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Worst Configurations (Instantons) for Compressed Sensing over Reals: a Channel Coding Approach

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***Abstract*— We consider Linear Programming (LP) solution of a Compressed Sensing (CS) problem over reals, also known as the Basis Pursuit (BasP) algorithm. The BasP allows interpretation as a channel-coding problem, and it guarantees the error-free reconstruction over reals for properly chosen measurement matrix and sufficiently sparse error vectors. In this manuscript, we examine how the BasP performs on a given measurement matrix and develop a technique to discover sparse vectors for which the BasP fails. The resulting algorithm is a generalization of our previous results on finding the most probable error-patterns, so called instantons, degrading performance of a finite size Low-Density Parity-Check (LDPC) code in the error-floor regime. The BasP fails when its output is different from the actual error-pattern. We design CS-Instanton Search Algorithm (ISA) generating a sparse vector, called CS-instanton, such that the BasP fails on the instanton, while its action on any modification of the CS-instanton decreasing a properly defined norm is successful. We also prove that, given a sufficiently dense random input for the error-vector, the CS-ISA converges to an instanton in a small finite number of steps. Performance of the CS-ISA is tested on example of a randomly generated $512 * 120$ matrix, that outputs the shortest instanton (error vector) pattern of length 11.**

I. Introduction

A. Background

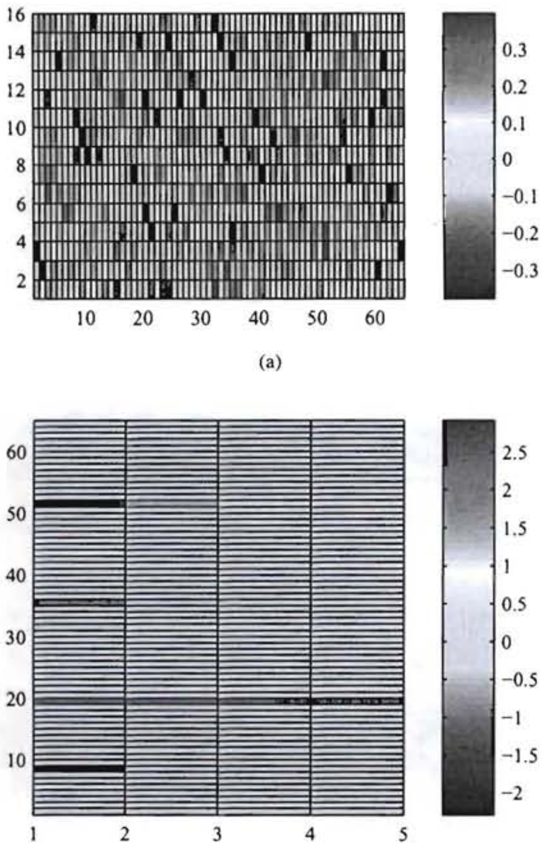
Compressed sensing (CS) [1] is a computational technique to recover a sparse signal from a small set of measurements. Given a set of measurements, described in terms of the so-called measurement matrix, the ideal CS looks for the sparsest signal, i.e. minimizing the ℓ_0-norm, consistent with the measurements. This ideal formulation is known to be NP-hard [2], [3]. A key practical observation of the CS theory came with the Basis Pursuit (BasP) algorithm of Chen, Donoho and Sanders [4] who suggested to relax the difficult ℓ_0-norm minimization (henceforth referred to as CS-OPT) to ℓ_1-norm minimization (henceforth referred to as BasP), which is convex and thus computationally tractable. The BasP heuristics has shown remarkable performance, that was theoretically explained in the breakthrough paper of Candes and Tao [5], stating the CS problem as a linear channel coding problem involving recovery of an input real vector from its corrupted image. In [5] it was proved that the ℓ_1 relaxation guarantees perfect recovery of the input vector for sufficiently sparse error vectors and properly chosen measurement matrix. The conditions on the measurement matrix were expressed in terms of the so-called Restricted Isometry Property (RIP). It was also shown in [5] that a matrix drawn from a proper random ensemble is RIP with high probability.

