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Some Considerations on Stochastic Neutron Populations

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Introduction

The neutron population in a multiplying body containing a weak random source may depart considerably from its average or expected value. The resulting behavior of the system is then unpredictable and a fully stochastic description of the neutron population becomes necessary. Stochastic considerations are especially important when dealing with pulsed reactors or in the case of criticality excursions in the presence of a weak source.

Using the theory of discrete-state continuous-time Markov processes, and subject to some physical approximations, Bell [1] obtained approximate solutions for the neutron number probability distributions (pdf), with and without an intrinsic random neutron source, that were valid at late times and large neutron populations. In recent work [4], we obtained exact solutions for Bell's model problem, and in this paper we use these exact probability distributions to: (1) assess the accuracy of Bell's asymptotic solutions and show how the latter follow from the exact solutions, (2) rigorously examine the probability of obtaining a divergent chain reaction, and (3) demonstrate the existence of an abrupt transition from a stochastic to a deterministic phase with increasing source strength.

Neutron Probability Distributions

We define $P_n(t)$ as the probability of finding n neutrons at time t in a lumped supercritical body in which a fission chain may be initiated by a single neutron or by an intrinsic random source. $P_n(t)$ satisfies a forward Chapmann-Kolmogorov equation [1, 2, 3, 4], which Bell solved for large neutron populations [1]. The exact solutions for the pdfs may also be obtained, as shown in [4]. For a single initiating neutron at $t = t_0$, the exact distribution is given by:

$$P_0(t) = 1 - \frac{a(t)}{1+b(t)}, \tag{1a}$$

$$P_n(t) = \frac{a(t)}{b^2(t)}\left[\frac{b(t)}{1+b(t)}\right]^{n+1}, \quad n \geq 1, \tag{1b}$$

where $P_0(t)$ is the extinction probability and where we have defined:

$$a(t) \equiv \overline{n}(t) = \exp\left[\int_{t_0}^{t} \alpha(t')dt'\right], \tag{2}$$

$$b(t) = \frac{\chi_2'}{2}\int_{t_0}^{t} \exp\left[\int_{t'}^{t} \alpha(t'')dt''\right] dt', \tag{3}$$

$\overline{n}(t)$ being the mean neutron population at time t. When an intrinsic random source is present, and there is no initiating neutron at $t = t_0$, the exact pdf is given by:

$$P_n(t) = \frac{1}{[1+b(t)]^{\eta}}\frac{\Gamma(\eta+n)}{\Gamma(\eta)\,n!}\left[\frac{b(t)}{1+b(t)}\right]^{n}, \; n \geq 0, \tag{4}$$

In the above, $\alpha(t) = [k(t)-1]/\tau > 0$, where τ is the neutron lifetime and $k(t)$ is the multiplication factor, χ_2' is a fission multiplicity parameter [1, 4] and $\Gamma(z)$ is the gamma function. Also, $\eta = 2S/\chi_2'$ where S is the strength of the random source. The pdfs given by Eqs.(1) & (4) are non-negative and properly normalized [4], and are valid for arbitrary neutron number $n \geq 0$ at all times $t \geq 0$. Bell's asymptotic approximations will be recovered below as limiting forms of these exact solutions, but first, we consider the probability of obtaining a divergent fission chain.

Probability of Divergent Chain Reaction

We examine in detail the source-free case and give the final result for a random source. To this end, we begin by expressing the pdf given by Eq.(1b) as a recurrence relation:

$$P_{n+1}(t) = \frac{b(t)}{1+b(t)}P_n(t) = h(t)P_n(t), \quad n \geq 1, \tag{5}$$

which defines the function $h(t)$ and where the initial value $P_1(t)$ is obtained by setting $n = 1$ in Eq.(1b). We are interested in establishing the conditions under which $P_\infty(t)$, the probability that a neutron chain diverges, i.e., grows to infinite size, is nonzero. Considering Eq.(5) as a sequence, $P_\infty(t)$ is the limiting value of this sequence for some fixed t. The existence of this limit implies the existence of a fixed point, i.e.,

$$\lim_{n\to\infty} P_n = P^*, \tag{6}$$

which from Eq.(5) satisfies:

