

Self-Dual Supergravity from $N = 2$ Strings

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ABSTRACT

A new heterotic $N = 2$ string with manifest target space supersymmetry is constructed by combining a conventional $N = 2$ string in the right-moving sector and a Green-Schwarz-Berkovits type string in the left-moving sector. The corresponding sigma model is then obtained by turning on background fields for the massless excitations. We compute the beta functions and we partially check the OPE's of the superconformal algebra perturbatively in α' , all in superspace. The resulting field equations describe $N = 1$ self-dual supergravity.

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1 Introduction

Supergravity and string theories are intimately related to each other. Supergravity theories, on the one hand, describe the worldsheet theories of strings and p -branes theories and, on the other hand, arise as the low energy theories of various superstring theories. Viewed differently, superstring theories serve as the definition of supergravity theories at the quantum level, as the field theories describing supergravity theories themselves often do not make sense at the quantum level. There are supergravity theories, however, whose quantum description in terms of a string (or other) theory are not known. The most notable example is eleven dimensional supergravity. The quantum theory having eleven-dimensional supergravity as its low-energy effective field theory has been coined M -theory. Recently, a description of M -theory has been proposed [1] in terms of the large N limit of a quantum-mechanical matrix model. A second outstanding example in this list of theories is four-dimensional self-dual supergravity (SD SUGRA) [2]. The search for a string theory that yields SD SUGRA as its low energy limit is the subject of this article.

Self-dual supergravity is interesting in its own right. Together with supersymmetric self-dual gauge theories (SD SYM), it is believed to generate, upon dimensional reduction, all integrable models in two dimensions[3, 4]. Furthermore, the role played by self-duality in four dimensions seems to be similar to the role played by conformal symmetry in two dimensions [5].

Self-dual theories in four dimensions are known to be closely related to $N = 2$ strings theories. These, though qualitatively quite different from $N = 0$ and $N = 1$ strings, have proven to constitute a rich family of string theories on their own, for reviews see e.g. [6, 7, 8]. After the introduction of the $N = 2$ string in [9, 10, 11] it took quite a while before it was shown in [12, 13, 14] that the target space theories described by $N = 2$ strings are versions of self-dual gravity, possibly coupled to self-dual Yang-Mills.

Recently, it was demonstrated in [15, 15, 17, 18] that heterotic $(1, 2)$ strings are also capable of having all known string theories as a background, so that it is in principle possible to recover arbitrary string theories by second quantizing $N = 2$ strings. A remarkable fact about the $N = 2$ strings is that their spectrum consist of only a finite number of massless particles, a fact that may make the problem of developing a string field theory tractable after all³. In addition, $(1, 2)$ strings seem to provide the low-energy world-volume description of M -theory. We shall argue in this article that a new $N = 2$ string, closely related to the $(2, 1)$ string, actually yields as a low energy effective theory $N = 1$ self-dual supergravity. This leads to the conclusion that the two outstanding examples of supergravity theories without known quantum definitions mentioned above are actually related to each other.

The conventional description of $(2, 1)$ strings has its drawbacks. Supersymmetry is not manifest and it is therefore quite hard to obtain the fermionic terms in the world-volume action of M theory. One has to resort to more cumbersome S-matrix techniques to get those terms. This led us to search for a more covariant formulation of $N = 2$ strings. The basic idea is to start with an heterotic $N = (1, 2)$ string theory, and to

³An interesting discussion of the string field theory description of the $N = 2$ string can be found in [19].

replace the $N = 1$ sector by a Green-Schwarz-Berkovits (GSB) type sigma model. For four-dimensional compactifications of the heterotic $N = (1, 0)$ string, a similar procedure leads to a full super-Poincaré invariant sigma model, which enables one to derive the low-energy effective action in superspace, keeping supersymmetry manifest all the time [20, 21, 22]. For $N = (1, 1)$ strings, one can simultaneously replace the left and right moving sectors by a Green-Schwarz-Berkovits type sigma model, yielding a covariant formulation of type II strings [20]. Here we extend this sequence to include heterotic $N = (1, 2)$ strings.

The Green-Schwarz-Berkovits (GSB) sigma model contains in addition to the coordinates of 4d superspace an extra set of fields that are the momenta conjugate to the anti-commuting coordinates of 4d superspace. Different versions of $N = 2$ strings that include these momenta in the world-sheet description have previously been considered in [23, 24]. The idea here, however, is different than the one employed in [23, 24]. Instead of replacing the conventional $N = 2$ string with a new string theory, we combine an $N = 2$ string in the right-moving sector with a manifestly supersymmetric left-moving sector. In other words, in our approach the right sector yields the self-duality and the left sector the supersymmetry. In addition, the sigma model includes a time-like chiral boson ρ . At the same time, the $N = 1$ algebra on the world-sheet is enlarged to an $N = 2$ algebra. This $N = 2$ formulation is related to the usual $N = 1$ formulation via a non-local field redefinition. Since each sector contains a differently realized $N = 2$ world-sheet supersymmetry it seems appropriate to call this model "type II heterotic". Because supersymmetry is kept manifest, there is no need to perform a GSO projection. The heterotic string with $N = (1, 2)$ world-sheet supersymmetry has a left-moving world-sheet algebra which must contain in addition to the $N = 1$ super Virasoro algebra a null current. In our case, we have to do the same, because generally speaking the number of bosonic currents in the world-sheet chiral algebra should be greater than or equal to the number of timelike coordinates in target space. For the left-moving sector this is three, two from the target space of signature $(2, 2)$, and the third one is the timelike boson ρ . We will choose a particular null current involving the extra momenta. It is not clear whether with this choice of null current the $N = 2$ string is equivalent to a known $N = 2$ string, and whether other consistent choices of null currents exist in our formulation.

$N = 2$ strings naturally live in a target space of dimension four with signature $(2, 2)$. To write down the world-sheet action, one has to choose a complex structure in target-space, thereby breaking the Lorentz group from $SO(2, 2)$ to $U(1, 1)$. The group $U(1, 1)$ is not able to distinguish between different spins of particles, so that every physical state transforms in the same way as a scalar. Furthermore, $N = 2$ strings have a $U(1)$ symmetry that make spectral flow into a gauge symmetry, making it difficult to determine even the target-space statistics of a physical state. For example, in [25] it is claimed that $N = 2$ strings lead to self-dual $N = 4$ Yang-Mills and self-dual $N = 8$ supergravity, whereas in [12, 13, 14] only bosonic states were found (although the Z_n models of [13] are somewhat reminiscent of theories involving superfields.) Also, in [26] fermions were found in the spectrum of $N = 2$ strings, but they seem to decouple from the theory. To some extent, these problems may be related to the precise definition of the $N = 2$ string one is employing, but for a better understanding of these issues a more covariant formulation of the $N = 2$ string is clearly desirable. In [19] it is suggested that in order to arrive at a covariant $N = 2$ string field theory, one has to attach Lorentz indices to the

string field, but it is not clear if this can be used to arrive at a more covariant world-sheet formulation.

A remarkable property of our formulation is that one can actually keep manifest $SO(2, 2)$ Lorentz invariance in target space. This is possible because the GSB sector has manifest $SO(2, 2)$ invariance in target space. By developing a sigma model that keeps these symmetries manifest we achieve results that are manifestly $SO(2, 2)$ covariant. This means, of course, that in the process we sacrifice manifest worldsheet supersymmetry which we now have to check by hand. We will check the closure of the GSB $N = 2$ superconformal algebra in the presence of curved background at tree and partially at one-loop level and the closure of the conventional $N = 2$ superconformal algebra at tree level. Checking the superconformal algebra in the GSB sector obviously does not break the target space Lorentz group. It is not obvious, though, that this will be true for the calculations in the right sector as well. Indeed, even to write down the superconformal generators one needs to choose a complex structure. So, any equation that follows from this sector will, in general, break the Lorentz invariance. It turns out, however, that this sector supplies no equations at all, and the $SO(2, 2)$ Lorentz group remains unbroken.