Derivation of the asymptotic limits on the fraction of tolerable errors for a random measurement matrix became an important consequence of the approach of Candes and Tao. However, the extension of this approach to a given finite measurement matrix is not easy, as checking if a given matrix is RIP becomes computationally hard. This aspect of the CS is reminiscent of similar statements made via the so-called expander graph based analysis [6] of the Low-Density Parity-Check (LDPC) codes [7]. In fact, expander based algorithms for CS have been investigated by a number of researchers [8], [9].

The relation between the theory of LDPC codes and CS was further explored in the work of Dimakis and Vontobel [10]. The basis of their investigation was the fact that on one hand, the BasP can be formulated as a Linear Program (LP) and on the other hand, LDPC codes can be decoded using the so-called LP decoder [11]. The LP decoder is a sub-optimal decoder whose performance is governed by the parity-check matrix used to define the constraints for the linear program. In [10] it was proved that binary matrix which is a good parity check matrix for LP decoding of the corresponding LDPC code is also a good measurement matrix for BasP over binary alphabet in CS. The authors of [10] also emphasized importance of addressing the relation between BasP and LP-LDPC over reals but did not give a hint on how to approach it. This challenge, of extending descriptive results from the 0-1-valued LDPC world into the world of real-valued CS reconstruction, has motivated this work.

B. Overview of Results

In this paper, we pose and suggest an algorithmic answer to the following question: *Given a measurement matrix good for the ℓ_1-norm recovery, how does one find the sparsest vector for which the ℓ_1-norm minimization fails?* Similar question arises in decoding of a finite LDPC code by Belief Propagation [12] or other suboptimal decodings, notably LP decoding, in the so-called error-floor regime [13], [14], [15]. Roughly speaking, error floor is an abrupt degradation in the frame error rate performance of a code in the high signal-to-noise ratio (SNR) regime and is caused by very few low-



(a)

(b)

Fig. 1. Fig. (a) shows example of a (15×64) measurement matrix, H, with orthonormal rows. Fig. (b) shows a sequence of error vectors representing the ISA for the measurement matrix from (a). The first vector (with 8 nonzero elements) was selected at random. It takes four iterations of the ISA (five columns in the Fig. (b)) to reach an instanton, i.e. the error-pattern for which BasP decoding fails, containing only two nonzero elements.

weight error patterns uncorrectable by the decoder. Algorithms to enumerate such rare noise configurations (known as instantons, i.e. instances in the space of noise configurations) are crucial to the design of better decoders as well as codes. Similar statements can be made in the CS setting, where identifying sparsest error vectors for which the BasP fails is important to evaluate a given measurement matrix as well guide design of better measurement matrices.

The main contribution of this paper is the formulation of the CS instanton search algorithm (CS-ISA) . Our approach to the problem of identifying instantons for CS consists in extending and adopting the ISA developed originally in [16], [17], [18] for analysis of the error-floor of a given LDPC code, to the problem of the worst configuration recovery in CS. Given a measurement matrix (e.g. the 64×15 matrix shown in Fig. 1(a)), the algorithm starts with a dense error vector for which the BasP fails and iteratively finds error vectors with smaller ℓ_0 norm that result in the failure of BasP. At some point, the algorithm finds a sparse error vector placed on the error surface i.e., any further reduction in the ℓ_0 norm of the error vector does not result in BasP failure. This progression of the algorithm is illustrated in Fig. 1(b). The method to find the intermediate error vectors and the identification of the instanton are key steps in the algorithm, a detailed exposition of which is presented in Section III.

We illustrate how the CS-ISA serves as an efficient algorithmic tool to test the quality of the measurement matrix, and thus guide construction of better CS codes. Running the ISA for different error-vector initiations samples the space of instantons efficiently. Adopting the l_0 norm for comparison of different instanton configurations, the instanton-spectrum of the measurement matrix represents distribution of instantons over their sparsity. (See the bar-diagram of Fig. 2 for an illustration.)

Finally, we also prove in Section V that for any sufficiently noisy initiation the CS-ISA converges in a finite number of steps, which is significantly smaller than the measurement matrix size. (Each step of the CS-ISA requires running a single instance of the BasP algorithm.)

The paper is organized as follows. Section II introduces the problem setting and related terminology. Section III describes the CS-ISA. Section IV illustrates performance of the CS-ISA in sampling the space of instantons of a randomly generated $(512*120)$ measurement matrix. We present a brief sketch of the theorem proving convergence of the algorithm in a small finite number of steps in Section V. Section VI is reserved for conclusions and discussions of the future work.