$$(1-h)P^* = 0, \tag{7}$$

where time appears parametrically and the limits are assumed to apply at each time. Thus, if $h \neq 1$, $P^* = 0$ is the only solution, indicating that the probability of a chain diverging is zero. However, a nonzero divergence probability, $P^* \neq 0$, is possible if $h = 1$, and from Eq.(5) we observe that this possibility depends on how $b(t)$ varies with time. It is clear from Eq.(3) that $b(t)$ increases monotonically between 0 and ∞ (for a supercritical system) for $0 \leq t \leq \infty$. It then follows from Eq.(4) that h increases monotonically from 0 to 1 for $0 \leq t \leq \infty$. Thus, a nonzero limit of the sequence in Eq.(5) is only possible at $t = \infty$ (when $h = 1$), and $P_\infty(\infty)$ is this limiting value. To obtain the latter, it is first necessary to consider the time-asymptotic limit of $P_n(t)$ for finite neutron populations. Noting that $\lim_{t\to\infty} P_1(t) = \lim_{t\to\infty} a(t)/[1+b(t)]^2 = 0$, it then follows from Eq.(5) that $P_n(\infty) = 0$ for $1 \leq n < \infty$. This means that as $t \to \infty$, the only neutron states that have nonzero probability are $n = 0$ and $n = \infty$. The normalization condition $\sum_{n=0}^{\infty} P_n(t) = 1$, which holds for all time, and Eq.(1a) then yields the explicit result:

$$P_\infty(\infty) = 1 - P_0(\infty) = \frac{a(\infty)}{1+b(\infty)}. \qquad (8)$$

Summarizing, given one initiating neutron, the neutron population in a source-free supercritical system will eventually (as $t \to \infty$) either become extinct with probability $P_0(\infty)$ or it will diverge with probability $1 - P_0(\infty)$. The finite-state ($0 < n < \infty$) occupation probabilities are all zero and, moreover, finite-time blow-up is not possible in such a system. For constant reactivity, Eq.(8) simplifies to:

$$P_\infty(\infty) = \frac{2(k-1)}{\chi_2' \tau} = \frac{2(k-1)}{k} \frac{\langle\nu\rangle}{\langle\nu^2\rangle - \langle\nu\rangle}, \qquad (9)$$

where $\langle\nu\rangle$ and $\langle\nu^2\rangle$ are the mean and mean-square number of neutrons per fission.

When a random source of constant strength is present, the corresponding pdf, Eq.(4), can be expressed as the following recurrence relation:

$$P_{n+1}(t) = \frac{b(t)}{1+b(t)} \left(\frac{\eta+n}{n+1}\right) P_n(t), \quad n \geq 0. \quad (10)$$

An analysis similar to that used for the source free case then shows that the divergence probability is unity. That is, the neutron population will always diverge when a source is present.

Time Asymptotic Solutions

Under the assumption of large t, so that the mean population is large, and large n, so that the actual population is also large, we have previously shown [4] that the discrete pdfs reduce to Bell's continuous distributions [1]. Here we follow a different approach and obtain an asymptotic result that is slightly more general than Bell's. For n large (and continuous), $P_{n+1}(t)$ can be Taylor expanded to first order, which transforms Eq.(5) to a differential equation that can be simply solved to obtain:

$$P_n(t) = A(t) \exp\left[-\frac{n}{1+b(t)}\right], \quad n \gg 1 \qquad (11)$$

where $A(t)$ is undetermined. Since the extinction probability is nonzero in this case, the complete pdf, now a probability density, can be written as:

$$P_n(t) = P_0(t)\delta(n) + A(t) \exp\left[-\frac{n}{1+b(t)}\right]. \quad (12)$$

Enforcing the normalization $\int_0^\infty P_n(t)dn = 1$ then gives $A(t)$ in terms of $a(t)$ and $b(t)$ and Eq.(12) becomes:

$$P_n(t) = P_0(t)\delta(n) + \frac{a(t)}{[1+b(t)]^2} e^{-\frac{n}{1+b(t)}}. \qquad (13)$$

Further assuming a constant reactivity and $t \gg 1/\tau$, which yields $b(t) \approx \overline{n}(t)/p_\infty$ we recover Bell's solution [1] :

$$P_n(t) = [1 - p_\infty]\delta(n) + \frac{p_\infty^2}{\overline{n}(t)} \exp\left[-\frac{np_\infty}{\overline{n}(t)}\right]. \quad (14)$$

where we have used the abbreviation p_∞ for $P_\infty(\infty)$. Eq.(13) provides a more accurate description than Bell's approximation Eq.(14) of the large n tail of the distribution at earlier times.

Similarly, when a random source is present, the recurrence relation in Eq.(10) can be converted to a differential equation for large n and solved to give:

$$P_n(t) = B(t) \frac{1}{\eta^{b(t)(1-\eta)/[1+b(t)]}} e^{-\frac{n}{1+b(t)}}. \qquad (15)$$

This solution can be normalized to unity to obtain $B(t)$, and, once again for $t \gg 1/\tau$, we recover Bell's solution [1]:

$$P_n(t) = \left[\frac{\eta\, n}{\bar{n}(t)}\right]^{\eta-1} \frac{\eta}{\bar{n}(t)\,\Gamma(\eta)}\, e^{-n\,\eta/\bar{n}(t)}. \qquad (16)$$

For fixed time t, Eq.(16) is a Gamma distribution in the normalized neutron population $n/\bar{n}(t)$, where $\bar{n}(t) = Sb(t)$ is the mean neutron number in this case. Fig.(1) displays the complement of the cdf, defined as the probability of the neutron number exceeding the mean at any time t, for the discrete and continuous distributions, Eq.(4) and Eq.(16). The relative error between these two distributions is shown in Fig. 2. It is observed that the relative error is less than 2.5% for a normalized neutron population less than approximately 0.01, and decreases to less than 0.1% when the normalized neutron population is greater that 0.3. We conclude that Bell's approximation is excellent, even for actual neutron population sizes considerably smaller than the mean.

