

PROPOSED SIAM PROBLEM

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ABSTRACT. Prove several conjectured evaluations of Bessel moment integrals.

1. BACKGROUND

A recent paper by the present authors, together with mathematical physicists David Broadhurst and M. Larry Glasser, explored Bessel moment integrals, namely definite integrals of the general form $\int_0^\infty t^m f^n(t) dt$, where the function $f(t)$ is one of the classical Bessel functions [2]. In that paper, numerous previously unknown analytic evaluations were obtained, using a combination of analytic methods together with some fairly high-powered numerical computations, often performed on highly parallel computers.

In several instances, while we were able to numerically discover what appears to be a solid analytic identity, based on extremely high-precision numerical computations, we were unable to find a rigorous proof. Thus we present here a brief list of some of these unproven but numerically confirmed identities. In the following, the functions $I_0(t)$ and $K_0(t)$ are the classical Bessel functions, as defined in [1, Chap. 15], while the function $\mathbf{K}(x)$ is the *complete elliptic integral* of the first kind, namely

$$\mathbf{K}(x) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - x^2 \sin^2 \phi}}.$$

These formulas also employ constants $K_3 := \mathbf{K}(k_3)$, $K'_3 = \sqrt{3}K_3$, $K_{15} := \mathbf{K}(k_{15})$, $K_{5/3} = \mathbf{K}(k_{5/3})$ and C , where

$$\begin{aligned} k_3 &= \frac{\sqrt{3}-1}{2\sqrt{2}} = \sin(\pi/12) \\ k_{15} &= \frac{(2-\sqrt{3})(\sqrt{5}-\sqrt{3})(3-\sqrt{5})}{8\sqrt{2}} \\ k_{5/3} &= \frac{(2-\sqrt{3})(\sqrt{5}+\sqrt{3})(3+\sqrt{5})}{8\sqrt{2}} \\ C &:= \frac{\pi}{16} \left(1 - \frac{1}{\sqrt{5}}\right) \left(1 + 2 \sum_{n=1}^{\infty} \exp(-n^2 \pi \sqrt{15})\right)^4. \end{aligned}$$

Alternatively

$$C = \frac{\sqrt{5}-1}{4\sqrt{5}\pi} K_{15}^2 = \frac{1}{2\sqrt{15}\pi} K_{15} K_{5/3}.$$

2. CONJECTURED IDENTITIES

Here are our selected conjectures. Can you find proofs for any (or all!) of these?

$$\begin{aligned}
 (2.1) \quad & \frac{1}{\pi^2} \int_0^\infty t I_0(t) K_0^4(t) dt \stackrel{?}{=} C \\
 (2.2) \quad & \frac{1}{\pi^2} \int_0^\infty t^3 I_0(t) K_0^4(t) dt \stackrel{?}{=} \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\
 (2.3) \quad & \frac{1}{\pi^2} \int_0^\infty t^5 I_0(t) K_0^4(t) dt \stackrel{?}{=} \left(\frac{4}{15}\right)^3 \left(43C - \frac{19}{40C}\right) \\
 (2.4) \quad & \frac{2}{\pi\sqrt{15}} \int_0^\infty t I_0^2(t) K_0^3(t) dt \stackrel{?}{=} C \\
 (2.5) \quad & \frac{2}{\pi\sqrt{15}} \int_0^\infty t^3 I_0^2(t) K_0^3(t) dt \stackrel{?}{=} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right) \\
 (2.6) \quad & \frac{2}{\pi\sqrt{15}} \int_0^\infty t^5 I_0^2(t) K_0^3(t) dt \stackrel{?}{=} \left(\frac{4}{15}\right)^3 \left(43C + \frac{19}{40C}\right) \\
 (2.7) \quad & \int_0^\infty t I_0^2(t) K_0^2(t) K_0(2t) dt \stackrel{?}{=} \frac{1}{12} K_3 K_3'.
 \end{aligned}$$

A number of other related experimentally discovered but as yet unproven identities are mentioned in [2]. A discussion of the relative difficulty of each of our list is discussed in [2].

REFERENCES

1. Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1965.
2. David H. Bailey, Jonathan M. Borwein, David Broadhurst and M. L. Glasser, "Elliptic Integral Evaluations of Bessel Moments," *Journal of Physics A: Mathematical and General*, vol. 41 (2008), pg. 205203, available at http://www.iop.org/Select/article/1751-8121/41/20/205203/a8_20_205203.pdf.

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