ax \cdot by = a\, b_{(1)} < x_{(1)}, b_{(2)} > \ x_{(2)}\, y.
\end{aligned}
\tag{1.54}
$$

That $\mathcal{A} \otimes \mathcal{U}$ is indeed an associative algebra with this multiplication follows from the Hopf algebra axioms; it is denoted $\mathcal{A} \rtimes \mathcal{U}$ and we call it the *quantized algebra of differential operators.* The commutation relation (1.53) should be interpreted as a product in $\mathcal{A} \rtimes \mathcal{U}$. (Note that we omit $\otimes$-signs wherever they are obvious, but we sometimes insert a product sign "$\cdot$" for clarification of the formulas.) Right actions and the corresponding commutation relations are also possible: $b \triangleleft \overleftarrow{x} = <\overleftarrow{x}, b_{(1)} > b_{(2)}$ and $b\,\overleftarrow{x} = \overleftarrow{x}_{(1)} < \overleftarrow{x}_{(2)}, b_{(1)} > b_{(2)}$.

Equation (1.53) can be used to calculate arbitrary inner products of $\mathcal{U}$ with $\mathcal{A}$, if we define [38] a *right vacuum* "$>$" to act like the counit in $\mathcal{U}$ and a *left vacuum* "$<$" to act like the counit in $\mathcal{A}$,

$$\begin{aligned} < x\, b > &= < b_{(1)} < x_{(1)}, b_{(2)} > x_{(2)} > \\ &= \epsilon(b_{(1)}) < x_{(1)}, b_{(2)} > \epsilon(x_{(2)}) \\ &= < \cdot(id \otimes \epsilon)\Delta(x),\ \cdot(\epsilon \otimes id)\Delta(b) > \\ &= < x, b >, \qquad \text{for } \forall\ \ x \in \mathcal{U},\, b \in \mathcal{A}. \end{aligned} \tag{1.55}$$

Using only the right vacuum we recover formula (1.42) for left actions,

$$\begin{aligned} x\, b > &= b_{(1)} < x_{(1)}, b_{(2)} > x_{(2)} > \\ &= b_{(1)} < x_{(1)}, b_{(2)} > \epsilon(x_{(2)}) \\ &= b_{(1)} < x, b_{(2)} > \\ &= x \triangleright b, \qquad \text{for } \forall\ \ x \in \mathcal{U},\, b \in \mathcal{A}. \end{aligned} \tag{1.56}$$

As an example we will write the preceding equations for A, L^+, and L^-:

$$L_2^+ A_1 = A_1 R_{12} L_2^+, \qquad \text{(commutation relation for } L^+ \text{ with } A), \tag{1.57}$$

$$L_2^- A_1 = A_1 R_{21}^{-1} L_2^-, \qquad \text{(commutation relation for } L^- \text{ with } A), \tag{1.58}$$

$$< A = I <, \qquad \text{(left vacuum for } A), \tag{1.59}$$

$$L^+ > = L^- > = > I, \qquad \text{(right vacua for } L^+ \text{ and } L^-). \tag{1.60}$$

Equation (1.56) is not the only way to define left actions of $\mathcal{U}$ on $\mathcal{A}$ in terms of the product in $\mathcal{A} \rtimes \mathcal{U}$. An alternate definition utilizing the coproduct and antipode in $\mathcal{U}$,

$$\begin{aligned} x_{(1)}\, b\, S(x_{(2)}) &= b_{(1)} < x_{(1)}, b_{(2)} > x_{(2)}\, S(x_{(3)})^{\dagger} \\ &= b_{(1)} < x_{(1)}, b_{(2)} > \epsilon(x_{(2)}) \\ &= b_{(1)} < x, b_{(2)} > \\ &= x \triangleright b, \qquad \text{for } \forall\ \ x \in \mathcal{U},\, b \in \mathcal{A}, \end{aligned} \tag{1.61}$$

is in a sense more satisfactory because it readily generalizes to left actions of $\mathcal{U}$ on $\mathcal{A} \rtimes \mathcal{U}$,

$$\begin{aligned} x \triangleright by &:= x_{(1)}\, by\, S(x_{(2)}) \\ &= x_{(1)}\, b\, S(x_{(2)})\, x_{(3)}\, y\, S(x_{(4)})^{\dagger} \\ &= (x_{(1)} \triangleright b)\, (x_{(2)} \overset{\mathrm{ad}}{\triangleright} y), \qquad \text{for } \forall\ \ x, y \in \mathcal{U},\, b \in \mathcal{A}, \end{aligned} \tag{1.62}$$

† Notation: $(\Delta \otimes id)\Delta(x) = (id \otimes \Delta)\Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = \Delta^2(x)$, $x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)} = \Delta^3(x)$, etc., see [33].

1.3 $\mathrm{SU}_q(2)$ and $\mathrm{E}_q(2)$

In this section we will present $\mathrm{SU}_q(2)$ and show how the deformed Euclidean group $\mathrm{E}_q(2)$ and its dual, the deformed Lie algebra $U_q\mathrm{su}(2)$, can be obtained from it by contraction. The Euclidean group E(2) is a simple example of an inhomogeneous group. Deformations of such groups in general have been studied in [36]. Celeghini *et al.* [37] found a deformation of $Ue(2)$ by contracting $U_q\mathrm{su}(2)$ and simultaneously letting the deformation parameter $h \equiv \ln q$ go to zero. Here we are interested in the case where q is left untouched.

1.3.1 $\mathrm{E}_q(2)$ by contraction of $\mathrm{SU}_q(2)$

The commutation relations for $\mathrm{SU}_q(2)$ [23, 38], may be written in compact matrix notation as

$$\begin{aligned} R_{12}T_1T_2 = T_2T_1R_{12}, \quad det_qT = 1, \quad T^\dagger = T^{-1}, \\ \Delta(T) = T\dot{\otimes}T, \quad \epsilon(T) = I, \quad S(T) = T^{-1}, \end{aligned} \tag{1.71}$$

where

$$T = \begin{pmatrix} \alpha & -q\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}, \quad R = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{1.72}$$

$\lambda = q - q^{-1}$ and $\bar{q} = q$. Now set

$$\alpha \equiv v, \quad \bar{\alpha} \equiv \bar{v}, \quad \gamma \equiv \ell\bar{n} \quad \text{and} \quad \bar{\gamma} \equiv \ell n,$$

where $\ell \in \mathbb{R} - \{0\}$ is a contraction parameter. Written in terms of v, $\bar{v}$, n and $\bar{n}$, relations (1.71) become

$$\begin{aligned} det_qT = v\bar{v} + q^2\ell^2 n\bar{n} = \bar{v}v + \ell^2\bar{n}n = 1, \\ n\bar{n} = \bar{n}n, \; vn = qnv, \; v\bar{n} = q\bar{n}v, \; etc. \end{aligned}$$

and give $\mathrm{E}_q(2)$ in agreement with [39] as a contraction of $\mathrm{SU}_q(2)$ in the limit $\ell \to 0$:

$$\begin{aligned} v\bar{v} = \bar{v}v = 1, \qquad n\bar{n} = \bar{n}n, \qquad vn = qnv, \\ n\bar{v} = q\bar{v}n, \qquad v\bar{n} = q\bar{n}v, \qquad \bar{n}\bar{v} = q\bar{v}\bar{n}, \\ \Delta(n) = n\otimes\bar{v} + v\otimes n, \quad \Delta(v) = v\otimes v, \end{aligned}$$

where we have introduced the left adjoint (inner) action in $\mathcal{U}$:

$$x \overset{\mathrm{ad}}{\triangleright} y = x_{(1)} y\, S(x_{(2)}), \qquad \text{for } \forall \;\; x, y \in \mathcal{U}. \tag{1.63}$$

1.2.3 Complex Structure

In the previous section we constructed a generalized semi-direct product algebra $\mathcal{A} \rtimes \mathcal{U}$ using commutation relations

$$x\, a = a_{(1)} < x_{(1)}, a_{(2)} > x_{(2)} \tag{1.64}$$

that allow ordering of all elements of $\mathcal{A} \rtimes \mathcal{U}$ in the form $\mathcal{A} \otimes \mathcal{U}$. After some easy manipulations we derive an alternative commutation relation

$$a\, x = x_{(2)} < S^{-1} x_{(1)}, a_{(2)} > a_{(1)}, \tag{1.65}$$

good for ordering in the form $\mathcal{U} \otimes \mathcal{A}$. We can now introduce complex conjugation on $\mathcal{A} \rtimes \mathcal{U}$ as an antimultiplicative involution, *i.e.*

$$\overline{a}\, \overline{x} = \overline{x a} = \overline{x_{(2)}} < x_{(1)}, a_{(2)} >^* \overline{a_{(1)}}. \tag{1.66}$$

Comparing this equation to equation (1.65) gives the following natural choices:

$$< x, a >^* = < S^{-1} \overline{x}, \overline{a} >, \tag{1.67}$$

$$\Delta(\overline{a}) = \overline{a_{(1)}} \otimes \overline{a_{(2)}}, \tag{1.68}$$

and hence

$$S^{-1} \overline{x} = \overline{S x}. \tag{1.69}$$

In this context let us also define a *unitary representation*: A unitary representation $T \in M_n(\mathcal{A})$ satisfies $T^\dagger \equiv \overline{T}^t = ST$ so that

$$< \overline{x}, T > = < x, \overline{ST} >^* = < x, T >^\dagger, \tag{1.70}$$

i.e. the matrix representing the complex conjugate of an element in $\mathcal{U}$ is equal to the adjoint of the matrix representing the original element.

In the next section we would like to give two examples to illustrate the material presented so far. The first one, $\mathrm{SU}_q(2)$, is by now the standard example for a quantum group; it is due to [34]. We pick it as a representative for the R-matrix approach to quantum groups. Dropping the reality and the unit determinant conditions one can obtain the further examples of $\mathrm{SL}_q(2)$ and $\mathrm{GL}_q(2)$ respectively. The second example is the Quantum Euclidean Group — we show how one can obtain it via a contraction procedure from $\mathrm{SU}_q(2)$; a more complete treatment of this original work can be found in [35].

$$\begin{aligned} \Delta(\bar{n}) &= \bar{n}\otimes v + \bar{v}\otimes \bar{n}, \quad \Delta(\bar{v}) = \bar{v}\otimes\bar{v}, \\ \epsilon(n) &= \epsilon(\bar{n}) = 0, \quad \epsilon(v) = \epsilon(\bar{v}) = 1, \\ S(n) &= -q^{-1}n, \qquad S(v) = \bar{v}, \\ S(\bar{n}) &= -q\bar{n}, \qquad S(\bar{v}) = v. \end{aligned} \tag{1.73}$$

It is convenient to introduce the operators θ, $\bar{\theta}$, m, and $\bar{m}$, defined by

$$v = e^{\frac{i}{2}\theta}, \ \ \bar{\theta} = \theta, \ \ m = nv, \ \ \bar{m} = \bar{v}\bar{n}. \tag{1.74}$$

In this basis, the coproducts take on the particularly nice form

$$\begin{aligned} \Delta(m) = &\ \ m\otimes 1 + e^{i\theta}\otimes m, \ \ \Delta(\bar{m}) = \bar{m}\otimes 1 + e^{-i\theta}\otimes\bar{m}, \\ \Delta(\theta) = &\ \ \theta\otimes 1 + 1\otimes\theta. \end{aligned} \tag{1.75}$$

The matrix E given by

$$E = \begin{pmatrix} e^{i\theta} & m \\ 0 & 1 \end{pmatrix} \tag{1.76}$$

satisfies the relations

$$\Delta(E) = E\dot{\otimes}E, \ \ S(E) = E^{-1}, \ \ \epsilon(E) = I. \tag{1.77}$$

These are exactly the relations one would expect for an element of a quantum matrix group. Notice that the action of E on the column vector $\begin{pmatrix} z \\ 1 \end{pmatrix}$, where z is a complex coordinate, is given by

$$z \mapsto e^{i\theta}z + m, \ \ \bar{z}\mapsto e^{-i\theta}\bar{z} + \bar{m}. \tag{1.78}$$

We may therefore identify E as an element of the deformed 2-dimensional Euclidean group $\mathrm{E}_q(2)$. $\mathrm{Fun}(\mathrm{E}_q(2))$ is the algebra of all C^∞ functions in the group parameters of $\mathrm{E}_q(2)$, *i.e.* the algebra spanned by ordered monomials in θ, m, and $\bar{m}$. Thus, $\mathrm{Fun}(\mathrm{E}_q(2))$ is taken to be $span\{\theta^a m^b \bar{m}^c \mid a,b,c = 0,1,\ldots\}$.

1.3.2 U_qe(2) by contraction of $U_q$su(2)

The deformed universal enveloping algebra $U_q\mathrm{su}(2)$, dual to $\mathrm{Fun}(\mathrm{SU}_q(2))$, is generated by hermitian operators H, X_+, X_- satisfying

$$\begin{aligned} [H, X_\pm] = \pm 2X_\pm, &\qquad [X_+, X_-] = \tfrac{q^H - q^{-H}}{q-q^{-1}}, \\ \Delta(H) = H\otimes 1 + 1\otimes H, &\ \ \Delta(X_\pm) = X_\pm\otimes q^{H/2} + q^{-H/2}\otimes X_\pm, \\ \epsilon(H) = \epsilon(X_\pm) = 0, & \\ S(H) = -H, &\qquad S(X_\pm) = -q^{\pm 1}X_\pm. \end{aligned} \tag{1.79}$$

Following [23] these relations can be rewritten as

$$\begin{aligned} R_{12}L_2^{\pm}L_1^{\pm} = L_1^{\pm}L_2^{\pm}R_{12}, \quad & R_{12}L_2^{+}L_1^{-} = L_1^{-}L_2^{+}R_{12}, \\ \Delta(L^{\pm}) = L^{\pm}\dot{\otimes}L^{\pm}, \quad & \epsilon(L^{\pm}) = I, \\ S(L^{\pm}) = (L^{\pm})^{-1}, & \end{aligned} \tag{1.80}$$

where $L^{\pm}$ are given by

$$L^{+} = \begin{pmatrix} q^{-H/2} & q^{-1/2}\lambda X_{+} \\ 0 & q^{H/2} \end{pmatrix}, \quad L^{-} = \begin{pmatrix} q^{H/2} & 0 \\ -q^{1/2}\lambda X_{-} & q^{-H/2} \end{pmatrix}. \tag{1.81}$$

Using this matrix notation, we can state the duality between the group and the algebra by means of commutation relations

$$L_1^{+}T_2 = T_2R_{21}L_1^{+}, \quad L_1^{-}T_2 = T_2R_{12}^{-1}L_1^{-}, \tag{1.82}$$

as explained in section 1.2.2. Equations (1.82) are not only consistent with the inner products

$$< L_1^{+}, T_2 >= R_{21}, \quad < L_1^{-}, T_2 >= R_{12}^{-1}, \tag{1.83}$$

given in [23] but also contain information about the coproducts of L^{+}, L^{-} and T so that equations (1.80) can actually be derived as consistency conditions to (1.71) and (1.82). Complex conjugation can be defined as an involution on the extended algebra generated by products of T and $L^{\pm}$. This agrees with

$$\Delta(\bar{h}) = \overline{\Delta(h)}, \quad \overline{S(\bar{h})} = S^{-1}(h) \tag{1.84}$$

and

$$< \bar{\chi}, h >=< \chi, S^{-1}(\bar{h}) >^{*} . \tag{1.85}$$

Unitarity of T then implies $(L^{+})^{\dagger} = (L^{-})^{-1}$, *i.e.* $\bar{H} = H$, $\overline{X_{\pm}} = X_{\mp}$. In the present case equations (1.82) become

$$\begin{gathered} Hv = vH - v, \quad X_{+}v = q^{1/2}vX_{+} - \ell qnq^{H/2}, \quad X_{-}v = q^{1/2}vX_{-}, \\ \ell H\bar{n} = \ell(\bar{n}H - \bar{n}), \quad \ell X_{+}\bar{n} = q^{1/2}\bar{n}\ell X_{+} + \bar{v}q^{H/2}, \quad \ell X_{-}\bar{n} = \ell q^{1/2}\bar{n}X_{-}, \end{gathered} \tag{1.86}$$

plus the complex conjugate relations.

The way that the deformation parameter ℓ appears in these relations suggests the definition of new operators

$$P_{+} \equiv \ell X_{+}, \quad P_{-} \equiv \overline{P_{+}} = \ell X_{-} \quad \text{and} \quad J \equiv H/2,$$

so that we will retain non-trivial commutation relations for $P_\pm$ and J with v, $\bar{v}$, n and $\bar{n}$ in the limit $\ell \longrightarrow 0$. Inserting $P_\pm$ and J into equation (1.79) we obtain $U_q\mathrm{e}(2)$ as a contraction of $U_q\mathrm{su}(2)$ in this limit: $\bar{J} = J$, $\overline{P_\pm} = P_\mp$, and

$$
\begin{aligned}
&[J, P_\pm] = \pm P_\pm, \qquad [P_+, P_-] = 0, \\
&\Delta(P_\pm) = P_\pm \otimes q^J + q^{-J} \otimes P_\pm, \quad \Delta(J) = J \otimes 1 + 1 \otimes J, \\
&\epsilon(P_\pm) = \epsilon(J) = 0, \\
&S(J) = -J, \qquad S(P_\pm) = -q^{\pm 1} P_\pm .
\end{aligned}
\tag{1.87}
$$

Note that the *algebra* obtained in (1.87) is the same as the classical 2-dimensional Euclidean algebra $e(2)$ (with $P_\pm = P_x \pm iP_y$ and J as hermitian generators) [37]. Note, however, as a *Hopf algebra* it is still deformed; the deformation parameter q remains unchanged.

It was shown by Paul Watts [35] that this Hopf algebra is identical to the one obtained by directly constructing the dual Hopf algebra of $Fun(E_q(2))$ using methods similar to [40]. The result was

$$
< \nu^k \mu^l \xi^n, \theta^a m^b \bar{m}^c > = [k]_q! [l]_{q^{-1}}! n! \delta_{na} \delta_{lb} \delta_{kc}, \quad [x]_q! \equiv \prod_{y=1}^{x} \frac{q^{2y} - 1}{q^2 - 1}, \tag{1.88}
$$

where $\{\nu^k \mu^l \xi^n \mid k, l, n = 0, 1, \ldots\}$ is a basis for $U_q e(2)$ which is related to our operators J, P_+, and P_- via

$$
J \equiv i\xi, \quad P_+ \equiv q q^{-i\xi} \nu, \quad P_- \equiv -q^{-1} \mu q^{-i\xi}. \tag{1.89}
$$

These two constructions are summarized in the following (commutative) diagram:

$$
\begin{tikzcd}
SU_q(2) \arrow[r, "\text{contraction}", "\ell \to 0"'] \arrow[d, "\text{dual}"] & E_q(2) \arrow[d, "\text{dual}"] \\
U_q su(2) \arrow[r, "\text{contraction}", "\ell \to 0"'] & U_q e(2)
\end{tikzcd}
$$

Chapter 2

Bicovariant Calculus

Having extended the left $\mathcal{U}$-module $\mathcal{A}$ to $\mathcal{A}\rtimes\mathcal{U}$through the construction of the cross product algebra, we would now like to also extend the definition of the coaction of $\mathcal{A}$ to $\mathcal{A}\rtimes\mathcal{U}$, making the quantized algebra of differential operators an $\mathcal{A}$-bicomodule.

2.1 Left and Right Covariance

In this section we would like to study the transformation properties of the differential operators in $\mathcal{A}\rtimes\mathcal{U}$ under left and right translations, i.e. the coactions ${}_{\mathcal{A}}\Delta$ and $\Delta_{\mathcal{A}}$ respectively. We will require,

$$\begin{aligned} {}_{\mathcal{A}}\Delta(by) &= {}_{\mathcal{A}}\Delta(b){}_{\mathcal{A}}\Delta(y) = \Delta(b){}_{\mathcal{A}}\Delta(y) \quad \in \mathcal{A}\otimes\mathcal{A}\rtimes\mathcal{U}, & (2.1)\\ \Delta_{\mathcal{A}}(by) &= \Delta_{\mathcal{A}}(b)\Delta_{\mathcal{A}}(y) = \Delta(b)\Delta_{\mathcal{A}}(y) \quad \in \mathcal{A}\rtimes\mathcal{U}\otimes\mathcal{A}, & (2.2)\end{aligned}$$

for all $b \in \mathcal{A}, y \in \mathcal{U}$, so that we are left only to define ${}_{\mathcal{A}}\Delta$ and $\Delta_{\mathcal{A}}$ on elements of $\mathcal{U}$. We already mentioned that we would like to interpret $\mathcal{U}$ as the algebra of *left invariant* vector fields; consequently we will try

$$ {}_{\mathcal{A}}\Delta(y) = 1\otimes y \qquad \in \mathcal{A}\otimes\mathcal{U}, \tag{2.3}$$

as a left coaction. It is easy to see that this coaction respects not only the left action (1.42) of $\mathcal{U}$ on $\mathcal{A}$,

$$\begin{aligned} {}_{\mathcal{A}}\Delta(x\triangleright b) &= {}_{\mathcal{A}}\Delta(b_{(1)}) < x, b_{(2)} > \\ &= 1\, b_{(1)}\otimes b_{(2)} < x, b_{(3)} > \\ &= x^{(1)'} b_{(1)}\otimes(x^{(2)}\triangleright b_{(2)}) \\ &=: {}_{\mathcal{A}}\Delta(x)\triangleright{}_{\mathcal{A}}\Delta(b), \end{aligned} \tag{2.4}$$

but also the algebra structure (1.53) of $\mathcal{A}\rtimes\mathcal{U}$,

$$\begin{aligned}
{}_{\mathcal{A}}\Delta(x\cdot b) &= {}_{\mathcal{A}}\Delta(b_{(1)}) < x_{(1)}, b_{(2)} > {}_{\mathcal{A}}\Delta(x_{(2)}) \\
&= b_{(1)}\,1\otimes b_{(2)} < x_{(1)}, b_{(3)} > x_{(2)} \\
&= 1\,b_{(1)}\otimes b_{(2)} < x_{(1)}, b_{(3)} > x_{(2)} \\
&= x^{(1)\prime} b_{(1)}\otimes (x^{(2)}\cdot b_{(2)}) \\
&=: {}_{\mathcal{A}}\Delta(x)\cdot {}_{\mathcal{A}}\Delta(b).
\end{aligned} \tag{2.5}$$

The right coaction, $\Delta_{\mathcal{A}}:\mathcal{U}\to\mathcal{U}\otimes\mathcal{A}$, is considerably harder to find. We will approach this problem by extending the commutation relation (1.53) for elements of $\mathcal{U}$ with elements of $\mathcal{A}$ to a generalized commutation relation for elements of $\mathcal{U}$ with elements of $\mathcal{A}\rtimes\mathcal{U}$,

$$x\cdot by =: (by)^{(1)} < x_{(1)}\,, (by)^{(2)\prime} > x_{(2)}, \tag{2.6}$$

for all $x, y\in\mathcal{U}$, $b\in\mathcal{A}$. In the special case $b=1$ this states,

$$x\cdot y = y^{(1)} < x_{(1)}\,, y^{(2)\prime} > x_{(2)}, \qquad x, y\in\mathcal{U}, \tag{2.7}$$

and gives an implicit definition of the right coaction $\Delta_{\mathcal{A}}(y)\equiv y^{(1)}\otimes y^{(2)\prime}$ of $\mathcal{A}$ on $\mathcal{U}$. Let us check whether $\Delta_{\mathcal{A}}$ defined in this way respects the left action (1.42) of $\mathcal{U}$ on $\mathcal{A}$:

$$\begin{aligned}
< z\otimes y, \Delta_{\mathcal{A}}(x\triangleright b) > &= < zy\,,\, x\triangleright b > \\
&= < zy\,,\, b_{(1)} >< x\,,\, b_{(2)} > \\
&= < zyx\,,\, b > \\
&= < z(x^{(1)} < y_{(1)}, x^{(2)\prime} > y_{(2)})\,,\, b > \\
&= < zx^{(1)}\otimes y_{(1)}\otimes y_{(2)}\,,\, b_{(1)}\otimes x^{(2)\prime}\otimes b_{(2)} > \\
&= < zx^{(1)}\otimes y\,,\, b_{(1)}\otimes x^{(2)\prime} b_{(2)} > \\
&= < z\otimes y\,,\, (x^{(1)}\triangleright b_{(1)})\otimes x^{(2)\prime} b_{(2)} > \\
&=: < z\otimes y\,,\, \Delta_{\mathcal{A}}(x)\triangleright\Delta_{\mathcal{A}}(b) >,
\end{aligned} \tag{2.8}$$

for all $x, y, z\in\mathcal{U}$, $b\in\mathcal{A}$, q.e.d. .
Given a linear basis $\{e_i\}$ of $\mathcal{U}$ and the dual basis $\{f^j\}$ of $\mathcal{A}=\mathcal{U}^*$, $< e_i, f^j >= \delta_i^j$, we can derive an explicit expression [41] for $\Delta_{\mathcal{A}}$ from (2.7):

$$\Delta_{\mathcal{A}}(e_i) = e_j \overset{\text{ad}}{\triangleright} e_i\otimes f^j, \tag{2.9}$$

or equivalently, by linearity of $\Delta_{\mathcal{A}}$:

$$\Delta_{\mathcal{A}}(y) = e_j \overset{\text{ad}}{\triangleright} y\otimes f^j, \qquad y\in\mathcal{U}. \tag{2.10}$$

It is then easy to show that,

$$(\Delta_{\mathcal{A}} \otimes id)\Delta_{\mathcal{A}}(y) = (id \otimes \Delta)\Delta_{\mathcal{A}}(y), \tag{2.11}$$

$$(id \otimes \epsilon)\Delta_{\mathcal{A}}(y) = y, \tag{2.12}$$

proving that $\Delta_{\mathcal{A}}$ satisfies the requirements of a coaction on $\mathcal{U}$, and,

$$\Delta_{\mathcal{A}}(xy) = \Delta_{\mathcal{A}}(x)\Delta_{\mathcal{A}}(y), \tag{2.13}$$

showing that $\Delta_{\mathcal{A}}$ is an $\mathcal{U}$-algebra homomorphism; $\Delta_{\mathcal{A}}$ is however in general not a $\mathcal{U}$-Hopf algebra homomorphism. Using the explicit expression for $\Delta_{\mathcal{A}}$ we can now prove that it respects the algebra structure of $\mathcal{A}\rtimes\mathcal{U}$:

$$\begin{aligned}
\Delta_{\mathcal{A}}(xa) &= \Delta_{\mathcal{A}}(a_{(1)} < x_{(1)}, a_{(2)} > x_{(2)}) \\
&= \Delta(a_{(1)}) < x_{(1)}, a_{(2)} > \Delta_{\mathcal{A}}(x_{(2)}) \\
&= (a_{(1)} \otimes a_{(2)})(< x_{(1)}, a_{(3)} > x_{(2)}{}^{(1)} \otimes x_{(2)}{}^{(2)'}) \\
&= (a_{(1)} \otimes a_{(2)})(< x_{(1)}, a_{(3)} > e_{i_{(1)}} x_{(2)} S e_{i_{(2)}} \otimes f^i) \\
&= a_{(1)} < x_{(1)}, a_{(3)} > e_{i_{(1)}} x_{(2)} S e_{i_{(2)}} \otimes a_{(2)} f^i S a_{(4)} a_{(5)} \\
&= a_{(1)} < e_k \otimes x_{(1)} \otimes S e_l, a_{(2)} \otimes a_{(3)} \otimes a_{(4)} > e_i \overset{ad}{\triangleright} x_{(2)} \otimes f^k f^i f^l a_{(5)} \\
&= a_{(1)} < e_{i_{(1)}} x_{(1)} S e_{i_{(3)}}, a_{(2)} > e_{i_{(2)}} \overset{ad}{\triangleright} x_{(2)} \otimes f^i a_{(3)} \\
&= a_{(1)} < e_{i_{(1)}} x_{(1)} S e_{i_{(4)}}, a_{(2)} > e_{i_{(2)}} x_{(2)} S e_{i_{(3)}} \otimes f^i a_{(3)} \\
&= e_{i_{(1)}} x S e_{i_{(2)}} a_{(1)} \otimes f^i a_{(2)} \\
&= (e_i \overset{ad}{\triangleright} x \otimes f^i)(a_{(1)} \otimes a_{(2)}) \\
&= \Delta_{\mathcal{A}}(x)\Delta_{\mathcal{A}}(a). \qquad \square
\end{aligned} \tag{2.14}$$

This not only proofs that $\Delta_{\mathcal{A}}$ is a $\mathcal{A}\rtimes\mathcal{U}$-algebra homomorphism but also that the algebra structure of $\mathcal{A}\rtimes\mathcal{U}$ is compatible with $\Delta_{\mathcal{A}}$*. Clearly a less complicated way to see this would be quite welcome. In the next section we will see that $\Delta_{\mathcal{A}}$ can be obtained for all elements of $\mathcal{A}\rtimes\mathcal{U}$ via conjugation by the canonical element $C \in \mathcal{U} \otimes \mathcal{A}$ so that the $\mathcal{A}\rtimes\mathcal{U}$-homomorphism property of $\Delta_{\mathcal{A}}$ is then obvious.

2.2 The Canonical Element

So far we have shown how the two dual Hopf algebras $\mathcal{A}$ "functions on the quantum group" and $\mathcal{U}$ "deformed universal enveloping algebra" can be combined into a new algebra, the cross product or generalized semi-direct product algebra $\mathcal{A}\rtimes\mathcal{U}$, and that

*In more mathematical terms: The two-sided ideal $I := xa - a_{(1)} < x_{(1)}, a_{(2)} > x_{(2)}$ that we factored out of $U(\mathcal{A} \otimes \mathcal{U})$ to obtain $\mathcal{A}\rtimes\mathcal{U}$ is invariant under $\Delta_{\mathcal{A}}$ in the sense $\Delta_{\mathcal{A}}(I) \subset I \otimes \mathcal{A}$.

this algebra may be viewed as consisting of bicovariant differential operators and the functions they act on. This algebra is not a Hopf algebra but it has $\mathcal{A}$ and $\mathcal{U}$ as Hopf subalgebras and can in principle be reconstructed from either one of them. As we shall show, the transformation properties of the elements of $\mathcal{A}\rtimes\mathcal{U}$ are simply given through conjugation by the *canonical element* C of $\mathcal{U}\otimes\mathcal{A}$ — furthermore, we can recover many of the familiar relations for quantum groups from the consistency relations which C satisfies in the case where $\mathcal{U}$ is quasitriangular [23, 38]. One could even take an extreme point of view and base everything on the canonical element C in $\mathcal{A}\rtimes\mathcal{U}$ and its commutation relations, making any explicit reference to the coalgebra structures (Δ, S, ϵ) of $\mathcal{A}$ and $\mathcal{U}$ superfluous.
The expression of the coaction in terms of the canonical element was found in collaboration with Paul Watts [46].

Definінition and Relations

So let us now introduce the canonical element C in $\mathcal{U}\otimes\mathcal{A}$

$$C \equiv e_i \otimes f^i. \tag{2.15}$$

C satisfies several relations; for instance, note that

$$\begin{aligned} ((S\otimes id)(C))\,C &= S(e_i)e_j \otimes f^i f^j \\ &= D_k^{ij} S(e_i)e_j \otimes f^k \\ &= (m\circ(S\otimes id)\circ\Delta)(e_k)\otimes f^k \\ &= 1_{\mathcal{U}}\epsilon(e_k)\otimes f^k \\ &= 1_{\mathcal{U}}\otimes E_k f^k \\ &= 1_{\mathcal{U}}\otimes 1_{\mathcal{A}}, \end{aligned} \tag{2.16}$$

where m is the multiplication map, D_k^{ij} is the matrix that describes the coproduct in $\mathcal{U}$ and E_k is the vector corresponding to the counit in $\mathcal{U}$, so

$$(S\otimes id)(C) = C^{-1}. \tag{2.17}$$

Similar calculations also give

$$(id\otimes S)(C) = C^{-1}, \tag{2.18}$$

as well as the following:

$$(\Delta\otimes id)(C) = C_{13}C_{23}, \tag{2.19}$$

$$(id\otimes\Delta)(C) = C_{12}C_{13}, \tag{2.20}$$

$$(\epsilon\otimes id)(C) = (id\otimes\epsilon)(C) = 1_{\mathcal{U}}\otimes 1_{\mathcal{A}}. \tag{2.21}$$

There is more to C than just the above relations; this is seen by computing the right coaction of a basis vector in $\mathcal{U}$. Using (2.10)

$$\begin{aligned}\Delta_{\mathcal{A}}(e_i) &= (e_j \triangleright e_i) \otimes f^j \\ &= (e_j)_{(1)} e_i S((e_j)_{(2)}) \otimes f^j \\ &= D_j^{mn} e_m e_i S(e_n) \otimes f^j \\ &= e_m e_i S(e_n) \otimes f^m f^n \\ &= (e_m \otimes f^m)(e_i \otimes 1_{\mathcal{A}})(S(e_n) \otimes f^n) \\ &= C(e_i \otimes 1_{\mathcal{A}})(S \otimes id)(C),\end{aligned} \tag{2.22}$$

so for any $x \in \mathcal{U}$,

$$\Delta_{\mathcal{A}}(x) = C(x \otimes 1)C^{-1}. \tag{2.23}$$

However, when we think of C as living in $(\mathcal{A} \rtimes \mathcal{U}) \otimes (\mathcal{A} \rtimes \mathcal{U})$, with e_i and f^i as the bases for the subalgebras $\mathcal{U}$ and $\mathcal{A}$ of $\mathcal{A} \rtimes \mathcal{U}$ respectively, further results follow. For instance, for $a \in \mathcal{A}$,

$$\begin{aligned}C(a \otimes 1)C^{-1} &= e_i a S(e_j) \otimes f^i f^j \\ &= (a_{(1)}(e_i)_{(2)} \left\langle (e_i)_{(1)}, a_{(2)} \right\rangle) S(e_j) \otimes D_k^{ij} f^k \\ &= a_{(1)} \left\langle (e_k)_{(1)}, a_{(2)} \right\rangle (e_k)_{(2)} S((e_k)_{(3)}) \otimes f^k \\ &= a_{(1)} \otimes \left\langle e_k, a_{(2)} \right\rangle f^k \\ &= a_{(1)} \otimes a_{(2)},\end{aligned} \tag{2.24}$$

(where $1 = 1_{\mathcal{A} \rtimes \mathcal{U}} \equiv 1_{\mathcal{A}} \otimes 1_{\mathcal{U}}$) so that

$$C(a \otimes 1)C^{-1} = \Delta(a). \tag{2.25}$$

Thus, the right coaction of $\mathcal{A}$ on $\mathcal{A} \rtimes \mathcal{U}$ is obtained through *conjugation* by C

$$\Delta_{\mathcal{A}}(\alpha) = C(\alpha \otimes 1)C^{-1} \tag{2.26}$$

for any $\alpha \in \mathcal{A} \rtimes \mathcal{U}$. This expression shows explicitly that $\Delta_{\mathcal{A}}$ is an algebra homomorphism

$$\begin{aligned}\Delta_{\mathcal{A}}(\alpha\beta) &= C(\alpha\beta \otimes 1)C^{-1} \\ &= C(\alpha \otimes 1)C^{-1}C(\beta \otimes 1)C^{-1} \\ &= \Delta_{\mathcal{A}}(\alpha)\Delta_{\mathcal{A}}(\beta)\end{aligned} \tag{2.27}$$

for $\alpha, \beta \in \mathcal{A} \rtimes \mathcal{U}$, and that it is consistent with the algebra structure of $\mathcal{A} \rtimes \mathcal{U}$

$$\begin{aligned}C(xa)C^{-1} &= C(a_{(1)} < x_{(1)}, a_{(2)} > x_{(2)} \otimes 1)C^{-1} \\ &= C(a_{(1)} \otimes 1)C^{-1} < x_{(1)}, a_{(2)} > C(x_{(2)} \otimes 1)C^{-1} \\ &= \Delta(a_{(1)}) < x_{(1)}, a_{(2)} > \Delta_{\mathcal{A}}(x_{(2)}) \\ &= \Delta_{\mathcal{A}}(xa).\end{aligned} \tag{2.28}$$

We can continue doing calculations along these lines, and we find

$$C^{-1}(1 \otimes x)C = \Delta(x) \tag{2.29}$$

for $x \in \mathcal{U}$. For elements of the cross product algebra this gives the left $\mathcal{U}$-coaction

$$_{\mathcal{U}}\Delta(\alpha) \equiv \alpha_{1'} \otimes \alpha_2 = C^{-1}(1 \otimes \alpha)C, \tag{2.30}$$

that appears in the general commutation relation

$$\alpha\beta = \beta^{(1)} < \alpha_{1'}, \beta^{(2)'} > \alpha_2. \tag{2.31}$$

Using these results, together with the coproduct relations for C, we obtain the equation

$$C_{23}C_{12} = C_{12}C_{13}C_{23}. \tag{2.32}$$

(Interestingly, this equation can be viewed as giving the multiplication on $\mathcal{A}\rtimes\mathcal{U}$ as defined in (3.16).)

Quasitriangular Case

In the case where $\mathcal{U}$ is a quasitriangular Hopf algebra with universal R-matrix $\mathcal{R}$, the coproduct relations involving C imply the following consistency conditions:

$$\begin{aligned} \mathcal{R}_{12}C_{13}C_{23} &= C_{23}C_{13}\mathcal{R}_{12}, \\ \mathcal{R}_{23}C_{12} &= C_{12}\mathcal{R}_{13}\mathcal{R}_{23}, \\ \mathcal{R}_{13}C_{23} &= C_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \end{aligned} \tag{2.33}$$

To see the added significance of these equations, note that

$$\langle C, a \otimes id \rangle = a, \tag{2.34}$$

where $a \in \mathcal{A}$, and we use the notation

$$\langle x, id \rangle = x \tag{2.35}$$

for $x \in \mathcal{U}$. Let $\rho : \mathcal{U} \to M_n(k)$ be a matrix representation of $\mathcal{U}$, and define the $n \times n$ matrices $A^i{}_j \in \mathcal{A}$ by

$$\left\langle x, A^i{}_j \right\rangle \equiv \rho^i{}_j(x). \tag{2.36}$$

(These $A^i{}_j$'s are what are usually viewed as the non-commuting matrix elements of the pseudo-matrix group associated with $\mathcal{U}$ [31].) Given ρ, we can define the $\mathcal{U}$-valued matrices

$$\begin{aligned} L^+ &\equiv (id \otimes \rho)(\mathcal{R}), \\ L^- &\equiv (\rho \otimes id)(\mathcal{R}^{-1}), \end{aligned} \tag{2.37}$$

and the numerical R-matrix

$$R \equiv (\rho \otimes \rho)(\mathcal{R}). \tag{2.38}$$

Furthermore, it is easily seen that $(\rho \otimes id)(C) = A$. Now let us apply $(\rho^i{}_k \otimes \rho^j{}_l \otimes id)$ to the first of equations (2.33); the left side gives

$$\begin{aligned}(\rho^i{}_k \otimes \rho^j{}_l \otimes id)(\mathcal{R}_{12}C_{13}C_{23}) &= (\rho^i{}_m \otimes \rho^j{}_n)(\mathcal{R})(\rho^m{}_k \otimes id)(C)(\rho^n{}_l \otimes id)(C) \\ &= R^{ij}{}_{mn}A^m{}_kA^n{}_l. \end{aligned}\tag{2.39}$$

The right hand side gives $A^i{}_mA^j{}_nR^{mn}{}_{kl}$, so using the usual notation, we obtain

$$RA_1A_2 = A_2A_1R, \tag{2.40}$$

which gives the commutation relations between the elements of A. Doing similar gymnastics with the other two equations in (2.33) gives

$$\begin{aligned} L_1^+A_2 &= A_2R_{21}L_1^+, \\ L_1^-A_2 &= A_2R^{-1}L_1^-, \end{aligned}\tag{2.41}$$

which give the commutation relations between elements of $\mathcal{U}$ and $\mathcal{A}$ within $\mathcal{A}\rtimes\mathcal{U}$. (Of course, we also have the commutation relations

$$\begin{aligned} RL_2^\pm L_1^\pm &= L_1^\pm L_2^\pm R, \\ RL_2^+L_1^- &= L_1^-L_2^+R, \end{aligned}\tag{2.42}$$

between elements of $\mathcal{U}$, obtained as above from $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$, the quantum Yang-Baxter equation.) Thus, we recover all the commutation relations between A and $L^\pm$ given in [38].