The outline of this paper is as follows. In section two we describe the new $N = 2$ string propagating in a flat background, consider its spectrum and the linearized equations of motion for the target space fields, and discuss their relation with self-dual supergravity. In these calculations the Lorentz symmetry is explicitly broken to $U(1, 1)$ simply by writing down the vertex operators. Only the σ -model calculations are explicitly $SO(2, 2)$ covariant. These are done in section three where we describe the new $N = 2$ string in a curved background, and confirm that the 4d sector of the string does indeed yield self-dual supergravity. Finally in section four we discuss possible implications of our results. The appendices describe our conventions and some more technical details and results.

2 A New $N = 2$ String

2.1 Flat Space

The usual critical $N = (2, 2)$ string propagating in a flat four-dimensional background with signature $(2, 2)$ is in superconformal gauge described by the following action (conventions are given in appendix A)

$$S = \frac{1}{\alpha'} \int d^2 z (\eta_{a\bar{a}} \partial \phi^a \bar{\partial} \phi^{\bar{a}} - \frac{1}{2} \eta_{a\bar{a}} \partial \psi^a \psi^{\bar{a}} - \frac{1}{2} \eta_{a\bar{a}} \bar{\partial} \chi^a \chi^{\bar{a}}), \quad (2.1)$$

where ϕ^a is a complex boson, and ψ^a, χ^a are complex fermions. The heterotic $N = (1, 2)$ string can be described by the same action plus an action describing only left-moving degrees of freedom which we denote by S_{int} . The left moving chiral algebra consists of an $N = 1$ super Virasoro algebra and a super null current, with components with conformal weight $1/2$ and 1 which we denote by $\mathcal{J}_{null}^{1/2, 1}$. They are given by

$$\begin{aligned} \mathcal{J}_{null}^{1/2} &= \lambda_a \chi^a + \lambda_{\bar{a}} \bar{\chi}^{\bar{a}} + \mathcal{J}_{int}^{1/2} \\ \mathcal{J}_{null}^1 &= i \lambda_a \partial \phi^a + i \lambda_{\bar{a}} \bar{\partial} \phi^{\bar{a}} + \mathcal{J}_{int}^1 \end{aligned} \quad (2.2)$$

and are required to have vanishing OPE's. There are two possibilities, one can either take $\mathcal{J}_{int}^{1/2,1}$ to be zero and λ^a to be a null vector in $(2,2)$ dimensions, or one can take $\mathcal{J}_{int}^{1/2,1}$ non-zero. Taking $\lambda^a = 0$ does not lead to a unitary theory. The central charge c_{int} of the internal theory S_{int} should be chosen such that it cancels the central charge $c = +6$ of the $4d$ matter fields and the central charges of the ghosts for the left-moving chiral algebra, $c = -26 + 11 - 2 - 1$, yielding $c_{int} = 12$.

Following [27, 28, 29, 20, 21] we now replace the left-moving four-dimensional sector by a Green-Schwarz-Berkovits type world-sheet theory. To make the right-moving $N = 2$ supersymmetry manifest, we write the action in $(0,2)$ superspace [30] with anti-commuting coordinates $\kappa, \bar{\kappa}$. Define the following $(0,2)$ superfields

$$\begin{aligned}\Phi_a &= \phi_a + \kappa\psi_a + \kappa\bar{\kappa}\bar{\partial}\phi_a \\ \Phi_{\bar{a}} &= \phi_{\bar{a}} + \kappa\psi_{\bar{a}} - \kappa\bar{\kappa}\bar{\partial}\phi_{\bar{a}} \\ \Delta_\alpha &= p_\alpha + \bar{\kappa}v_\alpha - \kappa\bar{\kappa}\bar{\partial}p_\alpha \\ \Theta^\alpha &= \theta^\alpha + \kappa w^\alpha + \kappa\bar{\kappa}\bar{\partial}\theta^\alpha \\ \Delta_{\dot{\alpha}} &= p_{\dot{\alpha}} + \bar{\kappa}v_{\dot{\alpha}} + \kappa\bar{\kappa}\bar{\partial}p_{\dot{\alpha}} \\ \Theta^{\dot{\alpha}} &= \theta^{\dot{\alpha}} + \kappa w^{\dot{\alpha}} - \kappa\bar{\kappa}\bar{\partial}\theta^{\dot{\alpha}}.\end{aligned}\tag{2.3}$$

We no longer see the left-moving fermions χ , but have instead the anti-commuting coordinates $\theta^\alpha, \theta^{\dot{\alpha}}$ of $4d$ superspace, their conjugate momenta $p_\alpha, p_{\dot{\alpha}}$ and some auxiliary fields. Out of these superfields we construct the following action

$$S = \frac{1}{2\alpha'} \int d^2z d^2\kappa (\eta_{a\bar{a}} \Phi^a \bar{\partial}\Phi^{\bar{a}} - \Delta_\alpha \Theta^\alpha + \Delta_{\dot{\alpha}} \Theta^{\dot{\alpha}}) - \frac{1}{2} \int d^2z \partial\rho \bar{\partial}\rho + S_{int},\tag{2.4}$$

where ρ is a left-moving chiral boson that is inert under the right-moving $N = 2$ supersymmetry. In components this action takes the form

$$\begin{aligned}S &= \frac{1}{\alpha'} \int d^2z (\eta_{a\bar{a}} \partial\phi^a \bar{\partial}\phi^{\bar{a}} - \frac{1}{2} \eta_{a\bar{a}} \partial\psi^a \psi^{\bar{a}} + p_\alpha \bar{\partial}\theta^\alpha + p_{\dot{\alpha}} \bar{\partial}\theta^{\dot{\alpha}}) \\ &\quad - \frac{1}{2} \int d^2z \partial\rho \bar{\partial}\rho + \frac{1}{2\alpha'} \int d^2z (v_\alpha w^\alpha + v_{\dot{\alpha}} w^{\dot{\alpha}}) + S_{int}.\end{aligned}\tag{2.5}$$

The commuting spinor fields $v_\alpha, v_{\dot{\alpha}}, w^\alpha, w^{\dot{\alpha}}$ are auxiliary and can be trivially integrated out. The left moving sector now coincides with that of the Green-Schwarz-Berkovits sigma model, except for S_{int} . The four-dimensional part of the left-moving sector has an $N = 2$ algebra, with central charge -3 , consisting of $+4$ from ϕ , -8 from (d, θ) and $+1$ from ρ . The $N = 2$ ghosts contribute $-26 + 22 - 2 = -6$. For unitarity, we still have to gauge a super null current, and the central charge of the internal sector is therefore given by $c_{int} = 9 - c_{null}$, where c_{null} is the central charge of the ghosts for the super null current. The null currents we will consider have components of weight $1/2$ and 1 . Such null currents can only exist if they form a chiral or anti-chiral multiplet with respect to the $N = 2$ algebra. Depending on whether the null current is bosonic or fermionic, we need to demand that S_{int} has a $c = 12$ or $c = 6$ $N = 2$ algebra respectively. This is a stronger condition than in the original heterotic $N = (1,2)$ string, where it was only required to have a $c = 12$ $N = 1$ algebra. As the sigma model above does have target space

supersymmetry, the requirement that the internal sector has an $N = 2$ supersymmetry is probably equivalent to the requirement of target space supersymmetry. The structure of the null current and internal sector, and possible generalizations will be discussed below.