II. Compressed Sensing Preliminaries: Problem Setting

In this Section, we discuss the BasP algorithm adopting formulation and terminology from [4], [5]. (The interested reader is referred to [19] for a comprehensive list of references on CS.) The problem setting is as follows. An original real-valued information vector $f \in \mathbb{R}^n$ is transformed (coded) into a longer vector $Af \in \mathbb{R}^m$, where $A \in \mathbb{R}^{m \times n}$ is a full rank matrix known as the generator matrix. The result, transmitted over the CS-channel, is received as $y = Af + e$, where $e \in \mathbb{R}^m$ is the unknown error vector assumed sparse. Recovering the error vector e is sufficient to reconstruct the information vector f, as knowledge of y along with e gives Af and hence f can be recovered straightforwardly as A is full rank. Now, consider a $p \times m$ matrix F such that $FA = 0$. It follows that $\tilde{y} = Fy = Fe$ and the problem of recovering the sparse error vector e is equivalent to the problem of finding the sparsest vector d subject to $Fd = \tilde{y}$. The CS-decoding/reconstruction is successful if $d = e$. The sparsest solution to this problem can be found by solving the following optimization problem (CS-OPT) [5]

$$\min_{d \in R^m} ||d||_{\ell_0}\Big|_{Fd=Fy}, \quad (1)$$

where the ℓ_0 norm of a vector d is its cardinality, i.e. the number of nonzero entries

$$||d||_{\ell_0} = |\{i : d_i \neq 0\}|.$$

As stated in Eq. (1) the problem is NP hard (of exponential complexity in m) [2], [3] and [4] suggested a relaxed (weaker but tractable) version of the CS-decoding, coined Basis Pursuit (BasP):

$$\min_{d \in R^m} ||d||_{\ell_1} \Big|_{Fd=\tilde{y}=Fe}, \quad (2)$$

where ℓ_0-norm is replaced by the ℓ_1-norm, $||d||_{\ell_1} = \sum_{i=1}^{m} |d_i|$. Eq. (2) can also be recast as an LP.

As shown in [5], the BasP is capable of exact reconstruction, i.e. $d = e$, under the conditions that (1) the measurement matrix obeys the RIP property, and (2) e is sufficiently sparse. Formally, [5] states that if the RIP constants δ_S, θ_S and $\theta_{S,2S}$ satisfy $\delta_S + \theta_S + \theta_{S,2S} < 1$, then for any error vector e with $||e||_{\ell_0} \le S$, e is the unique solution of both Eq. (1) and Eq. (2). However, finding the maximum value of S for which the condition $\delta_S + \theta_S + \theta_{S,2S} < 1$ holds is in general a difficult problem and most recent state-of-the-art results provide only an estimate for S/m in the asymptotic limit of large samples [5]. Moreover, these estimates for the RIP-constants are normally very pessimistic and cannot be used to evaluate the quality of reconstruction by practically important small-to-moderate sized matrixes.

On the other hand, brute force search techniques for finding sparse vectors making BasP to fail are prohibitively expensive. Indeed, BasP with a well-tuned measurement matrix performs on typical instances of the error-vector really well, thus using a standard (Monte Carlo) sampling techniques will typically not deliver a failure. Hence, there is a need to develop smart techniques sampling the space of failures (called instantons) of the given measurement matrix.

III. Instanton Search Algorithm for BasP

In this Section, we provide a formal description of the CS-ISA. We say that the BasP fails on a vector e if the solution d to Eq. (2) is not equal to e. We start with the following two definitions.

Definition 1 (Instanton): Let e be a k-sparse vector. (i.e. the number of nonzero entries in e is equal to k.) Consider an error-vector e', derived from e by replacing one of its nonzero component by zero (thus e' is $(k-1)$-sparse). e is an instanton if the BasP fails on e while it succeeds on any e' derived from e.

Definition 2 (Median): Let e be a vector and let t denote the smallest number such that the sum of t largest entries (in absolute value) of e is at least equal to $||e||_{\ell_1}/2$. Let $T = \{i_1, i_2, \dots, i_t\}$ denote the indices of the t largest entries of e. Then, the median of e is the vector $\hat{e}$ with support T and $\hat{e}_i = e_i$ for $i \in T$.