2.3 Bicovariant Vector Fields

The appearance of an infinite sum in equation (2.10) or for that matter (2.26) suggests that the elements of $\mathcal{U}$ have in general very complicated transformation properties. In contrast, the functions in $\mathcal{A}$, especially those constructed from the matrix elements of A, have very simple transformation properties given by the coproduct in $\mathcal{A}$ (1.27). We would like to show how to construct vector fields corresponding to — and inheriting the simple behavior of — these functions. This construction can then be used to find a basis of vector fields that closes under coaction and hence under (mutual) adjoint actions. First we need to proof the following lemma.

Lemma: *Let $\Upsilon \equiv \Upsilon_i \otimes \Upsilon^i \in \mathcal{U} \otimes \mathcal{U}$ such that $\Upsilon\Delta(x) = \Delta(x)\Upsilon$ for all $x \in \mathcal{U}$, then it follows that $\Upsilon_i \otimes (x \overset{\text{ad}}{\triangleright} \Upsilon^i) = (\Upsilon_i \overset{\text{ad}}{\triangleleft} x) \otimes \Upsilon^i$ with $\Upsilon_i \overset{\text{ad}}{\triangleleft} x \equiv S(x_{(1)})\Upsilon_i x_{(2)}$ for all $x \in \mathcal{U}$.*

Proof:

$$\begin{aligned} \Upsilon_i \otimes (x \overset{\text{ad}}{\triangleright} \Upsilon^i) &\equiv \Upsilon_i \otimes x_{(1)}\Upsilon^i S(x_{(2)}) \\ &= S(x_{(1)})x_{(2)}\Upsilon_i \otimes x_{(3)}\Upsilon^i S(x_{(4)}) \\ &= S(x_{(1)})\Upsilon_i x_{(2)} \otimes \Upsilon^i x_{(3)} S(x_{(4)}) \\ &= (\Upsilon_i \overset{\text{ad}}{\triangleleft} x) \otimes \Upsilon^i. \qquad \square \end{aligned} \tag{2.43}$$

For any function $b \in \mathcal{A}$, define

$$Y_b := \langle \Upsilon, b \otimes id \rangle \in \mathcal{U}. \tag{2.44}$$

Proposition: This vector field has the following transformation property:

$$\Delta_{\mathcal{A}}(Y_b) = Y_{b_{(2)}} \otimes S(b_{(1)})b_{(3)} \tag{2.45}$$

Proof:

$$\begin{aligned} \Delta_{\mathcal{A}}(Y_b) &= \langle \Upsilon_i, b\rangle\, (e_k \triangleright \Upsilon^i) \otimes f^k \\ &= \langle \Upsilon_i \triangleleft e_k, b\rangle\, \Upsilon^i \otimes f^k \\ &= \left\langle \Upsilon_i \otimes e_k, b_{(2)} \otimes S(b_{(1)})b_{(3)} \right\rangle \Upsilon^i \otimes f^k \\ &= Y_{b_{(2)}} \otimes S(b_{(1)})b_{(3)}. \qquad \square \end{aligned} \tag{2.46}$$

Example: *Let $\Upsilon := \mathcal{R}_{21}\mathcal{R}_{12}$ and $b := A^i{}_j$, then $Y^i{}_j := Y_{A^i{}_j} = \langle \mathcal{R}_{21}\mathcal{R}_{12}, A^i{}_j \otimes id\rangle$ is the well-known matrix of vector fields $L^+S(L^-)$ introduced in [43] with coaction: $\Delta_{\mathcal{A}}(Y^i{}_j) = Y^k{}_l \otimes S(A^i{}_k)A^l{}_j$.*

This last example may in some cases (when $\mathcal{U}$ is factorizable [47]) provide a way of computing the canonical element C from $\mathcal{R}_{21}\mathcal{R}_{12}$: Let μ be the map

$$\mu : \mathcal{A} \to \mathcal{U} : \ b \mapsto \langle \mathcal{R}_{21}\mathcal{R}_{12}, b \otimes id\rangle\,, \tag{2.47}$$

then $(id \otimes \mu)(C) = e_i \langle \mathcal{R}_{21}\mathcal{R}_{12}, f^i \otimes id\rangle = \mathcal{R}_{21}\mathcal{R}_{12}$ and, in cases where μ is invertible;

$$C = (id \otimes \mu^{-1})(\mathcal{R}_{21}\mathcal{R}_{12}). \tag{2.48}$$

In the next section we will elaborate more on elements like Υ and their connection to the "Pure Braid Group". There we will also proof the reverse of Proposition 2.45.

2.4 The Pure Braid Group

Introduction

In the classical theory of Lie algebras we start the construction of a bicovariant calculus by introducing a matrix $\Omega = A^{-1}\mathrm{d}A \in \Gamma$ of one-forms that is invariant under left transformations,

$$A \to A'A: \qquad \mathrm{d} \to \mathrm{d}, \quad \Omega \to \Omega, \tag{2.49}$$

and covariant under right transformations,

$$A \to AA': \qquad \mathrm{d} \to \mathrm{d}, \quad \Omega \to A'^{-1}\Omega A'. \tag{2.50}$$

The dual basis to the entries of this matrix Ω form a matrix X of vector fields with the same transformation properties as Ω:

$$\langle \Omega^i{}_j, X^k{}_l \rangle = \delta^i{}_l \delta^k{}_j \qquad \textit{(classical)}. \tag{2.51}$$

We find,

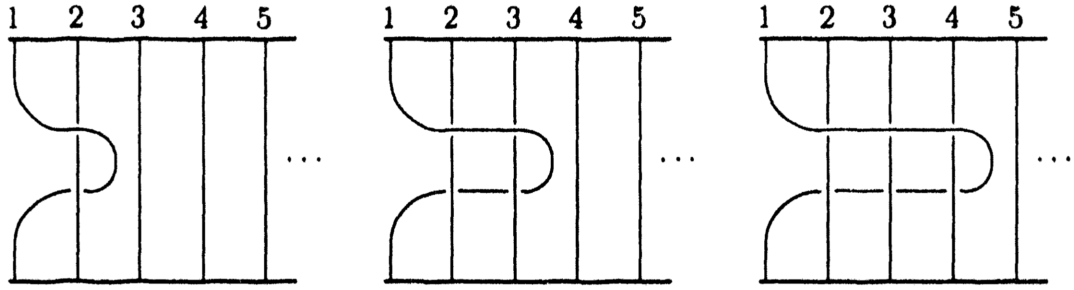
$$X = (A^T \frac{\partial}{\partial A})^T \qquad \textit{(classical)}. \tag{2.52}$$

Woronowicz [21] was able to extend the definition of a bicovariant calculus to quantum groups. His approach via differential forms has the advantage that coactions (transformations) ${}_{\mathcal{A}}\Delta : \Gamma \to \mathcal{A} \otimes \Gamma$ and $\Delta_{\mathcal{A}} : \Gamma \to \Gamma \otimes \mathcal{A}$ can be introduced very easily through,

$$\begin{aligned} {}_{\mathcal{A}}\Delta(\mathrm{d}a) &= (id \otimes \mathrm{d})\Delta a, & (2.53)\\ \Delta_{\mathcal{A}}(\mathrm{d}a) &= (\mathrm{d} \otimes id)\Delta a, & (2.54) \end{aligned}$$

where $\mathcal{A}$ is the Hopf algebra of 'functions on the quantum group', $a \in \mathcal{A}$ and Δ is the coproduct in $\mathcal{A}$. Equations (2.53,2.54) rely on the existence of an invariant map $\mathrm{d} : \mathcal{A} \to \Gamma$ provided by the exterior derivative. A construction of the bicovariant calculus starting directly from the vector fields is much harder because simple formulae like (2.53,2.54) do not seem to exist a priori. The properties of the element Υ that we introduced in the previous section however indicates exceptions: We will show that for Hopf algebras that allow "pure braid elements" Υ, like e.g. quasitriangular Hopf algebras, invariant maps from $\mathcal{A}$ to the quantized algebra of differential operators $\mathcal{A} \rtimes \mathcal{U}$ can indeed be constructed. Using these maps we will then construct differential operators with simple transformation properties and in particular a bicovariant matrix of vector fields roughly corresponding to (2.52).

In the next subsection we will hence describe a map, $\Phi : \mathcal{A} \to \mathcal{A} \rtimes \mathcal{U}$, that is invariant under (right) coactions and can be used to find $\Delta_{\mathcal{A}}$ on specific elements $\Phi(b) \in \mathcal{U}$ in terms of $\Delta_{\mathcal{A}}$ on $b \in \mathcal{A}$: $\Delta_{\mathcal{A}}(\Phi(b)) = (\Phi \otimes id)\Delta_{\mathcal{A}}(b)$.



$\mathcal{R}^{21}\mathcal{R}^{12}$ $\quad$ $\mathcal{R}^{21}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{12}$ $\quad$ $\mathcal{R}^{21}\mathcal{R}^{31}\mathcal{R}^{41}\mathcal{R}^{14}\mathcal{R}^{13}\mathcal{R}^{12}$

Figure 2.1: Generators of the pure braid group.

2.4.1 Invariant Maps and the Pure Braid Group

A basis of generators for the pure braid group B_n on n strands can be realized in $\mathcal{U}$, or for that matter $U_q\mathbf{g}$, as follows in terms of the universal $\mathcal{R}$:

$$\begin{array}{l} \mathcal{R}^{21}\mathcal{R}^{12}, \qquad \mathcal{R}^{21}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{12} \equiv (id \otimes \Delta)\mathcal{R}^{21}\mathcal{R}^{12}, \ \ldots \ , \\ \mathcal{R}^{21}\cdots\mathcal{R}^{n1}\mathcal{R}^{1n}\cdots\mathcal{R}^{12} \equiv (id^{(n-2)} \otimes \Delta)(id^{(n-3)} \otimes \Delta)\cdots(id \otimes \Delta)\mathcal{R}^{21}\mathcal{R}^{12}, \end{array}$$

and their inverses; see figure 2.1 and ref.[32]. All polynomials in these generators are central in $\Delta^{(n-1)}\mathcal{U} \equiv \{\Delta^{(n-1)}(x) \mid x \in \mathcal{U}\}$; in fact we can take,

$$\mathrm{span}\{B_n\} := \{\mathcal{Z}_n \in \mathcal{U}^{\hat{\otimes} n} | \mathcal{Z}_n\Delta^{(n-1)}(x) = \Delta^{(n-1)}(x)\mathcal{Z}_n, \ \ \mathrm{for}\forall x \in \mathcal{U}\}, \tag{2.55}$$

as a definition.

Remark: Elements of span$\{B_n\}$ do not have to be written in terms of the universal $\mathcal{R}$, they also arise from central elements and coproducts of central elements. This is particularly important in cases where $\mathcal{U}$ is not a quasitriangular Hopf algebra.

There is a map, $\Phi_n : \mathcal{A} \to \mathcal{A} \otimes \mathcal{U}^{\otimes(n-1)} \hookrightarrow (\mathcal{A} \rtimes \mathcal{U})^{\otimes(n-1)}$, associated to each element of span$\{B_n\}$:

$$\Phi_n(a) := \mathcal{Z}_n \triangleright (a \otimes id^{(n-1)}), \qquad \text{with } \mathcal{Z}_n \in \mathrm{span}\{B_n\}, \ a \in \mathcal{A}. \tag{2.56}$$

We will first consider the case $n = 2$. Let $\Upsilon \equiv \Upsilon_{1_\iota} \otimes \Upsilon_{2_\iota}$ be an element of span$\{B_2\}$ and $\Phi(b) = \Upsilon \triangleright (b \otimes id) = b_{(1)} < \Upsilon_{1_\iota}, b_{(2)} > \Upsilon_{2_\iota}$, for $b \in \mathcal{A}$. We compute,

$$\begin{aligned} x \cdot \Phi(b) &= \Delta(x) \triangleright \Phi(b) \\ &= \Delta(x)\Upsilon \triangleright (b \otimes id) \\ &= \Upsilon\Delta(x) \triangleright (b \otimes id) \\ &= \Upsilon \triangleright (x \cdot b) \\ &= \Phi(b_{(1)}) < x_{(1)}, b_{(2)} > x_{(2)}, \end{aligned} \tag{2.57}$$

which, when compared to the *generalized* commutation relation (2.6), i.e.

$$x \cdot \Phi(b) \quad = \quad [\Phi(b)]^{(1)} < x_{(1)}, [\Phi(b)]^{(2)'} > x_{(2)}, \tag{2.58}$$

gives,

$$\begin{aligned} &\Delta_{\mathcal{A}}(\Phi(b)) \equiv [\Phi(b)]^{(1)} \otimes [\Phi(b)]^{(2)'} = \Phi(b_{(1)}) \otimes b_{(2)} \\ &\Rightarrow \Delta_{\mathcal{A}}(\Phi(b)) = (\Phi \otimes id)\Delta_{\mathcal{A}}(b), \end{aligned} \tag{2.59}$$

as promised. However we are especially interested in the transformation properties of elements of $\mathcal{U}$, so let us define,

$$\Upsilon_b := < \Upsilon, b \otimes id > = < \Upsilon_{1_i}, b > \Upsilon_{1_i}, \tag{2.60}$$

for $\Upsilon \in \mathrm{span}(B_2)$, $b \in \mathcal{A}$. Using (2.2,2.59) we recover the result of Proposition 2.45

$$\Delta_{\mathcal{A}}(\Upsilon_b) = \Upsilon_{b_{(2)}} \otimes S(b_{(1)})b_{(3)}. \tag{2.61}$$

Let us now proof the reverse statement:

Proposition: If there is a linear map $\Upsilon : \mathcal{A} \to \mathcal{U}$, realized and labelled by some element $\Upsilon \in \mathcal{U}\hat{\otimes}\mathcal{U}$ via $b \mapsto \Upsilon_b \equiv < \Upsilon, b \otimes id >$, $\forall b \in \mathcal{A}$, such that the resulting element in $\mathcal{U}$ transforms like $\Delta_{\mathcal{A}}\Upsilon_b = \Upsilon_{b_{(2)}} \otimes Sb_{(1)}b_{(3)}$; then $\Upsilon \in \mathrm{span}(B_2)$, *i.e.* Υ must commute with all coproducts.

Proof: For all $x \in \mathcal{U}$ and $b \in \mathcal{A}$

$$\begin{aligned} < \Delta x \Upsilon, b \otimes id > &= < \Delta x, b_{(1)} \otimes id >< \Upsilon, b_{(2)} \otimes id > \\ &= < x_{(1)}, b_{(1)} > x_{(2)} \Upsilon_{b_{(2)}} \\ &= < x_{(1)}, b_{(1)} > \Upsilon_{b_{(3)}} < x_{(2)}, Sb_{(2)}b_{(4)} > x_{(3)} \\ &= \Upsilon_{b_{(3)}} < x_{(1)}, b_{(1)}Sb_{(2)}b_{(4)} > x_{(2)} \\ &= \Upsilon_{b_{(1)}} < x_{(1)}, b_{(2)} > x_{(2)} \\ &= < \Upsilon \Delta x, b \otimes id > . \qquad \Box \end{aligned} \tag{2.62}$$

From this follows an important **Corollary:**

If there exists a map $\phi : \mathcal{A} \to \mathcal{A} \rtimes \mathcal{U}$ such that $\Delta_{\mathcal{A}} \circ \phi = (\phi \otimes id) \circ \Delta$; then it follows that $\phi(b) = b_{(1)} < \Upsilon, b_{(2)} \otimes id >$ with $\Upsilon \in \mathrm{span}(B_2)$ for all $b \in \mathcal{A}$ and vice versa.

Here are a few important examples for "pure braid elements": For the simplest non-trivial example in the case of a quasitriangular Hopf algebra $\mathcal{Y} \equiv \mathcal{R}^{21}\mathcal{R}^{12}$ and $b \equiv A^i{}_j$, we obtain the 'reflection-matrix'[42] $Y \in \mathrm{M}_n(\mathcal{U})$, which has been introduced before by other authors [43, 44] in connection with integrable models and the differential calculus on quantum groups,

$$\begin{aligned} Y^i{}_j &:= Y_{A^i{}_j} \\ &= <\mathcal{R}^{21}\mathcal{R}^{12}, A^i{}_j \otimes id> \\ &= (<\mathcal{R}^{31}\mathcal{R}^{23}, A\dot{\otimes}A \otimes id>)^i{}_j \\ &= (<\mathcal{R}^{21}, A \otimes id><\mathcal{R}^{12}, A \otimes id>)^i{}_j \\ &= (L^+SL^-)^i{}_j, \end{aligned} \tag{2.63}$$

with transformation properties,

$$A \longrightarrow AA': \qquad Y^i{}_j \rightarrow \Delta_{\mathcal{A}}(Y^i{}_j) = Y^k{}_l \otimes S(A^i{}_k)A^l{}_j \equiv ((A')^{-1}YA')^i{}_j, \tag{2.64}$$

$$A \longrightarrow A'A: \qquad Y^i{}_j \rightarrow {}_{\mathcal{A}}\Delta(Y^i{}_j) = 1 \otimes Y^i{}_j. \tag{2.65}$$

The commutation relation (1.53) becomes in this case,

$$\begin{aligned} Y_2A_1 &= L_2^+SL_2^-A_1 \\ &= L_2^+A_1SL_2^-R_{21} \\ &= A_1R_{12}L_2^+SL_2^-R_{21} \\ &= A_1R_{12}Y_2R_{21}, \end{aligned} \tag{2.66}$$

where we have used (1.57), (1.58), and the associativity of the cross product (1.54); note that we did not have to use any explicit expression for the coproduct of Y. The matrix $\Phi(A^i{}_j) = A^i{}_kY^k{}_j$ transforms exactly like A, as expected, and interestingly even satisfies the same commutation relation as A,

$$R_{12}(AY)_1(AY)_2 = (AY)_2(AY)_1R_{12}, \tag{2.67}$$

as can be checked by direct computation. C. Chryssomalakos [45] found an "explanation" for this fact by expressing AY in terms of casimirs. We will come back to this in the next section.

The choice, $\mathcal{Y} \equiv (1 - \mathcal{R}^{21}\mathcal{R}^{12})/\lambda$, where $\lambda \equiv q - q^{-1}$, and again $b \equiv A^i{}_j$ gives us a matrix $X \in \mathrm{M}_n(\mathcal{U})$,

$$X^i{}_j := <(1 - \mathcal{R}^{21}\mathcal{R}^{12})/\lambda, A^i{}_j \otimes id> = ((I - Y)/\lambda)^i{}_j, \tag{2.68}$$

that we will encounter again in section 4.1. X has the same transformation properties as Y and is the quantum analog of the classical matrix (2.52) of vector fields.

Finally, the particular choice $b \equiv \det_q A$ in conjunction with $\mathcal{Y} \equiv \mathcal{R}^{21}\mathcal{R}^{12}$ can serve as the definition of the quantum determinant of Y,

$$\mathrm{Det}Y := Y_{\det_q A} = <\mathcal{R}^{21}\mathcal{R}^{12}, \det_q A \otimes id>; \tag{2.69}$$

we will come back to this in the next section, but let us just mention that this definition of $\mathrm{Det}Y$ agrees with,

$$\begin{aligned}
\det_q(AY) &= \det_q(A < \mathcal{R}^{21}\mathcal{R}^{12}, A \otimes id >) \\
&= \det_q A < \mathcal{R}^{21}\mathcal{R}^{12}, \det_q A \otimes id > \\
&= \det_q A \, \mathrm{Det}Y.
\end{aligned} \tag{2.70}$$

Before we can consider maps Φ_n for $n > 2$ we need to extend the algebra and coalgebra structure of $\mathcal{A}\rtimes\mathcal{U}$ to $(\mathcal{A}\rtimes\mathcal{U})^{\otimes(n-1)}$. It is sufficient to consider $(\mathcal{A}\rtimes\mathcal{U})^{\otimes 2}$; all other cases follow by analogy. If we let

$$(a \otimes b)(x \otimes y) \;=\; ax \otimes by, \qquad \text{for } \forall\, a, b \in \mathcal{A},\; x, y \in \mathcal{U}, \tag{2.71}$$

then it follows that

$$\begin{aligned}
x \cdot a \otimes y \cdot b &= a_{(1)} < x_{(1)}\,,\, a_{(2)} > x_{(2)} \otimes b_{(1)} < y_{(1)}\,,\, b_{(2)} > y_{(2)} \\
&= (a \otimes b)_{(1)} < (x \otimes y)_{(1)}\,,\, (a \otimes b)_{(2)} > (x \otimes y)_{(2)} \\
&= (x \otimes y) \cdot (a \otimes b), \qquad \text{for } \forall\, a, b \in \mathcal{A},\; x, y \in \mathcal{U},
\end{aligned} \tag{2.72}$$

as expected from a tensor product algebra. If we coact with $\mathcal{A}$ on $\mathcal{A}\rtimes\mathcal{U}^{\otimes 2}$, or higher powers, we simply collect all the contributions of $\Delta_{\mathcal{A}}$ from each tensor product space in one space on the right:

$$\begin{aligned}
&\Delta_{\mathcal{A}}(ax \otimes by) \;=\; (ax)^{(1)} \otimes (by)^{(1)} \otimes (ax)^{(2)'}(by)^{(2)'}, \\
&\text{for } \forall\, a, b \in \mathcal{A},\; x, y \in \mathcal{U}.
\end{aligned} \tag{2.73}$$

2.5 Casimirs

Casimirs play an important role in the theory of quantum groups, even more so than in classical group theory. They, or rather characters related to them, label representations; casimirs — in particular $\mathrm{tr}_q(Y)$ and $\mathrm{Det}_q(Y)$ show up as coefficients in the characteristic polynomial for the matrix of bicovariant generators Y and finally extra non-classical generators in Quantum Lie Algebras are given by casimirs. Here we want to collect some formulas for casimir operators and comment on a few of their uses.

Casimirs related to Δ^{Ad}-invariant elements of $\mathcal{A}$

Centrality of elements of $\mathcal{U}$ is synonymous to their invariance under the right $\mathcal{A}$-coaction because of

$$xy = y^{(1)} < x_{(1)}, y^{(2)'} > x_{(2)}, \qquad \Delta_{\mathcal{A}} y \equiv y^{(1)} \otimes y^{(2)'} \tag{2.74}$$

— if $\Delta_{\mathcal{A}} c = c \otimes 1$ then $xc = c < x_{(1)}, 1 > x_{(2)} = c\epsilon(x_{(1)})x_{(2)} = cx$. In the previous sections we have shown how to construct elements of $\mathcal{U}$ from elements of $\mathcal{A}$, preserving their transformation properties under adjoint coaction. The quantum determinant and the quantum traces are invariant under Δ^{Ad}, giving our first group of examples for casimir operators:

$$\mathrm{tr}_q(Y^k) = < \Upsilon, \mathrm{tr}_q(A^k) \otimes id >, \qquad \mathrm{Det}_q(Y) = < \Upsilon, \mathrm{Det}_q(A) \otimes id >, \tag{2.75}$$

where Υ is an element of the pure braid group, *i.e.* $\Upsilon\Delta(y) = \Delta(y)\Upsilon$ for all $y \in \mathcal{U}$. In the case of $\Upsilon = \mathcal{R}^{21}\mathcal{R}^{12}$ the first set of casimirs coincides with the ones given in [23]; there are in fact as many independent ones as the rank of the corresponding group.

Casimirs arising from the pure braid group

Let $\gamma := \Upsilon_i S \Upsilon^i$, where $\Upsilon \equiv \Upsilon_i \otimes \Upsilon^i$ is an element of the pure braid group. Here is a proof that γ is a casimir:

$$\begin{array}{rcl} \Upsilon_i y_{(1)} \otimes \Upsilon^i y_{(2)} & = & y_{(1)}\Upsilon_i \otimes y_{(2)}\Upsilon^i \\ \Leftrightarrow \Upsilon_i y_{(1)} S(y_{(2)}) S(\Upsilon^i) & = & y_{(1)}\Upsilon_i S(\Upsilon^i) S(y_{(2)}) \\ \Leftrightarrow \epsilon(y)\gamma & = & y_{(1)}\gamma S(y_{(2)}) \\ \Leftrightarrow y\gamma & = & \gamma y. \qquad \square \end{array} \tag{2.76}$$

More casimirs like $S(\Upsilon_i)\Upsilon^i$, $\epsilon(\Upsilon_i)\Upsilon^i$, $\Upsilon_i\epsilon(\Upsilon^i)$ can be obtained in similar ways.

Relation to Drinfeld's casimir c. Drinfeld [28, 19] showed that the S^2 automorphism is realized as conjugation by an element u in quasitriangular Hopf algebras. Let $\mathcal{R} = \alpha_i \otimes \beta_i$, then $u = S(\beta_i)\alpha_i$, $S(u) = \alpha_j S(\beta_j)$ and $c = uS(u)$. If we choose $\Upsilon = \mathcal{R}^{21}\mathcal{R}^{12}$ as our pure braid element, then

$$\begin{array}{rcl} \Upsilon_i S(\Upsilon^i) & = & \beta_i \alpha_j S(\beta_j) S(\alpha_j) \\ & = & \beta_i S(u) S(\alpha_j) \left(S(u)\right)^{-1} S(u) \\ & = & \beta_i S^{-1}(\alpha_i)\alpha_j S(\beta_j) \\ & = & uS(u) = c \end{array} \tag{2.77}$$

and similar $S(\Upsilon_i)\Upsilon^i = S^{-1}(c)$.

Extra Generators

Classically the commutator of Lie bracket of a casimir c and some vector field y vanishes because of the centrality of c; so casimirs do not play a role in classical Lie algebras. In the quantum case the commutator is replaced by the adjoint action and then

$$c \overset{\text{ad}}{\triangleright} y = c_{(1)} y S(c_{(2)}) \neq 0 \tag{2.78}$$

in general. We however still have

$$y \overset{\text{ad}}{\triangleright} c = y_{(1)} c S(y_{(2)}) = \epsilon(y) c, \tag{2.79}$$

which is zero if $\epsilon(y) = 0$, as is usually the case for a generator of a quantum Lie algebra.

Special properties of AY

We remarked earlier that $AY = AL^+SL^-$ satisfies the same algebra as A does. C.Chryssomalakos [45] found that this is also true for AY^k and gave a nice explanation for this fact that I would like to quote here: Using the coproduct of c

$$\Delta c = (\mathcal{R}^{21}\mathcal{R}^{12})^{-2}(c \otimes c) \tag{2.80}$$

one easily derives

$$AY = \alpha c^{-1} A c, \tag{2.81}$$

where $\alpha \delta_i^j = < c, A^j{}_i >$. In the case of a Ribbon Hopf Algebra [32, 19] there is a central element w that implements the square root of c; its coproduct is

$$\Delta w = (\mathcal{R}^{21}\mathcal{R}^{12})^{-1}(w \otimes w), \tag{2.82}$$

leading to

$$AY = \alpha^{-\frac{1}{2}} w^{-1} A w \tag{2.83}$$

and more general

$$AY^k = \alpha^{-\frac{k}{2}} w^{-k} A w^k. \tag{2.84}$$

In the case that we are not dealing with a ribbon Hopf algebra, there is an alternative expression [45] based on another algebra homomorphism $A \mapsto AD^{-1}$, where $D = < u, A >$,

$$AY = \alpha u A D^{-1} u^{-1}. \tag{2.85}$$

From the form of these equations it is clear that the map $Cr : A \mapsto AY^k$ is an algebra homomorphism. It also follows quite easily that this map is invariant in the sense $\Delta_{\mathcal{A}} \circ Cr = (Cr \otimes id) \circ \Delta$. This immediately poses the question of a relation to our theory of bicovariant generators and pure braid elements. For the "Ribbon" case we find

$$\begin{aligned} Y^k &= \alpha^{-\frac{k}{2}} S(A) w^{-k} A w^k \\ &= \alpha^{-\frac{k}{2}} < (w^{-k})_{(1)}, A > (w^{-k})_{(2)} w^k, \end{aligned} \tag{2.86}$$

so that $Y^k =< \Upsilon, A \otimes id >$ with the pure braid element

$$\Upsilon = \alpha^{-\frac{k}{2}} \Delta(w^{-k})(1 \otimes w^k). \tag{2.87}$$

The "Non-Ribbon" case gives

$$\begin{aligned} Y &= \alpha S(A) u A D^{-1} u^{-1} \\ &= \alpha < u_{(1)}, A > u_{(2)} < u^{-1}, A > u^{-1} \\ &= \alpha < u_{(1)} u^{-1}, A > u_{(2)} u^{-1}, \end{aligned} \tag{2.88}$$

such that again $Y =< \Upsilon', A \otimes id >$ with another pure braid element

$$\Upsilon' = \alpha \Delta(u)(u^{-1} \otimes u^{-1}). \tag{2.89}$$

Both examples are hence as expected special cases of the pure braid formulation.

Chapter 3

$\mathcal{R}$ - Gymnastics

In this chapter we would like to study for the example of $Y \in M_N(\mathcal{U})$ the matrix form of $\mathcal{U}$ as introduced at the end of section 1.1.2. Let us first derive commutation relations for Y from the quantum Yang-Baxter equation (QYBE): Combine the following two copies of the QYBE,

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}, \text{ and } \mathcal{R}^{21}\mathcal{R}^{31}\mathcal{R}^{32} = \mathcal{R}^{32}\mathcal{R}^{31}\mathcal{R}^{21},$$

resulting in,

$$\mathcal{R}^{21}\mathcal{R}^{31}\underline{\mathcal{R}^{32}\mathcal{R}^{12}\mathcal{R}^{13}}\mathcal{R}^{23} = \mathcal{R}^{32}\underline{\mathcal{R}^{31}\mathcal{R}^{21}\mathcal{R}^{23}}\mathcal{R}^{13}\mathcal{R}^{12},$$

and apply the QYBE to the underlined part to find,

$$\mathcal{R}^{21}(\mathcal{R}^{31}\mathcal{R}^{13})\mathcal{R}^{12}(\mathcal{R}^{32}\mathcal{R}^{23}) = (\mathcal{R}^{32}\mathcal{R}^{23})\mathcal{R}^{21}(\mathcal{R}^{31}\mathcal{R}^{13})\mathcal{R}^{12},$$

which, when evaluated on $< \, . \, , A_1 \otimes A_2 \otimes id >$, gives:

$$R_{21}Y_1R_{12}Y_2 = Y_2R_{21}Y_1R_{12}. \tag{3.1}$$

3.1 Higher Representations and the •-Product

As was pointed out in section 1.1.2, tensor product representations of $\mathcal{U}$ can be constructed by combining A-matrices. This product of A-matrices defines a new product for $\mathcal{U}$ which we will denote "•". The idea is to combine Y-matrices (or L^+, L^- matrices) in the same way as A-matrices to get higher dimensional matrix representations,

$$\begin{aligned} Y_1 \bullet Y_2 &:= < \mathcal{R}^{21}\mathcal{R}^{12}, A_1A_2 \otimes id >, & (3.2)\\ L_1^+ \bullet L_2^+ &:= < \mathcal{R}^{21}, A_1A_2 \otimes id >, & (3.3)\\ SL_1^- \bullet SL_2^- &:= < \mathcal{R}^{12}, A_1A_2 \otimes id > . & (3.4) \end{aligned}$$

Let us evaluate (3.2) in terms of the ordinary product in $\mathcal{U}$,

$$
\begin{aligned}
Y_1 \bullet Y_2 &= < (\Delta \otimes id)\mathcal{R}^{21}\mathcal{R}^{12}, A_1 \otimes A_2 \otimes id > \\
&= < \mathcal{R}^{32}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{23}, A_1 \otimes A_2 \otimes id > \\
&= < (\mathcal{R}^{-1})^{12}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{12}\mathcal{R}^{32}\mathcal{R}^{23}, A_1 \otimes A_2 \otimes id > \\
&= R_{12}^{-1} Y_1 R_{12} Y_2,
\end{aligned}
\tag{3.5}
$$

where we have used,

$$
\begin{aligned}
\mathcal{R}^{32}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{23} &= ((\mathcal{R}^{-1})^{12}\underline{\mathcal{R}^{12})\mathcal{R}^{32}\mathcal{R}^{31}}\mathcal{R}^{13}\mathcal{R}^{23} \\
&= (\mathcal{R}^{-1})^{12}\mathcal{R}^{31}\underline{\mathcal{R}^{32}\mathcal{R}^{12}\mathcal{R}^{13}}\mathcal{R}^{23} \\
&= (\mathcal{R}^{-1})^{12}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{12}\mathcal{R}^{32}\mathcal{R}^{23}.
\end{aligned}
$$

Similar expressions for L^+ and SL^- are:

$$
L_1^+ \bullet L_2^+ = L_2^+ L_1^+, \tag{3.6}
$$

$$
SL_1^- \bullet SL_2^- = SL_1^- SL_2^-. \tag{3.7}
$$

All matrices in $M_N(\mathcal{U})$ satisfy by definition the same commutation relations (1.30) as A, when written in terms of the $\bullet$- product,

$$
R_{12}L_1^+ \bullet L_2^+ = L_2^+ \bullet L_1^+ R_{12} \;\Leftrightarrow\; R_{12}L_2^+ L_1^+ = L_1^+ L_2^+ R_{12}, \tag{3.8}
$$

$$
R_{12}SL_1^+ \bullet SL_2^+ = SL_2^+ \bullet SL_1^+ R_{12} \;\Leftrightarrow\; R_{12}SL_1^+ SL_2^+ = SL_2^+ SL_1^+ R_{12}, \tag{3.9}
$$

$$
\begin{aligned}
R_{12}Y_1 \bullet Y_2 = Y_2 \bullet Y_1 R_{12} \;&\Leftrightarrow\; R_{12}(R_{12}^{-1} Y_1 R_{12} Y_2) \\
&\qquad = (R_{21}^{-1} Y_2 R_{21} Y_1) R_{12} \\
&\Leftrightarrow\; R_{21} Y_1 R_{12} Y_2 = Y_2 R_{21} Y_1 R_{12}.
\end{aligned}
\tag{3.10}
$$

Remark: Equations incorporating the $\bullet$-product are mathematically very similar to the expressions introduced in ref.[48] for braided linear algebras — our analysis was in fact motivated by that work — but on a conceptional level things are quite different: We are not dealing with a braided algebra with a braided multiplication but rather with a rule for combining matrix representations that turns out to be very useful, as we will see, to find conditions on the matrices in $M_N(\mathcal{U})$ from algebraic relations for matrices in $M_N(\mathcal{A})$.

3.1.1 Multiple $\bullet$-Products

We can define multiple (associative) $\bullet$-products by,

$$
Y_1 \bullet Y_2 \bullet \ldots \bullet Y_k := < \mathcal{R}^{21}\mathcal{R}^{12}, A_1 A_2 \cdots A_k \otimes id >, \tag{3.11}
$$

but this equation is not very useful to evaluate these multiple $\bullet$-products in practice. However, the "big" $\boldsymbol{R}$-matrix of equation (1.32) can be used to calculate multiple $\bullet$-products recursively: Let $\boldsymbol{Y}_I \equiv Y_{1'} \bullet Y_{2'} \bullet \ldots \bullet Y_{n'}$ and $\boldsymbol{Y}_{II} \equiv Y_1 \bullet Y_2 \bullet \ldots \bullet Y_m$, then:

$$\boldsymbol{Y}_I \bullet \boldsymbol{Y}_{II} = \boldsymbol{R}_{I,II}{}^{-1} \boldsymbol{Y}_I \boldsymbol{R}_{I,II} \boldsymbol{Y}_{II}; \tag{3.12}$$

compare to (1.33) and (3.5). The analog of equation (3.10) is also true:

$$\boldsymbol{R}_{I,II} \boldsymbol{Y}_I \bullet \boldsymbol{Y}_{II} = \boldsymbol{Y}_{II} \bullet \boldsymbol{Y}_I \boldsymbol{R}_{I,II} \tag{3.13}$$

$$\Leftrightarrow \quad \boldsymbol{R}_{II,I} \boldsymbol{Y}_I \boldsymbol{R}_{I,II} \boldsymbol{Y}_{II} = \boldsymbol{Y}_{II} \boldsymbol{R}_{II,I} \boldsymbol{Y}_I \boldsymbol{R}_{I,II}\,. \tag{3.14}$$

The $\bullet$-product of three Y-matrices, for example, reads in terms of the ordinary multiplication in $\mathcal{U}$ as,

$$\begin{aligned} Y_1 \bullet (Y_2 \bullet Y_3) &= \boldsymbol{R}_{1,(23)}^{-1} Y_1 \boldsymbol{R}_{1,(23)} (Y_2 \bullet Y_3) \\ &= (R_{12}^{-1} R_{13}^{-1} Y_1 R_{13} R_{12})(R_{23}^{-1} Y_2 R_{23}) Y_3. \end{aligned} \tag{3.15}$$

This formula generalizes to higher $\bullet$-products,*

$$\begin{aligned} &\boldsymbol{Y}_{(1\ldots 2)} \equiv \prod_{i=1}^{k} \bullet Y_i = \prod_{i=1}^{k} Y_{1\ldots k}^{(i)}, \qquad \text{where:} \\ &Y_{1\ldots k}^{(i)} = \begin{cases} R_{i\,(i+1)}^{-1} R_{i\,(i+2)}^{-1} \cdots R_{i\,k}^{-1} Y_i R_{i\,k} \cdots R_{i\,(i+1)}, & 1 \le i < k, \\ Y_k, \quad i = k. \end{cases} \end{aligned} \tag{3.16}$$

3.2 Quantum Determinants

Assuming that we have defined the quantum determinant $\det_q A$ of A in a suitable way — e.g. through use of the quantum ε_q-tensor, which in turn can be derived from the quantum exterior plane — we can then use the invariant maps Φ_n for $n = 2$ to find the corresponding expressions in $\mathcal{U}$; see (2.69). Let us consider a couple of examples:

$$\mathrm{Det} Y := < \mathcal{R}^{21} \mathcal{R}^{12}, \det_q A \otimes id >, \tag{3.17}$$

$$\mathrm{Det} L^+ := < \mathcal{R}^{21}, \det_q A \otimes id > \tag{3.18}$$

$$\mathrm{Det} SL^- := < \mathcal{R}^{12}, \det_q A \otimes id > . \tag{3.19}$$

*All products are ordered according to increasing multiplication parameter, e.g.

$$\prod_{i=1}^{k} \bullet Y_i \equiv Y_1 \bullet Y_2 \bullet \ldots \bullet Y_k.$$

Because of equations (3.6) and (3.7) we can identify,

$$\mathrm{Det}L^{+} \equiv \mathrm{det}_{q^{-1}}L^{+}, \quad \mathrm{Det}SL^{-} \equiv \mathrm{det}_{q}SL^{-}. \tag{3.20}$$

Properties of $\mathrm{det}_q A$, namely:

$$\begin{aligned} A\,\mathrm{det}_q A &= \mathrm{det}_q A\, A \qquad \textit{(central)}, & (3.21)\\ \Delta(\mathrm{det}_q A) &= \mathrm{det}_q A \otimes \mathrm{det}_q A \qquad \textit{(group-like)}, & (3.22)\end{aligned}$$

translate into corresponding properties of "Det". For example, here is a short proof of the centrality of $\mathrm{Det}Y \equiv Y_{\mathrm{det}_q A}$ based on equations (2.7) and (2.61):[†]

$$\begin{aligned} x\,Y_b &= Y_{b_{(2)}} < x_{(1)}\,,\, S(b_{(1)})b_{(3)} > x_{(2)}, \qquad \forall x \in \mathcal{U};\\ \Rightarrow x\,Y_{\mathrm{det}_q A} &= Y_{\mathrm{det}_q A} < x_{(1)}\,,\, S(\mathrm{det}_q A)\mathrm{det}_q A > x_{(2)}\\ &= Y_{\mathrm{det}_q A} < x_{(1)}\,,\, 1 > x_{(2)}\\ &= Y_{\mathrm{det}_q A}\, x, \qquad \forall x \in \mathcal{U}. \end{aligned} \tag{3.23}$$

The determinant of Y is central in the algebra, so its matrix representation must be proportional to the identity matrix,

$$< \mathrm{Det}Y, A >= \kappa I, \tag{3.24}$$

with some proportionality constant κ that is equal to one in the case of special quantum groups; note that (3.24) is equivalent to:

$$\mathrm{det}_1(R_{21}R_{12}) = \kappa I_{12}, \tag{3.25}$$

where det_1 is the ordinary determinant taken in the first pair of matrix indices. We can now compute the commutation relation of $\mathrm{Det}Y$ with A [24],

$$\begin{aligned} \mathrm{Det}Y A &= A < \mathrm{Det}Y, A > \mathrm{Det}Y\\ &= \kappa A\, \mathrm{Det}Y, \end{aligned} \tag{3.26}$$

showing that in the case of special quantum groups the determinant of Y is actually central in $\mathcal{A}\rtimes\mathcal{U}$.[‡]

Using (3.22) in the definition of $\mathrm{Det}Y$,

$$\begin{aligned} \mathrm{Det}Y &= < \mathcal{R}^{21}\mathcal{R}^{12}, \mathrm{det}_q A \otimes id >\\ &= < \mathcal{R}^{31}\mathcal{R}^{23}, \Delta(\mathrm{det}_q A) \otimes id >\\ &= < \mathcal{R}^{31}\mathcal{R}^{23}, \mathrm{det}_q A \otimes \mathrm{det}_q A \otimes id >\\ &= \mathrm{det}_{q^{-1}}L^{+} \cdot \mathrm{det}_q SL^{-}, \end{aligned} \tag{3.27}$$

[†]This proof easily generalizes to show the centrality of *any* (right) invariant $c \in \mathcal{U}$, $\Delta_{\mathcal{A}}(c) = c \otimes 1$, an example being the invariant traces $\mathrm{tr}(D^{-1}Y^k)$ [23].