Up to an overall factor of $1/\alpha'$, the right-moving $N = 2$ algebra of (2.5) has generators

$$\bar{T} = -\frac{1}{2}\eta_{\underline{a}\underline{b}}(\bar{\partial}\phi^{\underline{a}}\bar{\partial}\phi^{\underline{b}} + \psi^{\underline{a}}\bar{\partial}\psi^{\underline{b}}) \quad (2.6)$$

$$H = i\sqrt{2}\eta_{\underline{a}\underline{b}}\psi^{\underline{a}}\bar{\partial}\phi^{\underline{b}} \quad (2.7)$$

$$\bar{H} = i\sqrt{2}\eta_{\underline{a}\underline{b}}\psi^{\underline{a}}\bar{\partial}\phi^{\underline{b}} \quad (2.8)$$

$$\bar{J} = \eta_{\underline{a}\underline{b}}\psi^{\underline{a}}\psi^{\underline{b}} \quad (2.9)$$

whereas the left-moving $N = 2$ algebra has generators

$$\begin{aligned} J &= -i\partial\rho + J_{int} \\ G &= \frac{1}{i\alpha'\sqrt{8\alpha'}}e^{i\rho}d^\alpha d_\alpha + G_{int} \\ \bar{G} &= \frac{1}{i\alpha'\sqrt{8\alpha'}}e^{-i\rho}d^{\dot{\alpha}} d_{\dot{\alpha}} + \bar{G}_{int} \\ T &= \frac{1}{\alpha'}\left(-\frac{1}{2}\eta_{\underline{a}\underline{b}}\partial\phi^{\underline{a}}\partial\phi^{\underline{b}} - p_\alpha\partial\theta^\alpha - p_{\dot{\alpha}}\partial\theta^{\dot{\alpha}}\right) + \frac{1}{2}\partial\rho\partial\rho + T_{int}. \end{aligned} \quad (2.10)$$

Here, we defined

$$\begin{aligned} d_\alpha &= p_\alpha + i\theta^{\dot{\alpha}}\partial\phi_{\alpha\dot{\alpha}} + \frac{1}{2}\theta^{\dot{\alpha}}\theta_{\dot{\alpha}}\partial\theta_\alpha - \frac{1}{4}\theta_\alpha\partial(\theta^{\dot{\alpha}}\theta_{\dot{\alpha}}) \\ d_{\dot{\alpha}} &= p_{\dot{\alpha}} + i\theta^\alpha\partial\phi_{\alpha\dot{\alpha}} + \frac{1}{2}\theta^\alpha\theta_\alpha\partial\theta_{\dot{\alpha}} - \frac{1}{4}\theta_{\dot{\alpha}}\partial(\theta^\alpha\theta_\alpha). \end{aligned} \quad (2.11)$$

The target space supersymmetry generators are given by

$$\begin{aligned} q_\alpha &= \oint \frac{dz}{2\pi i} \left(p_\alpha - i\theta^{\dot{\alpha}}\partial\phi_{\alpha\dot{\alpha}} + \frac{1}{4}\theta_\alpha\partial(\theta^{\dot{\alpha}}\theta_{\dot{\alpha}}) \right) \\ q_{\dot{\alpha}} &= \oint \frac{dz}{2\pi i} \left(p_{\dot{\alpha}} - i\theta^\alpha\partial\phi_{\alpha\dot{\alpha}} + \frac{1}{4}\theta_{\dot{\alpha}}\partial(\theta^\alpha\theta_\alpha) \right). \end{aligned} \quad (2.12)$$

Notice that these generators only act on the left-moving sector, and they (trivially) commute with the right moving sector. In particular, both $\bar{\partial}\phi^{\alpha\dot{\alpha}}$ and $\psi^{\alpha\dot{\alpha}}$ are (trivially) inert under the target space SUSY. One can check that d_α and $d_{\dot{\alpha}}$ are also inert. To make the target space SUSY manifest we further define the following variables that also commute with target space SUSY

$$\begin{aligned} \Pi_{\alpha\dot{\alpha}} &= \partial\phi_{\alpha\dot{\alpha}} - i\theta_\alpha\partial\theta_{\dot{\alpha}} + i\partial\theta_\alpha\theta_{\dot{\alpha}} \\ \Pi_{\underline{\alpha}} &= \partial\theta_{\underline{\alpha}} \end{aligned} \quad (2.13)$$

These new variables are in fact given by

$$\Pi^A = \partial z^M E_M{}^A \quad (2.14)$$

where $z = \{\phi^{\mu\dot{\mu}}, \theta^\mu, \theta^{\dot{\mu}}\}$ are the superspace coordinates and E_M^A are the vielbeins of flat superspace that can be used to convert curved indices into flat indices. The vielbeins are one along the diagonal and the only non-zero off-diagonal elements are

$$E_\mu^{\alpha\dot{\alpha}} = i\delta_\mu^{\dot{\alpha}}\theta^\alpha, \quad E_\mu^{\alpha\dot{\alpha}} = i\delta_\mu^\alpha\bar{\theta}^{\dot{\alpha}}. \quad (2.15)$$

With these variables one may rewrite the flat space model in a manifestly supersymmetric form,

$$S = \frac{1}{\alpha'} \int d^2z \left(\frac{1}{2} \Pi^{\alpha\dot{\alpha}} \bar{\Pi}_{\alpha\dot{\alpha}} + d_\alpha \bar{\Pi}^\alpha + d_{\dot{\alpha}} \bar{\Pi}^{\dot{\alpha}} + \frac{1}{2} \bar{\Pi}^A \Pi^B B_{BA} - \frac{\alpha'}{2} \bar{\partial} \rho \partial \rho - \frac{1}{2} \eta_{a\bar{a}} \partial \psi^a \bar{\psi}^{\bar{a}} \right) + S_{int} \quad (2.16)$$

where we introduced an anti-symmetric tensor field B_{BA} whose only non-zero components are

$$\begin{aligned} B_{\alpha\dot{\alpha},\beta} &= iC_{\beta\dot{\alpha}}\theta_\alpha & B_{\alpha\dot{\alpha},\beta} &= iC_{\beta\alpha}\bar{\theta}_{\dot{\alpha}} \\ B_{\dot{\beta},\alpha\dot{\alpha}} &= -iC_{\beta\dot{\alpha}}\theta_\alpha & B_{\beta,\alpha\dot{\alpha}} &= -iC_{\beta\alpha}\bar{\theta}_{\dot{\alpha}} \end{aligned} \quad (2.17)$$

Covariant derivatives are in flat superspace given by $\nabla_A = E_A^M \partial_M$, where the inverse vielbein E_A^M has one on the diagonal and off-diagonal components

$$E_{\dot{\alpha}}^{\mu\dot{\mu}} = -i\delta_{\dot{\alpha}}^{\dot{\mu}}\theta^\mu, \quad E_{\alpha}^{\mu\dot{\mu}} = -i\delta_{\alpha}^{\mu}\bar{\theta}^{\dot{\mu}}, \quad (2.18)$$

as follows from (2.15). From this we deduce that the non-zero torsion (see appendix A) of flat superspace is in our conventions given by

$$T_{\alpha,\dot{\alpha}}^{\beta\dot{\beta}} = -2i\delta_{\alpha}^{\beta}\delta_{\dot{\alpha}}^{\dot{\beta}}. \quad (2.19)$$

Associated to the anti-symmetric tensor field in (2.17) is a non-zero field strength H_{ABC} . With curved indices it is given by $H_{MNP} = \frac{1}{4}\partial_{[M}B_{NP]}$ which in terms of flat indices becomes

$$H_{ABC} = \frac{1}{4}\nabla_{[A}B_{BC]} - \frac{1}{4}T_{[AB}{}^DB_{D|C]}. \quad (2.20)$$

The only non-zero components of H are

$$H_{\alpha,\dot{\alpha},\beta\dot{\beta}} = -iC_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}} \quad (2.21)$$

and its permutations.

