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On the Mixing Time of Geographical Threshold Graphs

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Abstract In this paper, we study the mixing time of random graphs generated by the geographical threshold graph (GTG) model, a generalization of random geometric graphs (RGG). In a GTG, nodes are distributed in a Euclidean space, and edges are assigned according to a threshold function involving the distance between nodes as well as randomly chosen node weights. The motivation for analyzing this model is that many real networks (e.g., wireless networks, the Internet, etc.) need to be studied by using a "richer" stochastic model (which in this case includes both a distance between nodes and weights on the nodes). We specifically study the mixing times of random walks on 2-dimensional GTGs near the connectivity threshold. We provide a set of criteria on the distribution of vertex weights that guarantees that the mixing time is $\Theta(n \log n)$.

Key words: geographical threshold graph, mixing time, cover time.

1 Introduction

In recent years, we have witnessed the development of numerous approaches to study the structure of large real-world technological and social networks, and to optimize processes on these networks. Large networks, such as the Internet, World Wide Web, phone call graphs, infections disease contacts and financial transactions, have provided new challenges for modeling and analysis [Bon05]. As an example, Web graphs may have billions of nodes and edges, which implies that processing and extracting information on these large sets of data, is 'hard' [APR02]. Extensive theoretical and experimental research has been done in web-graph modeling, attempting to capture both the structure and dynamics of the web graph [KRR$^+$00,BA99,ACL00,BRST01,CF01].

In general, a particularly fertile approach has been to consider the network as an instance of an ensemble, arising from a suitable random generative model. Since the seminal papers on the evolution of uniform random graph model [ER59,ER60], many other models have been proposed to better capture the structure seen in real-world networks, which are systematically covered in [Dur06]. One straightforward example is the random geometric graphs (RGG) model, where nodes are placed uniformly at random in a Euclidean space and edges are placed between any two nodes within a threshold distance. For further study of RGGs, see the monograph by Penrose [Pen03]. The RGGs have the advantage of describing many aspects of systems such as sensor networks, while avoiding unnecessary detail. However, they fail to capture heterogeneity in the network.

Geographical threshold graphs (GTGs) are a generalization of RGGs. Heterogeneity in the network is provided via a richer stochastic model that nevertheless preserves much of the simplicity of the RGG model. GTGs assign to nodes both a location and a weight, which may represent a quantity such as transmission power in a wireless network or influence in a social network. Edges are placed between two nodes if a symmetric function of their weights and the distance between them exceeds a certain threshold [BK07].

Structural properties of GTGs, such as connectivity, clustering coefficient, degree distribution, diameter, existence and absence of the giant component, chromatic number have been recently analyzed [BHP09,BHP07,BMP09]. These properties are not merely of theoretical importance, but also play an important role in applications. In communication networks, connectivity implies the ability to reach all parts of the network. In packet routing, diameter gives the minimal number of hops needed for transmission between two arbitrary nodes. In the case of epidemics, the existence or absence of the giant component controls whether the epidemic spreads or is contained. When treating the vertex colors as the different radio channels or frequencies, the chromatic number gives the minimal number of channels needed so that neighboring radios do not interfere with each other.

Random walks (or more formally, Markov chains) on large networks have many applications. For example, random walks model the spread of disease or the dispersion of information [BGPS06]. The *mixing time* of a random walk is the expected number of random steps that are required to guarantee that the current distribution is close to the stationary distribution. Mixing times are an essential tool in both theory and practice: for example, see the recent survey of Diaconis [Dia09] on Markov chain Monte Carlo methods.

The mixing time for RGG at the connectivity threshold has been determined. For the 2-dimensional RGG, Avin and Ercal [AE07] showed that this mixing time is $\Theta(n \log n)$. More recently, Cooper and Frieze [CF09] proved the analogous result for $d \geq 3$ and actually determine the asymptotically correct constant. In this paper, we study the mixing times of random walks on 2-dimensional GTGs near the connectivity threshold. We provide a set of criteria on the distribution of vertex weights that guarantees that the mixing time is $\Theta(n \log n)$.

2 Model

The GTG model is constructed from of a set of n nodes placed independently in $\mathbf{R}^d$ according to a Poisson point-wise process. A non-negative weight w_i, taken randomly and independently from a probability distribution function $f(w) : \mathbf{R}_0^+ \to \mathbf{R}_0^+$, is assigned to each node v_i for $i \in \{1, 2, \ldots, n\}$. Let $F(x) = \int_0^x f(w)dw$ be the cumulative density function. For two nodes i and j at distance r, the edge (i, j) exists if and only if the following connectivity relation is satisfied:

$$G(w_i, w_j)h(r) \geq \theta_n , \tag{1}$$

where θ_n is a given threshold parameter that depends on the size of the network. The function $h(r)$ is assumed to be decreasing in r. We use $h(r) = r^{-\alpha}$, for some positive

α, which is typical for e.g., the path-loss model in wireless networks [BK07]. The interaction strength between nodes $G(w_i, w_j)$ is usually taken to be symmetric and either multiplicatively or additively separable, i.e., in the form of $G(w_i, w_j) = g(w_i)g(w_j)$ or $G(w_i, w_j) = g(w_i) + g(w_j)$.

Some basic results have already been shown. For the case of uniformly distributed nodes over a unit space it has been shown [MMK05,BK07] that the expected degree of a node with weight w is

$$\mathrm{E}[k(w)] = \frac{n\pi^{d/2}}{\Gamma(d/2+1)} \int_{w'} f(w')\big(h^{-1}(\theta_n/G(w,w'))\big)^d dw', \tag{2}$$

where h^{-1} is the inverse of h. The degree distribution has been studied for specific weight distribution functions $f(w)$ [MMK05]. In both the multiplicative and additive case of $G(w, w')$, questions of diameter, connectivity, and topology control have been addressed [BK07].