[‡]The invariant traces are central only in $\mathcal{U}$ because they are not group-like.

we see that "Det$Y$" coincides with the definition of the determinant of Y given in [38].

A practical calculation of Det$Y$ in terms of the matrix elements of Y starts from,

$$\det_q A \ \ \varepsilon_q^{i_1 \cdots i_N} = \left(\prod_{k=1}^{N} A_k \right)^{i_1 \cdots i_N}{}_{j_1 \cdots j_N} \ \varepsilon_q^{j_1 \cdots j_N}, \tag{3.28}$$

and uses Det$Y = \det_q \bullet Y$, i.e. the q-determinant with the $\bullet$-multiplication:

$$\mathrm{Det}Y \ \ \varepsilon_q^{i_1 \cdots i_N} = \left(\prod_{k=1}^{N} \bullet Y_k \right)^{i_1 \cdots i_N}{}_{j_1 \cdots j_N} \ \varepsilon_q^{j_1 \cdots j_N}. \tag{3.29}$$

Now we use equation (3.16) and get:

$$\begin{aligned} &\mathrm{Det}Y \ \ \varepsilon_q^{i_1 \cdots i_N} = \left(\prod_{k=1}^{N} Y^{(k)}_{1 \ldots N} \right)^{i_1 \cdots i_N}{}_{j_1 \cdots j_N} \ \varepsilon_q^{j_1 \cdots j_N}, \qquad \text{where:} \\ &Y^{(i)}_{1 \ldots k} = \begin{cases} R^{-1}_{i\,(i+1)} R^{-1}_{i\,(i+2)} \cdots R^{-1}_{i\,k} Y_i R_{i\,k} \cdots R_{i\,(i+1)}, & 1 \leq i < k, \\ Y_k, \quad i = k. \end{cases} \end{aligned} \tag{3.30}$$

It is interesting to see what happens if we use a matrix $T \in M_N(\mathcal{A})$ with determinant $det_q T = 1$, e.g. $T := A/(det_q A)^{1/N}$, to define a matrix $Z \in M_N(\mathcal{U})$ [24] in analogy to equation (2.63),

$$Z := < \mathcal{R}^{21} \mathcal{R}^{12}, T \otimes id >; \tag{3.31}$$

we find that Z is automatically of unit determinant:

$$\begin{aligned} DetZ &:= < \mathcal{R}^{21} \mathcal{R}^{12}, det_q T \otimes id > \\ &= < \mathcal{R}^{21} \mathcal{R}^{12}, 1 \otimes id > \\ &= (\epsilon \otimes id)(\mathcal{R}^{21} \mathcal{R}^{12}) \ = \ 1. \end{aligned} \tag{3.32}$$

3.3 An Orthogonality Relation for Y

If we want to consider only such transformations

$$x \mapsto {}_{A}\Delta(x) = A \dot{\otimes} x, \qquad x \in \mathbf{C}_q^N, \ A \in M_N(\mathcal{A}), \tag{3.33}$$

of the quantum plane that leave lengths invariant, we need to impose an orthogonality condition on A; see [23]. Let $C \in M_N(k)$ be the appropriate metric and $x^T C x$ the length squared of x then we find,

$$A^T C A = C \qquad (orthogonality), \tag{3.34}$$

as the condition for an invariant length,

$$x^T C x \mapsto {}_{\mathcal{A}}\Delta(x^T C x) = 1 \otimes x^T C x. \tag{3.35}$$

If we restrict A — and thereby $\mathcal{A}$ — in this way we should also impose a corresponding orthogonality condition in $\mathcal{U}$. Use of the $\bullet$-product makes this, as in the case of the quantum determinants, an easy task: we can simply copy the orthogonality condition for A and propose,

$$\begin{aligned} (L^+)^T \bullet C L^+ &= C \;\Rightarrow\; L^+ C^T (L^+)^T = C^T, && (3.36) \\ (SL^-)^T \bullet C SL^- &= C \;\Rightarrow\; (SL^-)^T C SL^- = C, && (3.37) \\ Y^T \bullet C Y &= C, \quad \text{(matrix multiplication understood)}, && (3.38) \end{aligned}$$

as orthogonality conditions in $\mathcal{U}$. The first two equations were derived before in [23] in a different way. Let us calculate the condition on Y in terms of the ordinary multiplication in $\mathcal{U}$,

$$\begin{aligned} C_{ij} &= Y^k{}_i \bullet C_{kl} Y^l{}_j \\ &= C_{kl} (Y_1 \bullet Y_2)^{kl}{}_{ij} \\ &= C_{kl} (R_{12}^{-1} Y_1 R_{12} Y_2)^{kl}{}_{ij}, \end{aligned} \tag{3.39}$$

or, using $C_{ij} = q^{(N-1)} R^{lk}{}_{ij} C_{kl}$:

$$C_{ij} = q^{(N-1)} C_{mn} (Y_1 R_{12} Y_2)^{nm}{}_{ij}. \tag{3.40}$$

Remark: Algebraic relations on the matrix elements of Y like the ones given in the previous two sections also give implicit conditions on $\mathcal{R}$; however we purposely did not specify $\mathcal{R}$, but rather formally assume its existence and focus on the numerical R-matrices that appear in all final expressions. Numerical R-matrices are known for most deformed Lie algebras of interest [23] and many other quantum groups. One could presumably use some of the techniques outlined in this article to actually derive relations for numerical R-matrices or even for the universal $\mathcal{R}$.

3.4 About the Coproduct of Y

It would be nice if we could express the coproduct of Y,

$$\Delta(Y) = < (id \otimes \Delta)\mathcal{R}^{21}\mathcal{R}^{12}, A \otimes id >, \tag{3.41}$$

in terms of the matrix elements of the matrix Y itself, as it is possible for the coproducts of the matrices L^+ and L^-. Unfortunately, simple expressions have only been found in some special cases; see e.g. [49, 50, 51]. A short calculation gives,

$$\Delta(Y^i{}_j) = (\mathcal{R}^{-1})^{12} (1 \otimes Y^i{}_k) \mathcal{R}^{12} (Y^k{}_j \otimes 1); \tag{3.42}$$

this could be interpreted as some kind of braided tensor product [48, 52],

$$\Delta(Y^i{}_j) =: Y^i{}_k \tilde{\otimes} Y^k{}_j, \tag{3.43}$$

but for practical purposes one usually introduces a new matrix,

$$O_{(ij)}{}^{(kl)} := (L^+)^i{}_k S(L^-)^l{}_j \quad \in M_{N\times N}(\mathcal{U}), \tag{3.44}$$

such that,

$$\Delta(Y_A) = O_A{}^B \otimes Y_B, \tag{3.45}$$

where capital letters stand for pairs of indices. The coproduct of $X^i{}_j = (I - Y)^i{}_j/\lambda$ is in this notation:

$$\Delta(X_A) = X_A \otimes 1 + O_A{}^B \otimes X_B. \tag{3.46}$$

We will only use $O_A{}^B$ in formal expressions involving the coproduct of Y. It will usually not show up in any practical calculation, because commutation relation (2.66) already implicitly contains $\Delta(Y)$ and all inner products of Y with strings of A-matrices following from it.

Chapter 4

Vectorfields on Quantum Groups

In this chapter we are trying to find quantum analogs of two important and closely related concepts in the classical theory of Lie groups: Lie algebras of left-invariant vector fields and general vector fields over the group manifold. We will come back to both subjects in part 2, after developing the additional structure of an exterior differential calculus. Our approach will be heuristic in nature; stress is on formation of concepts (Begriffsbildung). The concept of vector fields can also be approached from differential forms, see [53].

4.1 Quantum Lie Algebras

4.1.1 Adjoint Action and Jac 'ɔi Identities

Classically the (left) adjoint actions of the generators χ_i of a Lie algebra g on each other are given by the cor.mutators,

$$\chi_i \overset{\text{ad}}{\triangleright} \chi_j = [\chi_i, \chi_j] = \chi_k f_i{}^k{}_j, \tag{4.1}$$

expressible in terms of the structure constants $f_i{}^k{}_j$, whereas the (left) adjoint action of elements of the corresponding Lie group G is given by conjugation,

$$h \overset{\text{ad}}{\triangleright} g = hgh^{-1}, \qquad h, g \in \mathrm{G}. \tag{4.2}$$

Both formulas generalize in Hopf algebra language to the same expression,

$$\begin{aligned} &\chi_i \overset{\text{ad}}{\triangleright} \chi_j = \chi_{i_{(1)}} \chi_j S(\chi_{i_{(2)}}), \qquad \text{with: } S(\chi) = -\chi, \\ &\Delta(\chi) \equiv \chi_{(1)} \otimes \chi_{(2)} = \chi \otimes 1 + 1 \otimes \chi, \qquad \text{for } \forall \chi \in \mathrm{g}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &h \overset{\text{ad}}{\triangleright} g = h_{(1)} g S(h_{(2)}), \qquad \text{with: } S(h) = h^{-1}, \\ &\Delta(h) \equiv h_{(1)} \otimes h_{(2)} = h \otimes h, \qquad \text{for } \forall h \in \mathrm{G}, \end{aligned} \tag{4.4}$$

and agree with our formula (1.63) for the (left) adjoint action in $\mathcal{U}$. We can derive two *generalized Jacobi identities* for double adjoint actions,

$$\begin{aligned} x \overset{\text{ad}}{\triangleright} (y \overset{\text{ad}}{\triangleright} z) &= (xy) \overset{\text{ad}}{\triangleright} z \\ &= ((x_{(1)} \overset{\text{ad}}{\triangleright} y) x_{(2)}) \overset{\text{ad}}{\triangleright} z \\ &= (x_{(1)} \overset{\text{ad}}{\triangleright} y) \overset{\text{ad}}{\triangleright} (x_{(2)} \overset{\text{ad}}{\triangleright} z), \end{aligned} \tag{4.5}$$

and,

$$\begin{aligned} (x \overset{\text{ad}}{\triangleright} y) \overset{\text{ad}}{\triangleright} z &= (x_{(1)} y S(x_{(2)})) \overset{\text{ad}}{\triangleright} z \\ &= x_{(1)} \overset{\text{ad}}{\triangleright} (y \overset{\text{ad}}{\triangleright} (S(x_{(2)}) \overset{\text{ad}}{\triangleright} z)). \end{aligned} \tag{4.6}$$

Both expressions become the ordinary Jacobi identity in the classical limit and they are not independent: Using the fact that $\overset{\text{ad}}{\triangleright}$ is an action they imply each other.

In the following we would like to derive the quantum version of (4.1) with "quantum commutator" and "quantum structure constants". The idea is to utilize the (passive) transformations that we have studied in some detail in sections 2.1 and 2.4.1 to find an expression for the corresponding active transformations or actions. The effects of passive transformations are the inverse of active transformations, so here is the inverse or right adjoint action for a group:

$$h^{-1} \overset{\text{ad}}{\triangleright} g = g \overset{\text{ad}}{\triangleleft} h = S(h_{(1)}) g h_{(2)}. \tag{4.7}$$

This gives rise to a (right) adjoint coaction in Fun(G):

$$A \mapsto S(A')AA', \qquad \text{i.e.}$$

$$\mathrm{Fun}(G_q) \ni \quad A^i{}_j \mapsto A^k{}_l \otimes S(A^i{}_k) A^l{}_j \quad \in \mathrm{Fun}(G_q) \otimes \mathrm{Fun}(G_q); \tag{4.8}$$

here we have written "Fun(G_q)" instead of "Fun(G)" because the coalgebra of Fun(G_q) is in fact the same undeformed coalgebra as the one of Fun(G). In section 2.4.1 we saw that the Y-matrix has particularly nice transformation properties:

$$\begin{aligned} A &\mapsto S(A')A: & Y &\mapsto 1 \otimes Y, \\ A &\mapsto AA': & Y &\mapsto S(A')YA'. \end{aligned}$$

It follows that:

$$A \mapsto S(A')AA': \qquad Y^i{}_j \mapsto Y^k{}_l \otimes S(A^i{}_k) A^l{}_j. \tag{4.9}$$

This is the "unspecified" adjoint right *coaction* for Y; we recover the "specific" left adjoint *action*,

$$x \overset{\text{ad}}{\triangleright} Y^i{}_j = x_{(1)} Y^i{}_j S(x_{(2)}),$$

of an arbitrary $x \in U_q\mathbf{g}$ by evaluating the second factor of the adjoint coaction (4.9) on x:

$$x \overset{\text{ad}}{\triangleright} Y^i{}_j = Y^k{}_l < x \,,\, S(A^i{}_k)A^l{}_j >, \qquad \text{for } \forall x \in U_q\mathbf{g}. \tag{4.10}$$

At the expense of intuitive insight we can alternatively derive a more general formula directly from equations (1.63), (2.7), and (2.61),

$$\begin{aligned} x \overset{\text{ad}}{\triangleright} Y_b &= x_{(1)} Y_b S(x_{(2)}) \\ &= (Y_b)^{(1)} < x_{(1)}, (Y_b)^{(2)'} > x_{(2)} S(x_{(3)}) \\ &= (Y_b)^{(1)} < x_{(1)}, (Y_b)^{(2)'} > \epsilon(x_{(2)}) \\ &= (Y_b)^{(1)} < x, (Y_b)^{(2)'} > \\ &= Y_{b_{(2)}} < x, S(b_{(1)}) b_{(3)} >; \end{aligned} \tag{4.11}$$

note the appearance of the (right) adjoined coaction [21] in $\mathrm{Fun}(G_q)$,

$$\Delta^{\mathrm{Ad}}(b) = b_{(2)} \otimes S(b_{(1)}) b_{(3)}, \tag{4.12}$$

in this formula.

We have found exactly what we were looking for in a *quantum Lie algebra*; the adjoint action (4.10) or (4.11) — which is the generalization of the classical commutator — of elements of $U_q\mathbf{g}$ on elements in a certain subset of $U_q\mathbf{g}$ evaluates to a *linear* combination of elements of that subset. So we do not really have to use the whole universal enveloping algebra when dealing with quantum groups but can rather consider a subset spanned by elements of the general form $Y_b \equiv < \mathcal{Y}, b \otimes id >$, $\mathcal{Y} \in \mathrm{span}\{B_2\}$; we will call this subset the "quantum Lie algebra" $\mathbf{g}_q$ of the quantum group. Now we need to find a basis of generators with the right classical limit.

4.1.2 R-Matrix Approach

Let us first evaluate (4.10) in the case where x is a matrix element of Y. We introduce the short hand,

$$\mathbf{A}^{(kl)}{}_{(ij)} \equiv S(A^i{}_k)A^l{}_j, \tag{4.13}$$

for the adjoint representation and find,

$$Y_A \overset{\text{ad}}{\triangleright} Y_B = Y_C < Y_A, \mathbf{A}^C{}_B >, \tag{4.14}$$

where, again, capital letters stand for pairs of indices. The evaluation of the inner product $< Y_A, \mathbf{A}^C{}_B > =: C_A{}^C{}_B$ is not hard even though we do not have an explicit expression for the coproduct of Y; we simply use the commutation relation (2.66) of Y with A and the left and right vacua defined in section 1.2.2:

$$\begin{aligned}
< Y_1, SA_2^T A_3 > &= < Y_1 SA_2^T A_3 > \\
&= < SA_2^T (R_{21}^{-1})^{T_2} Y_1 A_3 (R_{12}^{T_2})^{-1} > \\
&= < SA_2^T (R_{21}^{-1})^{T_2} A_3 R_{31} Y_1 R_{13} (R_{12}^{T_2})^{-1} > \\
&= (R_{21}^{-1})^{T_2} R_{31} R_{13} (R_{12}^{T_2})^{-1}, \\
\Rightarrow C_{(ij)}{}^{(kl)}{}_{(mn)} &= \left((R_{21}^{-1})^{T_2} R_{31} R_{13} (R_{12}^{T_2})^{-1} \right)^{ikl}{}_{jmn}.
\end{aligned} \tag{4.15}$$

The matrix Y becomes the identity matrix in the classical limit, so $X \equiv (I - Y)/\lambda$ is a better choice; it has the additional advantage that it has zero counit and its coproduct (3.46) resembles the coproduct of classical differential operators and therefore allows us to write the adjoint action (4.3) as a *generalized commutator*:

$$\begin{aligned}
Y_A \overset{\mathrm{ad}}{\triangleright} X_B &= Y_{A(1)} X_B S(Y_{A(2)}) \\
&= O_A{}^D X_B S(Y_D) \\
&= O_A{}^D X_B S(O_D{}^E) (\underbrace{I_E - \lambda X_E}_{Y_E} + \lambda X_E) \\
&= Y_A X_B + (O_A{}^E \overset{\mathrm{ad}}{\triangleright} X_B) \lambda X_E \\
&= Y_A X_B + \lambda < O_A{}^E, \mathbf{A}^D{}_B > X_D X_E, \\
&\text{with: } O_D{}^E I_E = Y_D, \quad S(O_D{}^E) Y_E = I_D; \\
\Rightarrow X_A \overset{\mathrm{ad}}{\triangleright} X_B &= X_A X_B - < O_A{}^E, \mathbf{A}^D{}_B > X_D X_E.
\end{aligned} \tag{4.16}$$

Following the notation of reference [25] we introduce the $N^4 \times N^4$ matrix,

$$\hat{\mathbb{R}}^{DE}{}_{AB} := < O_A{}^E, \mathbf{A}^D{}_B >, \tag{4.17}$$

$$\hat{\mathbb{R}}^{(mn)(kl)}{}_{(ij)(pq)} = \left((R_{31}{}^{-1})^{T_3} R_{41} R_{24} (R_{23}{}^{T_3})^{-1} \right)^{ilmn}{}_{kjpq}, \tag{4.18}$$

but realize when considering the above calculation that $\mathbb{R}$ is not the "R-matrix in the adjoint representation" — that would be $< \mathcal{R}, \mathbf{A}^E{}_A \otimes \mathbf{A}^D{}_B >$ — but rather the R-matrix for the braided commutators of $\mathfrak{g}_q$, giving the commutation relations of the generators a form resembling an (inhomogeneous) quantum plane.

Now we can write down the generalized Cartan equations of a quantum Lie algebra $\mathfrak{g}_q$:

$$X_A \overset{\mathrm{ad}}{\triangleright} X_B = X_A X_B - \hat{\mathbb{R}}^{DE}{}_{AB} X_D X_E = X_C f_A{}^C{}_B, \tag{4.19}$$

where, from equation (4.15),

$$f_A{}^C{}_B = (I_A I^C I_B - C_A{}^C{}_B)/\lambda. \tag{4.20}$$

4.1.3 General Case

Equation (4.19) is strictly only valid for systems of N^2 generators with an $N^2 \times N^2$ matrix $\hat{\rm I\!R}$ because $X \in M_N(g_q)$ in our construction. Some of these N^2 generators and likewise some of the matrix elements of $\hat{\rm I\!R}$ could of course be zero, but let us anyway consider the more general case of equation (4.11). We will assume a set of n generators X_{b_i} corresponding to a set of n linearly independent functions $\{b_i \in \mathrm{Fun}(G_q) \,|\, i = 1, \dots, n\}$ and an element of the pure braid group $\mathrm{X} \in \mathrm{span}(\mathcal{B}_2)$ via:

$$X_{b_i} = < \mathrm{X}, b_i \otimes id > . \tag{4.21}$$

We will usually require that all generators have vanishing counit. A sufficient condition on the b_i's ensuring linear closure of the generators X_{b_i} under adjoint action (4.11) is,

$$\Delta^{\mathrm{Ad}}(b_i) = b_j \otimes \mathrm{M}^j{}_i + k_l \otimes k_i^l, \tag{4.22}$$

where $\mathrm{M}^j{}_i \in M_n(\mathrm{Fun}(G_q))$ and k_l, $k_i^l \in \mathrm{Fun}(G_q)$ such that $< \mathrm{X}, k_l \otimes id >= 0$. The generators will then transform like,

$$\Delta_{\mathcal{A}}(X_{b_i}) = X_{b_j} \otimes \mathrm{M}^j{}_i; \tag{4.23}$$

from $(\Delta_{\mathcal{A}} \otimes id)\Delta_{\mathcal{A}}(X_{b_i}) = (id \otimes \Delta)\Delta_{\mathcal{A}}(X_{b_i})$ and $(id \otimes \epsilon)\Delta_{\mathcal{A}}(X_{b_i}) = X_{b_i}$ immediately follows* $\Delta(\mathrm{M}) = \mathrm{M}\dot{\otimes}\mathrm{M}$, $\epsilon(\mathrm{M}) = I$ and consequently $S(\mathrm{M}) = \mathrm{M}^{-1}$. M is the *adjoint matrix representation.* We find,

$$X_{b_k} \overset{\mathrm{ad}}{\triangleright} X_{b_i} = X_{b_j} < X_{b_k}, \mathrm{M}^j{}_i >, \tag{4.24}$$

as a generalization of (4.19) with structure constants $f_k{}^j{}_i =< X_{b_k}, \mathrm{M}^j{}_i >$. Whether $X_{b_k} \overset{\mathrm{ad}}{\triangleright} X_{b_i}$ can be reexpressed as a deformed commutator depends on the coproducts of the X_{b_i}'s and hence on the particular choice of X and $\{b_i\}$.

Equations (4.9) and (4.13) – (4.20) apply directly to $Gl_q(N)$ and $Sl_q(N)$ and other quantum groups in matrix form with (numerical) R-matrices. Such quantum groups have been studied in great detail in the literature; see e.g. [23, 25, 26] and references therein. In the next subsection we would like to discuss the 2-dimensional quantum euclidean algebra as an example that illustrates some subtleties in the general picture.

4.1.4 Bicovariant Generators for $e_q(2)$

In [39] Woronowicz introduced the functions on the deformed $E_q(2)$. This and the corresponding algebra $U_q(e(2))$ were explicitly constructed in chapter 1 using a contraction procedure; here is a short summary: m, $\overline{m}$ and $\theta = \overline{\theta}$ are generating elements

*This assumes that the X_{b_i}'s are linearly independent.

of the Hopf algebra $\mathrm{Fun}(E_q(2))$, which satisfy:

$$\begin{array}{lll} m\overline{m} = q^2\overline{m}m, & e^{i\theta}m = q^2 m e^{i\theta}, & e^{i\theta}\overline{m} = q^2\overline{m}e^{i\theta}, \\ \Delta(m) = m\otimes 1 + e^{i\theta}\otimes m, & & \Delta(\overline{m}) = \overline{m}\otimes 1 + e^{-i\theta}\otimes\overline{m}, \\ \Delta(e^{i\theta}) = e^{i\theta}\otimes e^{i\theta}, & S(m) = -e^{-i\theta}m, & S(\overline{m}) = -e^{i\theta}\overline{m}, \\ S(\theta) = -\theta, & \epsilon(m) = \epsilon(\overline{m}) = \epsilon(\theta) = 0. & \end{array} \tag{4.25}$$

$\mathrm{Fun}(E_q(2))$ coacts on the complex coordinate function z of the euclidean plane as $\Delta_{\mathcal{A}}(z) = z\otimes e^{i\theta} + 1\otimes m$; i.e. θ corresponds to rotations, m to translations. The dual Hopf algebra $U_q(e(2))$ is generated by $J = \overline{J}$ and $P_\pm = \overline{P_\mp}$ satisfying:

$$\begin{array}{lll} [J, P_\pm] = \pm P_\pm, & [P_+, P_-] = 0, & \\ \Delta(P_\pm) = P_\pm\otimes q^J + q^{-J}\otimes P_\pm, & & \Delta(J) = J\otimes 1 + 1\otimes J, \\ S(P_\pm) = -q^{\pm 1}P_\pm, & S(J) = -J, & \epsilon(P_\pm) = \epsilon(J) = 0. \end{array} \tag{4.26}$$

The duality between $\mathrm{Fun}(E_q(2))$ and $U_q(e(2))$ is given by:

$$\begin{array}{l} < {P_+}^k {P_-}^l q^{mJ}\ ,\ e^{i\theta a} m^b \overline{m}^c >= \\ \qquad (-1)^l q^{-1/2(k-l)(k+l-1)+l(k-1)} q^{(k+l-m)a}[k]_q![l]_{q^{-1}}!\delta_{lb}\delta_{kc}, \end{array} \tag{4.27}$$

where $k, l, b, c \in \mathbf{N}_0$, $m, a \in \mathbf{Z}$, and,

$$[x]_q! = \prod_{y=1}^{x}\frac{q^{2y}-1}{q^2-1}, \qquad [0]_q! = [1]_q! = 1.$$

Note that P_+P_- is central in $U_q(e(2))$; i.e. it is a casimir operator. $U_q(e(2))$ does not have a (known) universal $\mathcal{R}$, so we have to construct an element X of $\mathrm{span}(B_2)$ from the casimir P_+P_-:

$$\begin{array}{rcl} \mathrm{X} & := & \frac{1}{q-q^{-1}}\{\Delta(P_+P_-) - (P_+P_-\otimes 1)\} \\ & = & \frac{1}{q-q^{-1}}\{P_+P_-\otimes(q^{2J}-1) + P_+q^{-J}\otimes q^J P_- \\ & & + P_-q^{-J}\otimes q^J P_+ + q^{-2J}\otimes P_+P_-\}. \end{array} \tag{4.28}$$

X commutes with $\Delta(x)$ for all $x \in U_q(e(2))$ because P_+P_- is a casimir. We introduced the second term $(P_+P_-\otimes 1)$ in X to ensure $(id\otimes\epsilon)\mathrm{X} = 0$ so that we are guaranteed to get bicovariant generators with zero counit. Now we need a set of functions which transform like (4.22). A particular simple choice is $a_0 := e^{i\theta} - 1$, $a_+ := m$, and $a_- := e^{i\theta}\overline{m}$. These functions transform under the adjoint coaction as:

$$\Delta^{\mathrm{Ad}}(a_0, a_+, a_-) = (a_0, a_+, a_-)\dot{\otimes}\begin{pmatrix} 1 & e^{-i\theta}m & -e^{i\theta}\overline{m} \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}. \tag{4.29}$$

Unfortunately we notice that a_0 and thereby X_{a_0} are invariant, forcing X_{a_0} to be a casimir independent of the particular choice of X. Indeed we find $X_{a_0} = qP_+P_-$, $X_{a_+} = -\sqrt{q}/(q-q^{-1})q^JP_+$, and $X_{a_-} = q/(q-q^{-1})q^JP_-$, making this an incomplete choice of bicovariant generators for $e_q(2)$. An ansatz with four functions $b_0 := (e^{i\theta}-1)^2$, $b_1 := -me^{i\theta}\overline{m}$, $b_+ := -(e^{i\theta}-1)m$, and $b_- := q^{-2}(e^{i\theta}-1)e^{i\theta}\overline{m}$ gives:

$$\Delta^{\mathrm{Ad}}(b_0,b_1,b_+,b_-) = (b_0,b_1,b_+,b_-)\dot{\otimes}\begin{pmatrix} 1 & \overline{m}m & -e^{-i\theta}m & -q^{-2}e^{i\theta}\overline{m} \\ 0 & 1 & 0 & 0 \\ 0 & -\overline{m} & e^{-i\theta} & 0 \\ 0 & -m & 0 & e^{i\theta} \end{pmatrix}. \tag{4.30}$$

The corresponding bicovariant generators are:

$$\begin{aligned} &X_{b_0} = q(q^2-1)P_+P_-, \qquad X_{b_1} = (q-q^{-1})^{-1}(q^{2J}-1), \\ &X_{b_+} = q^JP_+, \qquad X_{b_-} = qq^JP_-. \end{aligned} \tag{4.31}$$

In the classical limit ($q \longrightarrow 1$) these generators become "zero", J, P_+, and P_- respectively†. The coproducts of the bicovariant generators have the form expected for differential operators

$$\Delta\begin{pmatrix} X_0 \\ X_1 \\ X_+ \\ X_- \end{pmatrix} = \begin{pmatrix} X_0 \\ X_1 \\ X_+ \\ X_- \end{pmatrix}\otimes 1 + \begin{pmatrix} \lambda SX_1+1 & \lambda X_0 & \lambda(\lambda SX_1+1) & \lambda X_+(\lambda SX_1+1) \\ 0 & \lambda X_1+1 & 0 & 0 \\ 0 & \lambda X_+ & 1 & 0 \\ 0 & \lambda X_- & 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} X_0 \\ X_1 \\ X_+ \\ X_- \end{pmatrix}. \tag{4.32}$$

The commutation relations of the generators follow directly from (4.26), their adjoint actions are calculated from (4.24), (4.27), and (4.30) and finally the commutation relations of the generators with the functions can be obtained from (1.53), (4.25) and (4.26).

4.2 General Vector Fields

In this section we will give a "quantum geometric" construction of the action of general, *i.e.* neither necessarily left or right invariant, vector fields, thereby justifying the form of the action that we used in the construction of the cross-product algebra of differential operators.

4.2.1 Classical Left Invariant Vector Fields

First, recall the left-invariant classical case: The Lie algebra is spanned by left-invariant vector fields on the group manifold of a Lie group G. These are uniquely

†The same generators and their transformation properties can alternatively be obtained by contracting the bicovariant calculus on $SU_q(2)$.

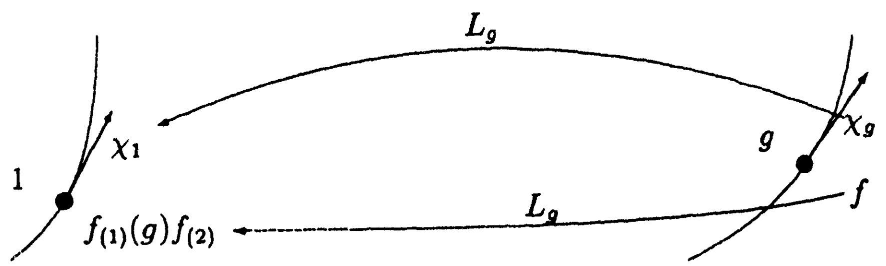
determined by the tangent space at 1 (the identity of G). Curves on G can be naturally transported by left (or right) translation i.e. $h \mapsto gh$ ($h \mapsto hg$). This defines a left transport $L_{g^{-1}}$ of the tangent vectors: $L_{g^{-1}}(\chi_1) = \tilde{\chi}_g$. χ_1 is the vector field χ at the identity of the group and $\tilde{\chi}_g$ is the new vector field $\tilde{\chi}$ evaluated at the point of the group manifold corresponding to the group element g; if χ is left invariant then $\chi = \tilde{\chi}$ and in particular

$$L_{g^{-1}}(\chi_1) = \tilde{\chi}_g = \chi_g. \tag{4.33}$$

An inner product for a vector field χ with a function f can be defined by acting with the vector field on the function and evaluating the resulting function at the identity of the group:

$$< \chi, f > := \chi_1 \triangleright f|_1 \in k. \tag{4.34}$$

If we know these values for all functions, we can reconstruct the action of χ on a function f, $\chi_g \triangleright f|_g$, at any (connected) point of the group manifold. The construction goes as follows (see figure):



We start at the point g, transport f and χ back to the identity by left translation and then evaluate them on each other. The result, being a number, is invariant under translations and hence gives the desired quantity. The left translation $Lg(f)$ of a function, implicitly defined through $L_g(f)(h) = f(gh)$, finds an explicit expression in Hopf algebra language

$$L_g(f) = f_{(1)}(g)f_{(2)}, \tag{4.35}$$

that we now use to express

$$\begin{aligned} \chi_g \triangleright f|_g &= L_g(\chi)_1 \triangleright f_{(1)}(g)f_{(2)}\big|_1 \\ &= \chi_1 \triangleright f_{(1)}(g)f_{(2)}\big|_1 \\ &= f_{(1)}(g) < \chi, f_{(2)} >, \end{aligned} \tag{4.36}$$

for a left-invariant vector field χ. If the drop g, we obtain the expression for the action of a vector field on a function valid on the whole group manifold

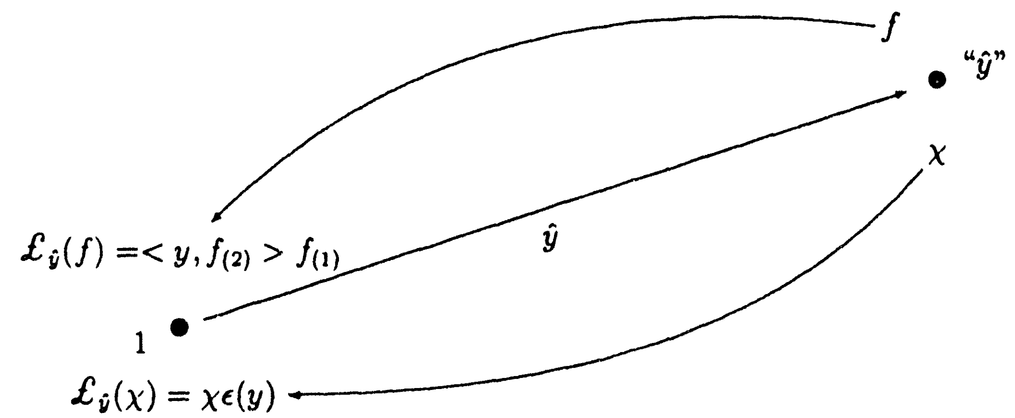
$$\chi \triangleright f = f_{(1)} < \chi, f_{(2)} >, \tag{4.37}$$

already familiar from the first chapter. The left and right vacua by the way find the following 'geometric' interpretation:

> left vacuum <: "Evaluate at the identity (of the group)."
> right vacuum >: "Evaluate on the unit function."

4.2.2 Some Quantum Geometry

Group elements (g) do not exist for quantum groups, everything has to be formulated in terms of a Hopf algebra of functions. The group operation is replaced by the coproduct of functions. A quantum group has only few classical points. These correspond to elements of $\mathcal{U}$ with group-like coproducts, e.g. the quantum determinant of Y in $Gl_q(2)$: $\Delta \det_q Y = \det_q Y \otimes \det_q Y$. If we take care only to speak about functions in $\mathcal{A}$ and its dual Hopf algebra $\mathcal{U}$, we can, however, still develop a geometric picture for vector fields on quantum groups. "Points" will be *labeled* by elements of $\tilde{\mathcal{U}}$, which is the same as $\mathcal{U}$ but has the opposite multiplication; elements of $\tilde{\mathcal{U}}$ are *right*-invariant. Lie derivatives along elements of $\tilde{\mathcal{U}}$ take the place of left translations, while Lie derivatives along elements of $\mathcal{U}$ correspond to right translations. Here is the quantum picture of the classical construction given in the previous section:



Note that $\mathcal{L}_{\hat{y}}(x) = x\epsilon(\hat{y})$ because x is left-invariant. (More precise definitions of these Lie derivatives in connection with right-projectors will be given in section 4.2.4). Before we can read any equations off the picture we have to invent a rule for multiple appearances of the same Hopf algebra element in the same term:

> Multiple occurrences of the same Hopf algebra element in a single term are not allowed. One should use the parts of the coproduct of this element instead — starting with the last part of the coproduct and collecting terms

from the right to the left as one moves along the path that the function is transported.

Now can compute $x \triangleright f$ in complete analogy to the classical case

$$
\begin{aligned}
x \triangleright f|_{``\hat{y}"} &= \mathcal{L}_{\hat{y}_{(1)}}(x) \triangleright \mathcal{L}_{\hat{y}_{(2)}}(f)\Big|_1 \qquad (\equiv \mathcal{L}_{\hat{y}}(x \triangleright f)\Big|_1) \\
&= \epsilon(y_{(1)}) x \triangleright \mathcal{L}_{\hat{y}_{(2)}}(f)\Big|_1 \\
&= x \triangleright \mathcal{L}_{\hat{y}}(f)\Big|_1 \\
&= x \triangleright < y, f_{(1)} > f_{(2)}\Big|_1 \\
&= < y, f_{(1)} >< x, f_{(2)} >
\end{aligned} \tag{4.38}
$$

or, for arbitrary $\hat{y}$:

$$x \triangleright f = f_{(1)} < x, f_{(2)} >, \tag{4.39}$$

giving a geometric justification for the left action of $\mathcal{U}$ on $\mathcal{A}$ that we had introduced in chapter 1.

Now we would like to study the adjoint action in $\mathcal{U}$, which can be interpreted as a quantum Lie bracket as we shall see. Recall the classical construction: Functions and hence curves on a group manifold can be transported along a vector field. With the curves we implicitly also transport their tangent vectors. This transport is called the Lie derivative of a (tangent) vector along a vector field. Classically we find it to be equal to the commutator (Lie bracket) of the two vector fields. Here is how the computation goes in practice: Let y be the vector field along which the functions are transported and let x be the "tangent" vector field. Consider a function f on the new curve and transport it along the following two equivalent paths:

1. Go back along y to the old curve, follow the old curve along x and finally return along y to the new curve.

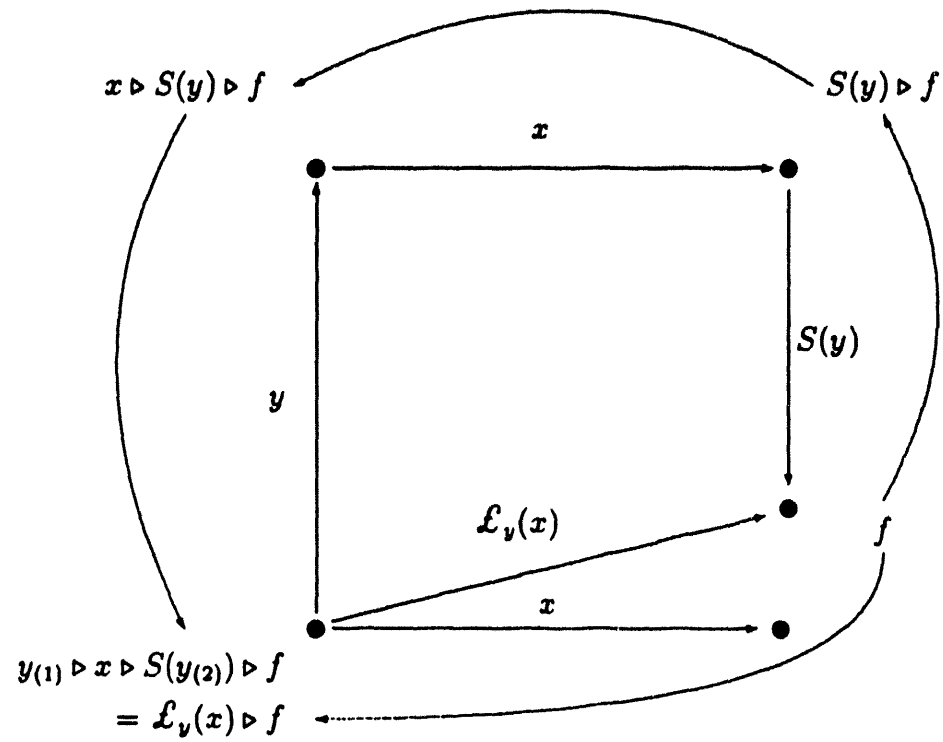
2. Follow the new curve along $\mathcal{L}_y(x)$.