Now we are ready to describe the **Instanton Search Algorithm**:

- Initialization ($l = 0$) step: Initialize the algorithm to a vector $e^{(0)}$ of length m with sufficient number of errors such that BasP fails on $e^{(0)}$, i.e. the BasP produces another vector $\bar{e}^{(0)} \ne e^{(0)}$.
- $l \ge 1$ step: Consider the vector $\tilde{e}^{(l-1)} = e^{(l-1)} - \bar{e}^{(l-1)}$, where $\bar{e}^{(l-1)}$ denotes the output of BasP on $e^{(l-1)}$. Let $\hat{e}^{(l-1)}$ denote the median of $\tilde{e}^{(l-1)}$. Only two cases arise (see Lemma 1):
(i) If $||\hat{e}^{(l-1)}||_{\ell_0} < ||e^{(l-1)}||_{\ell_0}$, then $e^{(l)} = \hat{e}^{(l-1)}$ is the l-th step output/$(l+1)$-th step input.
(ii) If $||\hat{e}^{(l-1)}||_{\ell_0} = ||e^{(l-1)}||_{\ell_0}$, define $L = \{i_1, i_2, \dots, i_{k_l}\}$ as the support of $\hat{e}^{(l-1)}$. Let $L_{i_t} = L \backslash i_t$ for some $i_t \in L$. Let r^{i_t} be a vector such that $r_j^{i_t} = \hat{e}_j^{(l-1)}, j \in L_{i_t}$ and contain zero elements elsewhere. Apply the BasP to all r^{i_t} and denote the i_t output by $\bar{r}^{i_t}$. If $\bar{r}^{i_t} = r^{i_t}$ for all i_t, then $\hat{e}^{(l-1)}$ is the desired instanton and the algorithm halts. Else, if $\bar{r}^{i_t} \ne r^{i_t}$, $e^{(l)}$ is set to r^{i_t}.

The algorithm aims, starting with some random non-sparse initiation of the error vector for which the BasP fails, to get iteratively as close as possible to the zero vector while keeping the failed status for BasP acting on the current vector. In the ISA (which should in fact, for the sake of accurateness, be called ISA for the l_0-norm channel (3)) this aim is achieved by alternating the BasP- and the median-steps. The logic behind the alternation is obvious: when BasP fails we apply the median-step to reduce the size of $\tilde{e}$, defined as the difference between the original e and its image under the BasP $\bar{e}$.

We will prove in Section V that the ISA algorithm defined above outputs an instanton in a small number of steps. However, prior to that we find it useful to illustrate the algorithm performance on an example in the next Section.

IV. Example of the Error-Surface Exploration

In this Section we first consider a 64×15 measurement matrix, used to illustrate the ISA in Fig. 1. This toy example, described in details in the next paragraph, while not useful in practice, allows a reasonable visualization and thus serves us mainly for illustration purposes. Afterwards, and focused on a tour-de-force demonstration, we discuss a more realistic 512×120 example

The orthonormal rows of the 64×15 measurement matrix are drawn from a Gaussian distribution and the measurement matrix is shown in Fig. 1(a). (The measurement matrices are generated in the following manner [20]: we first generate a 64×15 matrix H with independent entries drawn from a zero-mean unit variance Gaussian distribution. We then find the ortonormal basis for the range of H^T which gives a matrix with orthonormal columns and transposing this matrix yields the required measurement matrix.) The dynamics of the ISA is illustrated in Fig. 1(b). The error vector initiation is random and it happens to be with support of size 15 for the illustrative example. For this example locations of the nonzero components of the initiation are, [1 5 8 12 19 31 34 35 37 43 45 51 57 59 62], and their respective values are, [1.1738 0.6554 − 2.2990 − 0.1783 − 1.1907 − 0.4254 − 0.4768 2.0385 − 0.0695 0.6997 0.4137 2.9185 0.6545 1.1149 0.6789]. BasP fails on this error pattern producing the output $\bar{e}^0$. On step 1, we compute the median of $\hat{e}^0$ resulting in a vector with support [1 19 35 51 56 62] and components [1.1738 − 1.1907 1.1590 1.3883 1.0888 0.6789]. Next we set $e^{(1)} = \hat{e}^{(0)}$. On step 2, we compute the median of $\hat{e}^{(1)}$ resulted