It remains to define the null current for (2.5) to make the theory complete. It seems very complicated to keep track of the null current in the field redefinition that leads one from the RNS to the Green-Schwarz-Berkovits formulation. Therefore, our approach will be to find a null current that satisfies the required properties. As mentioned above, the null currents should be part of a chiral or anti-chiral multiplet of the $N = 2$ algebra. In addition, they should preserve target space supersymmetry. This more or less uniquely fixes the null currents to be of the form

$$\begin{aligned} \mathcal{J}_{null}^{1/2} &= e^{i\rho} u^\alpha d_\alpha \\ \mathcal{J}_{null}^1 &= u^\alpha \Pi_\alpha. \end{aligned} \quad (2.22)$$

The spinor u^α can be either bosonic or fermionic. If it is bosonic, the central charge of the internal sector has to be $c_{int} = 6$, if it is fermionic $c_{int} = 12$. Naively, one would think

that the internal sector has to be a meromorphic conformal field theory, in order to have a modular invariant partition function. No such theories with $c = 6$ are known; for $c = 12$ one particular example has been constructed in [31], see also [15, 16, 17, 18]. It consists of 8 fermions and 8 bosons compactified on an E_8 lattice. Therefore, one would be more inclined to consider only the case where u^α is fermionic. On the other hand, from the point of view of the generators (2.22) a bosonic u^α seems more natural. In addition, the modular transformation properties of the four-dimensional part of the theory are already quite intricate, and it is not clear what the precise conditions on the internal sector are in order to guarantee modular invariance. We will therefore in the remainder consider the case of a bosonic spinor u^α , and discuss the modifications that have to be made for a fermionic spinor in section 4.

A bosonic spinor u^α is automatically null, and gives rise to the required vanishing OPE between $\mathcal{J}_{null}^{1/2}$ and $\mathcal{J}_{null}^1$. The latter contains a term $\sim e^{i\rho} u^\alpha u_\alpha / (z - w)^2$. Such terms cannot be cancelled by anything from the internal sector. Therefore, (2.22) cannot be modified so as to include any contribution from the internal sector. It is at present not clear whether there is a, perhaps different, possibility to construct null currents that do include contributions from the internal sector. Alternatively, one could imagine introducing a null current that forms an unconstrained representation of the world-sheet $N = 2$ algebra, in which case the internal sector would have to be modified. An example of a full null current that does not break target space SUSY is given by $U^{\alpha\dot{\alpha}}(\Pi_{\alpha\dot{\alpha}}, d_\alpha \Pi_{\dot{\alpha}} e^{i\rho}, d_{\dot{\alpha}} \Pi_\alpha e^{-i\rho}, \Pi_\alpha \Pi_{\dot{\alpha}})$, where $U^{\alpha\dot{\alpha}}$ is a null vector. This is the supersymmetrization of the usual choice of null current in conventional $N = 2$ strings, our chiral null current being sort of its square root. It turns out, however, that introducing such a full null current trivializes the theory.

2.2 Vertex Operators

We now turn to a description of the spectrum of the $N = 2$ string. The spectrum of $N = 2$ strings has been studied in detail, see e.g. [12, 13, 14, 32, 26, 33, 34]. These results need to be combined with the results for the spectrum of the Green-Schwarz-Berkovits [29, 20]. The spectrum of the pure $N = 2$ string consists of one scalar, the Lee-Yang scalar that describes self-dual gravity. In our case this scalar field is promoted to a scalar superfield $W(x, \theta)$. Due to the presence of the null-current, it is not an unconstrained superfield. The constraints will be discussed below. Let $G, \bar{G}$ be the world-sheet supersymmetry generators of the left-movers, and $H, \bar{H}$ the world-sheet supersymmetry generators of the right-movers. Then the 4-D integrated vertex operators corresponding to $W(x, \theta)$ reads

$$V = \int d^2 z \{ G, \{ \bar{G}, \{ H^+, \{ H^-, W(x, \theta) \} \} \} \}, \quad (2.23)$$

where $\{A, B\}$ denotes the single pole in the OPE of A with B . In addition, there are states in the theory coming from the internal sector. These are described by integrated vertex operators of the form

$$V^i = \int d^2 z \{ H^+, \{ H^-, \{ G, M_1^i \Omega \} + \{ \bar{G}, M_2^i \bar{\Omega} \} \} \} \quad (2.24)$$

where $M_1^i(x, \theta)$ is a real chiral superfield, $M_2^i(x, \theta)$ is a real anti-chiral superfield (in space with signature (2,2) one can have real chiral superfields [2]) and $\Omega, \bar{\Omega}$ are chiral (anti-chiral) primaries of the internal $N = 2$ superconformal algebra with conformal weight

1/2. If the internal sector is represented by two free $N = 2$ superfields, there are 2 chiral and 2 anti-chiral primaries of conformal weight 1/2, and i runs from 1 to 2. For other internal sectors, needed e.g. for alternative choices of null current, the index i may take on a different set of values. We will suppress the superscript i for the time being. The W, M_1, M_2 are regular superfields in target-space, and are not allowed to depend on ψ^a . The on-shell conditions for the vertex operators yield the following linearized equations of motion,

$$D^2 W = \bar{D}^2 W = \square W = 0 \quad (2.25)$$

and

$$D^2 M_1 = \bar{D}^2 M_2 = 0. \quad (2.26)$$

Here, $D_\alpha, D_{\dot{\alpha}}$ are the standard (rigid) covariant derivatives in superspace. In addition, the vertex operators (2.23) and (2.24) have certain gauge invariances. The ones associated with the right-moving sector imply that W, M_1 and M_2 can be chosen not to depend on $\bar{\Pi}^A$, a fact already used above, whereas the ones associated with the left-moving sector read

$$\delta W = \Lambda + \bar{\Lambda} \quad (2.27)$$

where Λ and $\bar{\Lambda}$ are chiral and anti-chiral superfields respectively.

This is not yet the full story, because the vertex operators should also preserve the world-sheet gauge invariance generated by the null currents. The spin-one component of the null current generates the transformation $\delta d^\alpha = \epsilon u^\alpha$ and leaves all other world-sheet fields invariant. Thus $\delta\{\bar{G}, \{G, W(x, \theta)\}, \} \sim \{\bar{G}, e^{i\rho} u^\alpha D_\alpha W(x, \theta)\}$, and this vanishes if

$$u^\alpha D_\alpha W(x, \theta) = 0. \quad (2.28)$$

In a similar way we find that M_1 should satisfy

$$u^\alpha D_\alpha M_1(x, \theta) = 0, \quad (2.29)$$

and that there is no new equation emerging for M_2 . In addition, the Hilbert space of the theory should be identified with its spectral flow generated by the $U(1)$ current. This leads to extra gauge invariances⁴

$$\delta W(x, \theta) = u^\alpha D_\alpha Y(x, \theta), \quad \delta M_1(x, \theta) = u^\alpha D_\alpha Y_1(x, \theta) \quad (2.30)$$

with Y_1 such that it preserves the chirality of M_1 .