We have to invent a new rule for backward transport:[‡]

> Moving a function back along a vector field y is the same as moving forward along the antipode $S(y)$ of that vector field. (When moving a 1-form, one should use the inverse antipode.)

[‡]Note that we follow the path of the transported function; forward hence means "opposite to the direction that the vector is pointing", backward means "along the direction that the vector is pointing".

The following picture illustrates the geometric construction of the quantum Lie derivative of a left-invariant vector field along another left-invariant vector field:



We read off this picture that

$$\begin{aligned} \mathcal{L}_y(x) \triangleright f &= y_{(1)} \triangleright x \triangleright S(y_{(2)}) \triangleright f \\ &=: (y_{(1)} x S(y_{(2)})) \triangleright f, \end{aligned} \tag{4.40}$$

i.e. $\mathcal{L}_y(x) = y_{(1)} x S(y_{(2)}) = y \overset{\text{ad}}{\triangleright} x$.

4.2.3 Action of General Vector Fields

Our derivation of the action of a vector field on a function in the previous section relied on the use of left translations in conjunction with left-invariant vector fields. In this section we would like to free ourselves from this limitation and show how to derive the action of a general vector field — neither necessarily left or right invariant — on a function using alternatively left or right translations.

Left and right coactions ${}_{\mathcal{A}}\Delta$, $\Delta_{\mathcal{A}}$ contain the information about transformation properties of vector fields. Here is how a vector field transforms (classically) if we *left*-transport it from a point g on the group manifold back to the identity

$$\chi|_g \mapsto \chi^{(1)'}(g)\cdot\chi^{(2)}|_1, \qquad {}_{\mathcal{A}}\Delta(\chi) \equiv \chi^{(1)'}\otimes\chi^{(2)}; \tag{4.41}$$

here is the behavior under a *right* translation:

$$\chi|_g \mapsto \chi^{(1)}\cdot\chi^{(2)'}(g)|_1, \qquad \Delta_{\mathcal{A}}(\chi) \equiv \chi^{(1)}\otimes\chi^{(2)'}. \tag{4.42}$$

If we now redo the construction of the previous section for general vector fields χ, both for left and right translations, we get the following two equivalent results for actions on functions:

$$\boxed{\chi(f) = \underbrace{<\chi^{(1)},f_{(1)}>\chi^{(2)'}f_{(2)}}_{\text{from right translation}} = \underbrace{\chi^{(1)'}f_{(1)}<\chi^{(2)},f_{(2)}>}_{\text{from left translation}}}. \tag{4.43}$$

Technically there is an ordering ambiguity for f and the primed parts of χ, but this can be easily resolved by requiring $a(f) = af$ for $a \in \mathcal{A}$ in both cases; both expressions are written as left actions. From this equation we can derive the following relations between left and right coactions for $\chi \in \mathcal{A}^*$:

$$\begin{aligned}\Delta_{\mathcal{A}}\chi &= e_i\otimes\chi^{(1)'}f^i_{(1)}<\chi^{(2)},f^i_{(2)}>Sf^i_{(3)}\\ &= e_{i(1)}\chi^{(2)}Se_{i(2)}\otimes\chi^{(1)'}f^i\\ &= (e_i\overset{\mathrm{ad}}{\triangleright}\otimes id)\,\tau({}_{\mathcal{A}}\Delta\chi)(1\otimes f^i),\end{aligned} \tag{4.44}$$

$$\begin{aligned}{}_{\mathcal{A}}\Delta\chi &= \chi^{(2)}f^i_{(3)}<\chi^{(1)},f^i_{(2)}>S^{-1}f^i_{(1)}\otimes e_i\\ &= \chi^{(2)'}f^i\otimes S^{-1}(e_{i(2)})\chi^{(1)}e_{i(1)}\\ &= (id\otimes S^{-1}e_i\overset{\mathrm{ad}}{\triangleright})\,\tau(\Delta_{\mathcal{A}}\chi)(f^i\otimes 1).\end{aligned} \tag{4.45}$$

In this thesis we choose the *convention* that elements in $\mathcal{U}\cong\mathcal{A}^*$ be left-invariant.

4.2.4 Right and Left Projectors

In this section we will show how to obtain right-invariant vector fields from left-invariant ones by allowing functional coefficients. These right-invariant vector fields will live in $\mathcal{A}\rtimes\mathcal{U}$ — recall that elements of $\mathcal{U}$ were chosen to be left-invariant. Let x be the left-invariant vector field and $\hat{x}$ the corresponding right-invariant vector field. These vector fields should coincide at the identity, i.e. for any function f

$$\epsilon(\hat{x}) = \epsilon(x), \qquad <\hat{x},f> = <x,f>. \tag{4.46}$$

For this to make sense we have to extend the definition of the inner product a little bit to allow elements of $\mathcal{A}\rtimes\mathcal{U}$ in the first space. Recalling the geometrical definition

$$<\phi,f> := \phi\triangleright f|_1, \qquad \phi\in\mathcal{A}\rtimes\mathcal{U},\ f\in\mathcal{A} \tag{4.47}$$

this is not hard: $(a,f\in\mathcal{A},\ x\in\mathcal{U})$

$$
\begin{aligned}
<ax,f> &:= \epsilon(a)<x,f>, &(4.48)\\
<xa,f> &:= <x,af>, &(4.49)
\end{aligned}
$$

in perfect agreement with the formulation in terms of vacua. Let $\Delta_{\mathcal{A}}(x)=x^{(1)}\otimes x^{(2)'}\in\mathcal{U}\otimes\mathcal{A}$; it is not hard to see that

$$
\hat{x}:=S^{-1}(x^{(2)'})x^{(1)} \tag{4.50}
$$

has the required properties and is right invariant

$$
\begin{aligned}
\Delta_{\mathcal{A}}(\hat{x}) &= (S^{-1}x^{(2)'})_{(1)}x^{(1)(1)}\otimes(S^{-1}x^{(2)'})_{(2)}x^{(1)(2)'}\\
&= S^{-1}(x^{(2)'})_{(3)}x^{(1)}\otimes S^{-1}(x^{(2)'})_{(2)}x^{(2)'}{}_{(1)}\\
&= S^{-1}(x^{(2)'})_{(2)}x^{(1)}\otimes 1\epsilon(x^{(2)'}{}_{(1)}) \qquad (4.51)\\
&= S^{-1}(x^{(2)'})x^{(1)}\otimes 1\\
&= \hat{x}\otimes 1,
\end{aligned}
$$

but (of course) no longer left-invariant:

$$
\begin{aligned}
{}_{\mathcal{A}}\Delta(\hat{x}) &= (S^{-1}x^{(2)'})_{(1)}\otimes(S^{-1}x^{(2)'})_{(1)}x^{(1)}\\
&= S^{-1}x^{(2)'}\otimes\widehat{x^{(1)}}.
\end{aligned} \tag{4.52}
$$

We define $\hat{\mathcal{U}}$ to be the space $\{\hat{x}|x\in\mathcal{U}\}$. It turns out that the $\hat{}$-operation is a projection operator from $\mathcal{A}\rtimes\mathcal{U}$ to $\hat{\mathcal{U}}$; we will call it the right projector. Three explicit expressions for such right-invariant vector fields can be quickly derived:

$$
\begin{aligned}
\hat{x} &= f^i(S^{-1}(e_i)\overset{\text{ad}}{\triangleright}x)\\
&= f^k_{(3)}S^{-1}f^k_{(1)}<x,f^k_{(2)}>e_k
\end{aligned} \tag{4.53}
$$

and, for $\Upsilon_b=<\Upsilon,b\otimes id>$ with Υ being a pure braid element,

$$
\widehat{\Upsilon_b}=S^{-1}(b_{(3)})b_{(1)}\Upsilon_{b_{(2)}}. \tag{4.54}
$$

Left- and right-invariant vector fields commute:

$$
\begin{aligned}
y\hat{x} &= \hat{x}^{(1)}<y_{(1)},\hat{x}^{(2)'}>y_{(2)}\\
&= \hat{x}<y_{(1)},1>y_{(2)}\\
&= \hat{x}y.
\end{aligned} \tag{4.55}
$$

The right projector is an antimultiplicative operation:

$$
\begin{aligned}
\widehat{xy} &= S^{-1}\left((xy)^{(2)'}\right)(xy)^{(1)}\\
&= S^{-1}y^{(2)'}S^{-1}x^{(2)'}x^{(1)}y^{(1)}\\
&= S^{-1}y^{(2)'}\ \underbrace{\hat{x}y^{(1)}}_{\text{commute}} \qquad (4.56)\\
&= S^{-1}y^{(2)'}y^{(1)}\hat{x}\\
&= \hat{y}\hat{x}.
\end{aligned}
$$

The right invariant vector fields form a Hopf algebra with the same coproduct as $\mathcal{U}$ because of (4.46), but opposite antipode and multiplication:

$$\epsilon(\hat{x}) = x, \qquad \Delta\hat{x} = \widehat{x_{(1)}} \otimes \widehat{x_{(2)}}, \qquad S(\hat{x}) = \widehat{S^{-1}x}. \tag{4.57}$$

The Lie derivative of — or the adjoint action on — an element ϕ of $\mathcal{A}\rtimes\mathcal{U}$ along a right invariant vector field comes out formally equivalent to the left invariant version, when expressed in terms of the new Δ and S:

$$\begin{aligned} \mathcal{L}_{\hat{x}}(\phi) &= \hat{x} \overset{\mathrm{ad}}{\triangleright} \phi \\ &= \hat{x}_{(1)}\phi S\hat{x}_{(2)} \\ &= \widehat{x_{(1)}}\phi \widehat{S^{-1}x_{(2)}}. \end{aligned} \tag{4.58}$$

It immediately follows that

$$\mathcal{L}_{\hat{x}}(y) = 0, \qquad \text{for } y \in \mathcal{U}, \tag{4.59}$$

in agreement with the geometrical picture. Let us now compute the action of a right-invariant vector field on a function a, using only the algebraic relations of the cross product algebra and the right vacuum:

$$\begin{aligned} \hat{x}a> &= S^{-1}f^i a_{(1)} < e_i \overset{\mathrm{ad}}{\triangleright} x, a_{(2)} > \\ &= S^{-1}f^i a_{(1)} < e_i \otimes x, a_{(2)}Sa_{(4)} \otimes a_{(3)} > \\ &= a_{(4)}S^{-1}a_{(2)}a_{(1)} < x, a_{(3)} > \\ &= a_{(2)} < x, a_{(1)} > \\ &= < \hat{x}, a_{(1)} > a_{(2)}, \end{aligned} \tag{4.60}$$

as expected from the geometrical considerations of the previous section. The Hopf algebra $\hat{\mathcal{U}}$ mimics $\mathcal{U}$ very closely. There is even a canonical element $\hat{\mathcal{C}}$ in $\mathcal{A}\otimes\hat{\mathcal{U}}$ that determines *left* coactions by conjugation:

$$_{\mathcal{A}}\Delta(\hat{x}) = \hat{\mathcal{C}}(1\otimes\hat{x})\widehat{\mathcal{C}^{-1}}, \qquad \hat{x}\in\hat{\mathcal{U}}, \tag{4.61}$$

$$_{\mathcal{A}}\Delta(a) = \hat{\mathcal{C}}(1\otimes a)\widehat{\mathcal{C}^{-1}} \qquad a \in \mathcal{A} \tag{4.62}$$

$$= a_{(1)} \otimes a_{(2)}. \tag{4.63}$$

By symmetry there is of course also a left projector ˇ

$$\check{\phi} = S(\phi^{(1)'})\phi^{(2)}, \tag{4.64}$$

that is most useful in the equality

$$x = x^{(2)'}\widehat{x^{(1)}}. \tag{4.65}$$

4.2.5 Applications

Here is an example of a typical manipulation using projectors onto right-invariant vector fields:

$$\begin{aligned} xa &= xa \\ \Leftrightarrow x^{(2)'}\widehat{x^{(1)}}a &= x_{(1)}(a)x_{(2)} \\ \Leftrightarrow x^{(2)'}\widehat{x^{(1)}}_{(1)}(a)\widehat{x^{(1)}}_{(2)} &= x_{(1)}(a)x_{(2)}{}^{(2)'}\widehat{x_{(2)}{}^{(1)}} \end{aligned}$$

Now use the $\mathcal{A}\rtimes\hat{\mathcal{U}} \cong \mathcal{A}\otimes\hat{\mathcal{U}}$ isomorphism, remove the "$\hat{\ }$" over the second space and switch spaces:

$$\begin{aligned} \Leftrightarrow x^{(1)}{}_{(2)} \otimes x^{(2)'}\widehat{x^{(1)}}_{(1)}(a) &= x_{(2)}{}^{(1)} \otimes x_{(1)}(a)x_{(2)}{}^{(2)'} \\ \Leftrightarrow x^{(1)}{}_{(2)} \otimes x^{(2)'} < x^{(1)}{}_{(1)}, a_{(1)} > a_{(2)} &= x_{(2)}{}^{(1)} \otimes x_{(1)}(a)x_{(2)}{}^{(2)'} \end{aligned} \tag{4.66}$$

The expression that we have just derived is incidentally equivalent to a proof that $\Delta_{\mathcal{A}}$ is a $\mathcal{A}\rtimes\mathcal{U}$-algebra homomorphism, only this time we did not need to make any reference to linear infinite bases $\{e_i\}$ and $\{f^i\}$ of $\mathcal{U}$ and $\mathcal{A}$, that do not necessarily exist. Let us now complete the proof: Using the fact that $a \in \mathcal{A}$ was arbitrary, we take it to be the second part of the coproduct of some other element $b \in \mathcal{A}$ and multiply our expression by $b_{(1)}$ in the first space

$$\begin{aligned} \Leftrightarrow x^{(1)}{}_{(1)}(b_{(1)})x^{(1)}{}_{(2)} \otimes x^{(2)'}b_{(2)} &= b_{(1)}x_{(2)}{}^{(1)} \otimes x_{(1)}(b_{(2)})x_{(2)}{}^{(2)'} \\ \Leftrightarrow \Delta_{\mathcal{A}}(x)\Delta_{\mathcal{A}}(b) &= \Delta_{\mathcal{A}}(x_{(1)}(b))\Delta_{\mathcal{A}}(x_{(2)}) = \Delta_{\mathcal{A}}(xa). \qquad \square \end{aligned} \tag{4.67}$$

This example shows that the projections introduced in this section are powerful tools in formal computations. The manipulations in the given example were not quite as elegant as the corresponding ones using the canonical element, but the projectors are much more versatile tools and they do not require the existence of linear countable infinite bases that were implicitly assumed for the canonical element.

For further applications please see the covariance proofs in part II of this thesis.

Chapter 5

A Quantum Mechanical Model

In this chapter we would like to illustrate at the example of a simple toy model one possible way how quantum groups might find use in physics. Quantum mechanics is a remarkably good theory as far as experimental verification is concerned, so we will not attempt to modify its most basic features as for instance the canonical commutation relations. We instead want to focus on a generalization of unitary transformations. These transformations form groups in quantum mechanics; we will investigate — at the example of time evolution — what happens if we generalize these transformations to be elements of Hopf algebras. The introduction of deformed *Poincare* symmetry in physics is expected to lead to similar new phenomena. We will in particular embed the operator algebra of a simple quantum mechanical model in a Hopf algebra with possibly non-trivial coproduct and propose generalized time evolution equations. We find that probability is conserved in this formulation but pure states can evolve into mixed ones (and vice versa); microscopic entropy is only conserved for a special *stable* state. The theory could be interpreted as quantum mechanics for open systems.

Introduction

There have been a number of proposals for a deformation of ordinary quantum mechanical systems using quantum groups. In particular systems with quantum group symmetries *e.g.* [55, 56] and with deformed canonical commutation relations *e.g.* [57, 58] have been investigated in some detail. Here we would like to focus on deformed time evolution equations, i.e. deformations of the Heisenberg equations of motion (Heisenberg picture) and of the Liouville equation for the density operator (Schrödinger picture). It turns out to be fruitful to consider both pictures (H.p./ S.p.) simultaneously. Let us list some basic requirements on time evolution equations:

- The equations have to be linear.
- Time evolution should be multiplicative.
- Hermiticity must be preserved.
- Probability must be preserved, i.e. the trace of the density matrix must be constant.
- Probabilities must be positive at all times.

All these requirements are fulfilled by unitary time evolution:

$$X(t) = [U(t)]^{-1}X(0)U(t), \qquad \text{(H.p.)}, \tag{5.1}$$

$$\rho(t) = U(t)\rho(0)[U(t)]^{-1}, \qquad \text{(S.p.)}, \tag{5.2}$$

with $[U(t)]^{+} = [U(t)]^{-1}$. In this paper we would like to argue that the above equations are not the only possible ones satisfying all the listed requirements; in order to find more general equations we, however, need to extend the operator algebra to a Hopf algebra.

Generalizations of unitary time evolution have been studied before in the 70's in the context of completely positive maps and dynamical semi-groups. Lindblad [54] found the general form for generators of such semi-groups, however, without being able to give a *cause* for the modified time evolution equation because he does not make any reference to an underlying structure — like Hopf algebras or non-commutative geometry in our case.

5.1 Schrödinger Picture

Let us briefly review density matrices in "classical" quantum mechanics: All observables are described by operators X constant in time, states are given as time dependent density matrices $\rho(t)$. Expectation values are calculated as usual via

$$< X >_{\rho(t)} = tr(X\rho(t)), \tag{5.3}$$

where the trace is cyclic ($tr(xy) = tr(yx)$). The eigenvalues of the density matrix are the probabilities of the pure components of the mixed state. In a diagonal basis

$$\rho = \sum_i p_i |i><i|, \qquad 0 \le p_i \le 1. \tag{5.4}$$

The sum of the eigenvalues of the density matrix must hence be one

$$tr(\rho) = \sum_i p_i = 1, \qquad \textit{(normalization)} \tag{5.5}$$

independent of time. Note that

$$tr(\rho^2) = \sum_i p_i^2 \leq 1; \tag{5.6}$$

the equality is only satisfied for a pure ($\rho = |\psi><\psi|$) state. All mixed states have $tr(\rho^2) < 1$. Unitary time evolution not only preserves the trace of ρ, *i.e.* it conserves probability,

$$tr(U\rho U^{-1}) = tr(\rho) = 1 \tag{5.7}$$

but also conserves entropy:

$$tr\left((U\rho U^{-1})(U\rho U^{-1})\right) = tr(\rho^2). \tag{5.8}$$

It preserves hermiticity of ρ because of $U^\dagger = U^{-1}$ and is multiplicative: $U(t_1 + t_2) = U(t_1) \cdot U(t_2)$. Our task is now to find a generalized time evolution for the density matrix with all those properties except for the conservation of entropy. To satisfy *linearity* and *multiplicativity* we choose time evolution to be realized through the action (see chapter 1) of some new time evolution operator $\tilde{U}$

$$\rho(t) = \tilde{U} \overset{\text{op}}{\triangleright} \rho, \qquad \tilde{U} = \tilde{U}(t). \tag{5.9}$$

To leave freedom for deformations we ask $\tilde{U}$ to be an element of a Hopf algebra $\mathcal{U}$ (rather than a group) and propose the following left action:

$$\boxed{\rho(t) = \tilde{U}_{(2)}\rho S(\tilde{U}_{(1)})}. \tag{5.10}$$

Due to $S(\tilde{U}_{(1)})\tilde{U}_{(2)} = \epsilon(\tilde{U})$ and the cyclicity of the trace this time evolution equation *conserves probability*

$$tr(\tilde{U}_{(2)}\rho S(\tilde{U}_{(1)})) = tr(S(\tilde{U}_{(1)})\tilde{U}_{(2)}\rho) = tr(\rho) \cdot \epsilon(\tilde{U}), \tag{5.11}$$

if we impose

$$\boxed{\epsilon(\tilde{U}) = 1}. \tag{5.12}$$

In order to *conserve hermiticity* we have to impose

$$\boxed{\tilde{U}^\dagger = S(\tilde{U})}, \tag{5.13}$$

because then

$$\begin{aligned}(\rho(t))^\dagger &= \left(\tilde{U}_{(2)}\rho S(\tilde{U}_{(1)})\right)^\dagger \\ &= S^{-1}(\tilde{U}^\dagger_{(1)})\rho^\dagger \tilde{U}^\dagger_{(2)} \\ &= S^{-1}(\tilde{U}^\dagger) \overset{\mathrm{op}}{\triangleright} \rho^\dagger \\ &= \tilde{U} \overset{\mathrm{op}}{\triangleright} \rho^\dagger.\end{aligned} \tag{5.14}$$

Entropy is however no longer necessarily conserved:

$$tr\,(\rho(t)\cdot\rho(t)) \neq tr(\rho_0\cdot\rho_0), \qquad \textit{in general.} \tag{5.15}$$

Example: "Classical" Quantum Mechanics is a special case with

$$\begin{gathered}\Delta(U) = U\otimes U, \qquad S(U) = U^{-1}, \qquad \epsilon(U) = 1, \\ \rho(t) = U_{(2)}\rho S(U_{(1)}) = U\rho U^{-1}, \\ U^\dagger = S(U) = U^{-1}.\end{gathered} \tag{5.16}$$

5.2 Heisenberg Picture

Now we stick all the time evolution into the observables, leaving the density matrix time invariant. The time evolution equation for the operators easily follows from the one for the density matrix using the cyclic nature of the trace and the fact that the time evolution of the expectation values should be independent of the particular picture. We find:

$$\boxed{X(t) = X(0) \overset{\mathrm{ad}}{\triangleleft} \tilde{U}(t) \equiv S(\tilde{U}_{(1)})X(0)\tilde{U}_{(2)}}. \tag{5.17}$$

Two consistency requirements give the same conditions on $\tilde{U}$

$$1(t) = 1 \;\Rightarrow\; \epsilon(\tilde{U}) = 1, \tag{5.18}$$

$$(X(t))^\dagger = X^\dagger \overset{\mathrm{ad}}{\triangleleft} \tilde{U} \;\Rightarrow\; \tilde{U}^\dagger = S(\tilde{U}), \tag{5.19}$$

as were already obtained in the previous section.

5.3 Infinitesimal Transformation

One great thing about working with Hopf algebras is that finite and infinitesimal transformations are unified in the sense that they have the exact same form. The infinitesimal version of our time evolution equation must have the form

$$\frac{d\rho}{dt} = \frac{\tilde{H}}{i\hbar} \overset{\mathrm{op}}{\triangleright} \rho = \frac{1}{i\hbar}\tilde{H}_{(2)}\rho S(\tilde{H}_{(1)}), \tag{5.20}$$

where i is purely conventional and we have inserted $\hbar$ to give $\tilde{H}$ units of energy. The conditions on $\tilde{H}$ are slightly different from the ones on $\tilde{U}$:*

$$\begin{aligned} \epsilon(\tilde{H}) &= 0 && (5.21) \\ \tilde{H}^{\dagger} &= -S(\tilde{H}). && (5.22) \end{aligned}$$

How do we obtain the time evolution operator $\tilde{H}$ from the (hermitian) Hamiltonian H? Here is a **Conjecture:**

$$\tilde{H} = \frac{1}{2}\left(H - S^{-1}(H)\right), \qquad H^{\dagger} = H. \tag{5.23}$$

(The 2 might be a "quantum-2".) This choice for $\tilde{H}$ will automatically satisfy both conditions. Finite time translations can be recovered by Taylor expansion

$$\rho(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^n \rho}{dt^n}\bigg|_{t=0} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\left(\frac{\tilde{H}}{i\hbar}\right) \overset{\text{op}}{\triangleright}\right)^n \rho = e^{\frac{\tilde{H}}{i\hbar}t} \overset{\text{op}}{\triangleright} \rho, \tag{5.24}$$

where we have used the multiplicative properties of successive actions. Note that this is an ordinary exponential function, not a q-deformed one.

In the following section we will study a system with a finite number of eigenstates. In this case equation (5.20) can be converted into a matrix equation by taking the inner product with a matrix $A \in M_n(\mathcal{A})$ as follows:

$$\begin{aligned} \frac{d < \rho, A^i{}_l >}{dt} &= \tfrac{1}{i\hbar} < \tilde{H}_{(2)} \rho S \tilde{H}_{(1)}, A^i{}_l > \\ &= \tfrac{1}{i\hbar} < \tilde{H}, S(A^k{}_l) A^i{}_j >< \rho, A^j{}_k > \end{aligned} \tag{5.25}$$

or, in a short hand,

$$\frac{d\rho^{(il)}}{dt} = \frac{1}{i\hbar} \tilde{H}^{(il)}{}_{(jk)} \rho^{(jk)}. \tag{5.26}$$

This matrix equation can easily be exponentiated to give an explicit solution

$$\rho^{(il)}(t) = \exp\left(\frac{\tilde{H}}{i\hbar}\right)^{(il)}{}_{(jk)} \rho_0^{(jk)} \tag{5.27}$$

for ρ. In practice one would now express ρ in terms of eigenvectors of $\tilde{H}^{(il)}{}_{(jk)}$ so that the matrix exponential diagonalizes with the exponentials of $\frac{1}{i\hbar}$ times $\tilde{H}$'s eigenvalues along its diagonal.

5.4 A Simple 2-Level System

Consider a single particle in a double well potential (Fig. 5.1) with a barrier of height

*The first condition may possibly be interpreted as requiring a zero energy ground state.

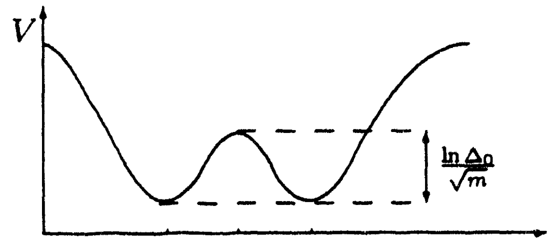


Figure 5.1: Double Well

$\sim \frac{\ln\Delta_0}{\sqrt{m}}$. If we are only interested in which dip the particle is localized then we are dealing with a 2-level system. A phenomenological hamiltonian that describes tunneling through the barrier is easily written down in terms of the x-Pauli matrix:

$$H = \Delta_0\sigma_x = \Delta_0(\sigma_+ + \sigma_-). \qquad \textit{(tunneling only)} \tag{5.28}$$

Instead of viewing the Pauli matrices as the fundamental representation of $su(2)$ we would like to consider $su_q(2)$ with $q \in (0,1]$ as given in (1.79). All irreducible representations of $su_q(2)$, e.g.

$$\begin{aligned}
&\text{2-dim:}\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
&\text{3-dim:}\quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \; J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \\
&\vdots
\end{aligned} \tag{5.29}$$

are undeformed. This makes it easy to derive a matrix representation of the time evolution operator:

$$\tilde{H} = \frac{1}{2}(H - S^{-1}(H)) = \frac{1}{2}(\sigma_+ + \sigma_- + q^{-1}\sigma_+ + q\sigma_-)\Delta_0 \propto \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}. \tag{5.30}$$

We will ignore the proportionality constant because it can always be incorporated in Δ_0. The time evolution equation in matrix form is

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}\left(\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}\rho\begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}\rho\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\right), \tag{5.31}$$

which reduces to the correct classical limit

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[\sigma_x, \rho], \qquad \textit{(classical)} \tag{5.32}$$

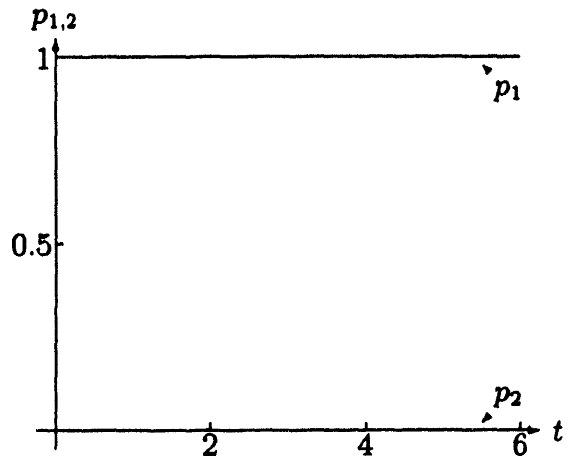


Figure 5.2: q = 1

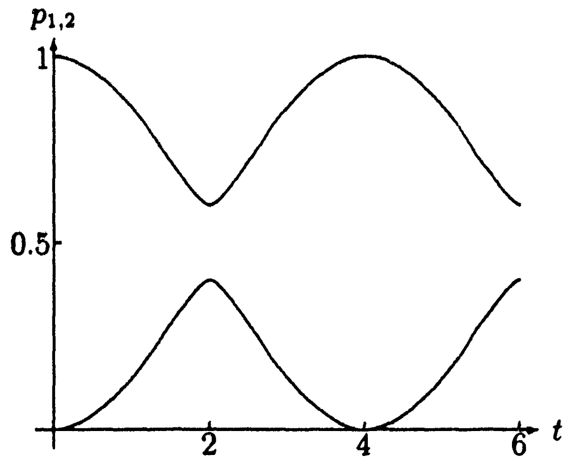


Figure 5.3: $q^2 = 0.7$

as $q \rightarrow 1$. Plugging the hamiltonian H into the matrix time evolution equation in the Heisenberg picture,

$$\frac{dX}{dt} = \frac{1}{i\hbar}\left(\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} X \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix} X \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}\right), \tag{5.33}$$

gives incidentally

$$\frac{dH}{dt} = 0, \tag{5.34}$$

i.e. energy *is* conserved in our toy model.

5.4.1 Time Evolution and Mixing

It is instructive to look at an actual computation of the evolution of a system that is in an eigenstate $|+>$ of σ_z at $t = 0$; the corresponding density matrix

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \textit{(initial pure state)} \tag{5.35}$$

is that of a pure state ($tr(\rho) = tr(\rho^2) = 1$). Interesting are the eigenvalues p_1,p_2 of $\rho(t)$ as a function of time. They are the probabilities of the respective pure states in the mixture. For $q = 1$ (Fig. 5.2) nothing much happens, but for e.g. $q = \sqrt{0.7} \approx 0.5$ (Fig. 5.3) the system oscillates between a pure and a partially mixed state. A behavior like that does not appear in ordinary quantum case and opens up interesting possibilities for, say in the present case, a phenomenological quantum mechanical description of just one part of a coupled system. Here we do not want to plunge too deep into possible interpretations but would just like to point out some

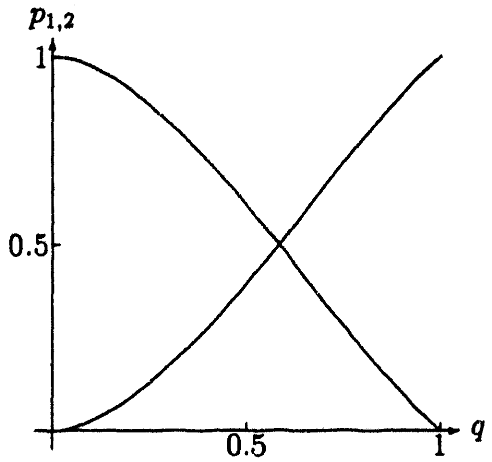


Figure 5.4: $t \approx 1.9$

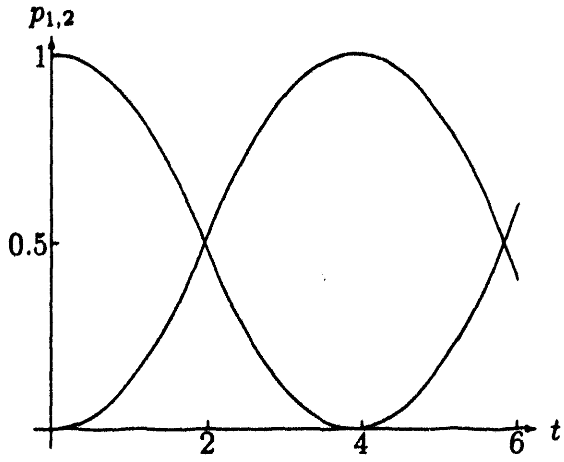


Figure 5.5: $q^2 = 1/3$

new phenomena that appear when laws of physics are deformed. Just out of curiousity let us find the q for which the system becomes totally mixed. Plotting $p_{1,2}$ against q (Fig. 5.4) at fixed time $t \approx 1.9$† we find $q_{critical} = \sqrt{1/3}$; see also Fig. 5.5. The significance of this number is unknown.

5.4.2 Stable State

An interesting question is whether there exists a stable (mixed) state that is invariant under the deformed time evolution. This is indeed the case and has to do with the square of the antipode: The square of the antipode is an inner automorphism in $U_q(su(2))$ implemented by elements u and $v = S(u)$ via conjugation [28]

$$S^2(x) = uxu^{-1} = v^{-1}xv, \qquad \forall x \in U_q(su(2)). \tag{5.36}$$

Let us try v as a density operator:

$$\begin{aligned} v(t) &:= \tilde{U}_{(2)} v S(\tilde{U}_{(1)}) \\ &= v S^2(\tilde{U}_{(2)}) S(\tilde{U}_{(1)}) \\ &= v \epsilon(\tilde{U}) \\ &= v. \end{aligned} \tag{5.37}$$

Thus v has the desired properties. Its 2-dimensional matrix representation

$$v = \frac{1}{1+q^2} \begin{pmatrix} q^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.38}$$

†This value was found by iteration.

looks like a thermal state for a hamiltonian with dominant part proportional to σ_z, *i.e.*

$$H = \Delta_1\sigma_z + \Delta_0\sigma_x, \qquad \Delta_1 \gg \Delta_0{}^{\ddagger}, \tag{5.39}$$

and suggests

$$q = \exp(-\Delta_1/kT), \qquad q \in (0,1]. \tag{5.40}$$

Higher matrix representations of v give additional support for this hypothesis:

$$\text{3-dim:} \qquad v \propto \begin{pmatrix} q^4 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \qquad e.t.c. \tag{5.41}$$

‡Or: Time average of $\Delta_0 \approx 0$.

Part II

Differential Geometry on Quantum Spaces

Chapter 6

Quantum Spaces

6.1 Quantum Planes

A classical plane can be fully described by the commutative algebra of (coordinate) functions over it. This algebra is typically covariant under the action of some symmetry group, and derivatives on it satisfy an undeformed product rule. A quantum plane in contrast to this is covariant under a quantum group whose non-commutative algebra of functions $\mathcal{A}$ also forces the algebra of functions on the q-plane Fun($\mathbf{M}_q$) to seize to commute. The transformations of Fun($\mathbf{M}_q$) and of the dual algebra of quantum derivatives $\mathcal{T}(\mathbf{M}_q)$ is most easily described in terms of $\mathcal{A}$-coactions on coordinate functions and partial derivatives

$$\Delta_{\mathcal{A}} x^i = x^j \otimes S t^i{}_j, \tag{6.1}$$

$$\Delta_{\mathcal{A}} \partial_i = \partial_j \otimes S^2 t^j{}_i, \tag{6.2}$$

which we sometimes write in short matrix form as

$$x \rightarrow t^{-1} \cdot x, \tag{6.3}$$

$$\partial \rightarrow \partial \cdot S^2 t. \tag{6.4}$$

Remark: The "S" was inserted here to make these transformations *right* coactions, the S^2 is needed for covariance (see below).
Remark: One can use $t^j{}_i$ in place of $St^i{}_j$. Then $x \rightarrow x \cdot t$ and $\partial \rightarrow St \cdot \partial$. The choice is purely conventional.

6.1.1 Product Rule for Quantum Planes

Having made the ring of functions non-commutative, we must now also modify the product rule in order to retain covariant equations. We make the following ansatz (see [22])

$$\partial_i x^k = \partial_i(x^k) + L_i{}^j(x^k)\partial_j, \tag{6.5}$$

where $\partial_i(x^k) = \delta_i^k$ and $L_i{}^j$ is a linear operator that describes the braiding of ∂_i as it moves through x^k. In place of the coordinate function x^k one could write any other function in $\mathrm{Fun}(\mathbf{M}_q)$ and in particular (formal) power series in the coordinate functions. When we consider products of coordinate functions we immediately see that L satisfies

$$L_i{}^j(xy) = L_i{}^l(x)L_l{}^j(y), \qquad L_i{}^j(1) = \delta_i^j, \tag{6.6}$$

which can be reinterpreted in Hopf algebra language as $\Delta L = L\dot{\otimes}L$ and $\epsilon(L) = I$; $SL = L^{-1}$ follows naturally. We are hence let to believe that L should belong to some Hopf algebra, the Braiding Hopf Algebra. In the case of linear quantum groups L is for instance an element of the quasitriangular Hopf algebra $\mathcal{U}$ of the quantum symmetry group. Considering multiple derivatives gives additional conditions that can be summarized by requiring that

$$_{\mathcal{U}}\Delta\partial_i = L_i{}^j \otimes \partial_j \tag{6.7}$$

be a Hopf algebra coaction, *i.e.*

$$_{\mathcal{U}}\Delta(\partial\partial') = {}_{\mathcal{U}}\Delta(\partial)_{\mathcal{U}}\Delta(\partial'), \quad (id \otimes {}_{\mathcal{U}}\Delta)_{\mathcal{U}}\Delta = (\Delta \otimes id)_{\mathcal{U}}\Delta, \quad (\epsilon \otimes id)_{\mathcal{U}}\Delta = id. \tag{6.8}$$

For arbitrary functions f and derivatives ∂ we find a generalized product rule

$$\boxed{\partial f = \partial(f) + \partial_{1'}(f)\partial_2}, \tag{6.9}$$

where $_{\mathcal{U}}\Delta\partial \equiv \partial_{1'}\otimes\partial_2$. Covariance of the product rule (6.5) under coactions is expected to give strong conditions on $L_i{}^j$.

Remark: The formula for the product rule (6.5) was inspired by the form of the multiplication of two elements ξ, ϕ in the cross product algebra $\mathcal{A}\rtimes\mathcal{U}$

$$\xi\phi = \phi^{(1)} < \xi_{1'}, \phi^{(2)'} > \xi_2, \tag{6.10}$$

where $\Delta_{\mathcal{A}}(\phi) = \phi^{(1)} \otimes \phi^{(2)'}$ and $_{\mathcal{U}}\Delta(\xi) = \xi_{1'} \otimes \xi_2$ (see chapter 2).

6.1.2 Covariance of: $\partial_i f = \partial_i(f) + L_i{}^j(f)\partial_j$

We need to use an **inductive** approach: We start by requiring that

$$\Delta_{\mathcal{A}}(\partial_i(x^j)) = \Delta_{\mathcal{A}}\partial_i(\Delta_{\mathcal{A}}x^j). \qquad \textit{(anchor)} \tag{6.11}$$

This is in fact satisfied, because we already have $\Delta_{\mathcal{A}} x^j = x^l \otimes St^j{}_l$ and iff $\Delta_{\mathcal{A}} \partial_j = \partial_l \otimes S^2 t^l{}_j$ then: $\Delta_{\mathcal{A}} \partial_i (\Delta_{\mathcal{A}} x^j) = \partial_k(x^l) \otimes S^2 t^k{}_i St^j{}_l = \delta^l_k \otimes S^2 t^k{}_i St^j{}_l = \delta^j_i \otimes 1$, in agreement with $\Delta_{\mathcal{A}}(\partial_i(x^j)) = \Delta_{\mathcal{A}}(\delta^j_i) = \delta_i^j \otimes 1$. That was the anchor; now the induction to higher powers in the coordinate functions: Assume that the action of ∂_i on f is covariant:

$$\Delta_{\mathcal{A}}(\partial_i(f)) = \Delta_{\mathcal{A}} \partial_i (\Delta_{\mathcal{A}} f), \tag{6.12}$$

where f is a function of the coordinate functions x^i. Try to proof covariance of the ∂_i–f commutation relation, *i.e.*

$$(6.12) \overset{?}{\Rightarrow} \Delta_{\mathcal{A}}(\partial_i f) = \Delta_{\mathcal{A}}(\partial_i) \cdot \Delta_{\mathcal{A}}(f). \qquad \textit{(induction)} \tag{6.13}$$

After some computation we find

$$\Delta_{\mathcal{A}}(L_i{}^j(f))(1 \otimes S^2 t^k{}_j) \overset{!}{=} L_l{}^k(f^{(1)}) \otimes S^2 t^l{}_i f^{(2)\prime}, \tag{6.14}$$

where $\Delta_{\mathcal{A}}(f) \equiv f^{(1)} \otimes f^{(2)\prime}$. This simplifies further if we know how $L_i{}^j$ acts on f. If the braiding Hopf algebra acts like the covariance quantum group, then $L_i{}^j(f) = f^{(1)} < L_i{}^j, f^{(2)\prime} >$, $L_i{}^j \in \mathcal{A}^*$ and (6.14) becomes

$$\left(L_i{}^j(f^{(2)\prime}) S^2 t^k{}_j - S^2 t^l{}_i \widehat{L_l{}^k}(f^{(2)\prime})\right) \otimes f^{(1)} = 0, \tag{6.15}$$

where $\hat{}\ : \mathcal{A} \rtimes \mathcal{U} \to \mathcal{A} \rtimes \mathcal{U}$ is the projector onto right-invariant vector fields: $\hat{x} = S^{-1}(x^{(2)\prime}) x^{(1)}$ with $\Delta_{\mathcal{A}}(x) \equiv x^{(1)} \otimes x^{(2)}$, such that $\widehat{L_l{}^k}(f^{(2)\prime}) = < L_l{}^k, f^{(2)\prime} > f^{(3)\prime}$. This is satisfied if

$$L_i{}^j(a) S^2 t^k{}_j = S^2 t^l{}_i \widehat{L_l{}^k}(a), \qquad \forall a \in \mathcal{A}. \tag{6.16}$$

(The reverse is true only if $\mathcal{A}$ is generated by $[St^i{}_j]$ — or $[t^i{}_j]$, if the choose the convention $\Delta_{\mathcal{A}}(x) = x \cdot t$.) In the case where the braiding Hopf algebra is quasitriangular, there are (exactly) two natural choices

$$L_j{}^i \propto \begin{cases} S^{-1} L^{-i}{}_j \equiv < \mathcal{R}, S^2 t^i{}_j \otimes id > \\ S^{-1} L^{+i}{}_j \equiv < \mathcal{R}, id \otimes St^i{}_j > \end{cases} \tag{6.17}$$

that satisfy the above equation and all other requirements (coproduct, *e.t.c.*).