Here we restrict ourselves to the case of $g(w) = w$, $\alpha = 2$, and nodes distributed uniformly over a two-dimensional space. For analytical simplicity we take the space to be a unit torus. We concentrate on the analysis of the additive model, i.e., when the connectivity relation for two nodes i and j is given by

$$\frac{w_i + w_j}{r_{ij}^2} \geq \theta_n. \tag{3}$$

3 Bounds on the maximal weight and on the degrees of the nodes in GTG

In Subsection 3.1 we firstly state the upper and lower bounds on the maximal weight in GTG. Then, in Subsection 3.2, we proceed by deriving the upper and lower bounds on the degrees of the nodes in GTG.

3.1 Bounds on the maximal weight

In this subsection we bound the maximal weight of the nodes in the graph. The maximal weight satisfies $\Pr[\max W \leq x] = F(x)^n$, since the weights are distributed independently, where $F(x) = \Pr[W \leq x]$ denotes the cumulative density function of the weight distribution.

In the special case of the exponential weight distribution $f(x) = e^{-x}$ we have: $F(x) = 1 - e^{-x}$ and $\Pr[\max W \leq x] = (1 - e^{-x})^n$. Let us choose $x = c\log n$, then it follows

$$\begin{aligned}\Pr[\max W \geq c\log n] &= 1 - (1 - e^{-c\log n})^n \\ &= 1 - \Big(1 - \frac{1}{n^c}\Big)^n \\ &\to 1 - e^{-n^{1-c}}.\end{aligned}$$

The last expression tends to: $1 - 1/e, 0, 1$, for $c = 1, c > 1, c < 1$, respectively. Thus, for any $\epsilon > 0$, we have $\Pr[\max W \geq (1+\epsilon)\log n] = o(1)$ and $\Pr[\max W \leq (1-\epsilon)\log n] = o(1)$. That is, for any $\epsilon > 0$, $\Pr[\max W \in (1 \pm \epsilon)\log n] \to 1$.

Now we consider a general density function $f(w)$. The goal is to bound $\max W$, that is, to find threshold weights w_t' and w_t'', such that $\Pr[\max W \leq w_t'] = o(1)$ and $\Pr[\max W \geq w_t''] = o(1)$. Let us define the function $\rho(x) := -\log(1 - F(x))$ (that is, $F(x) = 1 - e^{-\rho(x)}$). If $F(x)$ is continious and increasing (in the case of continuous wights without mass points) , then it follows that $\rho(x)$ is continious and increasing, and furthermore, $\rho(0) = 0$ and $\rho(\infty) = \infty$. The following is satisfied

$$\begin{aligned}\Pr[\max W \leq x] &= (1 - e^{-\rho(x)})^n \\ &\to \exp\Big(-n/e^{\rho(x)}\Big) \\ &= \exp\Big(-\exp(\log n - \rho(x))\Big),\end{aligned}$$

for $x \to \infty$, since $e^{\rho(x)} \to \infty$. The last expression takes values: $0, 1$ for $\rho(x) = \log n - \omega(1)$, $\rho(x) = \log n + \omega(1)$, respectively. Now, let us "invert" $\rho(x) = \log n \pm \omega(1)$, in order to obtain thresholds w_t' and w_t''. From the definition $\rho(x) = -\log(1 - F(x))$ it follows $\rho^{-1}(y) = F^{-1}(1 - e^{-y})$. That is, the thresholds are given by:

$$w_t' = F^{-1}(1 - e^{-(\log n - \omega(1))}) = F^{-1}\Big(1 - \frac{\omega(1)}{n}\Big), \tag{4}$$

$$w_t'' = F^{-1}(1 - e^{-(\log n + \omega(1))}) = F^{-1}\Big(1 - \frac{1}{n\omega(1)}\Big). \tag{5}$$

Finally, we have derived w_t' and w_t'', and it follows

$$\lim_{n \to +\infty} \Pr[w_t' \leq \max W \leq w_t''] = 1.$$

Specifically for the exponential weight distribution, as a double check, $F(x) = 1 - e^{-x}$, $F^{-1}(x) = -\log(1-x)$, $\rho(x) = x$, and $\rho^{-1}(x) = x$. This gives, $w_t' = -\log(1 - (1 - \omega(1)/n)) = -\log(\omega(1)/n) = \log n - \omega(1)$, and $w_t'' = \log n + \omega(1)$.

3.2 Bounds on the degrees of the nodes in GTG

Let us assume that weight distribution satisfies $\Pr[W \geq x] = O(x^{-\gamma})$, for some $\gamma > 1$. Then it follows that whp all nodes in the graph have weights bounded by $O(n/\log n)$. That is, $\Pr[W \geq \Theta(n/\log n)] = O((\log n/n)^\gamma)$, and by union bound we obtain

$$\Pr[\exists v \in V : W_v \geq n/\log n] = O\Big(\frac{\log^\gamma n}{n^{\gamma-1}}\Big) = o(1).$$

This means that the formula for degree distribution $d(v|w) \sim \mathrm{Bin}(n-1, p(w))$, where probability $p(w) = \frac{\pi}{\theta_n}(w+\mu)$ is the function of weight w, is valid for $w = O(n/\log n)$.