Let us first analyze the internal sector. The second equation in (2.26) imply that the component fields of real anti-chiral superfield M_2 satisfy the usual linearized field equations. We shall now show that the equation (2.29) together with the gauge invariance in (2.30) completely eliminate M_1 . Indeed, equation (2.29) is solved by

$$M_1 = u^\alpha D_\alpha N \quad (2.31)$$

where N is such that M_1 is chiral but otherwise arbitrary. One may clearly choose Y_1 such that M_1 is gauged away.

⁴The null vector is associated with a first class constraint and, therefore, should "kill twice", once by imposing the constraint and once by the gauge invariance it generates

Let us now consider the 4D vertex operator. One can go from the gauge variant prepotential W to a gauge invariant field strength W_α in the usual way

$$W_\alpha = \bar{D}^2 D_\alpha W; \quad W_{\dot{\alpha}} = D^2 \bar{D}_{\dot{\alpha}} W \quad (2.32)$$

Equations (2.25) imply

$$D^2 W_\alpha = 0; \quad \bar{D}^2 W_{\dot{\alpha}} = 0, \quad \square W_\alpha = 0. \quad (2.33)$$

In addition, we have the equation (2.28) that now becomes

$$u^\alpha W_\alpha = 0 \quad (2.34)$$

This equation is solved by

$$W_\alpha = u_\alpha X \quad (2.35)$$

where X satisfies the equations that follow from (2.33) but it is otherwise arbitrary. The null gauge invariance (2.30), in turn, reads

$$\delta W_\alpha = \frac{1}{2} u_\alpha \bar{D}^2 D^2 Y; \quad \delta W_{\dot{\alpha}} = -2i u^\alpha \partial_{\alpha\dot{\alpha}} D^2 W = 0 \quad (2.36)$$

From this one can derive that the null gauge invariance can be used to gauge away W_α . A somewhat more detailed analysis of both the spacetime and internal sector in components can be found in appendix B.

We, thus, see that in the 4D sector the linearized field equation are given by the equation $W_\alpha = 0$. This equation describes $U(1)$ self-dual SYM theory [2], and as such contains 1 + 1 degrees of freedom. The fermionic one is the Majorana-Weyl fermion λ_α and the bosonic one is the Yang-Lee scalar describing the self-dual field strength. We shall shortly see that these degrees of freedom can be consistently identified with one helicity component of the gravitino and graviton, respectively.

To summarize, the spectrum consist of 1+1 components coming from the 4D part, and $q(1+1)$ real components from M_2 , where q is the number of (anti)chiral primaries of weight 1/2 of the internal sector. There is no conventional target space action that can describe these degrees of freedom since one cannot write down a fermionic kinetic term for any of these multiplets.

Let us now show that the equations of motion we have found are consistent with those of self-dual supergravity. For this, we want to show that the $N = 1$ SD SUGRA equation $W_{\alpha\beta\gamma} = 0$ (but $W_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \neq 0$) [2] reduces after we choose a complex structure, at the linearized level and in a specific gauge and Lorentz frame to the equation $W_\alpha = 0$ (but $W_{\dot{\alpha}} \neq 0$) satisfied by the vertex operators.

The gravitino field strength at the linearized level is given by the following formula (up to irrelevant numerical factors; see [35], equation (5.2.4))

$$W_{\alpha\beta\gamma} = \bar{D}^2 D_{(\alpha} \partial_{\beta}^{\dot{\beta}} \hat{H}_{\gamma)\dot{\beta}}, \quad (2.37)$$

$$W_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = D^2 \bar{D}_{(\dot{\alpha}} \partial_{\dot{\beta}}^{\beta} \hat{H}_{|\beta|\dot{\gamma})}, \quad (2.38)$$

where $D_\alpha, \bar{D}_{\dot{\alpha}}$ are the flat superspace covariant derivatives (we are working at the linearized level) and $\hat{H}_{\underline{a}}$ is the SUGRA prepotential (in [35] this prepotential is denoted by $H_{\underline{a}}$). These equations are invariant under the following gauge transformations

$$\delta \hat{H}_{\underline{a}} = \partial_{\underline{a}} \lambda \quad (2.39)$$

We fix this gauge invariance by setting

$$\partial^{\underline{a}} \hat{H}_{\underline{a}} = 0 \quad (2.40)$$

One can solve this relation by expressing $\hat{H}_a$ in terms of a prepotential $B_{\underline{ab}}$,

$$\hat{H}_{\underline{a}} = \partial^{\underline{b}} B_{\underline{ba}}, \quad (2.41)$$

Expressing $B_{\underline{ab}}$ (which is antisymmetric in $\underline{a}, \underline{b}$) in terms of its Lorentz irreducible (i.e. symmetric in their indices) components $B_{\alpha\beta}$ and $B_{\dot{\alpha}\dot{\beta}}$ in the usual way we get

$$\hat{H}_{\alpha\dot{\beta}} = \partial_\alpha{}^{\dot{\gamma}} B_{\dot{\beta}\dot{\gamma}} + \partial^{\dot{\gamma}}{}_{\dot{\beta}} B_{\alpha\gamma} \quad (2.42)$$

Then, the gravitino field strength takes the form

$$W_{\alpha\beta\gamma} = \bar{D}^2 D_{(\alpha} \partial_{\beta}{}^{\dot{\beta}} \partial_{\gamma)}{}^{\dot{\gamma}} B_{\dot{\beta}\dot{\gamma}} - \frac{1}{2} \square \bar{D}^2 D_{(\alpha} B_{\beta\gamma)}, \quad (2.43)$$

$$W_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = D^2 \bar{D}_{(\dot{\alpha}} \partial_{\dot{\beta}}{}^{\beta} \partial_{\dot{\gamma})}{}^{\gamma} B_{\beta\gamma} - \frac{1}{2} \square D^2 \bar{D}_{(\dot{\alpha}} B_{\dot{\beta}\dot{\gamma})}. \quad (2.44)$$

Let us now consider the specific Lorentz frame in which only $p^1 - p^3 \neq 0$ (p^1 and p^3 are real). Such a choice is possible since we are on-shell. In complex coordinates (with the complex structure of appendix A), we have

$$p_{++} = -p_{--} = -ip_{+-} = -ip_{-+} \quad (2.45)$$

One may now work out $W_{\alpha\beta\gamma}$ and $W_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$. The result is

$$\begin{aligned} W_{+++} &= -3\bar{D}^2 D_+ W \\ W_{++-} &= -\bar{D}^2 D_- W - 2i\bar{D}^2 D_+ W \\ W_{+--} &= \bar{D}^2 D_+ W - 2i\bar{D}^2 D_- W \\ W_{---} &= 3\bar{D}^2 D_- W \end{aligned} \quad (2.46)$$

where

$$W = 2p_{++}^2 (B_{++} - B_{--} + 2iB_{+-}) \quad (2.47)$$

To get the result for $W_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$ one simply replaces dotted by undotted indices and vice versa. One may verify that the reality of $\hat{H}_{\underline{a}}$ implies

$$\overline{B_{++}} = -B_{--}; \quad \overline{B_{+-}} = -B_{-+} \quad (2.48)$$

and similar results for $B_{\dot{\alpha}\dot{\beta}}$. With these reality properties W is real.

Let us now define

$$W_\alpha = \bar{D}^2 D_\alpha W \quad (2.49)$$

(one can similarly define $W_{\dot{\alpha}}$). Obviously the condition $W_{\alpha\beta\gamma} = 0$ implies precisely $W_\alpha = 0$ and in addition $W_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$ and $W_{\dot{\alpha}}$ are not set equal to zero. Hence, the SD equation $W_{\alpha\beta\gamma} = 0$ after choosing a complex structure, at the linearized level and in a specific gauge and Lorentz frame, is precisely the equation satisfied by the VO's.

