FINAL REPORT
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Project director/principal investigator:
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1 Executive summary

Two important general problems of interest in plasma physics that may be addressed successfully by Magnetohydrodynamics (MHD) are: (1) Find magnetic field configurations capable of confining a plasma in equilibrium. (2) Study the stability properties of each such an equilibrium. It is often found that the length scale of many instabilities and waves that are able to grow or propagate in a system, are comparable with plasma size, such as in magnetically confined thermonuclear plasmas or in astrophysical accretion disks. Thus MHD is able to provide a good description of such large-scale disturbances. The Magnetorotational instability (MRI) is one particular instance of a potential instability. The project involved theoretical work on fundamental aspects of plasma physics. Researchers at the Princeton Plasma Physics Laboratory (PPPL) began to perform a series of liquid metal Couette flow experiments between rotating cylinders. Their purpose was to produce MRI, which they had predicted theoretically 2002, but was only observed in the laboratory since this project began. The personnel on the project consisted of three persons: (1) The PI, who was partially supported on the budget during each of four summers 2005-2008. (2) Two graduate research assistants, who worked consecutively on the project throughout the years 2005-2009. As a result, the first student, Fritzner Soliman, obtained an M.S. degree in 2006; the second student, Pablo Suarez obtained the Ph.D. degree in 2009. The work was in collaboration with scientists in Princeton, periodic trips were made by the PI as part of the project. There were 4 peer-reviewed publications and one book produced.

2 Original goals

Much has been learned since Chandrasekhar published his first influential work on the theory of magnetohydrodynamic (MHD) stability [1]. However, the flow regimes for which most of this research was carried out assumed that the magnetic Prandtl number P_m was negligible, but that the flow was Rayleigh-unstable. The contention of the MRI theory is that instability may occur in the Rayleigh-stable regime when P_m is not negligible. Inspired by significant work by Goodman & Ji [2] of the Princeton Plasma Physics Laboratory (PPPL), the short-term goal of the proposed research was to provide theoretical support and counterpoint to the latter researchers' proposed experiments. The long-term goal of the principal investigator was to perform theoretical work on fundamental aspects of plasma physics. Thus, being directly related to the PPPL project, the investigation had several parts. First, the relevant MHD physics of liquid metals was reviewed. In particular, questions had been raised concerning the proper boundary conditions to apply to the governing equations. These boundary conditions depend parametrically on the electromagnetic properties of the material walls and the liquid inside. The materials to be used in the containers, the liquids to be tested in the experiments, and dimensions of the actual apparatus to be employed were the sources for the parameters and geometry of the mathematical model. Secondly, once the model was to be established, it was to be solved analytically. In this connection, explanations of how the small$-P_m$ approximation suppresses the MRI ([2]) was be analyzed in detail. One reason for providing such an analysis, was that Goodman & Ji worked in the narrow-gap limit. The proposed research was carried out for the **finite-gap**. To start, the analysis was to be for infinitely-long cylinders, which may not accurately reflect the apparatus. Any attempt to do so is likely to involve certain perturbation/asymptotic approximations. Then, a detailed computation of the model was be carried out with the aid of nonlinear MHD codes provided by the Rensselaer Scientific Computation Research Center (SCOREC). The analytic solutions found previously were to provide input for the computations. Finally, the model calculations were to be compared with the experiments at PPPL. This result was a part of an emerging field of study, the case of *helical* MRI [10]

3 Research plan and its accomplishments

3.1 The proposed objectives were:

- The MHD equations and foundational results
- What are the correct boundary conditions?
- Why does the small$-P_m$ approximation suppresses the MRI?
- The theoretical counterpart to the experimental set-up
- Future Research: Stability of hydromagnetic Couette flow

3.1.1 The MHD equations and foundational results

The project was able to show how the approach of Herron [4] could be adapted to the Rayleigh-stable regime for which MRI would be expected. This was apparent in several of the publications which came from the project [5], [6].

3.1.2 What are the correct boundary conditions?

We examined the boundary conditions posed by Roberts [11] and by Goodman & Ji [2] and we were able to reconcile these conditions [5].

3.1.3 Why does the small$-P_m$ approximation suppresses the MRI?

We were able to add significantly to the understanding of the answer to this question [5] and determine conditions under which MRI would not be suppressed. That is, we were able to provide a gauge by which MRI is to be expected [8].

3.1.4 The theoretical counterpart to the experimental set-up

Owing to the untimely illness of our intended collaborator, Prof. Joseph E. Flaherty of the Department of Computer Science, we were not able to provide from this project a complete solution to the experimental setup. However, through a collaboration with Dr. Sylvain Nintcheu-Fata, of Oak Ridge National Laboratory, the doctoral work of Pablo Suarez developed an ideal case suggested by the experiment was solved [12].

3.1.5 Unexpected findings: Stability of hydromagnetic Couette flow

The following results not predicted from the original proposal were obtained relating to **helical MRI.**

1. The configuration considered by Chandrasekhar was generalized to the case of a *circular* magnetic field for both perfectly conducting and non-conducting walls. This was an M.S. project for Fritzner Soliman [6]. In collaboration with the graduate student, F. Soliman, the PI was able to establish the linear stability of Couette flow in a toroidal magnetic field. This result is described as follows. The stability of the hydromagnetic Couette flow was investigated when a constant current is applied along the axis of the cylinders. It was proved mathematically that if the resulting toroidal magnetic field depends only on this current, no linear instability to axisymmetric disturbances is possible. This was demonstrated using a method of quadratic functionals popularized by Chandrasekhar, originally used by Synge. Along with this, an operator notation was introduced, which aided in keeping track of the functionals involved. Thus the theoretical results we have obtained indicate experimental regimes in which not to expect MRI.