We consider the GTG 'around the connectivity regime', where $\theta_n = cn/\log n$, that is $p(w) = \frac{\pi\mu}{c}\frac{\log n}{n}(1 + \frac{w}{\mu})$. Then, for a given weight w, the expected degree of a node v is $\mathrm{E}[d(v|w)] = (n-1)p(w)$. By applying Chernoff bound the bound on the degree distribution follows.

Claim. If $c_0 > 2c/\left(1-\sqrt{\frac{c}{\mu\pi}}\right)$, then all nodes in GTG have degrees $\geq \frac{2\pi\mu}{c_0}\log n$.

Proof. The degree distribution of a node v, for a given its weight w, is $\text{Bin}(n-1, p(w))$. Applying Chernoff bound it follows $\Pr\left[d(v|w) \leq \frac{2\pi\mu}{c_0}\log n\right] \leq \exp(-\text{E}[X]\delta^2/2)$, where $\delta = 1 - \frac{2c/c_0}{1+\frac{w}{\mu}} > 0$. Furthermore

$$\begin{aligned}\Pr\left[d(v|w) \leq \frac{2\pi\mu}{c_0}\log n\right] &\leq \exp(-\text{E}[X]\delta^2/2)\\ &\leq \exp\left(-\frac{\mu\pi}{c}\frac{\log n}{n}(n-1)(1+\frac{w}{\mu})\left(1-\frac{2c/c_0}{1+\frac{w}{\mu}}\right)^2\right)\\ &= n^{-\frac{\mu\pi}{c}(1-\frac{1}{n})(1+\frac{w}{\mu})\left(1-\frac{2c/c_0}{1+\frac{w}{\mu}}\right)^2}. \end{aligned} \tag{6}$$

We now find conditions such that Eq. (6) is $o(1/n)$ for all $w \geq 0$ and n sufficiently big. For the sake of simplicity let us denote $x = 1 + w/\mu \geq 1$, and consider $\phi(x) = \frac{\mu\pi}{c}x(1-\frac{2c}{c_0 x})^2$. It follows that the minimum of $\phi(x)$ is attained at $x = 2c/c_0$, and $\phi(x)$ is strictly decreasing in $(0, 2c/c_0)$, and furthermore, strictly increasing in $(2c/c_0, +\infty)$. Now, taking $2c/c_0 \leq 1$ and $\phi(1) = \frac{\pi\mu}{c}(1-2c/c_0)^2 > 1$, or equivalently $c_0 > 2c/\left(1-\sqrt{\frac{c}{\mu\pi}}\right)$, it follows $\phi(x) > 1$ for $x \geq 1$. That is, Eq. (6) is $o(1/n)$, for sufficiently large n.

Thus, the degree distribution satisfies

$$\begin{aligned}\Pr\left[d(v) \leq \frac{2\pi\mu}{c_0}\log n\right] &= \int dw f(w)\Pr\left[d(v|w) \leq \frac{2\pi\mu}{c_0}\log n\right]\\ &\leq \int dw f(w) n^{-\frac{\mu\pi}{c}(1-\frac{1}{n})(1+\frac{w}{\mu})\left(1-\frac{2c/c_0}{1+\frac{w}{\mu}}\right)^2}\\ &= o\Big(\int dw f(w) n^{-1}\Big)\\ &= o(1/n).\end{aligned}$$

Now, by the union bound the claim follows.

Here we derive the interval I_{GTG}, such that all degrees belong to I_{GTG} whp, that is,

$$\forall v : d(v) \in I_{GTG}, \quad \text{whp.} \tag{7}$$

In Section 3.1, we have derived the bounds on the maximal weight Eq. (4) and Eq. (5),

$$F^{-1}(1-\frac{\omega(1)}{n}) \leq \max W \leq F^{-1}(1-\frac{1}{n\omega(1)}).$$

Furthermore, from continuity of $F(x)$, that is, from continuity of it inverse $F^{-1}(x)$, it follows that for any $\epsilon > 0$ and any function $\omega(1)$, there is sufficiently large $n = n(\epsilon)$, such that the upper and lower bounds on $\max W$ are arbitrarily close $|w_t'' - w_t'| \leq \epsilon$.

In Claim 3.2, we have derived the lower bound on the degrees in GTG. We now obtain the upper bound on the degrees. The degree, of a node having the maximal weight, satisfies Binomial distribution $\mathrm{Bin}(n-1, (\pi/\theta_n)(\max W + \mu))$, which is concentrated around its mean $(\pi/c)(1-1/n)(\max W + \mu)\log n$. Thus, whp all degrees of GTG belong to the following interval

$$I_{GTG} = \Big[a \log n, \log n \frac{\pi}{c} F^{-1}\Big(1 - \frac{1}{n\omega(1)}\Big)(1+o(1))\Big]. \quad (8)$$

Concretely, we present the following two examples.

Example 1. I_{GTG} for: (1) exponential weight pdf and (2) power-law ccdf.

1. For $f(x) = e^{-x}$ it follows

$$I_{GTG} = [a \log n, \frac{\pi}{c}(1+o(1))\log^2 n], \quad (9)$$

2. For $F(x) = 1 - x^{-\gamma}$ where $\gamma > 1$, it follows

$$I_{GTG} = [a \log n, \frac{\pi}{c}\log n \cdot n^{1/\gamma}\omega(1)^{1/\gamma}(1+o(1))] = [a \log n, \frac{\pi}{c} n^{1/\gamma + o(1)}]. \quad (10)$$

4 On the number of the nodes of the certain degree in GTG

Analogously as in [CF09], let us denote $D(k)$ the number of vertices v with $d(v) = k$ in GTG, and let $\mathrm{E}[D(k)]$ be its expected value. Despite the fact that in [CF09] all degrees belong to $I_c = [a_1 \log n, a_2 \log n]$ (see page 4, [CF09]), here, in the case of GTG, we have that degrees satisfy Eq. (8). That is, the length of I_{GTG} is not $\Theta(\log n)$, and thus we need some division different then intervals K_0, K_1, K_2 as in [CF09] (see page 5, [CF09]).