Finally, we give the relation between the components of the SUGRA multiplet and the components of the linearized analysis (the components of the SUGRA are given in [35], equation (5.2.8))

$$\lambda_{\dot{\alpha}} = D^2 \bar{D}_{\dot{\alpha}} W = 2p^\alpha \bar{D}^2 \bar{D}_{\dot{\alpha}} \hat{H}_{\alpha\dot{\alpha}} \sim p^\alpha \psi_{\underline{a}\dot{\alpha}} \quad (2.50)$$

$$f_{\alpha\beta} = D_{(\alpha} W_{\beta)} = 4ip_{-\dot{\alpha}} p_{(\alpha}^{\dot{\beta}} [D_{\beta)}, D_{\dot{\beta}}] \hat{H}_{-\dot{\alpha}} \sim p_{-\dot{\alpha}} p_{(\alpha}^{\dot{\beta}} h_{\beta)\dot{\beta}, -\dot{\alpha}} \quad (2.51)$$

$$D = D^\alpha W_\alpha = 2p_{-\dot{\beta}} D^\alpha \bar{D}^2 D_\alpha \hat{H}_{-\dot{\beta}} \sim p_{-\dot{\alpha}} A_{-\dot{\alpha}} \quad (2.52)$$

where $h_{\alpha\dot{\alpha}}$ is the conformal graviton, $\psi_{\alpha\dot{\alpha},\beta}$ is the conformal gravitino and $A_{\underline{a}}$ is the auxiliary vector of conformal supergravity. (Notice that the projection definition of $A_{\underline{a}}$ involves also a term proportional to ϵ_{abcd} . This term, however, drops out in our gauge).

3 The New $N = 2$ String in an Arbitrary Background

So far we have focused on the vertex operators of the theory, and the linearized equations of motion they satisfy. In this section we will couple the new $N = 2$ string to an arbitrary curved background. In order for the background to be compatible with the world-sheet symmetries, several constraints have to be satisfied. The main result will be that the constraints are exactly those that describe self-dual supergravity, confirming the results from the previous section.

The action (2.16) can easily be put in a curved background, by simply taking the vielbeins and anti-symmetric tensor to be arbitrary rather than those of flat superspace. Upon expanding them to first order around a flat background we should recover the massless vertex operators of the previous section. However, one then sees that (2.16) is not yet complete, because the massless vertex operators do contain terms bilinear in the right-moving fermions ψ , and we need to include such terms in the action as well. These terms are completely fixed by requiring the right-moving sector to have $N = 2$ supersymmetry, and we find for the action in a curved background (with the background fields coming from the internal sector turned off)

$$\begin{aligned} S = & \frac{1}{\alpha'} \int d^2 z \left(\frac{1}{2} \eta_{\underline{ab}} \Pi^{\underline{a}} \bar{\Pi}^{\underline{b}} + d_\alpha \bar{\Pi}^\alpha + d_{\dot{\alpha}} \bar{\Pi}^{\dot{\alpha}} + \frac{1}{2} \bar{\Pi}^A \Pi^B B_{BA} \right. \\ & \left. - \frac{\alpha'}{2} \bar{\partial} \rho \partial \rho + \eta_{\underline{ab}} \psi^{\underline{a}} \nabla \psi^{\underline{b}} + \Pi^{\underline{a}} \psi^{\underline{b}} \psi^{\underline{c}} T_{\underline{cb}\underline{a}} + d^{\underline{a}} \psi^{\underline{b}} \psi^{\underline{c}} T_{\underline{cb}\underline{a}} \right) + S_{int}. \end{aligned} \quad (3.1)$$

Here, ∇ is the pull-back of the target space covariant derivative to the world-sheet,

$$\nabla \psi^{\underline{a}} = \partial \psi^{\underline{a}} + \Pi^B [\omega_{B\beta}{}^\gamma M_\gamma{}^\beta + \omega_{B\dot{\beta}}{}^{\dot{\gamma}} M_{\dot{\gamma}}{}^{\dot{\beta}} + \Gamma_B Y, \psi^{\underline{a}}], \quad (3.2)$$

where ω and Γ are the spin and $U(1)$ connections in target space. One could also imagine adding a term $\Pi^\alpha \psi^b \psi^c T_{cba}$ to the action. However, such a term is incompatible with the local target space $U(1)$ invariance given by

$$\delta \Pi^a = 0, \delta \Pi^\alpha = -\frac{1}{2} \Lambda \Pi^\alpha, \delta \Pi^{\dot{\alpha}} = \frac{1}{2} \Lambda \Pi^{\dot{\alpha}}, \delta d_\alpha = \frac{1}{2} \Lambda d_\alpha, \delta d_{\dot{\alpha}} = -\frac{1}{2} \Lambda d_{\dot{\alpha}}, \delta \psi^a = 0 \quad (3.3)$$

and, in addition, breaks $N = 2$ world-sheet supersymmetry.

The generators of the world-sheet chiral and anti-chiral algebras read as follows. The generators of the left-moving $N = 2$ are

$$\begin{aligned} J &= -i\partial\rho \\ G &= \frac{1}{i\alpha'\sqrt{8\alpha'}} e^{i\rho} d^\alpha d_\alpha \\ \bar{G} &= \frac{1}{i\alpha'\sqrt{8\alpha'}} e^{-i\rho} d^{\dot{\alpha}} d_{\dot{\alpha}} \\ T &\rightarrow \frac{1}{\alpha'} \left(-\frac{1}{2} \Pi^{\alpha\dot{\alpha}} \Pi_{\alpha\dot{\alpha}} - d_\alpha \Pi^\alpha - d_{\dot{\alpha}} \Pi^{\dot{\alpha}} + \frac{\alpha'}{2} \partial\rho \partial\rho \right). \end{aligned} \quad (3.4)$$

The generators of the null current are still of the form (2.22),

$$\begin{aligned} \mathcal{J}_{null}^{1/2} &= e^{i\rho} u^\alpha d_\alpha \\ \mathcal{J}_{null}^1 &= u^\alpha \Pi_\alpha. \end{aligned} \quad (3.5)$$

The generators of the right-moving $N = 2$ algebra read

$$\begin{aligned} \bar{T} &= -\frac{1}{\alpha'} (\eta_{ab} (\frac{1}{2} \bar{\Pi}^a \bar{\Pi}^b + \psi^a \bar{\nabla} \psi^b) + \bar{\Pi}^a \psi^b \psi^c T_{cba}) \\ H^+ &= \frac{i}{\sqrt{2}\alpha'} ((\eta_{ab} + I_{ab}) \psi^a \bar{\Pi}^b + \psi^a \psi^b \psi^c C_{cba}^+) \\ H^- &= \frac{i}{\sqrt{2}\alpha'} ((\eta_{ab} - I_{ab}) \psi^a \bar{\Pi}^b + \psi^a \psi^b \psi^c C_{cba}^-) \\ \bar{J} &= \frac{1}{2\alpha'} I_{ab} \psi^a \psi^b. \end{aligned} \quad (3.6)$$

where I_{ab} is the complex structure that satisfies

$$I_{ab} = -I_{ba}; \quad I_{ab} I^{bc} = \delta_a^c \quad (3.7)$$

and C_{abc}^+, C_{abc}^- are tensors to be determined and that vanish in the flat space limit. One may easily check that in the flat space limit this algebra correctly reduces to the correct $N = 2$ algebra.