For the Wess-Zumino quantum plane [22] the action of L on the coordinate functions is linear and of first degree in those functions, so we can use the coaction $\Delta_{\mathcal{A}}$ to express it:

$$L_i{}^j(x^k) = < L_i{}^j, St^k{}_l > x^l \propto \begin{cases} r^{kj}{}_{li} x^l \\ (r^{-1})^{jk}{}_{il} x^l \end{cases} \tag{6.18}$$

in perfect agreement with [22]. (The overall multiplicative constant $(\frac{1}{q})$ is not fixed by covariance considerations but is given by the characteristic equation of $\hat{r}$ and the requirement that $\widehat{C}^{kj}{}_{li} \equiv < L_i{}^k, St^j{}_l >$ should have an eigenvalue -1.)

6.2 Quantum Groups

A quantum group is a quantum plane covariant under itself. However, it has more structure and the coactions $\Delta_{\mathcal{A}}$ and ${}_{\mathcal{U}}\Delta$ are now completely determined by the multiplication in $\mathcal{U}$ and $\mathcal{A}$: Let $\phi \in \mathcal{A} \rtimes \mathcal{U}$ and $\Delta_{\mathcal{A}}\phi \equiv \phi^{(1)} \otimes \phi^{(2)'}$; then

$$\mathcal{L}_{\chi}(\phi) = \chi \overset{\mathrm{ad}}{\triangleright} \phi \equiv \chi_{(1)}\phi S\chi_{(2)} = \phi^{(1)} < \chi, \phi^{(2)'} >, \qquad \forall \chi \in \mathcal{U} \tag{6.19}$$

determines $\Delta_{\mathcal{A}}$. The coaction ${}_{\mathcal{U}}\Delta$ is simply the coproduct $\Delta : \mathcal{U} \rightarrow \mathcal{U} \otimes \mathcal{U}$, so that the product rule becomes

$$xa = a_{(1)} < x_{(1)}, a_{(2)} > x_{(2)}, \tag{6.20}$$

where $x \in \mathcal{U}$, $a \in \mathcal{A}$. This defines the multiplicative structure in the so called cross product algebra [60] $\mathcal{A} \rtimes \mathcal{U}$. Interestingly, equation (6.18) does not apply in the case of a quantum group: In that case t is replaced by the adjoint representation T and L becomes O, a part in the coproduct of the basic generators. Not all elements of T are linearly independent. There is a trivial partial sum $T^{(ii)}{}_{(kl)} = 1\delta_{(kl)}$; the same sum for O, $O^{(ii)}{}_{(kl)} =: Y_{(kl)}$, is in general non-trivial thus leading to a contradiction. An explanation for this is that quantum groups have more structure than quantum planes. They already contain an intrinsic braiding and do not leave any freedom for external input such as $\mathcal{R}$ in equation (6.18); the product rule is in fact automatically covariant by the construction of the cross product algebra. There are, however, some indications that O and T might be related to a universal $\tilde{\mathcal{R}}$ that lives in the sub-Hopf algebra of $\mathcal{A}$ generated by the elements of T.

From the discussion of the quantum planes we would like to keep the idea of a finite number of so-called bicovariant generators χ_i that close under adjoint action $\chi_i \overset{\mathrm{ad}}{\triangleright} \chi_j = \chi_k f_i{}^k{}_j$ and span an invariant subspace of $\mathcal{U}$, *i.e.* $\Delta_{\mathcal{A}}\chi_j = \chi_k \otimes T^k{}_j$. We call quantum groups with such generators Quantum Lie Algebras. In following section we will give more precise definitions of quantum Lie algebras.

Chapter 7

Cartan Calculus

7.0.1 Cartan Identity

The central idea behind Connes Universal Calculus [2] in the context of non-commutative geometry was to retain from the classical differential geometry the nilpotency of $\mathbf{d}$

$$\mathbf{d}^2 = 0 \tag{7.1}$$

and the undeformed Leibniz rule for $\mathbf{d}$*

$$\mathbf{d}\alpha = \mathbf{d}(\alpha) + (-1)^p \alpha \mathbf{d} \tag{7.2}$$

for any p-form α. The exterior derivative $\mathbf{d}$ is a scalar making this equation hard to deform, except for a possible multiplicative constant in the second term. Here we want to base the construction of a differential calculus on quantum groups on two additional classical formulas: to extend the definition of a Lie derivative from functions and vector fields to forms we postulate

$$\pounds \circ \mathbf{d} = \mathbf{d} \circ \pounds; \tag{7.3}$$

this is essential for a geometrical interpretation along the lines of chapter 4. The second formula that we can — somewhat surprisingly — keep undeformed in the quantum case is originally due to Henri Cartan

$$\pounds_{\chi_i} = \boldsymbol{i}_{\chi_i}\mathbf{d} + \mathbf{d}\boldsymbol{i}_{\chi_i}, \qquad \textit{(Cartan Identity)} \tag{7.4}$$

*We use parentheses to delimit operations like $\mathbf{d}$, $\boldsymbol{i}_x$ and $\pounds_x$, *e.g.* $\mathbf{d}a = \mathbf{d}(a) + a\mathbf{d}$. However, if the limit of the operation is clear from the context, we will suppress the parentheses, *e.g.* $\mathbf{d}(\boldsymbol{i}_x \mathbf{d}a) \equiv \mathbf{d}(\boldsymbol{i}_x(\mathbf{d}(a)))$.

where χ_i are the generators of some quantum Lie algebra. The only possibility to deform this equation and not violate its covariance is to introduce multiplicative deformation parameters κ, λ for the two terms on the right hand side of (7.4) such that now $\pounds_{\chi_i} = \kappa \boldsymbol{i}_{\chi_i}\mathbf{d} + \lambda \mathbf{d}\boldsymbol{i}_{\chi_i}$. For a function $a \in \mathcal{A}$ that gives

$$\pounds_{\chi_i}(a) = \kappa \boldsymbol{i}_{\chi_i}(\mathbf{d}a)$$

($\boldsymbol{i}_{\chi_i}$ vanishes on functions), for $\mathbf{d}a$ we find

$$\pounds_{\chi_i}(\mathbf{d}a) = \lambda \mathbf{d}(\boldsymbol{i}_{\chi_i}(\mathbf{d}a))$$

and finally together

$$\pounds_{\chi_i}(\mathbf{d}a) = \frac{\lambda}{\kappa}\mathbf{d}(\pounds_{\chi_i}(a)),$$

in contrast to (7.3) unless $\frac{\lambda}{\kappa} = 1$, in which case we can easily absorb either κ or λ into $\boldsymbol{i}_{\chi}$. Being now (hopefully) convinced of our two basic equations (7.3) and (7.4) we want to turn to the generators χ_i next.
Several discussions with P. Aschieri helped clarifying the relation between the material presented in the next section and Woronowicz's theory.

7.1 Quantum Lie Algebras

A quantum Lie algebra is a Hopf algebra $\mathcal{U}$ with a finite-dimensional biinvariant sub vector space $\mathcal{T}_q$ spanned by generators $\{\chi_i\}$ with coproduct

$$\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j. \tag{7.5}$$

More precisely we will call this a quantum Lie algebra of **type II**. Let $\{\omega^j \in \mathcal{T}_q{}^*\}$ be a dual basis of 1-forms corresponding to a set of functions $b^j \in \mathcal{A}$ via $\omega^j \equiv Sb^j_{(1)}\mathbf{d}b^j_{(2)}$; *i.e.*

$$\begin{aligned}
{}_{\mathcal{A}}\Delta(\chi_i) &= 1 \otimes \chi_i, \\
\Delta_{\mathcal{A}}(\chi_i) &= \chi_j \otimes T^j{}_i, \quad T^j{}_i \in \mathrm{Fun}(G_q), && (7.6)\\
\boldsymbol{i}_{\chi_i}(\omega^j) &= -<\chi_i, Sb^j> = \delta_i^j, && (7.7)\\
{}_{\mathcal{A}}\Delta(\omega^i) &= 1 \otimes \omega^i, && (7.8)\\
\Delta_{\mathcal{A}}(\omega^i) &= \omega^j \otimes S^{-1}T^i{}_j. && (7.9)
\end{aligned}$$

If the functions b^i also close under adjoint coaction $\Delta^{Ad}(b^i) = b^j \otimes S^{-1}T^i{}_j$, we will call the corresponding quantum Lie algebra one of **type I**. Getting a little ahead of ourself's let us mention that we can derive an expression for the exterior derivative of a function from the Cartan identity (7.4) in terms of these bases

$$\mathbf{d}(a) = \omega^i(\chi_i \triangleright a) = \omega^i \pounds_{\chi_i}(a) \tag{7.10}$$

and that this leads to the following $f - \omega$ commutation relations [21]

$$f\omega^i = \omega^j({O_j}^i \triangleright f). \tag{7.11}$$

7.1.1 Generators, Metrics and the Pure Braid Group

How does one practically go about finding the basis of generators $\{\chi_i\}$ and the set of functions $\{b^i\}$ that define the basis of 1-forms $\{\omega^i\}$? Here we would like to present a method that utilizes pure braid group elements as introduced in the first part of this thesis.

Let us recall that a pure braid element Υ is an element of $\mathcal{U}\hat{\otimes}\mathcal{U}$ that commutes with all coproducts of elements of $\mathcal{U}$, *i.e.*

$$\Upsilon\Delta(y) = \Delta(y)\Upsilon, \qquad \forall y \in \mathcal{U}. \tag{7.12}$$

Υ maps elements of $\mathcal{A}$ to elements of $\mathcal{U}$ with special transformation properties under the right coaction:

$$\begin{aligned} \Upsilon : \mathcal{A} \rightarrow \mathcal{U} : \quad b \mapsto \Upsilon_b \equiv <\Upsilon, b \otimes id>; \\ \Delta_{\mathcal{A}}(\Upsilon_b) = \Upsilon_{b_{(2)}} \otimes S(b_{(1)})b_{(3)} = <\Upsilon \otimes id, \tau^{23}(\Delta^{Ad}(b) \otimes id)> . \end{aligned} \tag{7.13}$$

An element Υ of the pure braid group defines furthermore a bilinear quadratic form on $\mathcal{A}$

$$(\ ,\ ) : \mathcal{A} \otimes \mathcal{A} \rightarrow k : \qquad a \otimes b \mapsto (a, b) = - <\Upsilon, a \otimes S(b)> \in k, \tag{7.14}$$

with respect to which we can construct orthonormal $(b_i, b^j) = \delta_i^j$ bases $\{b_i\}$ and $\{b^j\}$ of functions that in turn will define generators $\chi_i := \Upsilon_{b_i}$ and 1-forms $\omega^j := S(b^j_{(1)})\mathbf{d}b^j_{(2)}$. Typically one can choose $\mathrm{span}\{b_i\} = \mathrm{span}\{b^j\}$; then one starts by constructing one set, say $\{b_i\}$, of functions that close under adjoint coaction

$$\Delta^{Ad} b_i = b_j \otimes {T^j}_i. \tag{7.15}$$

If the numerical matrix

$$\boxed{\eta_{ij} := - <\Upsilon, b_i \otimes Sb_j>} \qquad \textit{(metric)} \tag{7.16}$$

is invertible, *i.e.* $\det(\eta) \neq 0$, then we can use its inverse $\eta^{ij} := (\eta^{-1})_{ij}$ to raise indices

$$b^i = b_j \eta^{ji}. \tag{7.17}$$

This metric is invariant — or T is orthogonal — in the sense

$$\begin{aligned} \eta_{ji} &= - < S\chi_j, b_i > \\ &= - < S\chi_j, b_k > ST^k{}_l T^l{}_i \\ &= - < \chi_k, Sb_l > T^k{}_j T^l{}_i \\ &= \eta_{kl} T^k{}_j T^l{}_i, \end{aligned} \tag{7.18}$$

where we have used the Hopf algebraic identity

$$< \Delta_{\mathcal{A}}(\chi), Sa \otimes id >= S(< S\chi \otimes id, \Delta^{Ad}(a) >), \tag{7.19}$$

which we will proof in an appendix to this section. Once we have obtained a metric η, we can truncate the pure braid element Υ and work instead with:

$$\Upsilon \rightarrow \Upsilon_{trunc} = -S(\chi_i) \otimes \chi^i = -S(\chi_i) \otimes \chi_j \eta^{ji}, \qquad \textit{(truncated pure braid element)} \tag{7.20}$$

which also commutes with all coproducts. In part I of these thesis we have shown how to construct casimir operators from elements of the pure braid group. For the truncated pure braid element that gives the quadratic casimir:

$$[\cdot \circ \tau \circ (S^{-1} \otimes id)](\Upsilon_{trunc}) = \eta^{ji} \chi_j \chi_i. \qquad \textit{(casimir)} \tag{7.21}$$

Now we would like to show that we have actually obtained a quantum Lie algebra of type I:†

$$- < \chi_i, Sb^j >= - < \Upsilon, b_i \otimes Sb^j >= - < \Upsilon, b_i \otimes Sb_k > \eta^{kj} = \eta_{ik}\eta^{kj} = \delta_i^j, \tag{7.22}$$

$$\Delta_{\mathcal{A}}(\chi_i) = \Upsilon_{b_{i(2)}} \otimes S(b_{i(1)}) b_{i(3)} = \Upsilon_{b_j} \otimes T^j{}_i = \chi_j \otimes T^j{}_i \tag{7.23}$$

and

$$\Delta^{Ad}(b^i) = \Delta^{Ad}(b_j)\eta^{ji} = b_k \otimes T^k{}_j \eta^{ji} = b_k \otimes \eta^{kl}\eta_{ln} T^n{}_j \eta^{ji} = b^k \otimes S^{-1} T^i{}_k. \tag{7.24}$$

†Note, that Υ has to be carefully chosen to insure the correct number of generators. Furthermore, we still have to check the coproduct of the generators. If they are not of the form $\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j$ then we can still consider a calculus with deformed Leibniz rule (see next section).

Examples

The r-matrix approach: Often one can take $b_i \in \mathrm{span}\{t^n{}_m\}$, where $t^n{}_m$ is a quantum matrix in the defining representation of the quantum group under consideration. If we are dealing with a quasitriangular Hopf algebra, a natural choice for the pure braid element is

$$\Upsilon_r = \frac{1}{\lambda}\left(1 \otimes 1 - \mathcal{R}^{21}\mathcal{R}^{12}\right), \tag{7.25}$$

where the term $\mathcal{R}^{21}\mathcal{R}^{12}$ has been introduced and extensively studied by Reshetikhin & Semenov-Tian-Shansky [43] and later by Jurco [44], Majid [59] and Schupp, Watts & Zumino [60]. These choices of b_is and Υ lead to the r-matrix approach to differential geometry on quantum groups. The metric is

$$\eta = - < X_1, St_2 >= \frac{1}{\lambda}\left(\left[\left(r_{21}{}^{-1}\right)^{t_2}\left(r_{12}{}^{t_2}\right)^{-1}\right]^{t_2} - 1\right), \tag{7.26}$$

where $X_1 =< \Upsilon_r, t_1 \otimes id >$ and $r_{12} =< \mathcal{R}, t_1 \otimes t_2 >$. In the case of $\mathrm{GL}_q(2)$ we find‡

$$\eta_{\mathrm{GL}_q(2)} = -\begin{pmatrix} q^{-3} & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-3} & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \tag{7.27}$$

Now we will evaluate the metric in the case of $\mathrm{GL}_q(n)$. The $\hat{r}$-matrix of $\mathrm{GL}_q(n)$ satisfies a characteristic equation

$$\hat{r}^2 - \lambda\hat{r} - 1 = 0 \tag{7.28}$$

which we can use in the form

$$r_{21}^{-1} = r_{12} - \lambda P_{12}, \tag{7.29}$$

where $P^{ij}{}_{kl} = \delta^i_l \delta^j_k$ is the permutation matrix, to replace $(r_{21}^{-1})^{t_2}$ in equation (7.26). That gives

$$\begin{aligned} \eta_{12} &= -\left(P_{12}{}^{t_2}\left((r_{12}{}^{t_2})^{-1}\right)\right)^{t_2} \\ &= -\mathrm{tr}_3\left(P_{23}(r_{23}{}^{t_3})^{-1}\right)P_{12} \\ &= -D_2 P_{12}. \end{aligned} \tag{7.30}$$

‡In its reduced form, this matrix agrees [41] with a metric obtained along more standard lines from quantum traces (except perhaps in the casimir sector $X^1{}_1 + q^{-2}X^2{}_2$). The formulation in terms of the pure braid element has the great advantage that it does not require the existence of an element like u that implements the square of the antipode.

In the last step we have used

$$D \equiv < u, t > = \mathrm{tr}_2\left(P(r^{t_2})^{-1}\right), \tag{7.31}$$

where $u \equiv \cdot(S \otimes id)\mathcal{R}^{21}$ is the element of $\mathcal{U}$ that implements the square of the antipode. With the explicit formula ($\eta_{12} = -D_2 P_{12}$) for the metric we immediately find an expression [60] for the exterior derivative $\mathbf{d}$ on functions in terms of X and the Maurer-Cartan form $\Omega = t^{-1}\mathbf{d}t$:

$$\mathbf{d} = -\mathrm{tr}(D^{-1}\Omega X). \qquad \textit{(on functions)} \tag{7.32}$$

The pure braid approach to the construction of quantum Lie algebras is however particularly important in cases (like the 2-dim quantum euclidean group) where there is no quasitriangular Hopf algebra and where the b_is are not given by the elements of $t^i{}_j$.

The 2-dim quantum euclidean group is an example of a quantum Lie algebra that has no universal $\mathcal{R}$ and where the set of functions $\{b_i\}$ does not arise from the matrix elements of some quantum matrix. In section 4.1.4 we constructed such a set of functions

$$b_0 = (e^{i\theta} - 1)^2, \;\; b_1 = -me^{i\theta}\overline{m}, \;\; b_+ = -(e^{i\theta} - 1)m, \;\; b_- = q^{-2}(e^{i\theta} - 1)e^{i\theta}\overline{m}, \tag{7.33}$$

and a pure braid element

$$\Upsilon_e = \frac{1}{\lambda}\{P_+P_- \otimes (q^{2J} - 1) + P_+q^{-J} \otimes q^J P_- + P_- q^{-J} \otimes q^J P_+ + q^{-2J} \otimes P_+P_-\} \tag{7.34}$$

by hand. Now we can put the new machinery to work and calculate the (invertible) metric

$$\eta_{\mathrm{E}_q(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -q^{-2} & 0 \end{pmatrix}, \tag{7.35}$$

which immediately gives an expression for $\mathbf{d}$ on functions:

$$\mathbf{d} = \omega_0\chi_1 + \omega_1\chi_0 - q^2\omega_+\chi_- - \omega_-\chi_+. \tag{7.36}$$

7.1.2 Various Types of Quantum Lie Algebras

The functions $c^j := -Sb^j$ play the role of coordinate functions their span$\{c^j\} =: R^\perp$ is the vector space dual to the quantum tangent space $\mathcal{T}_q$, such that

$$\begin{aligned} 1 \oplus \mathcal{T}_q \oplus \mathcal{T}_q{}^\perp &= \mathcal{U} \\ 1 \oplus R^\perp \oplus R &= \mathcal{A} \end{aligned} \tag{7.37}$$

as vector spaces, with[§]

$$< \mathcal{T}_q, R >= 0, \qquad < \mathcal{T}_q{}^\perp, R^\perp >= 0. \tag{7.38}$$

Let $\widetilde{R^\perp}$ =span$\{b^i\}$ and $\tilde{R}$ be the spaces obtained from $R^\perp$ and R by application of S^{-1} on all of their elements. In the following we will state various desirable properties that different kinds of quantum Lie algebras might have; we will comment on their significance and we will derive the corresponding expressions in the dual space. The proofs are given in an appendix to this section.

$$i) \qquad \Delta_{\mathcal{A}}\mathcal{T}_q \subset \mathcal{T}_q \otimes \mathcal{A} \qquad \Leftrightarrow \qquad \Delta^{Ad}\tilde{R} \subset \tilde{R} \otimes \mathcal{A} \tag{7.39}$$

The left hand side states the right invariance of $\mathcal{T}_q$, which is important for the covariance of the cartan identity (7.4) and the invariance of the realization (7.10) of $\mathbf{d}$. The right hand side is essential to Woronowicz's formulation of the differential calculus because it allows to consistently set $\omega_{\tilde{R}} = 0$.

$$ii) \qquad \Delta\mathcal{T}_q \subset \mathcal{U} \otimes (\mathcal{T}_q \oplus 1) \qquad \Leftrightarrow \qquad \mathcal{A}R = R \tag{7.40}$$

The left hand side is necessary to ensure the existence of $f - \omega$ commutation relations that are consistent with an undeformed Leibniz rule for $\mathbf{d}$. It also implies a quadratic quantum commutator for the χ_i:

$$\chi_k \stackrel{ad}{\triangleright} \chi_l \equiv \pounds_{\chi_k}(\chi_l) = \chi_b\chi_c(\delta^c_k\delta^b_l - \hat{R}^{cb}{}_{kl}) = \chi_a < \chi_k, T^a{}_l >= \chi_a f_k{}^a{}_l, \tag{7.41}$$

where

$$\hat{R}^{cb}{}_{kl} =< O_k{}^b, T^c{}_l > \tag{7.42}$$

is the so-called "big R-matrix". If $ii)$ is not satisfied we have the choice of giving up the $f - \omega$ commutation relations, so that the algebra of forms Λ is only a left $\mathcal{A}$-module, or we can try a generalized Leibniz rule for $\mathbf{d}$. The right hand side of the equation is equivalent to $\tilde{R}\mathcal{A} = \tilde{R}$ and states that $\tilde{R}$ is a right $\mathcal{A}$-ideal; it is the second

[§]We write here vector spaces in place of their elements in an obvious notation.

fundamental ingredient of Woronowicz's theory. If the Leibniz rule is satisfied then *ii)* follows from $\omega_r = 0 \Rightarrow r \in \tilde{R} \oplus 1$: Let $a \in \mathcal{A}$, then

$$\omega_{ra} = S(a_{(1)})S(r_{(1)})\mathrm{d}(r_{(2)}a_{(2)}) = S(a_{(1)})\omega_r a_{(2)} + \epsilon(r)\omega_a = 0, \tag{7.43}$$

$\epsilon(ra) = \epsilon(r)\epsilon(a) = 0$ and hence $ra \in \tilde{R}$. $\mathcal{A}R = R$ is in agreement with the intuitive picture that the ideal R is spanned by polynomials in the c^i of order 2 or higher, *i.e.* $\mathrm{span}\{e_i\} \approx \{1, c^i, c^i c^j, \ldots\}$.

$$iii) \qquad \Delta^{Ad}\widetilde{R^\perp} \subset \widetilde{R^\perp} \otimes \mathcal{A} \qquad \Leftrightarrow \qquad \Delta_{\mathcal{A}}\mathcal{T}_q{}^\perp \subset \mathcal{T}_q{}^\perp \otimes \mathcal{A} \tag{7.44}$$

The right hand side keeps us out of trouble with covariance when we set $i_{\mathcal{T}_q{}^\perp} = 0$. The left hand side is a *sufficient* condition for $\Delta_{\mathcal{A}}(\mathcal{T}_q{}^*) \subset \mathcal{T}_q{}^* \otimes \mathcal{A}$. Quantum Lie algebras that satisfy *iii)* have particular nice properties in connection with pure braid elements and a (Killing) metric. That merits a special name for them:

Quantum Lie Algebra of **type I** : *i)*,*ii)*,*iii)*
Quantum Lie Algebra of **type II**: *i)*,*ii)*

We will mainly be dealing with type I, in fact, all examples of quantum group calculi known to me are of this type. Quantum Lie algebras of type II are mathematically equivalent to Woronowicz's [21] theory.

$$iv) \qquad \Delta R^\perp \subset \mathcal{A} \otimes (R^\perp \oplus 1) \qquad \Leftrightarrow \qquad \mathcal{U}\mathcal{T}_q{}^\perp = \mathcal{T}_q{}^\perp \tag{7.45}$$

The LHS enables us to define **partial derivatives** instead of left-invariant ones: It implies $\Delta c^i = M^i{}_j \otimes c^j + c^i \otimes 1$ with $\Delta M = M\dot{\otimes}M$, $SM = M^{-1}$, $\epsilon(M) = I$ and then $\chi_k c^i = M^i{}_k + M^i{}_j < O_k{}^l, c^j > \chi_l + c^i\chi_k$, such that $\partial_n := S^{-1}M^k{}_n\chi_k$ gives a commutation relation

$$\partial_n c^i = \delta^i_n + \left(S^{-1}M^k{}_n M^i{}_j < O_k{}^l, c^j > M^m{}_l + S^{-1}M^k{}_n c^i M^m{}_k\right)\partial_m \tag{7.46}$$

worthy of a partial derivative. (In the case of $\mathrm{GL}_q(n)$ we can use (7.30) to show that $c^{(mn)} = (D^{-1})^n{}_k St^k{}_m$, $M^{(mn)}{}_{(ij)} = St^i{}_m\delta^n_j$, and $\partial_{(ij)} = t^i{}_k X^k{}_j$.) The exterior derivative (on functions) becomes

$$\mathrm{d} = \omega^i\chi_i = \mathrm{d}(c^j)S^{-1}(M^i{}_j)M^n{}_i\partial_n = \mathrm{d}(c^n)\partial_n. \tag{7.47}$$

$$v) \qquad \Delta R^\perp \subset (R^\perp \oplus 1) \otimes \mathcal{A} \qquad \Leftrightarrow \qquad \mathcal{T}_q{}^\perp\mathcal{U} = \mathcal{T}_q{}^\perp \tag{7.48}$$

This and *ii)* imply quadratic $\chi - c$ commutation relations that close in terms of the elements of $\mathcal{T}_q$ and $R^\perp$. The right hand sides of *iv)* and *v)* state that $\mathcal{T}_q$ is a left (right) $\mathcal{U}$-ideal, which supports the picture of a Poincare-Birkhoff-deWitt type basis for $\mathcal{U}$ in terms of the χ_i, *i.e.* $\{1, \chi_i, \chi_i\chi_j, \ldots\}$. Here and in the discussion following *ii)* we have to be careful though with higher order conditions on the generators.

7.1.3 Universal Calculus

Given (infinite) linear bases $\{e_i\}$ of $\mathcal{U}$ and $\{f^i\}$ of $\mathcal{A}$ we can always construct new counit-free elements $\vec{e_i} := e_i - 1\epsilon(e_i)$ and $\vec{f^i} := f^i - 1\epsilon(f^i)$ that span (infinite) spaces ${\mathcal{T}_q}^u$ and $R^{\perp u}$ respectively satisfying properties i) through v); in fact $1 \oplus {\mathcal{T}_q}^u = \mathcal{U}$ and $1 \oplus R^{\perp u} = \mathcal{A}$ as vector spaces. The $f - \omega$ commutation relations, however, become trivial in that they are equivalent to the Leibniz rule for δ[¶]; we are hence dealing with a Connes type calculus [7], a "Universal Calculus on Hopf Algebras". It is interesting to see what happens to the formula for the partial derivatives in this limit:

A Subbialgebra and the Vacuum Projection Operator

To simplify notation we will assume that the infinite bases of $\mathcal{U}$ and $\mathcal{A}$ have been arranged in such a way that $e_0 = 1_{\mathcal{U}}$, $f^0 = 1^{\mathcal{A}}$ and e_i, f^i with $\epsilon(e_i) = \epsilon(f^i) = 0$ for $i = 1, \ldots, \infty$ span $\mathcal{T}_q$ and $R^{\perp}$ respectively. Greek indices $\alpha, \beta, \ldots$ will run from 0 to ∞ whereas Roman indices $i, j, k, \ldots$ will only take on values from 1 to ∞ unless otherwise stated. A short calculation gives

$$\Delta f^i = {M^i}_k \otimes f^k + f^i \otimes 1, \qquad {M^i}_k = f^i_{(1)} < e_k, f^i_{(2)} > \tag{7.49}$$

and

$$\Delta M = M \dot{\otimes} M, \qquad S(M) = M^{-1}, \qquad \epsilon(M) = I. \tag{7.50}$$

Using the definition from the previous section we will now write down partial derivatives

$$\partial_n = S^{-1}({M^l}_n) e_l, \qquad (l \geq 1!) \tag{7.51}$$

which take on a peculiar form when using the explicit expression for M

$$\begin{aligned} \partial_n &= S^{-1}(f^l_{(1)}) < e_n, f^l_{(2)} > e_l \\ &= S^{-1}(f^\alpha_{(1)}) < e_n, f^\alpha_{(2)} > e_\alpha \\ &= S^{-1}(f^\alpha) < e_n, f^\beta > e_\alpha e_\beta \\ &= S^{-1}(f^\alpha) e_\alpha e_n \\ &= E e_n, \end{aligned} \tag{7.52}$$

where we have introduced the "vacuum projector" E in the last step. It was first discovered (quite accidently) in collaboration with C. Chryssomalakos [46] and has

¶To distinguish this calculus from quantum Lie algebras we use the symbol δ inst[illegible] for the exterior derivative interesting properties like

$$\begin{aligned} Ea &= E\epsilon(a), \qquad a \in \mathcal{A}, &(7.53)\\ xE &= E\epsilon(x), \qquad x \in \mathcal{U}, &(7.54)\\ E^2 &= E. &(7.55)\end{aligned}$$

Prof. B. Zumino [7] pointed out that the classical expression of E is related to a Taylor expansion. Note also that

$$E = \partial_0 - 1. \tag{7.56}$$

As expected we can express $\boldsymbol{\delta}$ on functions in terms of partial derivatives

$$\boldsymbol{\delta}(f) = \boldsymbol{\delta}(f^i)\partial_i(f). \tag{7.57}$$

The partial derivatives are of course no longer left invariant, but it turns out that we can actually define a coproduct for them making the space $E\mathcal{U} = \{Ey; y \in \mathcal{U}\} \subset \mathcal{A} \rtimes \mathcal{U}$ a unital bialgebra. Inspired by

$$Eyf = < y_{(1)}, f > Ey_{(2)} = (Ey_{(1)})(f)Ey_{(2)} \tag{7.58}$$

we define

$$\Delta_E(Ey) = Ey_{(1)} \otimes Ey_{(2)}, \qquad \epsilon_E(Ey) = \epsilon(y), \qquad 1_E = E, \tag{7.59}$$

in consistency with the axioms for a bialgebra. $E\mathcal{U}$ is however not a Hopf algebra because it does not have an antipode — at least not with respect to the multiplication in $\mathcal{A} \rtimes \mathcal{U}$ — so $E\mathcal{U}$ might be of use as an example of a quantum plane.

Quantum Lie Algebras in a Universal Calculus

If the span $\mathcal{T}_q{}^u$ of the generators $\{e_a | a = 1, \ldots, \infty\}$ of the universal calculus contains a finite dimensional subspace, $\mathcal{T}_q$ spanned by $\{\chi_i | i = 1, \ldots, N\}$, that satisfies axioms *i*) and *ii*) then one may ask how to obtain the finite calculus from the infinite one. Let $\boldsymbol{\delta}$ be the exterior derivative of the universal calculus and $\mathbf{d}$ the exterior derivative of the finite calculus. One might be tempted to try an ansatz like

$$\boldsymbol{\delta} = \mathbf{d} + \mathbf{d}^{\perp}, \tag{7.60}$$

where $\boldsymbol{\delta} = \omega^a e_a$ and $\mathbf{d} = \omega^i \chi_i$ on functions. This equation is covariant if axiom *iii*) is also satisfied, but we run into problems with the $f - \omega$ commutation relations. From the Leibniz rule for $\boldsymbol{\delta}$ we obtain

$$f\omega^i = \omega^j O_j{}^i(f) + \omega^r \Theta_r{}^i(f), \qquad i = 1, \ldots, N; \;\; r = N+1, \ldots, \infty, \tag{7.61}$$

i.e. the $f-\omega$ commutation relations do not close within the finite calculus. So unless one decides to do without a bicovariant calculus we have to make the second term vanish. The naive choice is to try and set Θ equal to zero. This could be nicely expressed in terms of another axiom

$$\Delta \mathcal{T}_q^{\perp} \subset \mathcal{U} \otimes (\mathcal{T}_q^{\perp} \oplus 1) \qquad \Leftrightarrow \qquad \mathcal{A}R^{\perp} = R^{\perp},$$

but the right hand side neither has a classical limit nor does it lend itself to a description of $\mathcal{A}$ in terms of a Poincare-Birkhoff-deWitt basis. The only choice left is to set the forms ω^r corresponding to functions in R (recall: $< \mathcal{T}_q, R >= 0$) equal to zero. Following Woronowicz's approach we hence set

$$\omega_R = 0 \qquad \Rightarrow \qquad \delta \rightarrow \mathbf{d}. \tag{7.62}$$

Deformed Leibniz Rule?

Here we want to briefly mention what might happen if axiom *ii*) is not satisfied. We will still have $\omega_R = 0$ in consistency with axiom *i*) but the generators χ_i now have coproducts

$$\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j + \Theta_i{}^r \otimes e_r, \qquad i,j = 1,\ldots,N; \quad r = N+1,\ldots,\infty \tag{7.63}$$

that do not close in $\mathcal{U} \otimes (\mathcal{T}_q \oplus 1)$. After some thought we can convince ourselfs that we should use $f-\omega$ commutation relation

$$f\omega^i = \omega^j B_j{}^i(f), \tag{7.64}$$

with a braiding matrix $B_j{}^i \in \mathcal{U}$ that satisfies $\Delta(B) = B\dot{\otimes}B$, $S(B) = B^{-1}$, $\epsilon(B) = I$ and a bicovariance condition for all $f \in \mathcal{A}$

$$T^l{}_j \widehat{B_l{}^k}(f) = B_j{}^i(f) T^k{}_i, \tag{7.65}$$

where T is the adjoint representation. We will then need to change the Leibniz rule for $\mathbf{d}$ to

$$\mathbf{d}f = \mathbf{d}(f) + (SB_k{}^iO_i{}^j) \triangleright f\omega^k\chi_j + (SB_k{}^i\Theta_i{}^r) \triangleright f\omega^k e_r. \tag{7.66}$$

This is a fully bicovariant first order differential calculus with a deformed Leibniz rule. It might be of use in reducing the number of forms in quantum calculi to the classical number.

Appendix

Here we will give fairly detailed proofs of propositions *i*) and *ii*) and symbolic proofs of the related propositions *iii*) through *v*).