Let $l = \Theta(1)$ be a constant (l to be chosen later), such that $l = \frac{\log(bn/a\log n)}{\log r}$, which specifies $r = \sqrt[l]{\frac{bn}{a \log n}}$. Then we divide I_{GTG} into 'bins'

$$B_j = [ar^j \log n, ar^{j+1}\log n),$$

for $j = 0, 1, \ldots, l-1$. That is, the size $|B_j| = a\log n(r-1)r^j$. Now we define the intervals

$$K_j = \{k \in B_j : l_j \leq \mathrm{E}[D(k)] \leq u_j\}.$$

Applying Union bound and Markov inequality, as in [CF], we obtain

$$\begin{aligned}\Pr[\exists k \in K_j : D(k) \geq t_j] &\leq \sum_{k \in K_j} \Pr[D(k) \geq t_j] \\ &\leq \sum_{k \in K_j} \frac{\mathrm{E}[D(k)]}{t_j} \\ &= O\Big(|K_j|\frac{u_j}{t_j}\Big).\end{aligned}$$

Choosing bounds l_j, u_j on $\mathrm{E}[D(k)]$ in K_j and a bound t_j on $D(k)$ to be: $u_j = a \log nr^{j+1} r$ and $l_j = a \log nr^{j+1}/r$ and $t_j = (\log nr^{j+2})^2$, it gives

$$\Pr[\exists k \in K_j : D(k) \geq t_j] \leq O\Big(\frac{1}{r}\Big).$$

4.1 Approximations on $D(k)$ and $\mathrm{E}[D(k)]$ by using the saddle point method

From the previous discussion we have $\Pr[d(v|w) = k] = \binom{n-1}{k} p(w)^k (1-p(w))^{n-1-k}$. Thus, the following is satisfied for $D(k|w)$, $D(k)$ and $E[D(K)]$

$$D(k|w) = \sum_{v \in V} 1_{\{d(v|w)=k\}} \tag{11}$$

$$D(k) = \sum_{v \in V} \int_w dw f(w) 1_{\{d(v|w)=k\}} \tag{12}$$

$$\begin{aligned} \mathrm{E}[D(k|w)] &= \sum_{v \in V} \mathrm{E}[1_{\{d(v|w)=k\}}] \\ &= \sum_{v \in V} \Pr[d(v|w) = k] = n \binom{n-1}{k} p(w)^k (1-p(w))^{n-1-k} \end{aligned} \tag{13}$$

$$\mathrm{E}[D(k)] = \mathrm{E}_w[\mathrm{E}(D(k|w)] = n \binom{n-1}{k} \int_w dw f(w) p(w)^k (1-p(w))^{n-1-k} \tag{14}$$

In order to obtain the value of $\mathrm{E}[D(k)]$, we need to evaluate the integral

$$\int_w dw f(w) p(w)^k (1-p(w))^{n-1-k}.$$

To pursue the approximation, we use the saddle point approximation. Let $\phi(x) = \frac{k}{n-1} \ln x + (1 - \frac{k}{n-1}) \ln(1-x)$ and the probability density function $f(x)$ be continuous functions. By using the saddle point method [BO78], we approximate the integral $J(n)$,

$$J(n-1) := \int x^k (1-x)^{n-1-k} g(x) dx = \int f(x) e^{(n-1)\phi(x)} dx. \tag{15}$$

The following is satisfied $\phi'(x) = \frac{k}{n-1}\frac{1}{x} - (1 - \frac{k}{n-1})\frac{1}{1-x}$, and $\phi''(x) = -\frac{k}{n-1}\frac{1}{x^2} - (1 - \frac{k}{n-1})\frac{1}{(1-x)^2}$. The maximum of $\phi(x)$ is attained at $x_0 = k/(n-1)$ and $\phi(x_0) = -h(k/(n-1))$, $\phi''(x_0) = -\frac{(n-1)^2}{k(n-1-k)}$, where

$$h(x) = -x \log x - (1-x) \log(1-x), \tag{16}$$

is the entropy function (log denotes natural logarithm). By [BO78] Eq.(6.4.35) we have that

$$\begin{aligned} J(n-1) \approx \sqrt{\frac{2\pi}{(n-1)|\phi''(x_0)|}} e^{(n-1)\phi(x_0)} \cdot \Big\{ f(x_0) + \frac{1}{n-1} \Big(-\frac{f''(x_0)}{2\phi''(x_0)} \\ + \frac{f(x_0)\phi^{(4)}(x_0)}{8\phi''(x_0)^2} \frac{f'(x_0)\phi'''(x_0)}{2\phi''(x_0)^2} - \frac{5f'(x_0)\phi'''(x_0)^2 f(x_0)}{24\phi''(x_0)^2} \Big) \Big\}. \end{aligned} \tag{17}$$

It follows that

$$J(n-1) \approx f(x_0)e^{\phi(x_0)}\sqrt{\frac{2\pi}{(n-1)|\phi''(x_0)|}} = f(k/n)e^{-(n-1)h(k/(n-1))}\sqrt{2\pi\frac{k(n-1-k)}{(n-1)^3}}. \quad (18)$$