We will first analyze the constraints coming from $N = 2$ world-sheet supersymmetry, postponing the analysis of the null current to section 3.3. We find that the left-moving $N = 2$ supersymmetry imposes exactly the same set of constraints as in the case of the heterotic string (see [21]). The right movers yield only one constraint, namely $\nabla_\gamma I_{ab} = 0$,

and fix $C_{abc}^\pm$ to be of the form (3.31). This is a highly non-trivial calculation, the details of which follow below.

The equations of motion of (3.1) read

$$0 = \bar{\Pi}_{\underline{\alpha}} - \psi^{\underline{b}} \psi^{\underline{c}} T_{\underline{cb}\underline{\alpha}} = 0 \quad (3.8)$$

$$0 = \nabla \psi_{\underline{a}} + \Pi^{\underline{b}} \psi^{\underline{c}} T_{\underline{cab}} - d^{\underline{b}} \psi^{\underline{c}} T_{\underline{cab}} \quad (3.9)$$

The field equation for the fields $\Pi^{\underline{a}}$ and $d^{\underline{a}}$ differ from the field equations we found in [21] by terms bilinear in the fermion fields $\psi^{\underline{a}}$. This means that the tree level constraints that follow from the left-moving sector are the ones found in [21] plus possibly some new constraints due to the extra terms. The former ones supplemented by a maximal set of conventional constraints was solved in [21] to yield the following supergravity algebra

$$\{\nabla_{\alpha}, \nabla_{\beta}\} = 0, \quad (3.10)$$

$$\{\nabla_{\alpha}, \nabla_{\dot{\beta}}\} = -2i\nabla_{\alpha\dot{\beta}} - 4iH_{\dot{\beta}\gamma}M_{\alpha}^{\gamma} + 4iH_{\gamma\alpha}M_{\dot{\beta}}^{\gamma} + 4iH_{\dot{\beta}\alpha}Y, \quad (3.11)$$

$$\begin{aligned} [\nabla_{\alpha}, \nabla_{\dot{\beta}}] &= -2\nabla_{\beta}H_{\dot{\beta}\gamma}M_{\alpha}^{\gamma} \\ &+ [-2iC_{\alpha\beta}W_{\dot{\beta}\gamma}^{\delta} + C_{\dot{\beta}\gamma}(\nabla_{(\alpha}H_{\beta)}^{\delta} - \frac{1}{3}C_{\alpha\beta}\nabla^{\epsilon}H_{\epsilon}^{\delta})]M_{\dot{\delta}}^{\gamma} \\ &+ 2\nabla_{\beta}H_{\dot{\beta}\alpha}Y \end{aligned} \quad (3.12)$$

$$\begin{aligned} [\nabla_a, \nabla_b] &= \{-2H_{\dot{\alpha}\beta}\nabla_{\alpha\dot{\beta}} \\ &+ [\frac{i}{2}C_{\alpha\beta}\nabla_{(\dot{\alpha}}H_{\dot{\beta})}^{\gamma} + C_{\dot{\alpha}\dot{\beta}}(-\frac{i}{6}C^{\gamma}_{(\alpha|\nabla^{\epsilon}H_{\epsilon|\beta)} + W_{\alpha\beta}^{\gamma})]\nabla_{\gamma} \\ &+ [C_{\dot{\alpha}\dot{\beta}}(\frac{1}{24}\nabla_{(\alpha}W_{\beta\gamma)}^{\delta} + \frac{1}{4}(C^{\delta}_{\alpha}\nabla_{(\beta|\epsilon}H_{\epsilon|\gamma)} + \alpha \leftrightarrow \beta) \\ &+ \frac{i}{6}C_{\alpha\gamma}C^{\delta}_{\beta}\nabla^{\epsilon}\nabla^{\epsilon}H_{\epsilon\epsilon}) + \frac{i}{2}C_{\alpha\beta}\nabla_{\gamma}\nabla_{(\dot{\alpha}}H_{\dot{\beta})}^{\delta}]\nabla_{\delta}^{\gamma} \\ &- \frac{i}{2}C_{\alpha\beta}\nabla^{\delta}\nabla_{(\dot{\alpha}}H_{\dot{\beta})\delta}Y + \text{c.c.}\} \end{aligned} \quad (3.13)$$

where ‘c.c.’ denotes our definition of complex conjugation (not to be confused with the usual complex conjugation which in a space of signature (2, 2) does not mix dotted and undotted indices), which acts as follows on the various objects

$$(\nabla_{\alpha})^{\dagger} = \nabla_{\dot{\alpha}}; \quad (C_{\alpha\beta})^{\dagger} = C_{\dot{\alpha}\dot{\beta}}; \quad (M_{\gamma}^{\delta})^{\dagger} = M_{\dot{\gamma}}^{\dot{\delta}}; \quad (Y)^{\dagger} = -Y; \quad (3.14)$$

$$(\nabla_{\underline{a}})^{\dagger} = -\nabla_{\underline{a}}; \quad (H_{\dot{\beta}\alpha})^{\dagger} = H_{\dot{\alpha}\beta}; \quad (W_{\alpha\beta\gamma})^{\dagger} = W_{\dot{\alpha}\dot{\beta}\dot{\gamma}}. \quad (3.15)$$

and, in addition, does not reverse the order of factors, namely $(AB)^{\dagger} = A^{\dagger}B^{\dagger}$. $W_{\alpha\beta\gamma}$ is a completely symmetric chiral superfield,

$$\nabla_{\dot{\delta}}W_{\alpha\beta\gamma} = 0, \quad (3.16)$$

and $H_{\dot{\alpha}\beta}$ is defined in terms of H_{ABC} by

$$H_{abc} = C_{\gamma\alpha}C_{\dot{\gamma}\dot{\beta}}H_{\dot{\alpha}\beta} - C_{\gamma\beta}C_{\dot{\gamma}\dot{\alpha}}H_{\dot{\beta}\alpha}. \quad (3.17)$$

$W_{\alpha\beta\gamma}$ and $H_{\alpha\beta}$ satisfy the following differential relations

$$\nabla_{\underline{a}} H^{\underline{a}} = 0, \quad \nabla^\gamma W_{\gamma\alpha\beta} = \frac{i}{6} \nabla_{(\alpha} \nabla^\gamma H_{\gamma|\beta)} + \frac{i}{2} \nabla^\gamma \nabla_{(\alpha} H_{\gamma|\beta)}, \quad \nabla^\beta \nabla_\beta H_{\underline{a}} = 0. \quad (3.18)$$

Furthermore, all the components of the field strength H_{ABC} vanish except the ones that are given in (3.17) and (2.21). Compactly, we have

$$T_{ABc} + (-1)^{AB} 2H_{ABc} = 0. \quad (3.19)$$

The remaining field equations now take the form

$$\begin{aligned} \bar{\nabla} d_\alpha &= \psi^b \psi^c [\Pi^{\underline{d}} (R_{\underline{d}\alpha\underline{bc}} + 2iT_{\underline{cb}\delta} C_{\alpha\delta} + 2\nabla_\alpha H_{\underline{cbd}}) \\ &\quad + \Pi^\delta (R_{\delta\alpha\underline{bc}} - 4iH_{\underline{cb},\alpha\delta}) + d^\delta \nabla_\alpha T_{\underline{cb}\delta}] \\ \nabla \bar{\Pi}_{\underline{a}} &= -2\Pi^{\underline{e}} \bar{\Pi}^{\underline{b}} H_{\underline{bca}} - \bar{\Pi}^{\underline{e}} T_{\underline{ca}}{}^\beta d_\beta \\ &\quad + \psi^b \psi^c [\Pi^{\underline{d}} (R_{\underline{adb}c} + 2H_{\underline{e}[\underline{ab}H_{\underline{c}]\underline{de}} + 2\nabla_{\underline{a}} H_{\underline{cba}} - 2\nabla_{\underline{a}} H_{\underline{cbd}}) \\ &\quad + \Pi^\delta (R_{\underline{a}\delta\underline{bc}} + 2\nabla_\delta H_{\underline{cba}}) \\ &\quad + d^\delta (\nabla_{\underline{a}} T_{\underline{cb}\delta} - 4T_{\underline{be}\delta} H_{\underline{cea}})] \end{aligned} \quad (3.20)$$