Proof of i**).** We start by proofing a lemma about the relation of coactions in $\mathcal{U}$ and $\mathcal{A}$:

$$\begin{aligned} S^{-1}(x^{(2)'}) < x^{(1)}, Sa > &= S^{-1}(x^{(2)'})S(a_{(2)}) < x^{(1)}, Sa_{(1)} > a_{(3)} \\ &= S^{-1}(x^{(2)'})x^{(1)}(Sa_{(1)})a_{(2)} \\ &= \widehat{x}(Sa_{(1)})a_{(2)} \\ &= < x, Sa_{(2)} > Sa_{(1)}a_{(3)}. \qquad \Box \end{aligned} \tag{7.67}$$

Another useful identity:

$$< x^{(1)}, f > x^{(2)'} =< x, f_{(2)} > f_{(1)}S(f_{(3)}), \qquad \forall x \in \mathcal{U}, \quad f \in \mathcal{A}. \tag{7.68}$$

i) "$\Rightarrow$": Assume $\Delta_{\mathcal{A}}\mathcal{T}_q \subset \mathcal{T}_q \otimes \mathcal{A}$, then for $\forall x \in \mathcal{T}_q$, $S(a) \in R$

$$0 =< x^{(1)}, Sa > S^{-1}x^{(2)'} =< x, Sa_{(2)} > S(a_{(1)})a_{(3)}, \tag{7.69}$$

so that $Sa_{(2)} \otimes S(a_{(1)})a_{(3)} \subset (R \oplus 1) \otimes \mathcal{A}$, but $\epsilon(Sa_{(2)})S(a_{(1)})a_{(3)} = \epsilon(Sa) = 0$ and hence $Sa_{(2)} \otimes S(a_{(1)})a_{(3)} \subset R \otimes \mathcal{A}$, or

$$\Delta^{Ad}(a) \equiv a_{(2)} \otimes S(a_{(1)})a_{(3)} \subset \tilde{R} \otimes \mathcal{A}. \qquad \Box \tag{7.70}$$

i) "$\Leftarrow$": Assume $\Delta^{Ad}\tilde{R} \subset \tilde{R} \otimes \mathcal{A}$, then again for $\forall x \in \mathcal{T}_q$, $a \in \tilde{R}$

$$0 =< x, Sa_{(2)} > S(a_{(1)})a_{(3)} =< x^{(1)}, Sa > S^{-1}x^{(2)'}, \tag{7.71}$$

so that $x^{(1)} \otimes S^{-1}x^{(2)'} \subset (\mathcal{T}_q \oplus 1) \otimes \mathcal{A}$; with $0 =< x, 1 >=< x^{(1)}, 1 > x^{(2)'}$ from (7.68) that gives $x^{(1)} \otimes S^{-1}x^{(2)'} \subset \mathcal{T}_q \otimes \mathcal{A}$ and also

$$\Delta_{\mathcal{A}}x = x^{(1)} \otimes x^{(2)'} \subset \mathcal{T}_q \otimes \mathcal{A}. \qquad \Box \tag{7.72}$$

Proof of ii**).**

ii) "$\Rightarrow$": For all $x \in \mathcal{T}_q$, $a \in \mathcal{A}$ and $r \in R$ assume $\Delta x \in \mathcal{U} \otimes (\mathcal{T}_q \oplus 1)$, then

$$< x, ar >=< \Delta x, a \otimes r >= 0, \tag{7.73}$$

which implies $ar \in (R \oplus 1)$ or, taking into account that $\epsilon(ar) = \epsilon(a)\epsilon(r) = 0$,

$$ar \in R. \qquad \Box \tag{7.74}$$

ii) "$\Leftarrow$": Assume that for all $x \in \mathcal{T}_q$, $r \in R$ there exists a $r' \in R$ such that $r' = ar$; then we find

$$0 =< x, r' >=< x, ar >=< \Delta x, a \otimes r > \tag{7.75}$$

which can be restated as

$$\Delta x \in \mathcal{U} \otimes (\mathcal{T}_q \oplus 1). \qquad \square \tag{7.76}$$

Symbolic proof of *iii*).

$$0 =< \mathcal{T}_q{}^{\perp} \otimes id, (S \otimes id) \circ \Delta^{Ad}\widetilde{R^{\perp}} >=< (id \otimes S^{-1}) \circ \Delta_{\mathcal{A}} \mathcal{T}_q{}^{\perp}, S\widetilde{R^{\perp}} \otimes id > \tag{7.77}$$

Symbolic proof of *iv*).

$$0 =< R^{\perp}, \mathcal{T}_q{}^{\perp} >=< R^{\perp}, \mathcal{U}\mathcal{T}_q{}^{\perp} >=< \Delta R^{\perp}, \mathcal{U} \otimes \mathcal{T}_q{}^{\perp} >=< \mathcal{A} \otimes (R^{\perp} \oplus 1), \mathcal{U} \otimes \mathcal{T}_q{}^{\perp} > \tag{7.78}$$

Symbolic proof of *v*).

$$0 =< R^{\perp}, \mathcal{T}_q{}^{\perp} >=< R^{\perp}, \mathcal{T}_q{}^{\perp}\mathcal{U} >=< \Delta R^{\perp}, \mathcal{T}_q{}^{\perp} \otimes \mathcal{U} >=< (R^{\perp} \oplus 1) \otimes \mathcal{A}, \mathcal{T}_q{}^{\perp} \otimes \mathcal{U} > \tag{7.79}$$

7.2 Calculus of Functions, Vector Fields and Forms

The purpose of this section is to generalize the Cartan calculus of ordinary *commutative* differential geometry to the case of quantum Lie algebras. As in the classical case, the Lie derivative of a function is given by the action of the corresponding vector field, *i.e.*

$$\begin{aligned} \pounds_x(a) &= x \triangleright a = a_{(1)} < x, a_{(2)} >, \\ \pounds_x a &= a_{(1)} < x_{(1)}, a_{(2)} > \pounds_{x_{(2)}}. \end{aligned} \tag{7.80}$$

The action on products is given through the coproduct of x

$$x \triangleright ab = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b). \tag{7.81}$$

The Lie derivative along x of an element $y \in \mathcal{U}$ is given by the adjoint action in $\mathcal{U}$:

$$\pounds_x(y) = x \overset{\text{ad}}{\triangleright} y = x_{(1)} y S(x_{(2)}). \tag{7.82}$$

To find the action of i_{χ_i} we can now attempt to use the Cartan identity (7.4)[‖]

$$\begin{aligned} \chi_i \triangleright a &= \mathcal{L}_{\chi_i}(a) \\ &= i_{\chi_i}(\mathrm{d}a) + \mathrm{d}(i_{\chi_i} a). \end{aligned} \tag{7.83}$$

As the inner derivation i_{χ_i} contracts 1-forms and is zero on 0-forms like a, we find

$$i_{\chi_i}(\mathrm{d}a) = \chi_i \triangleright a = a_{(1)} < \chi_i, a_{(2)} > . \tag{7.84}$$

An equation like this could not be true for any $x \in \mathcal{U}$ because from the Leibniz rule for d we have $\mathrm{d}(1) = \mathrm{d}(1 \cdot 1) = \mathrm{d}(1)1 + 1\mathrm{d}(1) = 2\mathrm{d}(1)$ and any i_x that gives a non-zero result upon contracting $\mathrm{d}(1)$ will hence lead to a contradiction. From (7.84) we see that the troublemakers would be $x \in \mathcal{U}$ with $\epsilon(x) \neq 0$, but as $\epsilon(\chi_i) = 0$ we have nothing to worry about. Without loss of generality we can now set

$$\mathrm{d}(1) \equiv 0 \quad \text{and} \quad i_1 \equiv 0. \tag{7.85}$$

Next consider for any form α

$$\begin{aligned} \mathcal{L}_{\chi_i}(\mathrm{d}\alpha) &= \mathrm{d}(i_{\chi_i}\mathrm{d}\alpha) + i_{\chi_i}(\mathrm{d}\mathrm{d}\alpha) \\ &= \mathrm{d}(\mathcal{L}_{\chi_i}\alpha) + 0, \end{aligned} \tag{7.86}$$

which shows that Lie derivatives commute with the exterior derivative; $\mathcal{L}_{\chi_i}\mathrm{d} = \mathrm{d}\mathcal{L}_{\chi_i}$. We will later need to extend this equation to all elements of $\mathcal{U}$:

$$\mathcal{L}_x \mathrm{d} = \mathrm{d}\mathcal{L}_x. \tag{7.87}$$

From this and (7.80) we find

$$\mathcal{L}_x \mathrm{d}(a) = \mathrm{d}(a_{(1)}) < x_{(1)}, a_{(2)} > \mathcal{L}_{x_{(2)}}. \tag{7.88}$$

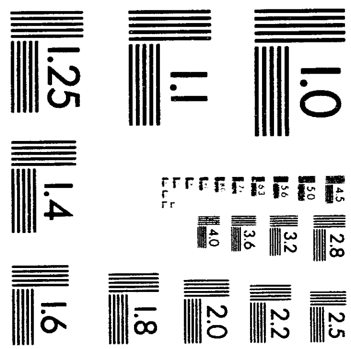
To find the complete commutation relations of i_{χ_i} with functions and forms rather than just its action on them, we next compute the action of $\mathcal{L}_{\chi_i}$ on a product of functions $a, b \in \mathcal{A}$

$$\begin{aligned} \mathcal{L}_{\chi_i}(ab) &= i_{\chi_i}\mathrm{d}(ab) \\ &= i_{\chi_i}(\mathrm{d}(a)b + a\mathrm{d}(b)) \end{aligned} \tag{7.89}$$

and compare with equation (7.81). Recalling that the χ_i have coproducts of the form

$$\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j, \qquad O_i{}^j \in \mathcal{U}, \tag{7.90}$$

[‖]The idea is to use this identity as long as it is consistent and modify it if needed.


1.0
1.1
1.25
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1.6
1.8
2.0
2.2
2.5
2.8
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3.6
4.0
4.5
5.0
5.6
6.3

we obtain

$$\begin{aligned} \boldsymbol{i}_{\chi_i} a &= (O_i{}^j \triangleright a)\, \boldsymbol{i}_{\chi_j} \\ &= \pounds_{O_i{}^j}(a)\, \boldsymbol{i}_{\chi_j}, \end{aligned} \tag{7.91}$$

if we assume that the commutation relation of $\boldsymbol{i}_{\chi_i}$ with $\mathbf{d}(a)$ is of the general form

$$\boldsymbol{i}_{\chi_i}\mathbf{d}(a) = \underbrace{\boldsymbol{i}_{\chi_i}(\mathbf{d}a)}_{\in\mathcal{A}} + \text{“braiding term”} \cdot \boldsymbol{i}_{\chi_?}\,. \tag{7.92}$$

A calculation of $\pounds_{\chi_i}(\mathbf{d}(a)\mathbf{d}(b))$ along similar lines gives in fact

$$\begin{aligned} \boldsymbol{i}_{\chi_i}\mathbf{d}(a) &= (\chi_i \triangleright a) - \mathbf{d}(O_i{}^j \triangleright a)\, \boldsymbol{i}_{\chi_j} \\ &= \boldsymbol{i}_{\chi_i}(\mathbf{d}a) - \pounds_{O_i{}^j}(\mathbf{d}a)\, \boldsymbol{i}_{\chi_j} \end{aligned} \tag{7.93}$$

and we propose for any p-form α:

$$\boldsymbol{i}_{\chi_i}\alpha = \boldsymbol{i}_{\chi_i}(\alpha) + (-1)^p \pounds_{O_i{}^j}(\alpha)\, \boldsymbol{i}_{\chi_j}. \tag{7.94}$$

Missing in our list are commutation relations of Lie derivatives with vector fields and inner derivations. It was shown earlier in chapter 2 that the product in $\mathcal{U}$ can be expressed in terms of a right coaction $\Delta_{\mathcal{A}} : \mathcal{U} \to \mathcal{U} \otimes \mathcal{A}$, denoted $\Delta_{\mathcal{A}}(y) = y^{(1)} \otimes y^{(2)\prime}$, such that $xy = y^{(1)} < x_{(1)}, y^{(2)\prime} > x_{(2)}$. In the context of (7.82), this gives

$$\pounds_x(y) = x_{(1)} y S(x_{(2)}) = y^{(1)} < x, y^{(2)\prime} >, \tag{7.95}$$

$$\pounds_x \pounds_y = \pounds_{\pounds_{x_{(1)}}(y)} \pounds_{x_{(2)}} = \pounds_{y^{(1)}} < x_{(1)}, y^{(2)\prime} > \pounds_{x_{(2)}}. \tag{7.96}$$

For the special case $\chi_i, \chi_j \in \mathcal{T}_q$ that becomes

$$\begin{aligned} \pounds_{\chi_i} \pounds_{\chi_k} &= \pounds_{\chi_i}(\pounds_{\chi_k}) + \pounds_{O_i{}^j}(\pounds_{\chi_k}) \pounds_{\chi_j} \\ &= \pounds_{\chi_l} f_i{}^l{}_k + \pounds_{\chi_a} \pounds_{\chi_b} \hat{R}^{ab}{}_{ik} \end{aligned} \tag{7.97}$$

and — using the Cartan identity —

$$\begin{aligned} \pounds_{\chi_i} \boldsymbol{i}_{\chi_k} &= \pounds_{\chi_i}(\boldsymbol{i}_{\chi_k}) + \pounds_{O_i{}^j}(\boldsymbol{i}_{\chi_k}) \boldsymbol{i}_{\chi_j} \\ &= \boldsymbol{i}_{\chi_l} f_i{}^l{}_k + \boldsymbol{i}_{\chi_a} \pounds_{\chi_b} \hat{R}^{ab}{}_{ik}, \end{aligned} \tag{7.98}$$

where

$$\hat{R}^{ab}{}_{ik} = < O_i{}^b, T^a{}_k > . \tag{7.99}$$

7.2.1 Maurer-Cartan Forms

The most general left-invariant 1-form can be written [21]

$$\omega_b := S(b_{(1)})\mathbf{d}(b_{(2)}) = -\mathbf{d}(S b_{(1)}) b_{(2)} \tag{7.100}$$

$$(\textit{left-invariance:}\ \ {}_{\mathcal{A}}\Delta(\omega_b) = S(b_{(2)})b_{(3)} \otimes S(b_{(1)})\mathbf{d}(b_{(4)}) = 1 \otimes \omega_b), \tag{7.101}$$

corresponding to a function $b \in \mathcal{A}$. If this function happens to be $t^i{}_k$, where $t \in M_m(\mathcal{A})$ is an $m \times m$ matrix representation of $\mathcal{U}$ with $\Delta(t^i{}_k) = t^i{}_j \otimes t^j{}_k$ and $S(t) = t^{-1}$, we obtain the well-known Cartan-Maurer form $\omega_t = t^{-1}\mathbf{d}(t) =: \Omega$. Here is a nice formula for the exterior derivative of ω_b:

$$\begin{aligned} \mathbf{d}(\omega_b) &= \mathbf{d}(Sb_{(1)})\mathbf{d}(b_{(2)}) \\ &= \mathbf{d}(Sb_{(1)})b_{(2)}S(b_{(3)})\mathbf{d}(b_{(4)}) \\ &= -\omega_{b_{(1)}}\omega_{b_{(2)}}. \end{aligned} \tag{7.102}$$

The Lie derivative is

$$\begin{aligned} \pounds_\chi(\omega_b) &= \pounds_{\chi_{(1)}}(Sb_{(1)})\pounds_{\chi_{(2)}}(\mathbf{d}b_{(2)}) \\ &= <\chi_{(1)}, S(b_{(1)})> S(b_{(2)})\mathbf{d}(b_{(3)}) <\chi_{(2)}, b_{(4)}> \\ &= \omega_{b_{(2)}} <\chi, S(b_{(1)})b_{(3)}> \\ &= <\chi_{(1)}, S(b_{(1)})> \omega_{b_{(2)}} <\chi_{(2)}, b_{(3)}>. \end{aligned} \tag{7.103}$$

For $\chi = \chi_i$ and $b = t^k{}_n$ this becomes a quantum commutator:

$$\begin{aligned} \pounds_{\chi_i}(t) &= <\chi_i, St> \cdot\Omega + <O_i{}^j, St> \cdot\Omega\cdot <S^{-1}\chi_j, St> \\ &= <\chi_i, St> \cdot\Omega - <O_i{}^j, St> \cdot\Omega\cdot <S^{-1}O_j{}^k, St> \cdot <\chi_k, St> \\ &= <\chi_i, St> \cdot\Omega - \pounds_{O_i{}^k}(\Omega)\cdot <\chi_k, St> \end{aligned} \tag{7.104}$$

and, if we denote a St-matrix representation for the moment by "$\tilde{\ }$",

$$\pounds_\chi(t) = \tilde{\chi}\cdot t - \tilde{O}\cdot t\cdot \tilde{O}^{-1}\cdot\tilde{\chi} =: [\tilde{\chi}, t]_q. \tag{7.105}$$

The contraction of left-invariant forms with $\boldsymbol{i}_\chi$ — *i.e.* by a *left-invariant* $x \in \mathcal{U}$ — gives a number in the field k rather than a function in $\mathcal{A}$ as was the case for $\mathbf{d}(a)$. (The result must be a number because the only invariant function is 1.)

$$\begin{aligned} \boldsymbol{i}_\chi(\omega_b) &= \boldsymbol{i}_\chi(-\mathbf{d}(Sb_{(1)})b_{(2)}) \\ &= -\boldsymbol{i}_\chi(\mathbf{d}Sb_{(1)})b_{(2)} \\ &= -<\chi, S(b_{(1)})> S(b_{(2)})b_{(3)} \\ &= -<\chi, S(b)>. \end{aligned} \tag{7.106}$$

As an exercise and to check consistency we will compute the same expression in a different way:

$$\begin{aligned} \boldsymbol{i}_{\chi_i}(\omega_b) &= \boldsymbol{i}_{\chi_i}(Sb_{(1)}\mathbf{d}(b_{(2)})) \\ &= <O_i{}^j, S(b_{(1)})> S(b_{(2)})\boldsymbol{i}_{\chi_j}(\mathbf{d}b_{(2)}) \\ &= <O_i{}^j, S(b_{(1)})> S(b_{(2)})b_{(3)} <\chi_j, b_{(4)}> \\ &= <O_i{}^j, S(b_{(1)})><\chi_j, b_{(2)}> \\ &= -<\chi_i, S(b)>. \end{aligned} \tag{7.107}$$

The Exterior Derivative on Functions

We would like to express the exterior derivative of a function f in terms of the basis of 1-forms $\{\omega^i\}$ with functional coefficients. There are two natural ansätze: $\mathbf{d}(f) = \omega^j a_j$ and $\mathbf{d}(f) = b_j\omega^j$ with appropriate $a_j, b_j \in \mathcal{A}$. Applying the Cartan identity (7.4) to f we find

$$\pounds_{\chi_i}(f) = a_i = \pounds_{O_i{}^j}(b_j),$$

giving two alternate expressions for $\mathbf{d}(f)$:

$$\mathbf{d}(f) = \omega^j \pounds_{\chi_j}(f) = -\pounds_{S\chi_j}(f)\omega^j. \tag{7.108}$$

The Woronowicz and Castellani groups use the second expression, while we prefer the first one because it allows us to write $\mathbf{d}$ as an operator $\omega^j\chi_j$ on $\mathcal{A}$. An operator expression just like this, but written in terms of partial derivatives, is at least classically valid on all forms. (For quantum planes that also holds [7]). Combining the two expressions for $\mathbf{d}$ one easily derives the well-known $f - \omega$ commutation relations

$$f\omega^i = \omega^j \pounds_{O_j{}^i}(f). \tag{7.109}$$

The classical limit is given by $O_j{}^i \to 1\delta^i_j$, so that forms commute with functions.

On the Invariance of $\mathbf{d} = \omega_{b^j}\chi_j$. Recall: $\Delta_{\mathcal{A}}(\omega^i) = \omega_{b^i_{(2)}} \otimes S(b^i_{(1)})b^i_{(3)} = -\omega^j\otimes < S\chi_j, b^i_{(2)} > Sb^i_{(1)}b^i_{(3)}$. Assuming $\Delta_{\mathcal{A}}\chi_i = \chi_j \otimes T^j{}_i$ (axiom i)) we would like to show

$$\Delta_{\mathcal{A}}(\omega_{b^i}\chi_i) = \omega_{b^i_{(2)}}\chi_i^{(1)} \otimes S(b^i_{(1)})b^i_{(3)}\chi_i^{(2)'} = \omega^i\chi_i \otimes 1, \tag{7.110}$$

i.e.

$$\Delta_{\mathcal{A}}(\omega^i) = \omega^j \otimes S^{-1}(T^i{}_j), \tag{7.111}$$

or equivalently

$$- < S\chi_k, b^i_{(2)} > S(b^i_{(1)})b^i_{(3)} = -S^{-1}(< \chi_k^{(1)}, Sb^i > \chi_k^{(2)'}). \tag{7.112}$$

This turns out to be a purely Hopf algebraic identity for *any* $x \in \mathcal{U}$, $a \in \mathcal{A}$ (see equation 7.67):

$$S^{-1}(x^{(2)'}) < x^{(1)}, Sa >=< x, Sa_{(2)} > Sa_{(1)}a_{(3)}. \tag{7.113}$$

7.2.2 Tensor Product Realization of the Wedge

From (7.103) and (7.106) we find commutation relations for i_{χ_i} with ω^j,

$$\begin{aligned} i_{\chi_i}\omega^j &= \delta_i^j - \pounds_{O_i{}^k}(\omega^j) i_{\chi_k} \\ &= \delta_i^j - \omega^m < O_i{}^k, S^{-1}(T^j{}_m) > i_{\chi_k}, \end{aligned} \tag{7.114}$$

which can be used to define the wedge product $\wedge$ of forms as some kind of antisymmetrized tensor product**: as in the classical case we make an ansatz for the product of two forms in terms of tensor products

$$\omega^i \wedge \omega^j = \omega^i \otimes \omega^j - \hat{\sigma}^{ij}{}_{mn}\omega^m \otimes \omega^n, \tag{7.115}$$

with as yet unknown numerical constants $\hat{\sigma}^{ij}{}_{mn} \in k$, and define i_{χ_i} to act on this product by contracting in the first tensor product space, *i.e.*

$$i_{\chi_i}(\omega^j \wedge \omega^k) = \delta_i^j \omega^k - \hat{\sigma}^{jk}{}_{mn}\delta_i^m \omega^n. \tag{7.116}$$

But from (7.114) we already know how to compute this, namely

$$\begin{aligned} i_{\chi_i}(\omega^j \wedge \omega^k) &= \delta_i^j \omega^k - \pounds_{O_i{}^\ell}(\omega^j)\delta_\ell^k \\ &= \delta_i^j \omega^k - \omega^m < O_i{}^k, S^{-1}(T^j{}_m) >, \end{aligned} \tag{7.117}$$

and by comparison we find

$$\hat{\sigma}^{ij}{}_{mn} = < O_m{}^j, S^{-1}(T^i{}_n) >, \tag{7.118}$$

or

$$\begin{aligned} \omega^i \wedge \omega^j &= \omega^i \otimes \omega^j - < O_m{}^j, S^{-1}(T^i{}_n) > \omega^m \otimes \omega^n \\ &= (I - \hat{\sigma})^{ij}{}_{mn}\omega^m \otimes \omega^n \\ &= \omega^i \otimes \omega^j - \omega^k \otimes \pounds_{O_k{}^j}(\omega^i). \end{aligned} \tag{7.119}$$

These equations can be used to obtain the (anti)commutation relations between the ω^is; by using the characteristic equation for $\hat{\sigma}$, projection matrices orthogonal to the antisymmetrizer $I - \hat{\sigma}$ can be found, and these will annihilate $\omega^i \wedge \omega^j$. The resulting equations will determine how to commute the 1-forms. In some rare cases the $\omega - \omega$ commutation relations are of higher than second order. We are then forced to consider orthogonal projectors to the operator W, introduced below. There is another reason why we want to emphasize the tensor product realization of the wedge product rather than commutation relations given in terms of projection operators: In the case of quantum groups in the A, B, C and D series $\hat{\sigma}$ typically has one

**So far we have suppressed the $\wedge$-symbol; to avoid confusion we will reinsert it in this paragraph.

eigenvalue equal to 1, so there is exactly one projection operator P_0 [41] orthogonal to $(1-\hat{\sigma})$, but while $(1-\hat{\sigma})$ has a sensible classical limit — it becomes $(1-P)$ where P is the permutation matrix — P_0, on the other hand might change discontinuously as q reaches 1 if $(1-\hat{\sigma})$ had other eigenvalues λ_i that become equal to 1 in that limit because the corresponding projection operators P_i will now *all* be orthogonal to $(1-P) = (1-\hat{\sigma})\big|_{q=1}$. The approach of the group in München trying to circumvent this problem in the case of $SO_q(3)$ was to impose additional conditions on the wedge product "by hand", requiring that all projection operators P_i (see above) vanish on it. In the present context we would have to simultaneously impose similar conditions on products of inner derivations *and* check consistency of the resulting equations on a case by case basis.

Example: Maurer-Cartan-Equation

$$
\begin{aligned}
\mathbf{d}\omega^j &= \mathbf{d}\omega_{b^j} = -\omega_{b^j_{(1)}} \wedge \omega_{b^j_{(2)}} \\
&= -\omega_{S^{-1}(Sb^j_{(1)}b^j_{(3)})} \otimes \omega_{b^j_{(2)}} \\
&= -\omega^k \otimes \omega^l < -S\chi_k, S^{-1}(Sb^j_{(1)}b^j_{(3)}) >< -S\chi_l, b^j_{(2)} > \\
&= -\omega^k \otimes \omega^l < \underbrace{(S^{-1}\chi_k)_{(1)}\chi_l S(S^{-1}\chi_k)_{(2)}}_{S^{-1}\chi_k \overset{\mathbf{ad}}{\triangleright} \chi_l}, Sb^j > \\
&= -f'^{\,j}_{k\ \ l}\,\omega^k \otimes \omega^l.
\end{aligned}
\tag{7.120}
$$

In the previous equation we have introduced the adjoint action of a left-invariant vector field on another vector field. A short calculation gives

$$
S^{-1}\chi_k \overset{\mathbf{ad}}{\triangleright} \chi_l = \chi_b\chi_c(\delta^c_k\delta^b_l - \hat{\sigma}^{cb}{}_{kl}) = \chi_a < S^{-1}\chi_k, T^a{}_l >= \chi_a f'_k{}^a{}_l \tag{7.121}
$$

as compared to

$$
\chi_k \overset{\mathbf{ad}}{\triangleright} \chi_l \equiv \pounds_{\chi_k}(\chi_l) = \chi_b\chi_c(\delta^c_k\delta^b_l - \hat{R}^{cb}{}_{kl}) = \chi_a < \chi_k, T^a{}_l >= \chi_a f_k{}^a{}_l, \tag{7.122}
$$

with $\hat{R}^{cb}{}_{kl} =< O_k{}^b, T^c{}_l >$. The two sets of structure constants are related by

$$
f_k{}^a{}_l = -f'^{\,a}_{i\ \ l} R^{ij}{}_{kl}. \tag{7.123}
$$

Please see [61] for a detailed discussion of such structure constants.

Using the same method as for ω we can also obtain a tensor product decomposition of products of inner derivations

$$
\boldsymbol{i}_{\chi_m} \wedge \boldsymbol{i}_{\chi_n} = \boldsymbol{i}_{\chi_m} \otimes \boldsymbol{i}_{\chi_n} - \hat{\sigma}^{ij}{}_{mn}\boldsymbol{i}_{\chi_i} \otimes \boldsymbol{i}_{\chi_j}, \tag{7.124}
$$

defined to act on 1-forms by contraction in the first tensor product space. This can again be used to find (anti)commutation relations for the $\boldsymbol{i}$s via projection matrices as mentioned above.

Remark: The tensor product decomposition of the wedge product is invariant under linear changes of the $\{\chi_i\}$ basis, but it does depend on our choice of quantum tangent bundle. With the extreme choice of $\mathcal{U}$ =span$\{e_i\}$ (viewed as a vector space) for instance we get a Connes type "Universal Cartan Calculus".

The "Anti-Wedge" Operator. There is actually an operator W that recursively translates wedge products into the tensor product representation:

$$\begin{array}{l} W : \Lambda_q^p \to \mathcal{T}_q^* \otimes \Lambda_q^{p-1}, \quad p \geq 1, \\ W(\alpha) = \omega^n \otimes \boldsymbol{i}_{\chi_n}(\alpha), \end{array} \tag{7.125}$$

for any p-form α. Two examples:

$$\begin{aligned} \omega^j \wedge \omega^k &= \omega^n \otimes \boldsymbol{i}_{\chi_n}(\omega^j \wedge \omega^k) \\ &= \omega^n \otimes (\delta_n^j \omega^k - \pounds_{O_n{}^m}(\omega^j)\delta_m^k) \\ &= \omega^j \otimes \omega^k - \omega^n \otimes \pounds_{O_n{}^k}(\omega^j) \\ &= \omega^j \otimes \omega^k - \omega^n \otimes \omega^m \hat{\sigma}^{jk}{}_{nm} \end{aligned} \tag{7.126}$$

and, after a little longer computation that uses W twice,

$$\begin{aligned} \omega^a \wedge \omega^b \wedge \omega^c &= \omega^a \otimes (\omega^b \wedge \omega^c) - \omega^i \otimes (\omega^j \wedge \omega^c)\hat{\sigma}^{ab}{}_{ij} \\ &\quad + \omega^i \otimes (\omega^j \wedge \omega^k)\hat{\sigma}^{al}{}_{ij}\hat{\sigma}^{bc}{}_{lk} \\ &= \omega^a \otimes \omega^b \otimes \omega^c - \omega^a \otimes \omega^j \otimes \omega^k \hat{\sigma}^{bc}{}_{jk} \\ &\quad - \omega^i \otimes \omega^j \otimes \omega^c \hat{\sigma}^{ab}{}_{ij} + \omega^i \otimes \omega^j \otimes \omega^k \hat{\sigma}^{lc}{}_{jk}\hat{\sigma}^{ab}{}_{il} \\ &\quad + \omega^i \otimes \omega^j \otimes \omega^k \hat{\sigma}^{al}{}_{ij}\hat{\sigma}^{bc}{}_{lk} - \omega^i \otimes \omega^j \otimes \omega^k \hat{\sigma}^{an}{}_{il}\hat{\sigma}^{bc}{}_{nm}\hat{\sigma}^{lm}{}_{jk}. \end{aligned} \tag{7.127}$$

In some cases this expression can be further simplified with the help of the characteristic equation of $\hat{\sigma}$.

7.2.3 Summary of Relations in the Cartan Calculus

Commutation Relations For any p-form α:

$$\mathrm{d}\alpha = \mathrm{d}(\alpha) + (-1)^p \alpha \mathrm{d} \tag{7.128}$$

$$\boldsymbol{i}_{\chi_i}\alpha = \boldsymbol{i}_{\chi_i}(\alpha) + (-1)^p \pounds_{O_i{}^j}(\alpha)\boldsymbol{i}_{\chi_j} \tag{7.129}$$

$$\pounds_{\chi_i}\alpha = \pounds_{\chi_i}(\alpha) + \pounds_{O_i{}^j}(\alpha)\pounds_{\chi_j} \tag{7.130}$$

Actions For any function $f \in \mathcal{A}$, 1-form $\omega_f \equiv Sf_{(1)}\mathrm{d}f_{(2)}$ and vector field $\phi \in \mathcal{A}\rtimes\mathcal{U}$:

$$\begin{aligned}
i_{\chi_i}(f) &= 0 && (7.131)\\
i_{\chi_i}(\mathrm{d}f) &= \mathrm{d}f_{(1)} < \chi_i, f_{(2)} > && (7.132)\\
i_{\chi_i}(\omega_f) &= - < \chi_i, Sf > && (7.133)\\
\mathcal{L}_{\chi}(f) &= \chi(f) = f_{(1)} < \chi, f_{(2)} > && (7.134)\\
\mathcal{L}_{\chi}(\omega_f) &= \omega_{f_{(2)}} < \chi, S(f_{(1)})f_{(3)} > && (7.135)\\
\mathcal{L}_{\chi}(\phi) &= \chi_{(1)}\phi S(\chi_{(2)}) && (7.136)
\end{aligned}$$

Graded Quantum Lie Algebra of the Cartan Generators

$$\begin{aligned}
\mathrm{dd} &= 0 && (7.137)\\
\mathrm{d}\mathcal{L}_{\chi} &= \mathcal{L}_{\chi}\mathrm{d} && (7.138)\\
\mathcal{L}_{\chi_i} &= \mathrm{d}i_{\chi_i} + i_{\chi_i}\mathrm{d} && (7.139)\\
\left[\mathcal{L}_{\chi_i}, \mathcal{L}_{\chi_k}\right]_q &= \mathcal{L}_{\chi_l} f_i{}^l{}_k && (7.140)\\
\left[\mathcal{L}_{\chi_i}, i_{\chi_k}\right]_q &= i_{\chi_l} f_i{}^l{}_k && (7.141)
\end{aligned}$$

The quantum commutator $[\,,\,]_q$ is here defined as follows

$$\left[\mathcal{L}_{\chi_i}, \Box\right]_q := \mathcal{L}_{\chi_i}\Box - \mathcal{L}_{O_i{}^j}(\Box)\mathcal{L}_{\chi_j}. \qquad (7.142)$$

This quantum Lie algebra becomes infinite dimensional as soon as we introduce derivatives along general vector fields (see below).

7.2.4 Braided Cartan Calculus

There are several graphical representations of the relations that we derived in the previous sections. One that emphasizes the nature of differential operators is illustrated here at the example of equation (7.130):

$$\begin{aligned}
\mathcal{L}_{\chi_i}\alpha\beta &= \mathcal{L}_{\chi_i}\ \alpha\ \ \beta \ + \ \mathcal{L}_{\chi_i}\ \alpha\ \ \beta\\
&= \mathcal{L}_{\chi_i}(\alpha)\,\beta \ + \ \mathcal{L}_{O_i{}^j}(\alpha)\mathcal{L}_{\chi_j}\ \beta\\
&= \mathcal{L}_{\chi_i}(\alpha)\,\beta \ + \ \mathcal{L}_{O_i{}^j}(\alpha)\mathcal{L}_{\chi_j}(\beta)
\end{aligned}$$

There is another graphical representation that is special in as it shows that we are in fact dealing with a graded and braided Lie algebra in the sense of [62]. Recall that in the braided setting the coproducts and antipodes of the generators $\{\chi_i\}$ take on the classical linear form

$$\Delta\chi_i = \chi_i \otimes 1 + 1 \otimes \chi_i, \qquad S\chi_i = -\chi_i \qquad \textit{(braided)}, \tag{7.143}$$

while the multiplication of tensor products acquires braiding

$$(a \otimes b) \cdot (c \otimes d) = a\Psi(b \otimes c)d \quad \in W \otimes V,$$

described by a "braided-transposition"[62] operator $\Psi_{V,W} : V \otimes W \to W \otimes V$. This notation suggests that the braiding is of a symmetric nature with respect to the two spaces V and W. In the present case it turns out to be more fruitful to assign all braiding to the generators χ_i — or linear combinations of them — as they move through various objects. The general braiding rule can be stated symbolically as

$$\Psi : \quad \chi_i \otimes \Box \mapsto \mathcal{L}_{O_i^j}(\Box) \otimes \chi_j, \tag{7.144}$$

where χ_i could be part of an object like $\mathcal{L}$ or $\boldsymbol{i}$. If χ_i is part of $\boldsymbol{i}$, *i.e.* of degree -1, there will be an additional $(-1)^p$ grading, depending on the degree p of $\Box$. Here is a summary of all braid relations involving Cartan generators: For $\Box \in \{\mathcal{L}_{\chi_k}, \boldsymbol{i}_{\chi_k}, \mathbf{d}, \text{vector fields}, \text{forms}, \text{functions}\}$

$$\Psi : \mathcal{L}_{\chi_i} \otimes \Box \mapsto \mathcal{L}_{O_i^j}(\Box) \otimes \mathcal{L}_{\chi_j}, \tag{7.145}$$

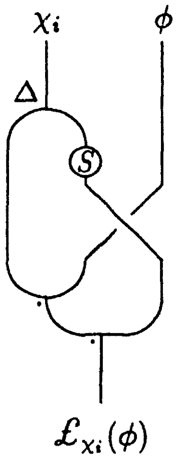
for $\Box \in \{\mathbf{d}, \text{vector fields}, \text{forms}, \text{functions}\}$

$$\Psi : \boldsymbol{i}_{\chi_i} \otimes \Box \mapsto (-1)^p \mathcal{L}_{O_i^j}(\Box) \otimes \boldsymbol{i}_{\chi_j}, \tag{7.146}$$

and finally

$$\Psi : \mathbf{d} \otimes \mathbf{d} \mapsto -\mathbf{d} \otimes \mathbf{d}. \tag{7.147}$$

Let us now look at the graphical representation of the adjoint action (7.136) $(\chi_i, \phi) \mapsto \mathcal{L}_{\chi_i}(\phi) = \chi_{i(1)}\phi S(\chi_{i(2)})$:

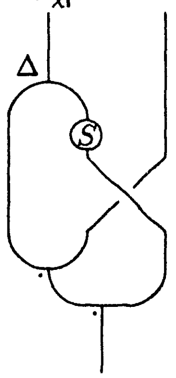


$$\begin{aligned}
&\chi_i \otimes \phi \\
&(\chi_i \otimes 1 + 1 \otimes \chi_i) \otimes \phi \\
&(\chi_i \otimes 1 + 1 \otimes -\chi_i) \otimes \phi \\
&\chi_i \otimes \phi \otimes 1 - 1 \otimes \mathcal{L}_{O_i^j}(\phi) \otimes \chi_j \\
&\chi_i\phi \otimes 1 - \mathcal{L}_{O_i^j}(\phi) \otimes \chi_j \\
&\chi_i\phi - \mathcal{L}_{O_i^j}(\phi)\chi_j
\end{aligned}$$

(In the right column we have translated the various graphical manipulations into their algebraic counterparts.) Taking this diagram as the definition of a braided (and graded) commutator we can now express all Cartan relations in graphical form:

Lie derivatives. Note that $\mathcal{L}_{O_i{}^j}(\mathbf{d}) = \delta_i^j \mathbf{d}$ because $\mathbf{d}$ is invariant.

$\mathcal{L}_{X_i}$ $\quad \square$

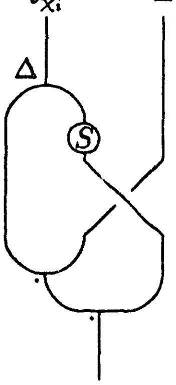


$$
\begin{array}{ll}
\mathcal{L}_{X_i} \otimes i_{X_k} & \mathcal{L}_{X_i} \otimes \mathbf{d} \\
(\mathcal{L}_{X_i} \otimes 1 + 1 \otimes \mathcal{L}_{X_i}) \otimes i_{X_k} & (\mathcal{L}_{X_i} \otimes 1 + 1 \otimes \mathcal{L}_{X_i}) \otimes \mathbf{d} \\
(\mathcal{L}_{X_i} \otimes 1 + 1 \otimes \mathcal{L}_{-X_i}) \otimes i_{X_k} & (\mathcal{L}_{X_i} \otimes 1 + 1 \otimes \mathcal{L}_{-X_i}) \otimes \mathbf{d} \\
\mathcal{L}_{X_i} \otimes i_{X_k} \otimes 1 - 1 \otimes i_{\mathcal{L}_{O_i{}^j}(X_k)} \otimes \mathcal{L}_{X_j} & \mathcal{L}_{X_i} \otimes \mathbf{d} \otimes 1 - 1 \otimes \delta_i^j \mathbf{d} \otimes \mathcal{L}_{X_j} \\
\mathcal{L}_{X_i} i_{X_k} \otimes 1 - \text{“}\mathcal{L}_{O_i{}^j}(i_{X_k})\text{”} \otimes \mathcal{L}_{X_j} & \mathcal{L}_{X_i}\mathbf{d} \otimes 1 - \mathbf{d} \otimes \mathcal{L}_{X_i} \\
\mathcal{L}_{X_i} i_{X_k} - \text{“}\mathcal{L}_{O_i{}^j}(i_{X_k})\text{”} \mathcal{L}_{X_j} & \mathcal{L}_{X_i}\mathbf{d} - \mathbf{d}\mathcal{L}_{X_i} \\
[\mathcal{L}_{X_i}, \square]_q \qquad = i_{X_l} f_i{}^l{}_k & = 0
\end{array}
$$

The relation $\left[\mathcal{L}_{X_i}, \mathcal{L}_{X_k}\right]_q = \mathcal{L}_{X_l} f_i{}^l{}_k$ has a very similar picture, so we did not show it here.

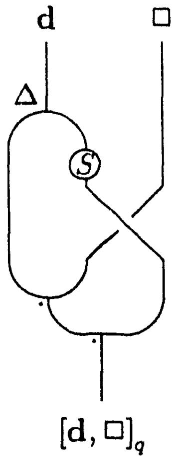
Inner derivations. α is a p-form here.

i_{X_i} $\quad \square$



$$
\begin{array}{ll}
i_{X_i} \otimes \alpha & i_{X_i} \otimes \mathbf{d} \\
(i_{X_i} \otimes 1 + 1 \otimes i_{X_i}) \otimes \alpha & (i_{X_i} \otimes 1 + 1 \otimes i_{X_i}) \otimes \mathbf{d} \\
(i_{X_i} \otimes 1 + 1 \otimes i_{-X_i}) \otimes \alpha & (i_{X_i} \otimes 1 + 1 \otimes i_{-X_i}) \otimes \mathbf{d} \\
i_{X_i} \otimes \alpha \otimes 1 - 1 \otimes (-1)^p \mathcal{L}_{O_i{}^j}(\alpha) \otimes i_{X_j} & i_{X_i} \otimes \mathbf{d} \otimes 1 + 1 \otimes \delta_i^j \mathbf{d} \otimes i_{X_j} \\
i_{X_i}\alpha \otimes 1 - (-1)^p \mathcal{L}_{O_i{}^j}(\alpha) \otimes i_{X_j} & i_{X_i}\mathbf{d} \otimes 1 + \mathbf{d} \otimes i_{X_i} \\
i_{X_i}\alpha - (-1)^p \mathcal{L}_{O_i{}^j}(\alpha) i_{X_j} & i_{X_i}\mathbf{d} + \mathbf{d} i_{X_i} \\
[i_{X_i}, \square]_q \qquad = i_{X_i}(\alpha) & = \mathcal{L}_{X_i}
\end{array}
$$

Exterior derivative. Here we use that $\mathbf{d}$ is a derivation in the sense “$\Delta(\mathbf{d})$” $= \mathbf{d} \otimes 1 + 1 \otimes \mathbf{d}$.