We now approximate Eq. (14). by firstly evaluating the integral

$$J = \int_0^{\frac{cn}{\pi \log n}-\mu} dw f(w)p(w)^k(1-p(w))^{n-1-k}.$$

Let us denote $J(x_0,\epsilon) = \int_{x_0-\epsilon}^{x_0+\epsilon} dw f(w)p(w)^k(1-p(w))^{n-1-k}$. From the continuity of $\phi(x)$ and uniqueness of its maximum, it can easily be shown that

$$|J - J(x_0,\epsilon)| \le e^{-(n-1)\delta},$$

where $\delta = \max_{x\in(x_0-\epsilon,x_0+\epsilon)}(f(x_0)-f(x))$. Now from the previous analysis for $\theta_n = cn/\log n$ it follows

$$\begin{aligned} J &\approx \int_{x_0-\epsilon}^{x_0+\epsilon} dw f(w)\Big(\frac{\pi}{\theta_n}(\mu+w)\Big)^k\Big(1-\frac{\pi}{\theta_n}(\mu+w)\Big)^{n-1-k} \\ &= \frac{\theta_n}{\pi}\int_{x_0-\epsilon}^{x_0+\epsilon} dx f\Big(x\frac{\theta_n}{\pi}-\mu\Big)x^k(1-x)^{n-1-k} \\ &\approx \frac{c}{\pi}\frac{n}{\log n}e^{-(n-1)H(k/(n-1))}\sqrt{2\pi\frac{k(n-1-k)}{(n-1)^3}}f\Big(\frac{\theta_n k}{\pi(n-1)}-\mu\Big) \end{aligned}$$

By $\frac{\theta_n k}{\pi(n-1)} = \frac{c}{\pi}\frac{k}{\log n}(1+\Theta(1/n))$ and using continuity of f, we finally obtain

$$\mathrm{E}[D(k)] \approx n\binom{n-1}{k}\frac{c}{\pi}\frac{n}{\log n}e^{-(n-1)H(k/(n-1))}\sqrt{2\pi\frac{k(n-1-k)}{(n-1)^3}}f\Big(\frac{c}{\pi}\frac{k}{\log n}-\mu\Big). \quad (19)$$

Example 2. (Exponential weight distribution) We discuss the example of the exponential weight distribution, i.e., where the weights are drawn from the exponential distribution $f(w) = e^{-w}$. Let us denote $\lambda = \frac{\pi n}{\theta_n}(\mu+w)$ and $\nu = \frac{\pi n}{\theta_n}$, then $w = \lambda/\nu - \mu$. By using Poisson approximation it follows

$$
\begin{aligned}
&= \int_0^\infty dw f(w) \binom{n}{k} \left(\frac{\pi}{\theta_n}(\mu + w)\right)^k \left(1 - \frac{\pi}{\theta_n}(\mu + w)\right)^{n-k} \\
&\to \int_0^\infty dw e^{-w} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \frac{1}{\nu} \int_{\mu\nu}^\infty d\lambda e^{-(\lambda/\nu - \mu)} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \frac{e^\mu}{\nu \cdot k!} \int_{\mu\nu}^\infty \lambda^k e^{-\lambda(1+1/\nu)} d\lambda \\
&= \frac{e^\mu}{\nu \cdot k!} (1 + 1/\nu)^{-(k+1)} \int_{\mu\nu(1+1/\nu)}^\infty (\lambda(1 + 1/\nu))^k e^{-\lambda(1+1/\nu)} d\lambda(1 + 1/\nu) \\
&= \frac{e^\mu}{\nu \cdot k!} (1 + 1/\nu)^{-(k+1)} \int_{\mu(1+\nu)}^\infty t^k e^{-t} dt \\
&= e^\mu \frac{\nu^k}{(1 + \nu)^{k+1}} \frac{\Gamma(k + 1; \mu(1 + \nu))}{\Gamma(k + 1)}.
\end{aligned}
$$

5 Mixing time bound via Canonical Paths

Our argument is similar to the one found in [CF09]. We use a canonical path argument, as introduced in [MS96]. For every vertex pair x, y we choose a canonical path γ_{ab} between them. We define

$$\rho = \max_{e=\{x,y\}\in E(G)} \frac{1}{\pi(x)P(x,y)} \sum_{\gamma_{ab}\ni e} \pi(a)\pi(b)|\gamma_{ab}|$$

where $|\gamma_{ab}|$ is the length of the canonical path between a and b. As per [MS96] Proposition 12.1, the mixing time from vertex x satisfies

$$\tau_x(\epsilon) \le \rho \left(\log \pi(x)^{-1} + \log \epsilon^{-1}\right).$$

5.1 Canonical paths on the toric grid

For the moment, let's talk about creating canonical paths on the toric $k \times k$ grid. We want to ensure that every edge appears in roughly the same number of paths. (We'll show how to adapt this to GTG later.) The (two phase) path from (a, b) to (c, d) will be

$$(a, b), (a + 1 \bmod k, b), \ldots, (c, b), (c, b + 1 \bmod k), \ldots, (c, d).$$

Note that we always increment the index by $+1$ (even if there is a shorter path). While there are k^4 canonical paths, each edge appears in at most k^3 of them. Indeed if the canonical path from (a, b) to (c, d) traverses the edge $((i, j), (i + 1, j)$ then $b = j$, leaving at most k^3 choices for a, c, d. Similarly, if we traverse the edge $((i, j), (i, j + 1)$ then $i = c$, leaving k^3 choices for a, b, d.

Furthermore, the maximum length of a path is $2k$.