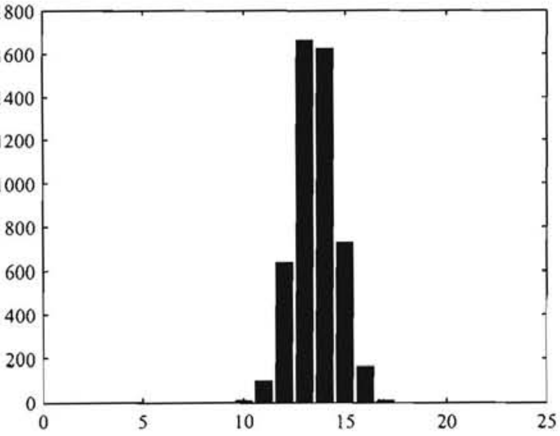


Fig. 2. Bar-diagram showing number of instance per recorded instanton length found in the result of 5000 trial runs of the ISA on a measurement matrix taken as a typical representative of the (512×120) GOE ensemble and with random initiations of cardinality (size of support) 40.

in a vector with support [1 19 51 56] and components [1.1738 −1.1907 1.0843 1.0888]. Hence, we set $e^{(2)} = \hat{e}^{(1)}$. On step 3, we compute $\hat{e}^{(2)}$ yielding the vector with support [1 19 56] and components [1.1738 − 1.1907 1.0888], thus resulted in $e^{(3)} = \hat{e}^{(2)}$. On step 4, the median of $\tilde{e}^{(3)}$ generates the vector with support [1 19 56] and components [1.1738 − 1.1907 1.0138]. Since, $||\hat{e}^{(3)}||_{\ell_0} = ||e^{(3)}||_{\ell_0}$, we consider all the error vectors derived from $\hat{e}^{(3)}$ by replacing one (of the three) nonzero components by zero. BasP applied to these derived vectors decodes correctly, and thus $\hat{e}^{(3)}$ is declared an instanton of the measurement matrix.

In order to illustrate effectiveness of the ISA for sampling the space of instantons, we consider a sample of (512×120) measurement matrix from the respective GOE ensemble. We run the ISA 5000 times generating initiation for the error-vector always with 40 initial errors and record the length of the instanton found for each trial. (40 is a sufficiently large integer chosen to guarantee that majority of random error-vector initiations leads to BasP failure and to an instanton configuration consequently, while only a small portion of initiations of cardinality 40 are decoded correctly by BasP, and these rare and not interesting instances are simply ignored. Note, that lowering the initiation cardinality will lead in wasting a lot of initiations as these would be typically decoded by BasP without an error.) The resulting distribution of the instanton lengths is shown in Fig. 2. The smallest length instanton discovered by the ISA for this example is 11.

The ISA was implemented in MATLAB and the BasP step in the algorithm was solved using the "l1-magic" package from [20]. For the aforementioned measurement matrix of size (512×120) and the initial error-vector of cardinality 40, the average time of the BasP and the ISA runs were 0.2 seconds and 4.2 seconds respectively on a (laptop) Intel core duo 2GHz processor.

Remark: The statistics of instantons for a given measurement matrix F can be translated into a statement about the error-floor, in the regime of weak-noise and the asymptote for the Frame-Error-Rate (FER) of the CS code performing over a formally defined channel can be characterized by the following error-probability,

$$p_q(e) \sim \exp(-q||e||_{\ell_0}), \tag{3}$$

where q plays the role of the Signal-to-Noise-Ratio (SNR) in the standard channel-coding setting [1].

V. Proof of the Correctness of the CS-IS Algorithm

In this section, we establish the correctness of the CS-ISA and prove that the ISA outputs an instanton in a fintie number of steps.