$\mathrm{d} \otimes \alpha$	$\mathrm{d} \otimes \mathrm{d}$
$(\mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}) \otimes \alpha$	$(\mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}) \otimes \mathrm{d}$
$(\mathrm{d} \otimes 1 + 1 \otimes -\mathrm{d}) \otimes \alpha$	$(\mathrm{d} \otimes 1 + 1 \otimes -\mathrm{d}) \otimes \mathrm{d}$
$\mathrm{d} \otimes \alpha \otimes 1 - 1 \otimes (-1)^p \alpha \otimes \mathrm{d}$	$\mathrm{d} \otimes \mathrm{d} \otimes 1 + 1 \otimes \mathrm{d} \otimes \mathrm{d}$
$\mathrm{d}\alpha \otimes 1 - (-1)^p \alpha \otimes \mathrm{d}$	$\mathrm{dd} \otimes 1 + \mathrm{d} \otimes \mathrm{d}$
$\mathrm{d}\alpha - (-1)^p \alpha \mathrm{d}$	$2\mathrm{dd}$
$= \mathrm{d}(\alpha)$	$= 0$

7.2.5 Lie Derivatives Along General Vector Fields

So far we have focused on Lie derivatives and inner derivations along *left-invariant* vector fields, *i.e.* along elements of $\mathcal{T}_q$. The classical theory allows functional coefficients, *i.e.* the vector fields need not be left-invariant. Here we may introduce derivatives along elements in the $\mathcal{A}{\rtimes}\mathcal{T}_q$ plane by the following set of equations valid on forms: (note: $\epsilon(\chi) = 0$ for $\chi \in \mathcal{T}_q$)

$$\begin{aligned} \boldsymbol{i}_{f\chi} &= f \boldsymbol{i}_\chi, & (7.148)\\ \boldsymbol{\pounds}_{f\chi} &= \mathbf{d}\boldsymbol{i}_{f\chi} + \boldsymbol{i}_{f\chi}\mathbf{d}, & (7.149)\\ \boldsymbol{\pounds}_{f\chi} &= f\boldsymbol{\pounds}_\chi + \mathrm{d}(f)\boldsymbol{i}_\chi, & (7.150)\\ \boldsymbol{\pounds}_{f\chi}\mathbf{d} &= \mathbf{d}\boldsymbol{\pounds}_{f\chi}. & (7.151) \end{aligned}$$

Equation (7.150) can be used to define Lie derivatives recursively on any form. There does not seem to be a way to generalize (7.162), *i.e.* to introduce Lie derivatives of *vector fields* along *arbitrary* elements of $\mathcal{A}{\rtimes}\mathcal{U}$ or $\mathcal{A}{\rtimes}\mathcal{T}_q$ in the quantum case. Exceptions are the right-invariant vector fields $\hat{x} \in \mathcal{A}{\rtimes}\mathcal{U}$, where

$$\boldsymbol{\pounds}_{\hat{x}}(\phi) = \widehat{x_{(1)}}\phi\widehat{S^{-1}x_{(2)}}, \qquad \text{for } \phi \in \mathcal{A}{\rtimes}\mathcal{U}. \qquad (7.152)$$

7.3 Universal Cartan Calculus

The equations presented in this section were obtained in collaboration with P. Watts starting directly from Hopf algebras without explicitly referring to any bases.

As we have already mentioned in the section on quantum Lie algebras, given (infinite) linear bases $\{e_i\}$ and $\{f^i\}$ of the Hopf algebras $\mathcal{U}$ and of $\mathcal{A}$, we can always construct new counit-free elements $\vec{e_i} := e_i - 1\epsilon(e_i)$ and $\vec{f^i} := f^i - 1\epsilon(f^i)$ that span (infinite) spaces $\mathcal{T}_q{}^u$ and $R^{\perp u}$ respectively satisfying properties i) through v); in fact $1 \oplus \mathcal{T}_q{}^u = \mathcal{U}$ and $1 \oplus R^{\perp u} = \mathcal{A}$ as vector spaces. Using some Schmidt orthogonalization procedure one can rearrange the infinite bases of $\mathcal{U}$ and $\mathcal{A}$ in such a way that $e_0 = 1_{\mathcal{U}}$, $f^0 = 1^{\mathcal{A}}$ and e_i, f^i with $\epsilon(e_i) = \epsilon(f^i) = 0$ for $i = 1, \ldots, \infty$ span $\mathcal{T}_q{}^u$ and $R^{\perp u}$ respectively. Greek indices $\alpha, \beta, \ldots$ will run from 0 to ∞, whereas roman indices $i, j, k, \ldots$ will only take on values from 1 to ∞, unless otherwise stated. To avoid confusion with the finite dimensional quantum Lie algebras we will use the symbol $\boldsymbol{\delta}$ instead of $\mathbf{d}$ for the exterior derivative.

Given orthonormal linear basis $\{e_i\}$ and $\{f^i\}$ of $\mathcal{T}_q{}^u$ and $\mathcal{R}^{\perp u}$ we can now express $\boldsymbol{\delta}$ on functions $a \in \mathcal{A}$ as

$$\boldsymbol{\delta}(a) = -\omega_{S^{-1}f^i} \mathcal{L}_{e_i - 1\epsilon(e_i)}(a); \tag{7.153}$$

note, however, that *all* of these $\omega_{S^{-1}f^i}$s are treated as linearly independent and even in the classical limit stay linearly independent because (7.153) in conjunction with the Leibniz rule for $\boldsymbol{\delta}$ only gives trivial commutation relations ($a\omega_b = \omega_{bS^{-1}a_{(2)}} a_{(1)} - \epsilon(b)\omega_{S^{-1}a_{(2)}} a_{(1)}$) for forms with functions that do not permit reorganization of the infinite set of $\omega_{S^{-1}f^i}$s into a finite basis of 1-forms. This is the case for Connes' non-commutative geometry ([10] and references therein) and is in contrast to the ordinary text book treatment of differential calculi that has forms commuting with functions.

Here is a summary of basis-free commutation relations for the Universal Cartan Calculus valid on any form. All of these equations are identical to the corresponding quantum Lie algebra relations when written in terms of the bases $\{e_\alpha\}$ and $\{f^\alpha\}$. $x, y \in \mathcal{U}$, $a \in \mathcal{A}$, α is a p-form and $v \in \mathcal{A} \rtimes \mathcal{U}$ is a vector field.

$$\begin{aligned}
\mathcal{L}_x a &= a_{(1)} < x_{(1)}, a_{(2)} > \mathcal{L}_{x_{(2)}} && (7.154)\\
\mathcal{L}_x \boldsymbol{\delta}(a) &= \boldsymbol{\delta}(a_{(1)}) < x_{(1)}, a_{(2)} > \mathcal{L}_{x_{(2)}} && (7.155)\\
\mathcal{L}_x \alpha &= \mathcal{L}_{x_{(1)}}(\alpha)\, \mathcal{L}_{x_{(2)}} && (7.156)\\
\boldsymbol{i}_x a &= a_{(1)} < x_{(1)}, a_{(2)} > \boldsymbol{i}_{x_{(2)}} && (7.157)\\
\boldsymbol{i}_x \boldsymbol{\delta}(a) &= a_{(1)} < x - 1\epsilon(x), a_{(2)} > - \boldsymbol{\delta}(a_{(1)}) < x_{(1)}, a_{(2)} > \boldsymbol{i}_{x_{(2)}} && (7.158)\\
\boldsymbol{i}_x \alpha &= \boldsymbol{i}_x(\alpha) + (-1)^p \mathcal{L}_{x_{(1)}}(\alpha)\, \boldsymbol{i}_{x_{(2)}} && (7.159)\\
\boldsymbol{\delta}\alpha &= \boldsymbol{\delta}(\alpha) + (-1)^p \alpha \boldsymbol{\delta} && (7.160)\\
\boldsymbol{\delta}\boldsymbol{\delta}(\alpha) &= -(-1)^p \boldsymbol{\delta}(\alpha)\boldsymbol{\delta} && (7.161)
\end{aligned}$$

$$\mathcal{L}_x(v) = x_{(1)} v S(x_{(2)}) \tag{7.162}$$

$$
\begin{aligned}
\boldsymbol{\delta}^2 &= 0 && (7.163)\\
\boldsymbol{\delta}\boldsymbol{\mathcal{L}}_x &= \boldsymbol{\mathcal{L}}_x\boldsymbol{\delta} && (7.164)\\
\boldsymbol{\mathcal{L}}_x &= \boldsymbol{\delta}\boldsymbol{i}_x + 1\epsilon(x) + \boldsymbol{i}_x\boldsymbol{\delta} \qquad \textit{(generalized Cartan identity)} && (7.165)\\
\boldsymbol{\mathcal{L}}_x\boldsymbol{\mathcal{L}}_y &= \boldsymbol{\mathcal{L}}_{y^{(1)}} < x_{(1)}, y^{(2)\prime} > \boldsymbol{\mathcal{L}}_{x_{(2)}} && (7.166)\\
\boldsymbol{\mathcal{L}}_x\boldsymbol{i}_y &= \boldsymbol{i}_{y^{(1)}} < x_{(1)}, y^{(2)\prime} > \boldsymbol{\mathcal{L}}_{x_{(2)}} && (7.167)
\end{aligned}
$$

The "generalized Cartan identity" is due to P. Watts.

Chapter 8

Quantum Planes Revisited

With the new tools that we have developed in the previous sections we are now ready to take a second look at quantum planes. The first two sections that follow will be devoted to the realization and action of quantum Lie algebra generators on a quantum plane. After introducing the basic equations we will spend some time on the important question of their covariance. The third section finally gives an introduction to the construction of a Cartan calculus on quantum planes with the surprising result — first observed by Prof. B. Zumino [7] in the example of the 2-dimensional quantum plane — that the $\pounds_\partial - x$ commutation must contain inner derivation terms in order to be consistent with a Lie derivative that commutes with $\mathbf{d}$. For simplicity we will however suppress these inner derivation terms in the following two sections.

8.1 Induced Calculus

In this section we wish to show how the calculus of the symmetry quantum group induces a calculus on the plane. Originally, I was interested in this topic trying to develop as general applicable a formalism for a calculus on quantum planes as we have presented it in part I in the case of quantum groups. As we have already mentioned, quantum planes do not have a Hopf algebra structure — at least not in the unbraided theory — and so we have to look for a different approach than the one that we used to construct the cross product algebra. Later it turned out that a better approach is based on $\mathcal{U}$-coactions leading to the introduction of the generalized product rule in the first section of this chapter. The material presented here is however of interest in its own right: We will study realizations of quantum group generators in terms of the calculus on a quantum plane. This will also give an explanation for the appearance of "inner derivation terms" in the generalized product rule.

The central idea of this section, inspired by a comment of Prof. B. Zumino, is to give the coordinate functions on the quantum plane functional coefficients in $\mathcal{A}$, i.e. to make them variable with respect to the action of vector fields in $\mathcal{U}$. Let $x_0^i \in \mathrm{Fun}(\mathbf{M}_q)$ be the "fixed" coordinate functions and define new variable ones via $x^i := (t^{-1})^i{}_j x_0^j$. Instead of the differentials $\mathbf{d}x_0^i$ we will use $\boldsymbol{\delta} x^i = -(\Omega x)^i$ because

$$\boldsymbol{\delta} x = \boldsymbol{\delta} t^{-1} \cdot x_0 = t^{-1} \cdot t \cdot \boldsymbol{\delta}(t^{-1}) \cdot x_0 = -t^{-1} \cdot \boldsymbol{\delta}(t) \cdot t^{-1} \cdot x_0 = -\Omega \cdot x, \tag{8.1}$$

where $\boldsymbol{\delta}$ is the exterior derivative on the quantum group and $\Omega = t^{-1}\boldsymbol{\delta} t$ is the Maurer-Cartan Matrix. By "pullback" the group derivative will become the derivative on the plane, inducing a differential calculus there. It then immediately follows that $\Delta_{\mathcal{A}}(\mathbf{d}x^i) = \mathbf{d}x^j \otimes (t^{-1})^j{}_i$, which will ultimately give us the desired commutation between Lie derivatives and $\mathbf{d}$.

Turn now to the quantum group. Reserving Latin indices $i, j, \ldots$ for the plane coordinates, let us use Greek indices for the adjoint representation of the quantum group. Let $\{v_\alpha\}$* be a basis of bicovariant generators with coproduct $\Delta v_\alpha = v^\alpha \otimes 1 + O_\alpha{}^\beta \otimes v_\beta$ spanning $\mathcal{T}_q \subset \mathcal{U}$ and let $\{\omega_\alpha\}$ be the dual basis of 1-forms; $\boldsymbol{i}_{v_\alpha}(\omega^\beta) = \delta_\alpha^\beta$, $\Omega^i{}_j = \omega^\alpha \boldsymbol{i}_{v_\alpha}(\Omega^i{}_j) = -\omega^\alpha < v_\alpha, (t^{-1})^i{}_j >$. Via the Cartan identity $\boldsymbol{\mathcal{L}}_v = \boldsymbol{i}_v \boldsymbol{\delta} + \boldsymbol{\delta} \boldsymbol{i}_v$ one computes actions of $\mathcal{T}_q$ on $\mathrm{Fun}(\mathbf{M}_q)$:

$$v_\alpha \triangleright x^i = \boldsymbol{i}_{v_\alpha}(\boldsymbol{\delta} x^i) =< v_\alpha, (t^{-1})^i{}_j > x^j. \tag{8.2}$$

Now we can make an ansatz for a realization of the group generators in terms of functions and derivatives on the plane†

$$v_\alpha \doteq J_\alpha^i \partial_i, \tag{8.3}$$

where $J_\alpha^i \in \mathrm{Fun}(\mathbf{M}_q)$ is easily computed, using $\partial_i(x^j) = \delta_i^j$ to be

$$J_\alpha^i = v_\alpha(x^i) =< v_\alpha, (t^{-1})^i{}_j > x^j. \tag{8.4}$$

In some lucky cases there is an inverse expression for the partial derivatives on the plane in terms of the group generators. With $\tilde{J}_i^\alpha \in \mathrm{Fun}(\mathbf{M}_q)$

$$\partial_i = \tilde{J}_i^\alpha \otimes v_\alpha \triangleright, \tag{8.5}$$

an expression that is classically only valid locally and may exclude some points unless we are dealing with an inhomogeneous group, but will give explicit $\partial - x$ commutation relations if it exists:

$$\partial_i x^j = \tilde{J}_i^\alpha v_\alpha x^j = \partial_i(x^j) + \underbrace{\tilde{J}_i^\alpha O_\alpha{}^\beta(x^j) J_\beta{}^k}_{L_i{}^k(x^j)} \partial_k. \tag{8.6}$$

*We write v instead of χ here to avoid confusion with coordinate functions $x \in \mathrm{Fun}(\mathbf{M}_q)$.

†$\doteq$ means: "equal when evaluated on $\mathrm{Fun}(\mathbf{M}_q)$"

Example: $\mathrm{GL}_{\frac{1}{q}}(2)$, Manin-Wess-Zumino Quantum Plane

The coordinate functions x, y of the Manin plane satisfy commutation relations $xy = qyx$ that are covariant under coactions of the quantum matrix group $\mathrm{GL}_{\frac{1}{q}}(2)$. This quantum group has four bicovariant generators v_1, v_2, v_+, v_-; we will focus on the last two for the moment, giving their fundamental t^{-1} representations

$$< v_+, t^{-1} >= \begin{pmatrix} 0 & q^3 \\ 0 & 0 \end{pmatrix}, \qquad < v_-, t^{-1} >= \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} \tag{8.7}$$

and the first tensor product representations

$$< \Delta v_+, t_1^{-1} \otimes t_2^{-1} >= \begin{pmatrix} 0 & q^4 & q^3 & 0 \\ 0 & 0 & 0 & q^5 \\ 0 & 0 & 0 & q^4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad < \Delta v_-, t_1^{-1} \otimes t_2^{-1} >= \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 0 \\ q & 0 & 0 & 0 \\ 0 & q^3 & q^2 & 0 \end{pmatrix}. \tag{8.8}$$

All these were obtained from

$$r_{\frac{1}{q}} = \begin{pmatrix} \frac{1}{q} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{q} \end{pmatrix}. \tag{8.9}$$

We immediately find

$$\partial_x = \tilde{J}_x^\alpha v_\alpha = q^{-3} y^{-1} v_+, \qquad \partial_y = q^{-1} x^{-1} v_-, \tag{8.10}$$

which we only have to check on pairs of functions because of the form of (8.6):

$$\partial_x \begin{pmatrix} xx \\ xy \\ yx \\ yy \end{pmatrix} = \begin{pmatrix} (1+q^2)x \\ q^2 y \\ qy \\ 0 \end{pmatrix}, \qquad \partial_y \begin{pmatrix} xx \\ xy \\ yx \\ yy \end{pmatrix} = \begin{pmatrix} 0 \\ qx \\ x \\ y + q^2 y \end{pmatrix}. \tag{8.11}$$

From this we read off the following $\partial - x$ commutation relations in perfect agreement with the results given in [22]

$$\partial_x x = 1 + q^2 x \partial_x + (q^2 - 1) y \partial_y, \tag{8.12}$$
$$\partial_x y = qy\partial_x, \tag{8.13}$$
$$\partial_y x = qx\partial_y, \tag{8.14}$$
$$\partial_y y = 1 + q^2 y \partial_y. \tag{8.15}$$

Using the other two generators v_1, v_2 gives identical results. This method works for all linear quantum planes [7] and can be formulated abstractly in terms of r-matrices. If one does not want to extend the algebra by introducing inverses y^{-1}, x^{-1} of the coordinate functions, it is also possible to obtain the above commutation relations as a vanishing ideal of xy thereby also avoiding the questionable use of $\tilde{J}$.

8.2 Covariance

Let us collect some of the equations valid on a quantum plane. Let $f, g \in \mathrm{Fun}(\mathbf{M}_q)$ be functions and ∂_i be derivatives on the quantum plane, let v_a be generators of the quantum Lie algebra — corresponding to the symmetry quantum group of the plane — with coproduct $\Delta v_a = v_a \otimes 1 + O_a{}^b \otimes v_b$, and let $L_i{}^j$ be a linear automorphism of $\mathrm{Fun}(\mathbf{M}_q)$:

$$\begin{aligned} v_a f &= v_a(f) + O_a{}^b(f) v_b, & (8.16)\\ v_a &\doteq J_a^i \partial_i,^{\ddagger} & (8.17)\\ \partial_i f &= \partial_i(f) + L_i{}^j(f)\partial_j. & (8.18) \end{aligned}$$

From this equations we can form a new one

$$J_a^i L_i^k(f) = O_a^b(f) J_b^k, \tag{8.19}$$

that can sometimes be rewritten as

$$L_i{}^k(f) = \tilde{J}_i^a O_a{}^b(f) J_b^k. \tag{8.20}$$

<u>Examples</u>

Quantum group as plane: $\mathrm{Fun}(\mathbf{M}_q) := \mathcal{A}$.

Left-Invariant Generators: $\partial_i := v_i \Rightarrow J_i^j = \delta_i^j,\quad L_i{}^j = O_i{}^j$.

Plane-Like Generators: $\partial_{(ij)} := t^i{}_k X^k{}_j \Rightarrow J_{(kl)}^{(ij)} = (t^{-1})^k{}_i \delta_l^j,\quad L_{(lj)}{}^{(nm)}(f) = t^l{}_i O_{(ij)}{}^{(km)}(f)(t^{-1})^k{}_n$.

‡Careful: An expression linear in the partials may not always exist, in particular for $e_q(2)$ we get a power series instead. It *does* exist for Wess-Zumino type quantum planes and then we have $J_a^i = < v_a, (t^{-1})^i{}_j > x^j$.

Linear quantum plane: The algebra of functions on the linear quantum plane is invariant under coactions of $\mathrm{GL}_q(N)$; $\Delta_{\mathcal{A}}(x^i) = x^j \otimes St^i{}_j$, $J^i_a =< v_a, St^i{}_j > x^j$. Using (8.19) we find

$$x^l(< v_a, St^i{}_l > L_i{}^k(x^j) - < O_a{}^b, St^j{}_l >< v_b, St^k{}_n > x^n) = 0,$$

so that $L_i{}^k(x^j)$ should be homogenous of first order in x, which suggests

$$L_i{}^k(x^j) =< L_i{}^k, St^j{}_l > x^l, \qquad L_i{}^k \in \mathcal{U}.$$

Covariance of: $v f = v_{(1)}(f) v_{(2)}$

Here: $v \in \mathcal{U}$, $f \in \mathrm{Fun}(\mathbf{M}_q)$ and $v(f) = f^{(1)} < v, f^{(2)'} >$.

Covariance of $v(f)$ alone:

$$\begin{aligned}\Delta_{\mathcal{A}}(v)\,(\Delta_{\mathcal{A}} f) &= f^{(1)} < v^{(1)}, f^{(2)'} > \otimes v^{(2)'} f^{(3)'} \\ &= f^{(1)} < v, f^{(3)'} > \otimes f^{(2)'} S(f^{(4)'}) f^{(5)'} \\ &= f^{(1)} \otimes v(f^{(2)'}) \\ &= \Delta_{\mathcal{A}}(v(f)), \qquad \square\end{aligned} \tag{8.21}$$

where we have used identity (7.68).

Covariance of the complete commutation relation:

$$\begin{aligned}\Delta_{\mathcal{A}} v \cdot \Delta_{\mathcal{A}} f &= f^{(1)} < v^{(1)}{}_{(1)}, f^{(2)'} > v^{(1)}{}_{(2)} \otimes v^{(2)'} f^{(3)'} \\ &= f^{(1)} \underline{v^{(1)}{}_{(2)} \otimes v^{(2)'}\, \widehat{v^{(1)}}_{(1)}(f^{(2)'})} \\ &= f^{(1)} \underline{v_{(2)}{}^{(1)} \otimes v_{(1)}(f^{(2)'})\, v_{(2)}{}^{(2)'}} \\ &= \Delta_{\mathcal{A}}(v_{(1)}(f)) \Delta_{\mathcal{A}}(v_{(2)}) \\ &\stackrel{def}{=} \Delta_{\mathcal{A}}(v f). \qquad \square\end{aligned} \tag{8.22}$$

The underlined parts were rewritten using a compatability relation between the right $\mathcal{A}$-coaction and the coproduct in $\mathcal{U}$:

$$v_{(2)}{}^{(1)} \otimes v_{(1)}(f^{(2)'})\, v_{(2)}{}^{(2)'} = v^{(1)}{}_{(2)} \otimes v^{(2)'}\, \widehat{v^{(1)}}_{(1)}(f^{(2)'}). \tag{8.23}$$

Please refer to section 4.2.4 for the definition of the right projector "$\hat{\ }$".

Covariance of: $\partial_i f = \partial_i(f) + L_i{}^j(f) \partial_j$

See section 6.1.2. The main result was the following condition on $L_i{}^j$:

$$\left(L_i{}^j(f^{(2)'}) S^2 t^k{}_j - S^2 t^l{}_i \widehat{L_l{}^k}(f^{(2)})\right) \otimes f^{(1)} = 0. \tag{8.24}$$

Covariance of: $J^i_a L_i{}^k(f) = O_a{}^b(f) J^k_b$

This proof is somewhat involved and we should keep in mind that equation $v_a f = v_a(f) + O_a{}^b(f) v_b$ is already based on $\Delta_{\mathcal{A}}$ being an algebra homomorphism; nevertheless, in several steps:

$\Delta_{\mathcal{A}}$ is a homomorphism of $\mathrm{Fun}(\mathbf{M}_q) \rtimes \mathcal{T}(\mathbf{M}_q)$. Proof on a function f:

$$\begin{aligned}
\Delta_{\mathcal{A}}(J^i_a \partial_i f) &= \Delta_{\mathcal{A}}(v_a f) \\
&= v_a^{(1)} f^{(1)} \otimes v_a^{(2)\prime} f^{(2)\prime} \\
&= \underline{< v_a^{(1)}, St^k{}_l >} \, x^l \partial_k f^{(1)} \otimes \underline{v_a^{(2)\prime}} f^{(2)\prime} \\
&= < v_a, St^s{}_r > x^l \partial_k f^{(1)} \otimes St^r{}_l S^2 t^k{}_s f^{(2)\prime} \\
&= \underline{x^l} \partial_k f^{(1)} \otimes \underline{St^r{}_l < v_a, St^s{}_r >} S^2 t^k{}_s f^{(2)\prime} \\
&= \Delta_{\mathcal{A}} J_a{}^s \, \Delta_{\mathcal{A}}(\partial_s f), \qquad \square
\end{aligned} \tag{8.25}$$

and also

$$\begin{aligned}
\Delta_{\mathcal{A}}(J^i_a \partial_i(f)) &= x^j \partial_i(f^{(1)}) \otimes St^s{}_j < v_a, St^r{}_s > S^2 t^i{}_r f^{(2)\prime} \\
&= \Delta_{\mathcal{A}} J_a{}^r \, \Delta_{\mathcal{A}}(\partial_i) \, (\Delta_{\mathcal{A}} f) \\
&= \Delta_{\mathcal{A}} J^r_a \, \Delta_{\mathcal{A}} \, (\partial_i(f)) \, . \qquad \square
\end{aligned} \tag{8.26}$$

A short aside, checking consistency of $O_a{}^b(f) J^k_b$ with $\Delta_{\mathcal{A}}$ being an algebra homomorphism of $\mathrm{Fun}(\mathbf{M}_q)$.

$$\begin{aligned}
\underline{\Delta_{\mathcal{A}}(O_a{}^b(f) J^i_b)} \, \Delta_{\mathcal{A}}(\partial_i) &\overset{def}{=} \Delta_{\mathcal{A}}(O_a{}^b(f) J^i_b \partial_i) \\
&= \Delta_{\mathcal{A}}(O_a{}^b(f) v_b) \\
&= \Delta_{\mathcal{A}}(O^b_a(f)) \, \Delta_{\mathcal{A}}(v_b) \\
&= \Delta_{\mathcal{A}}(O_a{}^b(f)) \, \Delta_{\mathcal{A}}(J^i_b \partial_i) \\
&\overset{def}{=} \underline{\Delta_{\mathcal{A}}(O_a{}^b(f)) \, \Delta_{\mathcal{A}}(J^i_b)} \, \Delta_{\mathcal{A}}(\partial_i). \qquad \square
\end{aligned} \tag{8.27}$$

Synthesis: Comparing

$$v_a f = v_a(f) + O_a{}^b(f) v_b$$

and

$$J^i_a \partial_i f = J^i_a \partial_i(f) + J^i_a L_i{}^j(f) \partial_j$$

we finally find:

$$\begin{aligned}
\Delta_{\mathcal{A}}(J^i_a L_i{}^k(f)) &= \Delta_{\mathcal{A}}(J^i_a) \, \Delta_{\mathcal{A}}(L_i{}^k(f)) \\
&= \Delta_{\mathcal{A}}(O_a{}^b(f) J_b{}^k) \\
&= \Delta_{\mathcal{A}}(O_a{}^b(f)) \, \Delta_{\mathcal{A}}(J_b{}^k). \qquad \square
\end{aligned} \tag{8.28}$$

Remark: Given a linear operator $L_i{}^j : \mathrm{Fun}(\mathbf{M}_q) \to \mathrm{Fun}(\mathbf{M}_q)$, satisfying the appropriate consistency conditions, — equation

$$J_a^i L_i{}^k(f) = O_a{}^b(f) J_b{}^k \tag{8.29}$$

could very well be used to give explicit covariant $x - x$ commutation relations.

8.3 Cartan Calculus on Quantum Planes

So far we have only dealt with functions and (partial) derivatives that we combined into an algebra of differential operators on the quantum plane via commutation relations

$$\partial_i f = \partial_i(f) + L_i{}^j(f)\partial_j, \qquad \partial_i \in \mathcal{T}(\mathbf{M}_q),\ f \in \mathrm{Fun}(\mathbf{M}_q). \tag{8.30}$$

Now we would like to construct differential forms through an exterior derivative $\mathbf{d}$: $\mathrm{Fun}(\mathbf{M}_q) \to \Lambda^1(\mathrm{Fun}(\mathbf{M}_q))$ that is nilpotent and satisfies the usual graded Leibniz rule. Lie derivatives are introduced next, recalling that they *act* on functions like the ordinary derivatives, that they correspond to $\pounds_{\partial_i}(f) = \partial_i(f)$, and requiring that they commute with the exterior derivative $\pounds_{\partial_i} \circ \mathbf{d} = \mathbf{d} \circ \pounds_{\partial_i}$. Just like it was the case for quantum Lie algebras, the linear operator $L_i{}^j$ should also act like a Lie derivative, *i.e.* we extend its definition from functions to forms by requiring that it commute with $\mathbf{d}$. Inner derivations $\boldsymbol{i}_{\partial_i}$ are defined as graded linear operators of degree -1 orthogonal to the natural basis $\xi^i := \mathbf{d}(x^i)$ of 1-forms: $\boldsymbol{i}_{\partial_i}(\xi^j) = \delta_i^j$ — in consistency with the Cartan identity

$$\pounds_{\partial_i} = \boldsymbol{i}_{\partial_i}\mathbf{d} + \mathbf{d}\boldsymbol{i}_{\partial_i} \tag{8.31}$$

that we want to postulate. For the exterior derivative of a function we can choose between two expansions in terms of 1-forms

$$\mathbf{d}(f) = \xi^i a_i = b_i \xi^i \tag{8.32}$$

that we contract with $\boldsymbol{i}_{\partial_j}$ to find

$$\partial_j(f) = a_j = \boldsymbol{i}_{\partial_j}(b_i\xi^i) \tag{8.33}$$

and

$$\mathbf{d}(f) = \xi^i \partial_i(f). \tag{8.34}$$

The second expression has to wait while we quickly derive $x - \xi$-commutation relations with the help of the first expression and the Leibniz rule for $\mathbf{d}$:

$$\begin{array}{rcl} \mathbf{d}f & = & \xi^i \partial_i f \\ & = & \xi^i \partial_i(f) + \xi^i L_i{}^j(f)\partial_j \\ = \mathbf{d}(f) + f\mathbf{d} & = & \xi^i\partial_i(f) + f\xi^j\partial_j, \end{array} \tag{8.35}$$

valid on any function and hence

$$f\xi^j = \xi^i L_i{}^j(f), \tag{8.36}$$

so that the second expression takes the (not so pretty) form

$$\mathrm{d}(f) = \left(SL_i{}^j \circ \partial_j\right)(f), \tag{8.37}$$

which, unlike in the quantum group case, does not simplify any further. Lie derivatives and inner derivations along arbitrary first order differential operators $f^i\partial_i$, $f^i \in \mathrm{Fun}(\mathrm{M}_q)$ are introduced by the following set of consistent equations:

$$\begin{aligned}
\boldsymbol{i}_{f^i\partial_i} &= f^i \boldsymbol{i}_{\partial_i}, && (8.38)\\
\boldsymbol{\mathcal{L}}_{f^i\partial_i} &= \mathrm{d}\boldsymbol{i}_{f^i\partial_i} + \boldsymbol{i}_{f^i\partial_i}\mathrm{d}, && (8.39)\\
\boldsymbol{\mathcal{L}}_{f^i\partial_i} &= f^i \boldsymbol{\mathcal{L}}_{\partial_i} + \mathrm{d}(f^i)\boldsymbol{i}_{\partial_i}, && (8.40)\\
\boldsymbol{\mathcal{L}}_{f^i\partial_i}\mathrm{d} &= \mathrm{d}\boldsymbol{\mathcal{L}}_{f^i\partial_i}. && (8.41)
\end{aligned}$$

We will not give a complete set of commutation relations here because the reader can easily obtain most of them from the quantum group treatment simply by replacing $\boldsymbol{\mathcal{L}}_{O_i{}^j} \to L_i{}^j$. The problem of defining a Lie bracket of vector fields on the quantum *plane* has, however, not found a satisfactory solution yet.

8.4 Induced Cartan Calculus

We would like to complete the program started in section 8.1, where we induced a calculus on the plane from the calculus on the symmetry quantum group of that plane using a realization $v_a \doteq J_a^i\partial_i$ of the bicovariant group generators in terms of functions and derivatives on the plane. From this expression we get the following two relations for the Cartan generators on the plane:

$$\begin{aligned}
\boldsymbol{i}_{v_a} &\doteq \boldsymbol{i}_{J_a^i\partial_i} = J_a^i \boldsymbol{i}_{\partial_i} && (8.42)\\
\boldsymbol{\mathcal{L}}_{v_a} &\doteq \boldsymbol{\mathcal{L}}_{J_a^i\partial_i} = J_a^i \boldsymbol{\mathcal{L}}_{\partial_i} + \mathrm{d}(J_a^i)\boldsymbol{i}_{\partial_i}. && (8.43)
\end{aligned}$$

Commutation relations for the inner derivation with functions are easily derived;

$$\boldsymbol{i}_{v_a} f = \boldsymbol{\mathcal{L}}_{O_a{}^b}(f)\boldsymbol{i}_{v_b} \tag{8.44}$$

and hence

$$J_a^i \boldsymbol{i}_{\partial_i} f = \boldsymbol{\mathcal{L}}_{O_a{}^b}(f) J_b^k \boldsymbol{i}_{\partial_k} \tag{8.45}$$

or, if a $\tilde{J}_i^a$ exists,

$$\boldsymbol{i}_{\partial_i} f = \tilde{J}_i^a \mathcal{L}_{O_a{}^b}(f) J_b^k \boldsymbol{i}_{\partial_k}, \tag{8.46}$$

and finally

$$\boldsymbol{i}_{\partial_i} f = L_i{}^k(f) \boldsymbol{i}_{\partial_k}. \tag{8.47}$$

Commutation relations for the Lie derivatives with functions can now be calculated using the Cartan identity. We will present the result of such a computation for Wess-Zumino type linear planes (where $\tilde{J}_i^a$ exists):

$$\begin{aligned} \mathcal{L}_{\partial_i} x^l &= \delta_i^l + \underbrace{\tilde{J}_i^a O_a{}^b(x^l) J_b^k}_{L_i{}^k(x^l)} \mathcal{L}_{\partial_k} \\ &\quad + \left(\mathrm{d}(\tilde{J}_i^a O_a{}^b(x^l) J_b^k) - \tilde{J}_i^a \mathrm{d}(O_a{}^b(x^l)) J_b^k\right) \boldsymbol{i}_{\partial_k}. \end{aligned} \tag{8.48}$$

Classically: $O_a{}^b(x^l) \longrightarrow \delta_a^b x^l$ and functions commute with functions and forms so that the last term in the above equation vanishes. The quantum case has a little surprise for us: As was first discovered by Prof. Zumino through purely algebraic considerations in the case of the $\mathrm{GL}_q(2)$-plane, an inner derivation term is necessary in the $\mathcal{L}_\partial - x$ commutation relations in order to get consistency with the undeformed Cartan identity. Let us illustrate this at our standard example.

Cartan Calculus for the 2-dimensional Quantum Plane. Using $x - \mathrm{d}(x)$ commutation relations from (8.36)

$$x\mathrm{d}(x) = q^2 \mathrm{d}(x) x, \tag{8.49}$$

$$x\mathrm{d}(y) = (q^2 - 1)\mathrm{d}(x) y + q\mathrm{d}(y) x, \tag{8.50}$$

$$y\mathrm{d}(x) = q\mathrm{d}(x) y, \tag{8.51}$$

$$y\mathrm{d}(y) = q^2 \mathrm{d}(y) y, \tag{8.52}$$

we obtain

$$\mathcal{L}_{\partial_x} x = 1 + q^2 x \mathcal{L}_{\partial_x} + (q^2 - 1) y \mathcal{L}_{\partial_y} + q\lambda \mathrm{d}(x) \boldsymbol{i}_{\partial_x} + \lambda^2 \mathrm{d}(y) \boldsymbol{i}_{\partial_y}, \tag{8.53}$$

$$\mathcal{L}_{\partial_x} y = q y \mathcal{L}_{\partial_x} + \lambda \mathrm{d}(y) \boldsymbol{i}_{\partial_x}, \tag{8.54}$$

$$\mathcal{L}_{\partial_y} x = q x \mathcal{L}_{\partial_y} + \lambda \mathrm{d}(x) \boldsymbol{i}_{\partial_y}, \tag{8.55}$$

$$\mathcal{L}_{\partial_y} y = 1 + q^2 y \boldsymbol{i}_{\partial_y} + q\lambda \mathrm{d}(y) \boldsymbol{i}_{\partial_y}, \tag{8.56}$$

directly from (8.48) after a lengthy computation. Alternatively, we could have started with $\boldsymbol{i}_\partial - x$ commutation relations

$$\boldsymbol{i}_{\partial_x} x = q^2 x \boldsymbol{i}_{\partial_x} + (q^2 - 1) y \boldsymbol{i}_{\partial_y}, \tag{8.57}$$

$$
\begin{aligned}
\boldsymbol{i}_{\partial_x} y &= q y \boldsymbol{i}_{\partial_x}, & (8.58)\\
\boldsymbol{i}_{\partial_y} x &= q x \boldsymbol{i}_{\partial_y}, & (8.59)\\
\boldsymbol{i}_{\partial_y} y &= q^2 y \boldsymbol{i}_{\partial_y}, & (8.60)
\end{aligned}
$$

which have the great advantage that they have the exact same form as the well-known $\partial - x$ relations. This also means that all of our covariance considerations are still valid here.

Chapter 9

A Torsion-free Tangent Bundle for $SU_q(2)$

Introduction

In the classical theory of Lie groups one can introduce a tangent bundle over the group manifold. There are two natural choices for the connection: Either one imposes the condition of zero curvature and then chooses a vanishing connection in an appropriate gauge — such that the torsion is given by the RHS of the Cartan-Maurer equation — or one can attempt to set the torsion equal to zero to obtain a (Riemannian or G-Structure type) non-vanishing curvature. The first scenario generalizes quite easily to the quantum group case. In this chapter we will try to generalize the more interesting case of vanishing torsion at the example of $SU_q(2)$.

To establish notation, a review (including some additional relevant material) of the theory of quantum Lie algebras is given in the next section, followed by the description of a tangent bundle structure over a quantum group. We then elaborate on the example of $SU_q(2)$ giving all R-matrices and structure constants explicitly.

9.1 Quantum Lie Algebras

Quantum Lie Algebras are Hopf algebras $U_q\mathbf{g}$ that contain a finite-dimensional sub vector space that closes under left and right coactions. Let $\{e_i\}$ be a linear basis of generators for this space* and $\{e^j\}$ a dual basis of 1-forms corresponding to a set of

*In this chapter we will not consider a linear basis of the whole Hopf algebra so there should not be any confusion from this notation.

functions $b^j \in \mathrm{Fun}(G_q)$ via $e^j \equiv Sb^j_{(1)}\mathbf{d}b^j_{(2)}$:

$$
\begin{aligned}
{}_{\mathcal{A}}\Delta(e_i) &= 1 \otimes e_i, \\
\Delta_{\mathcal{A}}(e_i) &= e_j \otimes T^j{}_i, \quad T^j{}_i \in \mathrm{Fun}(G_q), && (9.1)\\
\boldsymbol{i}_{e_i}(e^j) &= -< e_i, Sb^j >= \delta_i^j, && (9.2)\\
{}_{\mathcal{A}}\Delta(e^i) &= 1 \otimes e^i, && (9.3)\\
\Delta_{\mathcal{A}}(e^i) &= e^j \otimes S^{-1}T^i{}_j. && (9.4)
\end{aligned}
$$

The exterior derivative on functions can be expressed in terms of these bases as

$$\mathbf{d}(a) = e^i(e_i \triangleright a) = e^i \mathcal{L}_{e_i}(a). \tag{9.5}$$

The Leibniz rule for $\mathbf{d}$ requires that the generators $\{e_i\}$ have a coproduct of the form

$$\Delta(e_i) = e_i \otimes 1 + \theta_i{}^j \otimes e_j. \tag{9.6}$$

A Cartan calculus can be introduced on these quantum Lie algebras with equations like

$$
\begin{aligned}
\mathcal{L}_{e_i}\alpha &= \mathcal{L}_{e_i}(\alpha) + \mathcal{L}_{\theta_i{}^j}(\alpha)\mathcal{L}_{e_j} && (9.7)\\
\boldsymbol{i}_{e_i}\alpha &= \boldsymbol{i}_{e_i}(\alpha) + (-1)^p \mathcal{L}_{\theta_i{}^j}(\alpha)\boldsymbol{i}_{e_j} && (9.8)\\
\mathcal{L}_{e_i} &= \mathbf{d}\boldsymbol{i}_{e_i} + \boldsymbol{i}_{e_i}\mathbf{d} && (9.9)\\
e^i &= Sb^i_{(1)}\mathbf{d}b^i_{(2)} =: e_{b^i}, && (9.10)
\end{aligned}
$$

where α is a p-form, for a more complete list see section 7.2.3. As in the classical case we make an ansatz for the product of two forms in terms of tensor products

$$e^i \wedge e^j = e^i \otimes e^j - \hat{\sigma}^{ij}{}_{mn} e^m \otimes e^n, \tag{9.11}$$

with as yet unknown numerical constants $\hat{\sigma}^{ij}{}_{mn} \in k$ and define $\boldsymbol{i}_{e_i}$ to act on this product by contracting in the first tensor product space. This leads to the following explicit expression for $\hat{\sigma}^{ij}{}_{mn}$:

$$\hat{\sigma}^{ij}{}_{mn} =< S^{-1}\theta_m{}^j, T^i{}_n > \tag{9.12}$$

and, in a particular example that we will need later,

$$
\begin{aligned}
\mathbf{d}e^j \equiv \mathbf{d}e_{b^j} &= -e_{b^j_{(1)}} \wedge e_{b^j_{(2)}} \\
&= -e_{S^{-1}(Sb^j_{(1)}b^j_{(3)})} \otimes e_{b^j_{(2)}} \\
&= -e^k \otimes e^l < -Se_k, S^{-1}(Sb^j_{(1)}b^j_{(3)}) >< -Se_l, b^j_{(2)} > && (9.13)\\
&= -e^k \otimes e^l < \underbrace{(S^{-1}e_k)_{(1)}e_l S(S^{-1}e_k)_{(2)}}_{S^{-1}e_k \overset{\mathrm{ad}}{\triangleright} e_l}, Sb^j > .
\end{aligned}
$$

In the previous equation we have introduced the adjoined action of a left-invariant vector field on another vector field. A short calculation gives

$$S^{-1}e_k \overset{\mathrm{ad}}{\triangleright} e_l = e_b e_c(\delta^c_k\delta^b_l - \hat{\sigma}^{cb}{}_{kl}) = e_a < S^{-1}e_k, T^a{}_l >= e_a f'_k{}^a{}_l \tag{9.14}$$

and similarly

$$e_k \overset{\mathrm{ad}}{\triangleright} e_l \equiv \mathcal{L}_{e_k}(e_l) = e_b e_c(\delta^c_k\delta^b_l - \hat{R}^{cb}{}_{kl}) = e_a < e_k, T^a{}_l >= e_a f_k{}^a{}_l, \tag{9.15}$$

where

$$\hat{R}^{cb}{}_{kl} =< \theta_k{}^b, T^c{}_l > \tag{9.16}$$

is the so-called "big R-matrix" related to σ† by

$$\sigma^{ij}{}_{kl}R^{kl}{}_{mn} = \delta^i_m\delta^j_n. \tag{9.17}$$

A little more work gives

$$f_m{}^a{}_n = -f'^a_k{}_l R^{kl}{}_{mn}. \tag{9.18}$$

Were we to impose zero curvature now and chose a vanishing connection, then the right hand side of equation (9.13) would give the torsion two form.