5.2 Choosing the grid size

We assume that there are αn high weight vertices and $(1-\alpha)n$ low weight vertices. Let $r_c = \sqrt{\frac{\log n}{\alpha \pi n}}$ denote the critical radius for the high weight vertices. We divide the unit torus into a grid whose squares have side length $\frac{1}{\sqrt{5}} r_c$. This makes a $\Theta(r_c^{-1}) \times \Theta(r_c^{-1})$ grid.

Let S_i denote a square in this grid. Consider 'high-weighted' (blue) vertices ('high-weighted' vertices are the vertices with weights $\geq F^{-1}(1-\alpha)$). By the Chernoff bound in each S_i there are $B_i = \Theta(\log n)$ blue vertices whp

$$\Pr[B_i \geq (1-\delta)\alpha c_0 \log n] \geq 1 - n^{-\alpha c_0 \delta^2/2}. \tag{20}$$

Now, the probability that every S_i has at least $(1-\delta)\alpha c_0$ blue vertices, as $n \to \infty$, is given by

$$\Pr[\bigcap_i \{B_i \geq (1-\delta)\alpha c_0 \log n\}] \geq (1 - n^{-\alpha c_0 \delta^2/2})^{n/(c_0 \log n)} \tag{21}$$

$$\to \exp\{-\frac{n^{1-\alpha c_0 \delta^2/2}}{c_0 \log n}\}, \tag{22}$$

since the vertices are tossed over the squares independently. Since α, c_0, δ are constants, the last expression, Eq. (22) tends to 1, as far as

$$\alpha c_0 \delta^2 \geq 2. \tag{23}$$

Similarly, there are $\Theta(\log n)$ low weight nodes in each square. We summarize these two observations in the following claim. For each square S, let $L(S)$ denote the low weight nodes and let $H(S)$ denote the high weight nodes.

Claim. Whp, there exist constants c_0, c_1 such that every square S satisfies

$$c_0 \log n \leq |L(S)| \leq c_1 \log n$$
$$c_0 \log n \leq |H(S)| \leq c_1 \log n$$

We turn our attention to the connections between adjacent squares in our grid. High weight nodes in adjacent squares are adjacent in the GTG.

Claim. For any two 'neighboring' squares S_i and S_j there are $\Omega(\log^2 n)$ interconnections between them. That is, the number of edges that connect blue vertices from S_i and blue vertices from S_j is at least

$$\Big((1-\delta)\alpha c_0 \log n\Big)^2. \tag{24}$$

Proof. Let S_i and S_j be two neighboring squares, as stated above. Let us consider any blue vertex $b_i \in S_i$ and any blue vertex $b_j \in S_j$, with the weights w_{b_i} and w_{b_j}, respectively. The distance between b_i and b_j is at most $\sqrt{5}c_0 \log n/n$. Consider the connectivity relation

$$\frac{w_{b_i} + w_{b_j}}{r^2} \geq \frac{F^{-1}(1-\alpha) + F^{-1}(1-\alpha)}{(\sqrt{5}c_0 \log n/n)^2} = \frac{F^{-1}(1-\alpha)}{c_0} \frac{n}{\log n}. \tag{25}$$

Then it follows, that any two blue vertices $b_i \in S_i$, $b_j \in S_j$ are connected with probability one if

$$F^{-1}(1-\alpha)/c_0 \geq c. \tag{26}$$

5.3 Canonical paths for GTG

The general idea for GTG is to use the $k \times k$ toric grid with $k = 5r_c^{-1}$ as a guide for how to connect pairs of vertices. We turn these toric paths into canonical paths for out GTG by chosing a high random weight vertex in each square of the path. The independence of these choices guarantees that whp 'no particular edge is used more than $\Theta(k^3) = \Theta(r_c^{-3}) = \Theta((n/\log n)^{3/2})$ times.

Here is the randomized procedure that guarantees our bound on $\max_{e \in E} |\{\gamma_{ab} | e \in \gamma_{ab}\}|$. For each square S, partition $L(S)$ evenly into $H(S)$ sets. So each set in the partition will contain either $\lfloor c_1/c_0 \rfloor$ or $\lceil c_1/c_0 \rceil$ low weight vertices. Associate each set in the partition to a unique high weight vertex.

For every vertex pair (x, y):

1. Say $x \in S_{a,b}$ and $y \in S_{c,d}$:
2. Use the toric grid to identify the sequence of squares in the canonical path: $S_{a,b}, S_{a+1,b}, \ldots, S_{c,b}, S_{c,b+1}, \ldots S_{c,d}$. For simplicity, call these squares $S_0, S_1, \ldots, S_t$,
3. If x is a high weight vertex, set $x_0 = x$. Otherwise set x_0 to be the high weight vertex associated to x.
4. For $1 \leq i \leq t-1$, choose x_{i+1} to be a random high weight node in S_i.
5. If y is a high weight vertex, set $x_t = y$. Otherwise set x_t to be the high weight vertex associated to y.
6. Connect x to x_0 (if $x \neq x_0$).
7. For $0 \leq i < t$, connect x_i to x_{i+1}.
8. Connect x_t to y (if $y \neq x_t$).

Let the random variable Z_{xy} denote the number of times the edge xy is chosen. First, suppose that x is low weight and y is high weight. Unless x and y are in the same square, $Z_{xy} = 0$. When x and y are in the same square, then Z_{xy} is chosen if and only if the canonical path has x as one of its endpoints. There are $n-1 = o(r_c^{-3})$ such paths.

Now suppose that both x and y are high weight vertices. There are a number of cases.

Case 1: x and y are in the same square. There are at most $(1 + \lceil c1/c0 \rceil)^2 = O(1)$ canonical paths that use the edge xy. The endpoints of these paths must be x, y or one of the low weight nodes associated to either x or y.