Lemma 1: Let e be a vector and let $\bar{e}$ denote the output of BasP on e. Further, let $\tilde{e} = e - \bar{e}$ and denote the median of $\tilde{e}$ by $\hat{e}$. Then,

$$||\hat{e}||_{\ell_0} \leq ||e||_{\ell_0}$$

Proof: Let S_1 denote the support of e. Since $\bar{e}$ is the BasP output for e, we have

$$||e||_{\ell_1} \geq ||\bar{e}||_{\ell_1}$$

In other words,

$$\sum_{i \in S_1} |e_i| \geq \sum_{j \in J} |\bar{e}_j| \tag{4}$$

$$\Rightarrow \sum_{i \in S_1} (|e_i| - |\bar{e}_i|) \geq \sum_{j \in J \setminus S_1} |\bar{e}_j| \tag{5}$$

Now consider, $\sum_{i \in S_1} |\tilde{e}_i|$. We have

$$\begin{aligned}\sum_{i \in S_1} |\tilde{e}_i| &= \sum_{i \in S_1} |e_i - \bar{e}_i| \geq \sum_{i \in S_1} (|e_i| - |\bar{e}_i|) \geq \sum_{j \in J \setminus S_1} |\bar{e}_j| \\ &= \sum_{j \in J \setminus S_1} |e_j - \bar{e}_j| = \sum_{j \in J \setminus S_1} |\tilde{e}_j| \end{aligned} \tag{6}$$

Eq. (6) suggests that the sum of entries of $\tilde{e}$ in S_1 is at least equal to the sum of entries of $\tilde{e}$ over the remaining entries. This implies that the ℓ_0 norm of the median of $\tilde{e}$ cannot exceed $|S_1| = ||e||_{\ell_0}$. ∎

Corollary 1: The ISA converges to an instanton in finite number of steps.

Proof: At every step l of the ISA, we have $||e^{(l)}||_{\ell_0} < ||e^{(l-1)}||_{\ell_0}$. By construction, the output of the ISA is an instanton. ∎

VI. Summary and Conclusions

We envision further development of the Instanton Search technique along the following lines.

- The sampling of the instanton space provided by the ISA is not uniform. An important future task is to design a uniform-sampling modification of the ISA.

[1] Eq. (3) can be thought of as a real-valued version of the binary symmetric channel, thus relating a non-zero component of the error-vector to a bit flip in the standard binary setting. Note that the choice of the l_0-norm in Eq. (3) is kind of arbitrary, and some other examples can also be discussed.

- If the error-vector is almost sparse, or alternatively if the measurements are noisy the problem of exact CS reconstruction is replaced by an approximate reconstruction. One possible modification of the BasP algorithm, so-called Lasso algorithm [21], consists in adding to the l_1-norm of the signal (the original BasP) a part with the l_2-norm of the noise. With the proper (soft) definition of failure we should also be able to extend the ISA approach to find the worth (highest probability) configurations leading to Lasso's failures.
- The ISA technique should allow extension to the problem of Matrix Completion via the computationally tractable minimization of the matrix nuclear norm (replacing exact but not tractable minimization of the matrix rank) [22], [23]. In this setting a sparse and properly conditioned matrix we aim to reconstruct is fixed and one possible question becomes to explore the set of measurements required for exact success of the nuclear-norm minimization. Proper modification of the ISA should be capable of sampling most dangerous (instanton) configurations of measurement, for example defined as a set of measurements such that their nuclear-norm based decoding leads to a failure, while addition of any single non-zero measurements can be nuclear-norm decoded successfully.
- We can also adopt the ISA algorithm to sequential compressed sensing formulation of Maloutov et al [24]. In fact, the modification is rather straightforward as it only replaces BasP, as a sub-step of the ISA, by its sequential implementation (also imitating the order in which the measurements are received). Similar modification can-be implemented for any other LP-based modification of BasP, such as Subspace Pursuit algorithm discussed in [25]. Note, that this streamlining of BasP is somehow similar to the adaptive realization of the LP decoding of LDPC codes suggested in [26].
- However successful the BasP algorithm, it is still sub-optimal with respect to the exact l_0-norm minimization. It will thus be important to explore algorithmically how big is this gap between the exact and suboptimal decoding, in particular constructing the set of convex optimization improving performance of BasP. This sequence of approximations may be though of along the lines of discussion in [10] illustrated on example of the $0-1$ measurement matrix and error-vector that the LP-LDPC is stronger than respective BasP. The ISA-approach can be easily adopted to such convex improvements over BasP.
- The last but not the least, we envision that the most interesting (but also most challenging) application for this Instanton Search approach will be in designing a set of good measurement matrixes, in the spirit of how we can think of using the instanton search for selecting a good LDPC-parity check matrix (say picked from given random ensemble optimized with respect to its waterfall behavior) with the lowest error-floor.

VII. Acknowledgments

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