2. In collaboration with colleagues at PPPL the following results were obtained [10]. There was investigated theoretically and numerically a new type of MRI in a magnetized Couette flow in the presence of combined axial and azimuthal magnetic fields. Confirmed were the quantitative results given by Hollerbach and Rüdiger [**?**] for the onset of instability with periodic axial boundary conditions, but also uncovered were some other features of the new modes that cast doubt upon both their experimental realization and their relevance to astrophysical disks. The new mode is only propagating along the negative axial direction and any mode traveling along the positive axial direction decays. In the very resistive limit, the new mode is a weakly destabilized hydrodynamic inertial oscillation. In a non-periodic cylinder of finite length (the experimental apparatus), insulating end-caps tend to reduce the growth rate of the new modes and even prevent the new mode occurring in the very resistive limit. The new type of MRI in magnetized Taylor-Couette flow in the presence of combined axial and azimuthal magnetic fields is "helical" MRI (HMRI).

4 Activities and developments

4.1 Publications and collaborations

See references [5], [6], [10], [7], [12], and [8]. Also see http://mri.pppl.gov/ and http://mri.pppl.gov/. The principal investigator benefited from a sabbatical leave in the spring of 2006. This, with partial support from the U.S. Department of Energy in the summer of 2006, allowed for significant advance in the preparation of the manuscript of what has become a textbook *Partial Differential Equations in Fluid Dynamics,* with coauthor Michael R. Foster [7].

Beginning in the spring of 2003, the principal investigator began visits to the Princeton Plasma Physics Laboratory (PPPL), having been exposed to the significant work of Goodman & Ji [2]. The collaboration which began then led to the current research grant in 2005. As the work progressed, the principal investigator was invited to attend research workshops, the first in May 2007 held at Lawrence Livermore National Laboratory http://www.sc.doe.gov/ascr/WorkshopsConferences/AppMathPIMeet.html. There, the work of Dr. Sylvain Nintcheu-Fata of Oak Ridge National Laboratory came to his attention. This led to a visit by Dr. Nintcheu in the fall of 2007. It also permitted, graduate student, Pablo Suarez to become aware of his work. In the fall of 2008, Suarez paid a visit to Oak Ridge National Laboratory during which time he was able to work more closely with Dr. Nincheu. As a result of these collaborations Dr. Nintcheu served on the Ph.D. examination committee for Suarez in spring and summer of 2009. The PI also attended, October 15-17, 2008 the Applied Mathematics Principal Investigators Meeting at Argonne National Laboratory Argonne, IL

4.1.1 Selected reviews of publications

Reviews of the results of the published papers have begun to appear. For instance, the review of [6] is as follows MR2246187 (2008a:76060)

Herron, Isom(1-RSP); Soliman, Fritzner(1-RSP) The stability of Couette flow in a toroidal magnetic field. (English summary) Appl. Math. Lett. 19 (2006), no. 10, 1113–1117. 76E25 (76W05) "The authors study the linear stability under axisymmetric perturbations of a Couette flow of a viscous incompressible fluid between two rotating impervious partially electrically conducting cylinders subject to a toroidal magnetic field. The governing

model is a two-point eigenvalue problem containing eleven physical dimensional parameters, for a system of four ordinary differential equations with non-constant coefficients in the complex plane, in radial coordinates. Non-slip conditions are imposed. Some other boundary conditions are derived, either from the equations themselves or from the requirement that the perturbing magnetic fields match with the exterior solutions, which are bounded for very small or very large values of the radial variable. Appropriate integration of the uncoupled two-point problem for the radial magnetic field shows that this field vanishes. The remaining problem is reformulated by means of four positive definite operators. Then an orthogonal projection of the equations on the space of solutions shows the nonexistence of eigenvalues, whence the conclusion that no instability occurs and no marginal modes exist either."
Reviewed by Adelina Georgescu

The review of [4] is MR2053587 (2005b:76063)

Herron, Isom H.(1-RSP) Onset of instability in hydromagnetic Couette flow. (English summary) Anal. Appl. (Singap.) 2 (2004), no. 2, 145–159. 76E25 "The author studies the onset of instability for Couette flow of a viscous incompressible fluid in the presence of a constant axial magnetic field. For such flows, the principle of exchange of stabilities (PES) is investigated; this principle regulates how instability arises. In the case of zero magnetic Prandtl number, isolating boundary conditions and axisymmetric perturbations, it is stated that the first eigenvalue with zero real part has imaginary part also equal to zero. The method employed uses an abstract operator formulation, and the proof of PES is reduced to the proof of commutativity between certain operators, and positiveness of operators. Equations are presented in the narrow gap approximation. The abstract formulation is rather general." **Reviewed by Mariarosaria Padula**

The review of [5] is as follows. MR2244809 (2007b:76050)

Herron, Isom(1-RSP); Goodman, Jeremy(1-PRIN-OB) The small magnetic Prandtl number approximation suppresses magnetorotational instability. (English summary) Z. Angew. Math. Phys. 57 (2006), no. 4, 615–622. 76E25 (76U05 76W05) Summary: "The axisymmetric stability of a viscous resistive magnetized Couette flow is re-examined, with emphasis on flows that would be hydrodynamically stable according to Rayleigh's criterion: opposing gradients of angular velocity and specific angular momentum. In this regime, magnetorotational instability (MRI) may occur. The governing system in cylindrical coordinates is of tenth order. It is proved, by methods based on those of Synge and Chandrasekhar, that by dropping one term from the system, MRI is suppressed; in fact no instability at all occurs, with insulating boundary conditions. This term is often neglected because it has the magnetic Prandtl number, which is very small, as a factor; nevertheless it is crucially important."

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† Publications stemming from this research grant.