Case 2: $x \in S_1$ and $y \in S_2$ are in adjacent squares. Here there are a number of subcases for the path types. Let

$$\mathcal{P}(S_1, S_2) = \{\gamma_{ab} | \gamma_{ab} \text{ traverses from } S_1 \text{ to } S_2\}.$$

1. The canonical path starts and ends in S_1 and S_2. There are $(1+\lceil c1/c0\rceil)^2 = O(1)$ such paths (similar to Case 1)
2. The canonical path starts and ends outside of both S_1 and S_2. This means that both x and y were chosen randomly to realize the toric canonical path. Let Z'_{ab} denote the number of such events. Let

$$\mathcal{P}'(S_1, S_2) = \{\gamma_{ab} | \gamma_{ab} \in \mathcal{P}(S_1, S_2) \wedge a \notin S_1 \wedge b \notin S_2\}.$$

Then

$$E(Z'_{ab}) = \sum_{\gamma_{ab}\in\mathcal{P}'(S_1,S_2)} \Pr(xy \in \gamma_{ab}) \leq r_c^{-3}(c_1 \log n)^2 \cdot \frac{1}{(c_0 \log n)^2} = \left(\frac{c_1}{c_0}\right)^2 r_c^{-3}.$$

Indeed, the number of toric paths that pass from S_1 to S_2 is $O(r_c^{-3})$. Each toric path corresponds to at most $(c_1 \log n)^2$ paths in the GTG. Since both x and y are internal vertices of these paths, they were both chosen uniformly and independently with probability at most $1/c_0 \log n$.
Now we must use tight concentration and a union bound to show that all edges are used $O(r_c^{-3})$ times. Using Chernoff for this binomial distribution,

$$\begin{aligned}\Pr[|Z'_{ab} - E(Z'_{ab})| \geq \epsilon E(Z'_{ab})] &\leq 2\exp\left(-\frac{\epsilon^2}{3}E(Z'_{ab})\right)\\ &\leq 2\exp\left(-\frac{\epsilon^2}{3}r_c^{-3}\right)\\ &= 2\exp\left(-\frac{\epsilon^2}{3}\left(\frac{n}{\alpha\pi\log n}\right)^{3/2}\right)\\ &= o(e^{-n})\end{aligned}$$

The union bound now gives

$$\begin{aligned}&\Pr\left[\wedge_{a,b}\left(|Z'_{a,b} - E(Z'[a,b])| \geq \epsilon E(Z'_{a,b})\right)\right]\\ &\leq \sum_{a,b}\Pr\left[|Z'_{a,b} - E(Z'[a,b])| \geq \epsilon E(Z'_{a,b})\right]\\ &\leq n^2 o(e^{-n}) \to 0.\end{aligned}$$

Therefore whp, every edge between high weight vertices in adjacent squares is used by $(1\pm\epsilon)r_c^{-3} = \Theta((n/\log n)^{3/2})$ canonical paths.
3. The path starts in S_1 and does not end in S_2. The total number of such edges is $O(n)$. Indeed, the number of start vertices in S_1 who connect to S_2 via a fixed high weight vertex is at most $1+\lceil c_1/c_0\rceil$. There are $O(n)$ end vertices for paths that must leave S_1.
4. The path ends in S_2 and does not start in S_1. By an argument analogous to the previous case, the total number of such edges is $O(n)$.

Our conclusion is that the 2.2 case above is the dominant one. Therefore the maximum usage of any edge is $\Theta((n/\log n)^{3/2})$.

5.4 The dominant term for canonical paths

Let B denote the set of all low weight nodes in the GTG. Let $\rho(B, B)$ denote the contribution to ρ of paths between low weight nodes. We have

$$\begin{aligned}\rho(B,B) &\sim \frac{1}{n\log n}\left(\sqrt{\frac{\log n}{n}}\alpha n\log n\right)(\alpha n\log n)\left(\frac{1}{\log n}\right)^2\sqrt{\frac{n}{\log n}}\\ &= \frac{\alpha^2 n}{\log n}.\end{aligned}$$

Indeed, we have $1/2|E| = \Theta(nlogn)$,. A strip of width $\sqrt{\log n/n}$ contains $\sqrt{\frac{\log n}{n}}\alpha n \log n$ low weight nodes. These nodes have degree $\Theta(n)$. The next term states that every low weight node is a potential end point, each one contributing $\Theta(\log n)$ for its degree. The $(\log n)^{-2}$ reflects the random choice of edge to traverse from one square to the next. Finally, every canonical path has length $O(\sqrt{n/\log n})$.

We will see below that this is indeed the dominant term when calculating ρ. So we will obtain

$$\tau_x(\epsilon) \approx n + \frac{n}{\log n}\log \epsilon^{-1}.$$

6 Canonical paths argument for the exponential distribution

For now, we limit our argument about high weight nodes to the exponential distribution. We will extend this argument to hold for general distributions.

6.1 Using the Gamma Function to calculate the total number of edges

We consider $f(w) = e^{-w}$. We know that whp all weights less than $\log n$. This means that the whp the degrees are in

$$[\log n, (\log n)^2].$$

Each weight is governed by the exponential distribution $\Pr(w \le x) = 1 - e^{-x}$. So the sum of the weights is governed by the gamma distribution $\Gamma(n, 1)$. We have $\mu_\Gamma = n$ and $\sigma_\Gamma^2 = n$.

We know that $2|E| = \sum_v \deg(v) \approx \sum_v w_v \log n = \log n \sum_v w_v$.