The calculus on quantum Lie algebras is by construction covariant under left and right coactions. It has however a closely related additional symmetry: All equations that we have given are invariant under linear changes of the bases e_i and e^j:

$$e_i \rightarrow \chi_i = e_l M^l{}_i, \qquad e^i \rightarrow \tau^i = (M^{-1})^i{}_l e^l, \qquad M \in M_N(k). \tag{9.19}$$

The adjoined matrix representation T and the braiding operator θ transform as expected under this change of basis

$$T^i{}_j \rightarrow \bar{T}^i{}_j = (M^{-1})^i{}_l T^l{}_m M^m{}_j, \tag{9.20}$$

$$\theta_i{}^j \rightarrow \bar{\theta}_i{}^j = (M^{-1})^i{}_l \theta_m{}^l M^m{}_j, \tag{9.21}$$

such that now

$$\Delta_{\mathcal{A}}(\chi_i) = \chi_j \otimes \bar{T}^j{}_i \qquad \Delta_{\mathcal{A}}(\tau^i) = \tau^j \otimes S^{-1}\bar{T}^j{}_i \tag{9.22}$$

and

$$\Delta\bar{T} = \bar{T}\otimes\bar{T}, \quad \epsilon\bar{T} = I, \quad S\bar{T} = \bar{T}^{-1}, \tag{9.23}$$

i.e. $\bar{T}$ (like T) satisfies the appropriate relations for a matrix representation of $U_q\mathbf{g}$.

†The hat "$\hat{}$" denotes the action of the permutation matrix $P^{ij}{}_{kl} = \delta^i_l\delta^j_k$, *i.e.* $\hat{\sigma} \equiv P\sigma$.

9.2 Quantum Tangent Bundle, Torsion, Curvature

In this chapter we are going to use a formulation [63] of the theory of fiber bundles where all forms are pulled back to the base manifold. This formulation is well suited for the generalization to quantum groups because it makes it easier to keep track of subtle distinctions between the calculi of base vs. fiber.

The base manifold in the problem under consideration is a quantum group, implicitly defined by the Hopf algebra of functions $\mathrm{Fun}(G_q)$ on it. The typical fiber of the tangent bundle is the invariant space $\mathrm{span}\{e_i\}$, i.e. the "quantum Lie algebra". We chose a basis $\{\chi_i\}$ of sections on the tangent bundle and consider "pointwise" infinitesimal transformations within the fiber along elements A_μ of $U_q\mathbf{g}$

$$A_\mu \triangleright \chi_i = A_{\mu_{(1)}} \chi_i S A_{\mu_{(2)}} = \chi_j < A_\mu, \bar{T}^i{}_j >, \tag{9.24}$$

where we have used $\Delta_{\mathcal{A}}\chi_i = \chi_j \otimes \bar{T}^i{}_j$. In order to justify the word "infinitesimal" the A_μ should be linear combinations of the e_i and possibly $S^{-1}e_i$[‡]. These heuristic considerations suggest that the connection 1-form should have the following form

$$\omega = e^\mu A_\mu, \qquad \omega^j{}_i = e^\mu < A_\mu, \bar{T}^j{}_i > \tag{9.25}$$

which enters in the expression of the covariant derivative ∇ on the section basis:

$$\nabla \chi_i = \chi_j \otimes \omega^j{}_i. \tag{9.26}$$

This equation is basically a reformulation of (9.24) in differential form language and equation (9.26) replaces the metricity condition on ω in the sense of G-structures: In the classical theory we construct classes of G-bases fixing one orthogonal basis $\{\chi_i\}$ and getting all other orthogonal bases by transforming $\{\chi_i\}$ by a Lie subgroup of the general linear group. For quantum groups we choose transformation matrices of the form $< x, \bar{T} >$. Later we will come back to the question which metric — if any — is preserved by said transformations. Using properties of ∇ like

$$\begin{aligned} \nabla(\chi + \psi) &= \nabla\chi + \nabla\psi, & (9.27)\\ \nabla(f\psi) &= \mathrm{d}f \otimes f\nabla\psi, \qquad f \in \mathrm{Fun}(G_q), & (9.28)\\ \nabla_{fu+v}\psi &= f\nabla_u\psi + \nabla_v\psi, \qquad \nabla_u\psi \equiv \boldsymbol{i}_u(\psi), & (9.29) \end{aligned}$$

[‡]Higher powers of S do not result in new generators in the example under consideration in the next section.

we can easily calculate the covariant derivative of an arbitrary section $\psi = \chi_i \psi^i$:

$$\nabla\psi = \chi_i \otimes (\nabla\psi)^i = \chi_i \otimes (\mathbf{d}\psi^i + \omega^i{}_j \psi^j). \tag{9.30}$$

For section-valued p-forms we introduce an exterior covariant differentiation $\mathbf{D}$:

$$\mathbf{D}(\psi \otimes \alpha) := \nabla\psi \wedge \alpha + \psi \otimes \mathbf{d}\alpha \tag{9.31}$$

in accordance to the *undeformed* Leibniz rule.

The last ingredient, enabling us to define torsion, is the fusion form $\eta = \chi_i \otimes e^i$, viewed as a section valued 1-form. It effectively identifies elements in the fibers of the tangent bundle with the tangent space over the points of the base manifold. One usually takes the canonical element $e_i \otimes e^i$ as a natural choice for the fusion form, but $\eta = \chi_i \otimes e^i = e_l M^l{}_i \otimes e^i$, where $M^l{}_i$ is a constant numerical matrix that may however differ from $\delta^l{}_i$, is also a mathematically acceptable description and will in fact be quite important in the quantum case as we shall see. The torsion 2-form Θ is defined as the exterior covariant derivative of the fusion form

$$\begin{aligned} \Theta = \mathbf{D}\eta &= \nabla\chi_i \wedge e^i + \chi_i \otimes \mathbf{d}e^i \\ &= \chi_j \otimes (\omega^j{}_i \wedge e^i + \mathbf{d}e^j) \\ &=: \chi_j \otimes \Theta^j. \end{aligned} \tag{9.32}$$

We will later try to set $\Theta = 0$. The curvature 2-form of a section ψ is $\Omega = \mathbf{D}\nabla\psi$, i.e. the exterior covariant derivative of the section valued 1-form $\nabla\psi$. In terms of the section basis we find

$$\begin{aligned} \mathbf{D}\nabla\chi_i &= \mathbf{D}(\chi_j \otimes \omega^j{}_i) \\ &= \chi_k \otimes (\omega^k{}_j \wedge \omega^j{}_i + \mathbf{d}\omega^k{}_i) \\ &=: \chi_k \otimes \Omega^k{}_i. \end{aligned} \tag{9.33}$$

The Ricci tensor can also be defined in this context:

$$e^\mu R_{\mu i} := \boldsymbol{i}_{e_k} \Omega^k{}_i. \tag{9.34}$$

For simple Lie algebras it has the particularly simple form of the Killing metric times a constant.

Using tools from the previous section we can expand the torsion 2-form in terms of tensor products

$$\mathbf{d}e^j = -e^k \otimes e^l f'^{j}_{k\ l}, \tag{9.35}$$

$$\omega^j{}_i \wedge e^i = \omega_\mu{}^j{}_i e^\mu \wedge e^i = \omega_\mu{}^j{}_i e^k \otimes e^l (\delta^\mu_k \delta^i_l - \hat{\sigma}^{\mu i}{}_{kl}) \tag{9.36}$$

and the condition of zero torsion becomes

$$\omega_\mu{}^j{}_i(\delta^\mu_k\delta^i_l - \hat{\sigma}^{\mu i}{}_{kl}) = f'_k{}^j{}_l. \tag{9.37}$$

This is a set of linear equations for $\omega_\mu{}^j{}_i$ with non-trivial null space, i.e. we will get a solution $v_\mu{}^j{}_i$ and vectors $N_\mu{}^a{}_i$ with $N_\mu{}^a{}_i(\delta^\mu_k\delta^i_l - \hat{\sigma}^{\mu i}{}_{kl}) = 0$ such that

$$\omega_\mu{}^j{}_i = v_\mu{}^j{}_i + \sum_a n^j_a N_\mu{}^a{}_i, \quad n^j_a \in k. \tag{9.38}$$

To decide whether it is possible to find an $\omega_\mu{}^j{}_i$ that satisfies all conditions, in particular

$$\omega_\mu{}^j{}_i \stackrel{?}{=} < A_\mu, \bar{T}^j{}_i >, \tag{9.39}$$

it is now instructive to look at the concrete example of $SU_q(2)$.

9.3 Example $\mathbf{SU}_q(2)$

...or $Sl_q(2)$ if one modifies the reality condition. Recall [23], [38] the commutation relations for $SU_q(2)$, here written in compact matrix notation as

$$\begin{aligned} r_{12}t_1t_2 = t_2t_1r_{12}, \quad det_q t = 1, \quad t^\dagger = t^{-1}, \\ \Delta(t) = t\dot{\otimes}t, \quad \epsilon(t) = I, \quad S(t) = t^{-1}, \end{aligned} \tag{9.40}$$

where $t \in M_n(Fun(SU_q(2)))$ and r is the "small" r-matrix

$$r =< \mathcal{R}, t_1 \otimes t_2 >= \frac{1}{\sqrt{q}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - \frac{1}{q} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{9.41}$$

The deformed universal enveloping algebra $U_q su(2)$, dual to $Fun(SU_q(2))$, is generated by operators H, X_+, X_- satisfying

$$\begin{aligned} [H, X_\pm] = \pm 2X_\pm, \qquad & [X_+, X_-] = \tfrac{q^H - q^{-H}}{q - q^{-1}}, \\ \Delta(H) = H \otimes 1 + 1 \otimes H, \quad & \Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm, \\ \epsilon(H) = \epsilon(X_\pm) = 0, & \\ S(H) = -H, \qquad & S(X_\pm) = -q^{\pm 1} X_\pm. \end{aligned} \tag{9.42}$$

Following [23] these relations can be rewritten as

$$\begin{aligned} r_{12}L^\pm_2 L^\pm_1 = L^\pm_1 L^\pm_2 r_{12}, \quad & r_{12}L^+_2 L^-_1 = L^-_1 L^+_2 r_{12}, \\ \Delta(L^\pm) = L^\pm \dot{\otimes} L^\pm, \qquad & \epsilon(L^\pm) = I, \\ S(L^\pm) = (L^\pm)^{-1}, & \end{aligned} \tag{9.43}$$

where $L^{\pm}$ are given by

$$L^{+} =< \mathcal{R}, id \otimes t >= \begin{pmatrix} q^{-H/2} & \frac{1}{\sqrt{q}}\lambda X_{+} \\ 0 & q^{H/2} \end{pmatrix} \tag{9.44}$$

and

$$L^{-} =< \mathcal{R}, St \otimes id >= \begin{pmatrix} q^{H/2} & 0 \\ -\sqrt{q}\lambda X_{-} & q^{-H/2} \end{pmatrix}, \tag{9.45}$$

where $\lambda \equiv q - \frac{1}{q}$. Unitarity of T implies $(L^{+})^{\dagger} = (L^{-})^{-1}$, *i.e.* $\bar{H} = H, \overline{X_{\pm}} = X_{\mp}$.

Following the method described in section 2.4.1 we can construct a matrix of bicovariant generators corresponding to an element $1 \otimes 1 - \mathcal{R}^{21}\mathcal{R}$ of the "pure braid group":

$$\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} := \frac{1}{\lambda} < 1 \otimes 1 - \mathcal{R}^{21}\mathcal{R}, t \otimes id >= \frac{1}{\lambda} L^{+} S L^{-} =: X. \tag{9.46}$$

The right coaction is then

$$\Delta_{\mathcal{A}} X^{i}{}_{l} = X^{j}{}_{k} \otimes St^{i}{}_{j} t^{k}{}_{l}, \tag{9.47}$$

so that span$\{e_i\}$ forms an invariant subspace as required. $c := e_1 + q^{-2} e_4$ by the way is the casimir. The functions $b^i \in \mathrm{Fun}(G_q)$ see equation (9.2) can be chosen as linear combinations of the elements of t [35] because t (and St) form faithful (anti)representations of the e_is. Classical commutators become adjoint actions

$$e_k \overset{\mathrm{ad}}{\triangleright} e_l := e_{k_{(1)}} e_l S e_{k_{(2)}} = e_b e_c (\delta^c_k \delta^b_l - \hat{R}^{cb}{}_{kl}) = e_a f_k{}^a{}_l,$$

where the $\hat{R}$ and f can be calculated [60] from r (see section 4.1.2)

$$\hat{R}^{(mn)(kl)}{}_{(ij)(pq)} = \left((r_{31}{}^{-1})^{T_3} r_{41} r_{24} (r_{23}{}^{T_3})^{-1} \right)^{ilmn}{}_{kjpq} \tag{9.48}$$

and

$$f_k{}^a{}_l = \frac{1}{\lambda} \left(I_k \delta^a_l - \sum_i \hat{R}^{a(ii)}{}_{kl} \right). \tag{9.49}$$

Explicitly: $f^a{}_{(kl)}$

$$\begin{matrix} \frac{1-q^2}{q^3} & 0 & 0 & -\frac{1}{q}+q & 0 & 0 & \frac{1}{q} & 0 & 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1+q^2-q^4}{q^3} & 0 & 0 & -\frac{1}{q} & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & -q & 0 & 0 \\ 0 & 0 & -q & 0 & 0 & 0 & 0 & 0 & q^{-3} & 0 & 0 & -\frac{1}{q} & 0 & 0 & \frac{1}{q} & 0 \\ \frac{-1+q^2}{q^3} & 0 & 0 & \frac{1}{q}-q & 0 & 0 & -\frac{1}{q} & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \tag{9.50}$$

and $f'^a{}_{(kl)}$,

$$
\begin{array}{cccccccccccccccc}
\frac{1-q^2}{q^3} & 0 & 0 & -\frac{1}{q}+q & 0 & 0 & -\frac{1}{q} & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q & 0 & 0 & q^{-3} & 0 & 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 & \frac{1}{q} & 0 & 0 \\
0 & 0 & \frac{1+q^2-q^4}{q^3} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{q} & 0 & 0 & q & 0 & 0 & -q & 0 \\
\frac{-1+q^2}{q^3} & 0 & 0 & \frac{1}{q}-q & 0 & 0 & \frac{1}{q} & 0 & 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{9.51}
$$

obtained by similar methods. In both matrices rows are labeled by $a \in \{1,\ldots,4\}$ and columns are labeled by $(kl) \in \{(1,1),(1,2),\ldots,(4,4)\}$.

Using the explicit expressions for $\hat{\sigma}$ (see appendix) and $f'^a{}_{(kl)}$ we find the following particular solutions $v^j{}_{(\mu i)}$ of (9.38):

$$
\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -q^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{9.52}
$$

The null space of said linear equation, i.e. of $\hat{\sigma}-I$, is spanned by $N^a_{(\mu i)}$, $a = 1,\ldots,10$:

$$
\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1+q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-4}-q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{9.53}
$$

The fact that there are 10 null vectors shows by the way that the number of independent 2-forms is reduced from $4 \times 4 = 16$ to $16 - 10 = 6 = 4 \times 3/2$ as one would expect.

We will now investigate choices for ω of gradually increasing complexity starting with a simple ansatz with $M = I$

$$
\omega_\mu{}^j{}_i = < A_\mu, T^j{}_i > = A^\nu{}_\mu f_\nu{}^j{}_i + A'^\nu{}_\mu f'_\nu{}^j{}_i \tag{9.54}
$$

corresponding to

$$
A_\mu = A^\nu{}_\mu e_\nu + A'^\nu{}_\mu S^{-1} e_\nu. \tag{9.55}
$$

In the classical case we would find $A^{\nu}{}_{\mu} = -A'^{\nu}{}_{\mu} = \frac{1}{4}\delta^{\nu}_{\mu}$ as a solution. Explicit computation shows however that there are no solutions for $A^{\nu}{}_{\mu}$ and $A'^{\nu}{}_{\mu}$ in the quantum case. Next we try an ansatz with trivial $A^{\nu}{}_{\mu}$ and $A'^{\nu}{}_{\mu}$ in analogy to the classical solution, but we allow the basic generators e_{ν} and $S^{-1}e_{\nu}$ in (9.55) to be multiplied by elements $z, z' \in U_q\mathbf{g}$

$$\omega_{\mu}{}^{j}{}_{i} = < ze_{\mu} - z'S^{-1}e_{\mu}, T^{j}{}_{i} > = Z^{j}{}_{k} f_{\mu}{}^{k}{}_{i} + Z'^{j}{}_{l} f'{}_{\mu}{}^{l}{}_{i} \tag{9.56}$$

where $Z^{(\prime)j}{}_{k} = < z^{(\prime)}, T^{j}{}_{k} >$. Hence solving

$$v_{\mu}{}^{j}{}_{i} + n^{j}_{a} N_{\mu}{}^{a}{}_{i} = Z'^{j}{}_{l} f'{}_{\mu}{}^{l}{}_{i} + Z^{j}{}_{k} f_{\mu}{}^{k}{}_{i} \tag{9.57}$$

for $\{n^{j}_{a}, Z'^{j}{}_{l}, Z^{j}{}_{k}\}$ gives

$$\begin{array}{cccccccccc}
0 & \frac{1}{2q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{-q-q^3} & \frac{q}{1+q^2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{q+q^3} & -\frac{q}{1+q^2} & 0 & 0 & 0 \\
0 & \frac{-1}{2q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$
$$\begin{array}{cccccccc}
-\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & -\frac{q^2}{(1+q^2)^2} & 0 & 0 & 0 & \frac{q^2}{(1+q^2)^2} & 0 & 0 \\
0 & 0 & -\frac{q^2}{(1+q^2)^2} & 0 & 0 & 0 & \frac{q^2}{(1+q^2)^2} & 0 \\
\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0
\end{array} \tag{9.58}$$

as a particular solution and

$$\begin{array}{cccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array} \tag{9.59}$$

as the corresponding null space. The first 10 columns in both matrices are labeled by a, the next 4 columns are labeled by k, and the last 4 are labeled by l. j is the row index. Two comments about the null space are in order: Note that the first ten columns are zero. This means that n^{j}_{a} and hence $\omega_{\mu}{}^{j}{}_{i}$ are in fact uniquely determined by our ansatz. Note also that both f and f' and thereby e_{μ} *and* $S^{-1}e_{\mu}$ were necessary to satisfy the equation. All that remains is some arbitrariness in the definition of K and K'. This actually comes from the existence of an invariant form $e^1 + e^4$. Being invariant means $\mathrm{d}(e^1 + e^4) = 0$ or $f'{}_{\mu}{}^{1}{}_{i} + f'{}_{\mu}{}^{4}{}_{i} = 0$; by equation (9.18) the same is true for f. We use this remaining freedom to diagonalize

$$K = -K' = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{q^2}{(1+q^2)^2} & 0 & 0 \\ 0 & 0 & \frac{q^2}{(1+q^2)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{9.60}$$

corresponding for instance to

$$z = -z' = \frac{1}{4} + \frac{q(1-q^2)}{4(1+q^2)^2}(e_4 - S^{-1}e_4) \tag{9.61}$$

and

$$A_\mu = z e_\mu. \tag{9.62}$$

If z and z' had been invariant elements (casimirs) then A_μ would have had nice transformation properties. The way it is, the solution is somewhat unsatisfying. Luckily it turns out that z can be eliminated without having to change our solution for ω if we allow for a non-trivial M matrix. As can be seen by inspection of the explicit forms of f and f':

$$Z^j{}_l(f_\mu{}^l{}_i - f'_\mu{}^l{}_i) = (M^{-1})^j{}_k M^\nu{}_\mu M^h{}_i (f_\nu{}^k{}_h - f'_\nu{}^k{}_h), \tag{9.63}$$

where

$$M = \begin{pmatrix} \frac{q^2}{(1+q^2)^2} & 0 & 0 & 0 \\ 0 & \frac{q}{2(1+q^2)} & 0 & 0 \\ 0 & 0 & \frac{q}{2(1+q^2)} & 0 \\ 0 & 0 & 0 & \frac{q^2}{(1+q^2)^2} \end{pmatrix} \tag{9.64}$$

such that now

$$\omega_\mu{}^j{}_i = < \chi_\mu - S^{-1}\chi_\mu, \bar{T}^j{}_i >, \qquad i.e.\ A_\mu = \chi_\mu - S^{-1}\chi_\mu. \tag{9.65}$$

9.4 Appendix

$\hat{R}^{ij}{}_{kl}$:

$$\left(\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-q^{-2} & 0 & 0 & 1 & 0 \\
0 & 0 & 1-q^2 & 0 & 0 & 0 \\
1+q^{-4}-\frac{2}{q^2} & 0 & 0 & 2-q^{-2}-q^2 & 0 & 0 \\
0 & q^{-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1-q^{-2} & 0 & 0 & 1-q^2 & 0 & 0 \\
0 & -1+q^{-4}-q^{-2}+q^2 & 0 & 0 & 1-q^{-2} & 0 \\
0 & 0 & q^2 & 0 & 0 & 0 \\
-1+q^{-2} & 0 & 0 & -1+q^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1+q^{-2} & 0 & 0 & 0 & 0 \\
0 & 0 & -1+q^2 & 0 & 0 & 0 \\
-1-q^{-4}+\frac{2}{q^2} & 0 & 0 & \left(-\frac{1}{q}+q\right)^2 & 0 & 0
\end{array}\right.$$

$$
\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1+q^{-2} & 0 & 0 & 1-q^{-2} & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-q^{2} & 0 & 0 & 0 & 0 & 0 & q^{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-4}-q^{-2} & 0 & 0 & 1-q^{-2} & 0 & 0 & q^{-2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1-q^{-2} & 0 & 0 & -1+q^{-2} & 0 & 0 & 0 & 0 & 0 & 1
\end{matrix}\right) \tag{9.66}
$$

Here is the "big" $\hat{\sigma}$, which describes the $\wedge$ product

$$
\left(\begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-q^{2} & 0 & 0 & 1 & 0 \\
0 & 0 & 1-q^{-2} & 0 & 0 & 0 \\
1+q^{-4}-\frac{2}{q^{2}} & 0 & 0 & 2-q^{-2}-q^{2} & 0 & 0 \\
0 & q^{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1+q^{-2} & 0 & 0 & -1+q^{2} & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-4}-q^{-2} & 0 \\
0 & 0 & q^{-2} & 0 & 0 & 0 \\
1-q^{-2} & 0 & 0 & 1-q^{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1+q^{-4}-q^{-2}+q^{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1+q^{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -1+q^{-2} & 0 & 0 & 0 \\
-1-q^{-4}+\frac{2}{q^{2}} & 0 & 0 & -2+q^{-2}+q^{2} & 0 & 0
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-q^{-2} & 0 & 0 & -1+q^{-2} & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-q^{-2} & 0 & 0 & 1-q^{2} & 0 & 0 & q^{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1+q^{-2} & 0 & 0 & 1-q^{-2} & 0 & 0 & 0 & 0 & 0 & 1
\end{matrix}\right) \tag{9.67}
$$

Chapter 10

Toward a BRST Formulation of Quantum Group Gauge Theory

In this chapter we will give a brief introduction to a BRS type formalism for quantum gauge theories. All fields will live on the base manifold. A BRS formulation has two main advantages here: It can be formulated as a purely algebraic theory with abstract operators $\delta, \mathrm{d}, t, \ldots$ (see [64] for a beautiful example of the use of this abstract algebra in the context of anomalies) and it emphasizes the coalgebra aspect of the quantum structure group — which is undeformed in the case of matrix pseudo groups. This will lead to equations that are of virtually identical form as their classical counterparts; this was the base of Isaevs [65] approach to quantum group gauge theory. We will however go a step beyond this work in as we will give an interpretation of objects like $\mathrm{d}(t) \neq 0$, where d is the exterior derivative on the *base* manifold of a bundle with quantum group valued fiber, even though $t \in M_n(\mathcal{A})$ may not have any base-dependence, thereby justifying the coexistence of such different objects within one algebra. We will not attempt any further (physical) interpretations of *e.g.* the connection form here, because this subject is still controversial at the moment. Nevertheless we hope to give an easy-to-use formalism that could serve as a starting platform for further investigations. Articles of related interest are [66]; see [67] for an abstract treatment of quantum group gauge theory and many examples.

Let $\mathcal{A} = \mathrm{Fun}(G_q)$ be the algebra of functions on the quantum structure group and $\mathcal{B}$ =Fun(M) be the — possibly non-commutative — algebra of functions on the base manifold; for instance space-time. The symbol δ shall denote the exterior derivative of $\Lambda(\mathcal{A})$ and d ditto of $\Lambda(\mathcal{B})$ — classically: $\mathrm{d} = \mathrm{d}(x^\mu)\frac{\partial}{\partial x^\mu}$; we will require them to anticommute

$$\delta \mathrm{d} = -\mathrm{d}\delta. \tag{10.1}$$

The quantum matrix $(t^i{}_j)_{i,j=1}^n \in M_n(\mathcal{A})$ (in the fundamental representation) shall describe the gauge transformation of a column vector ψ_0 of fields, A_0 is the quantum Lie algebra valued matrix of connection 1-forms and v finally, the "ghost", is an abbreviation for the Cartan-Maurer form $t^{-1}\delta(t)$. As in the chapter on the induced calculus we make ψ_0 and A_0 "variable" with the help of $\mathcal{A}$-coactions:

$$\begin{aligned} \psi &:= t^{-1}\psi_0 = \text{“}\Delta_{\mathcal{A}}(\psi_0)\text{”}, && (10.2) \\ A &:= t^{-1}A_0 t + t^{-1}\mathrm{d}(t) = \text{“}\Delta_{\mathcal{A}}(A_0)\text{”}. && (10.3) \end{aligned}$$

To justify the name "coaction" for

$$\Delta_{\mathcal{A}}(A_0{}^i{}_l) = A_0{}^j{}_k \otimes S(t^i{}_j)t^k{}_l + 1 \otimes S(t^i{}_j)\mathrm{d}(t^j{}_l) \tag{10.4}$$

we have to extend the notion of the Hopf algebra $\mathcal{A}$ to a graded Hopf algebra $\mathcal{A} \oplus \mathcal{A} \otimes \mathrm{d}(\mathcal{A})$ via

$$\begin{aligned} \Delta \circ \mathrm{d} &:= (\mathrm{d} \otimes id + id \otimes \mathrm{d}) \circ \Delta, && (10.5) \\ \epsilon \circ \mathrm{d} &:= \mathrm{d} \circ \epsilon : \mathcal{A} \to \{0\}, && (10.6) \\ S \circ \mathrm{d} &:= \mathrm{d} \circ S. && (10.7) \end{aligned}$$

Consider *e.g.*

$$\epsilon(\mathrm{d}a) = \cdot(S \otimes id)\Delta(\mathrm{d}a) = S(\mathrm{d}a_{(1)})a_{(2)} + S(a_{(1)})\mathrm{d}(a_{(2)}) = 0, \qquad \textit{e.t.c.}. \tag{10.8}$$

It is straightforward to show that (10.4) does indeed satisfy the axioms of a coaction:

$$(\Delta_{\mathcal{A}} \otimes id)\Delta_{\mathcal{A}} = (id \otimes \Delta)\Delta_{\mathcal{A}}, \qquad (id \otimes \epsilon)\Delta_{\mathcal{A}} = id. \tag{10.9}$$

We are now ready to derive a set of BRS transformations

$$\begin{aligned} \delta(\psi) &= \delta(t^{-1}\psi_0) = t^{-1}t\delta(t^{-1})\psi_0 = -t^{-1}\delta(t)t^{-1}\psi_0 \\ &= -v\psi, && (10.10) \\ \delta(\mathrm{d}\psi) &= \mathrm{d}(v)\psi - v\mathrm{d}(\psi), && (10.11) \\ \delta(v) &= -v^2, && (10.12) \\ \delta(t) &= tv, && (10.13) \\ \delta(t^{-1}) &= -vt^{-1}, && (10.14) \\ \delta(t^{-1}\mathrm{d}(t)) &= -vt^{-1}\mathrm{d}(t) - t^{-1}\mathrm{d}(t)v - \mathrm{d}(v), && (10.15) \\ \delta(A) &= \delta(t^{-1})A_0 t - t^{-1}A_0\mathrm{d}(t) + \delta(t^{-1}\mathrm{d}(t)) \\ &= -vA - Av - \mathrm{d}(v), && (10.16) \end{aligned}$$

simply by applying δ and working out the algebra; the first and last lines should give a flavor of these computations. All these equations correspond via the Cartan identity $\mathcal{L}_{\chi_i} = \delta i_{\chi_i} + i_{\chi_i}\delta$ to infinitesimal gauge transformations. The δi_{χ_i} term is actually zero on functions and on left-invariant 1-forms like v, so we only need the second term $i_{\chi_i}\delta$, *i.e.* *all* gauge transformation information is already contained in the BRS δ; *e.g.*

$$\begin{aligned} \mathcal{L}_{\chi_i}(\psi) &= i_{\chi_i}(\delta\psi) \\ &= -i_{\chi_i}(v\psi) \\ &= <\chi_i, St> \psi =: \lambda_i\psi \end{aligned} \tag{10.17}$$

and

$$\begin{aligned} \mathcal{L}_{\chi_i}(v) &= \underbrace{\delta(i_{\chi_i}v)}_{0} + i_{\chi_i}(\delta v) \\ &= -i_{\chi_i}(v^2) \\ &= -i_{\chi_i}(v)v + \mathcal{L}_{O_i{}^j}(v)i_{\chi_j}(v) \\ &= \lambda_i v + \mathcal{L}_{O_i{}^j}(v)\lambda_j = \{\lambda_i, v\}_q \\ &= \lambda_i v + M_i{}^l v(M^{-1})_l{}^j\lambda_j, \end{aligned} \tag{10.18}$$

with $M_i{}^l \equiv < O_i{}^l, St >$. Next we introduce a covariant derivative D such that $D\psi$ transforms covariantly

$$\delta(D\psi) = -v(D\psi) \tag{10.19}$$

in analogy to $\delta(\psi) = -v\psi$. This is not really an extension of the algebra as $D = \mathbf{d}+A$ — in fact that is exactly what motivated A's transformation properties. From $\mathbf{d}$ and A we can construct another covariant tensor

$$F := \mathbf{d}(A) + AA, \tag{10.20}$$

the "field strength". A short (purely algebraic) computation gives

$$\delta(F) = -vF + Fv. \tag{10.21}$$

It is now time to give an interpretation to objects like $\mathbf{d}(t)$, where $\mathbf{d}$ is the exterior derivative on the *base* space so that we have to give $\mathcal{B}$-dependence to t in some way: *i*) It is always possible to construct a new explicitly $\mathcal{B}$-dependent $t_{W} \in M_n(\mathcal{B}\otimes\mathcal{A})$

$$t_W := W^{-1}tW, \tag{10.22}$$

where $W \in M_n(\mathcal{B})$ is a pointwise invertible Matrix of functions on the base space. Here we were careful not to destroy t's Hopf algebra properties that are reminiscent of a representation, *i.e* $\Delta t_W = t_W \otimes t_W$, $St_W = t_W^{-1}$ and $\epsilon t_W = I$. This type of $\mathcal{B}$-dependence is essentially classical because it could be obtained from the adjoint

action on t of an element $\gamma \in \mathcal{B} \otimes \mathcal{U}$ that is $\mathcal{B}$-dependent and group-like $\Delta\gamma = \gamma \otimes \gamma$: $\gamma \overset{\mathrm{ad}}{\triangleright} t =< \gamma^{-1}, t > t < \gamma, t >$; see also [68]. More important is:
ii) Implicit $\mathcal{B}$-dependence. Say, we have a $\mathcal{B}$-dependent gauge transformation g, *i.e.* $t(g) \in M_n(\mathcal{B})$, we then define $(\mathbf{d}t)$ *on* it by

$$(\mathbf{d}t)\,(g) := \mathbf{d}\,(t(g))\,; \tag{10.23}$$

that can be classically expressed as:

$$(\mathbf{d}t)\,(g(x)) := \mathbf{d}(x^\mu)\frac{\partial}{\partial x^\mu}\,(t\,(g(x)))\,. \qquad \textit{(classical)} \tag{10.24}$$

(It would be interesting to see whether one could actually rewrite $(\mathbf{d}t)$ as a matrix

$$\mathbf{d}t \approx \mathbf{d}(x^\mu)\phi^a_\mu(\chi_a \triangleright t) \qquad \in M_n(\Lambda^1(\mathcal{B}) \otimes \mathcal{B} \otimes \mathcal{A})$$

for every given choice of gauge, parameterized by $\phi^a_\mu \in \mathcal{B}$.)
Remark: In our formulation we are actually more interested in actions than contractions, but remembering $\Delta \circ \mathbf{d} = (\mathbf{d} \otimes 1 + 1 \otimes \mathbf{d}) \circ \Delta$, this is easily accomplished:

$$x \triangleright \mathbf{d}(a) = \mathbf{d}(a_{(1)}) < x, a_{(2)} > + a_{(1)}\mathbf{d}(< x, a_{(2)} >). \tag{10.25}$$

If we contract with an element x of $\mathcal{U}$ a *product* of say two functions in $\mathcal{A}$, we look at the coproduct of $\mathcal{U}$ to determine how to split up x into parts, each contracting its respective function: $< x, ab >=< x_{(1)}, a >< x_{(2)}, b >$. As soon as x becomes also a function on the base, say $x = \sum \beta^a \chi_a \in \mathcal{B} \otimes \mathcal{U}$, as is the case for local gauge transformations — and we are trying to contract things like $t^{-1}\mathbf{d}(t)$, we have a problem: we need to give rules for where to put the $\mathcal{B}$-dependence in coproducts like

$$< x, t^{-1}\mathbf{d}(t) > \stackrel{?}{=} < x_{(1)}, t^{-1} > \mathbf{d}(< x_{(2)}, t >) \tag{10.26}$$

because otherwise it might sneak past the $\mathbf{d}$ and escape to the left ... . There is an infinity of possible rules for Δx; $\beta^a \chi_{a(1)} \otimes \chi_{a(2)}$, $\chi_{a(1)} \otimes \beta^a \chi_{a(2)}$, $\beta^a \chi_a \otimes 1 + \chi_{a(1)} \otimes \beta^a(\chi_{a(2)} - 1\epsilon(\chi_{a(2)}))$, ... are examples. Luckily $\beta^a \in \mathcal{B}$ and *not* $\in k$, so that it need not commute with $\otimes$ and one has at least the opportunity to give rules. No matter what we choose, we must not violate the Leibniz rule, in particular we must be in consistency with $\mathbf{d}(1) = \mathbf{d}(11) = 2\mathbf{d}(1)$, which implies that only x with zero counit can have $\mathcal{B}$-dependence. In the classical case that singles out one natural choice:

$$\text{“}\Delta\text{”}\beta^a \chi_a = \beta^a \chi_a \otimes 1 + 1 \otimes \beta^a \chi_a.$$

This riddle is solved by extending the Cartan calculus to include Lie derivatives along elements of $\mathcal{B} \otimes \mathcal{T}_q$ via

$$\pounds_{\beta^a \chi_a} = \beta^a \pounds_{\chi_a} + \mathbf{d}(\beta^a)\mathbf{i}_{\chi_a}. \tag{10.27}$$

(Note the appearance of the exterior derivative $\mathbf{d}$ of the *base* and the corresponding inner derivation $\mathbf{i}$ in this equation.) Here is an example, showing how $t^{-1}\mathbf{d}(t)$ transforms under a gauge transformation along $\beta^a\chi_a$:

$$\begin{aligned}
\pounds_{\beta^a\chi_a}(t^{-1}\mathrm{d}t) &= \beta^a\pounds_{\chi_a}(t^{-1}\mathrm{d}t) + \mathrm{d}(\beta^a)\mathrm{i}_{\chi_a}(t^{-1}\mathrm{d}t) \\
&= \beta^a < \chi_{a(1)}, t^{-1} > t^{-1}(\mathrm{d}(t) < \chi_{a(2)}, t > + t\,\underbrace{\mathrm{d}(< \chi_{a(2)}, t >)}_{=0}) \\
&\quad + \mathrm{d}(\beta^a)\mathrm{i}_{\chi_a}(t^{-1}\mathrm{d}t) \\
&= \beta^a(< \chi_a, t^{-1} > t^{-1}\mathrm{d}(t) + < O_a{}^b, t^{-1} > t^{-1}\mathrm{d}(t) < \chi_b, t >) \\
&\quad + \mathrm{d}(\beta^a) < -\chi_a, t^{-1} > \\
&= \beta^a\left[\lambda_a, t^{-1}\mathrm{d}(t)\right]_q - \mathrm{d}(\beta^a)\lambda_a.
\end{aligned} \tag{10.28}$$

(Compare to (10.18).) This calculation implicitly used further relations of the extended Cartan calculus:

$$\pounds_\chi \mathrm{d} = \mathrm{d}\pounds_\chi \tag{10.29}$$

$$i_\chi \mathrm{d} = -\mathrm{d}i_\chi. \tag{10.30}$$

Before we leave the subject let us make a short remark about ordering problems. If our base space has more than 1+1 dimension we cannot define a physical (local) ordering on it; only a lexicographic ordering is possible. Does this lead to contradictions if we are dealing with non-commutative functions? Not necessarily, as long as we are ordering within the column vector of fields and otherwise use global commutation relations and in particular just one global copy of t. Consider for instance the quantum structure group $SU_q(2)$ and two column vectors ψ and ψ' at different points on the base space. They will satisfy the following four mixed commutation relations*

$$\psi_1\psi_2 = q\psi_2\psi_1, \quad \psi_1\psi'_2 = q\psi'_2\psi_1, \quad \psi'_1\psi_2 = q\psi_2\psi'_1, \quad \psi'_1\psi'_2 = q\psi'_2\psi'_1,$$

and they will both transform according to the same copy of t:

$$\psi \mapsto S^{-1}t\psi, \qquad \psi' \mapsto S^{-1}\psi'.$$

(An interesting idea would be to try and give $\mathcal{B}$-dependence to the braiding operator $O_a{}^b$, but that will affect the multiplication in $\mathcal{A}$ in a way that may lead to inconsistencies.) Are we dealing with a non-local theory because of the global commutation

*In a more conservative approach along the lines of the previous chapter the ψ_0 would be merely the (commuting) coefficients of a section basis — the ordering problem would then presumably show up somewhere else.

relations? The commutation relations of the fields contained in ψ are obviously non-local, however, the real physical observables are gauge invariant objects like $\mathrm{tr}_q(F)$ (see [69] for a discussion of such a set of observables) and those could very well be central in the algebra and in that sense "local". This subject matter is quite controversial, so we want to leave it at that for now — hoping that the new tools provided will be beneficial in future discussions.

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