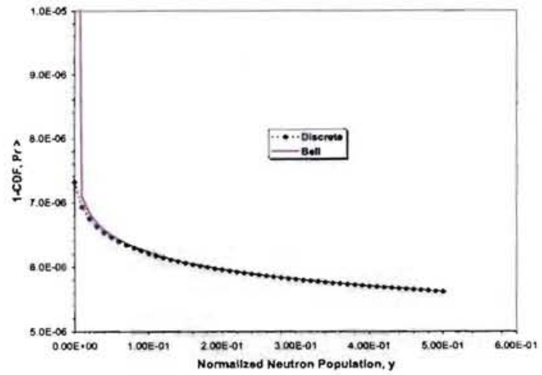

Fig. 1: Exact and Bell CDFs

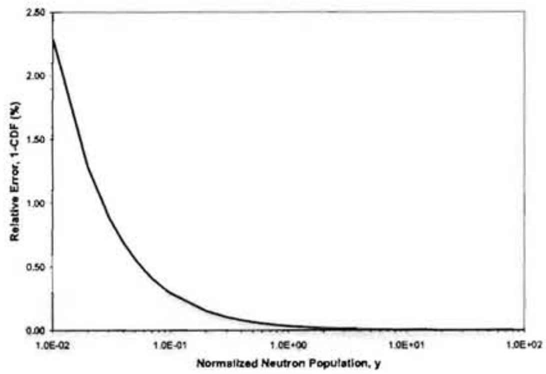

Fig. 2: Relative Accuracy of Bell Solution

Transition from a Stochastic to a Deterministic Phase

We now use the above solutions to investigate the behavior of the pdfs as a function of the source strength S or, equivalently, η. Assuming constant reactivity, for numerical expediency, Fig(3) shows the discrete pdf Eq.(4) plotted against the normalized neutron population $y = n/\bar{n}(t)$, for different source strengths. A drastic transition in the qualitative shape of the distribution is evident at a critical value of S which corresponds to $\eta = 1$. For $\eta < 1$, the pdf monotonically decreases with increasing y but becomes unimodal when $\eta > 1$, with an apparent maximum around $y = 1$. The distribution continues to sharpen around this value with increasing η, with the standard deviation decreasing as $1/\sqrt{S}$. Interestingly, this transition is explicit in Bell's asymptotic solution, Eq.(16), and arises from the change in sign of the exponent of $n^{\eta-1}$ as η increases past unity. Eq.(16) can also be used to show that the maximum occurs at $y = (\eta - 1)/\eta$, which very rapidly approaches unity as η increases, and, moreover that, as $\eta \to \infty$, $P_n(t) \to \delta[n - \bar{n}(t)]$ [4]. Fig(3) clearly shows that for a weak source, the neutron population remains strongly stochastic (the pdf is very broad with a long tail), no matter how large the mean neutron population. The source must be sufficiently strong for a deterministic phase to set in and for the point kinetics model to provide a valid and useful description. While this result is qualitatively well appreciated [5], our work demonstrates that the transition from a stochastic to a deterministic phase is in fact sharp and can be quantified.

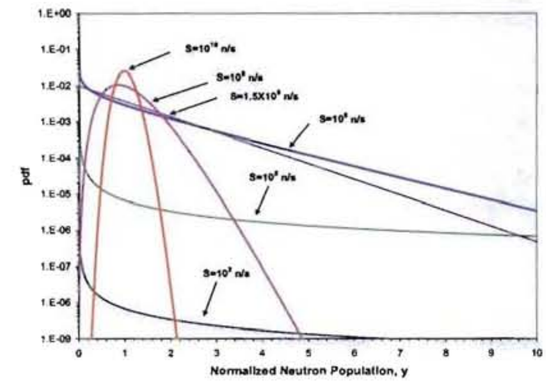

Fig. 3: PDFs for Various Source Strengths

Conclusions

We have used previously obtained [4] exact discrete pdfs of the neutron population in a supercritical body, with and without an intrinsic random source, to (1) numerically demonstrate the high accuracy of Bell's approximate solution [1], (2) establish rigorously the existence and magnitude of the probability of a divergent fission chain, and (3) demonstrate the existence of a "phase transition" in the pdf as the source strength increases past a critical value.

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