Using Chebycheff, we have $X = \sum_v w_v \sim \Gamma(n, 1)$ and

$$\Pr(|X - n| \ge n) \le \frac{1}{n}.$$

So whp,

$$2|E| = \Theta(n \log n).$$

6.2 Partitioning the weights

We know that whp, all weights are smaller than $2\log n$. We have picked $0 < \alpha < 1$ so that αn nodes have degree $\Theta(\log n)$. In particular, they all have weight no greater than $\log(1/\alpha) > 1$ whp. We are turn our attention to the nodes with weights in $[\log(1/\alpha), 2\log n]$.

We divide this interval into subregions as follows. Let

$$\begin{aligned} w_0 &= 2\log n \\ w_k &= \min\{2\log(a_{k-1}), \log(1/\alpha)\} \end{aligned}$$

The w_k are only defined until we reach $\log(1/\alpha)$. Call this final index M. The iterated log function $\log^* n$ is the number of times we must iteratively apply the log function so that the result is less than 1. This slowly growing function satisfies $\log^* n = o(\log\log n)$. Our partition consists of $M \leq \log^* n$ subintervals of the form $[w_k, w_{k-1}]$. Note that $\exp(w_k) = w_{k-1}^2$ for $1 \leq k \leq M$. Let A_k be the set of nodes with weights in interval $[w_k, w_{k-1}]$. Then

$$E(|A_k|) \leq ne^{-w_k} = \frac{n}{w_{k-1}^2}.$$

By the Chebychev inequality, the actual value is tightly concentrated around its mean. We also know that the degree of each of the nodes in A_k is $O((1+w)\log n) = O(w_{k-1}\log n)$.

6.3 Canonical paths involving high weight nodes

Our previous argument for low weight nodes still holds. These nodes have degree $\Theta(\log n)$ and the total number of edges is $O(n\log n)$. We must extend our argument to handle the high weight nodes. In particular, we show that the "low weight to low weight" nodes are the dominant case.

We focus on the usage of the randomly chosen edge between squares. Recall that

$$\begin{aligned} \rho &= \max_{e=\{x,y\}\in E(G)} \frac{1}{\pi(x)P(x,y)} \sum_{\gamma_{ab}\ni e} \pi(a)\pi(b)|\gamma_{ab}| \\ &= \max_{e=\{x,y\}\in E(G)} \frac{1}{2|E|} \sum_{\gamma_{ab}\ni e} \deg(a)\deg(b)|\gamma_{ab}| \end{aligned}$$

where we make use of the fact that for an undirected graph, the stationary distribution of a node is proportional to its degree.

We make a small digression. We have partitioned the square into a grid structure, where each subsquare has side length $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$. If we look at how many nodes in A_1 are in a given square, we get

$$O\left(\frac{\log n}{n}\frac{n}{w_0}\right) = O(1)$$

which is problematic. Luckily, we do not care about particular squares! For example, let us fix a horizontal edge (x, y) between squares. Let $u, v \in A_0$. If a canonical (u, v)-path is eligible to use this edge, the node u must be in the same row as both x and y. This means we are interested in a strip with area $\sqrt{\frac{\log n}{n}}$. The number of nodes in A_1 in such a strip is

$$O\left(\sqrt{\frac{\log n}{n}}\frac{n}{w_0}\right) = O\left(\sqrt{\frac{n}{\log n}}\right)$$

which is most certainly concentrated! As for the node v, we place no restrictions. So the contribution to ρ from paths between nodes in A_0 is roughly

$$\begin{aligned}\rho(A_0, A_0) &\sim \frac{1}{n\log n}\left(\sqrt{\frac{\log n}{n}}\frac{n}{w_0^2}(\log n)w_0\right)\left(\frac{n}{w_0^2}(\log n)w_0\right)\left(\frac{1}{\log n}\right)^2\sqrt{\frac{n}{\log n}}\\ &= \frac{n}{w_0^2\log n} \sim \frac{n}{(\log n)^3}.\end{aligned}$$

Now let's consider the more general case where $u \in A_r$ and $v \in A_s$.

$$\begin{aligned}\rho(A_r, A_s) &\sim \frac{1}{n\log n}\left(\sqrt{\frac{\log n}{n}}\frac{n}{w_r^2}(\log n)w_r\right)\left(\frac{n}{w_s^2}(\log n)w_s\right)\left(\frac{1}{\log n}\right)^2\sqrt{\frac{n}{\log n}}\\ &= \frac{n}{w_r w_s\log n}.\end{aligned}$$

The final new case to consider is when the first node is low weight and the second node is high weight. Let's use B to denote the set of low weight nodes.

$$\begin{aligned}\rho(B, A_s) &\sim \frac{1}{n\log n}\left(\sqrt{\frac{\log n}{n}}\alpha n\log n\right)\left(\frac{n}{w_s^2}(\log n)w_s\right)\left(\frac{1}{\log n}\right)^2\sqrt{\frac{n}{\log n}}\\ &= \frac{\alpha n}{w_s\log n}.\end{aligned}$$

Now all of these terms are $o(n/\log n)$. Furthermore, there are a total of $O((1 + \log^* n)^2)$ such pairings of various types of nodes. The total contribution is

$$\sim \frac{\alpha n}{\log n}\sum_{i=1}^{M}\frac{1}{w_i} + \frac{n}{\log n}\sum_{j=1}^{M}\sum_{k=1}^{M}\frac{1}{w_j w_k} = O\left(\frac{n}{\log n}\right).$$

In other words, the paths between low weight nodes are the dominant term in the calculation of ρ, and our previous result is still valid.

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