We want to determine whether (3.1) is indeed invariant under the symmetries generated by (3.4), (3.6). First, we check whether the terms bilinear in fermions bring in any new equations in the analysis of the left sector. Using the d field equation and ignoring the terms which are proportional to d^2 as these terms can be removed by modifying the transformation rule of the gravitini that couple to the supersymmetry currents (see [21]) we get that the following three equations should hold

$$R_{\underline{d}\alpha\underline{bc}} + 2iT_{\underline{cb}\delta} C_{\alpha\delta} + 2\nabla_\alpha H_{\underline{cbd}} = 0 \quad (3.21)$$

$$R_{\delta\alpha\underline{bc}} - 4iH_{\underline{cb},\alpha\delta} = 0 \quad (3.22)$$

$$\nabla_\alpha T_{\underline{cb}\gamma} = 0. \quad (3.23)$$

A direct computation, which consist of substituting the relevant expressions from the supergravity algebra for the tensors involved shows that all of them are automatically satisfied. This means that the extra terms bilinear in the fermions produce no new equations.

We now move to the analysis of the right moving sector. Conservation of the $U(1)$ current $\partial \bar{J} = 0$ implies the following three equations

$$\nabla_\gamma I_{\underline{ab}} = 0 \quad (3.24)$$

$$\nabla_{\underline{c}} I_{\underline{ab}} + 2I_{\underline{d}[\underline{a}} H_{\underline{b}]\underline{cd}} = 0 \quad (3.25)$$

$$I_{\underline{d}[\underline{a}} T_{\underline{b}]\underline{d}\gamma} = 0 \quad (3.26)$$

The last two conditions actually follow from the first one as we now show. Equation (3.25) simply follows from the equation

$$\{\nabla_\gamma, \nabla_{\dot{\gamma}}\} I_{\underline{ab}} = 0. \quad (3.27)$$

Differentiating (3.25) by ∇^γ yields equation (3.26). To get this result one may use the following identities

$$[\nabla_{\dot{\alpha}}, \nabla_{\dot{b}}]V_{\alpha} = -2iT_{\dot{a}\dot{b}}{}^{\gamma}V_{\gamma}, \quad (3.28)$$

$$[\nabla_{\alpha}, \nabla_{\dot{b}}]V_{\gamma} = 2C_{\gamma\alpha}\nabla_{\beta}H_{\dot{\beta}}{}^{\delta}V_{\delta}, \quad (3.29)$$

$$T_{\alpha\dot{\alpha},\beta\dot{\beta},\gamma} - T_{\gamma\dot{\alpha},\beta\dot{\beta},\alpha} = iC_{\alpha\gamma}\nabla_{\dot{\beta}}H_{\dot{\alpha}\beta} \quad (3.30)$$

In these identities $V_{\underline{\alpha}}$ is assumed to have the $U(1)$ charge of its index, namely $1/2$ if the index is undotted and $-1/2$ if the index is dotted. One may use the first two identities to derive how commutators act on tensors. (One has to be careful, however, when the $U(1)$ charge of the tensor is different from the one its indices indicate.)

The condition in (3.25) is the familiar condition that the complex structure should be covariantly constant with respect to a connection with torsion. In our case, however, this is a derived condition! As is usual in supersymmetric theories, there is a more fundamental spinorial condition in lower (mass) dimension. This condition states that the complex structure is both chiral and anti-chiral. In flat space this condition correctly implies that the complex structure is constant.

Having analyzed the $U(1)$ current we now turn to the two supersymmetry currents. The equation $\partial H^{\pm} = 0$ involves terms linear in the fermions $\psi^{\underline{a}}$ and terms cubic in the fermions $\psi^{\underline{a}}$. The ones linear in the fermions vanish upon using the equations (3.25) and (3.26). The ones cubic in the fermion field yield six equations that contain two unknowns, the tensors $C_{\underline{c}\underline{b}\underline{a}}^{\pm}$. It is an excellent consistency check that this system of equations does have a unique solution. The computations involved are similar to the ones we already presented so far but considerably more complicated. The details can be found in appendix C. Here we only give the final solution for $C_{\underline{c}\underline{b}\underline{a}}^{\pm}$,

$$C_{\underline{c}\underline{b}\underline{a}}^{\pm} = \frac{1}{3}(-2H_{\underline{c}\underline{b}\underline{a}} \pm I_{\underline{d}[\underline{c}}H_{\underline{b}\underline{a}]\underline{d}}) \quad (3.31)$$

This almost concludes the discussion of the anti-holomorphic $N = 2$ algebra. We only have to show that the stress-energy tensor $\bar{T}$ in (3.6) is conserved, $\partial\bar{T} = 0$. For this we do not have to do any complicated analysis, we simply observe $\bar{T}$ is the Noether current for the symmetry $\bar{z} \rightarrow \bar{z} + \bar{\epsilon}(\bar{z})$, which automatically guarantees $\partial\bar{T} = 0$.

3.1 The dilaton

In the previous section we have discussed how to couple the $N = 2$ string to an arbitrary curved background, except for the dilaton. Although we have not worked out the precise form of the dilaton vertex operator, it can be included in the action using a suitable generalization of the Fradkin-Tseytlin term [20, 21]. The form of this term can be determined by looking at the geometry of a super world-sheet. In the case of the heterotic string one has to consider a super world-sheet with $N = (2, 0)$ supersymmetries, in the case of the heterotic $N = 2$ string we have a holomorphic and anti-holomorphic $N = 2$ algebra and the relevant world-sheet geometry is that of $N = (2, 2)$ superspace. In the formulation of [36] there are four types of world-sheet supercurvatures, that are (anti)-chiral and twisted (anti)-chiral from the world-sheet point of view. In order to write down

a supersymmetric world-sheet action, these four types of curvatures have to couple to target space fields with the same world-sheet properties. (Anti)-chirality for the holomorphic $N = 2$ algebra translates directly into (anti)-chirality for the target space fields, as one sees from the OPE's $\{G, \phi\} \sim \nabla_\alpha \phi$, $\{\bar{G}, \phi\} \sim \bar{\nabla}_\alpha \phi$. Similarly, the OPE's show that (anti)-chirality with respect to the anti-holomorphic $N = 2$ translates into the property $(\eta_{ab} \pm I_{ab}) \nabla_b \phi = 0$. These are the (anti)-holomorphic derivatives in target space, and we see that a world-sheet chiral superfield is a target space holomorphic chiral superfield, a world-sheet twisted chiral superfield is a target space anti-holomorphic chiral superfield, etc. We denote the sum of the target space holomorphic and anti-holomorphic chiral superfields by ϕ , and the sum of the target space holomorphic and anti-holomorphic anti-chiral superfields by $\bar{\phi}$. Once the form of the Fradkin-Tseytlin term has been determined, one can go to superconformal gauge. Then there is no dilaton-dependent term in the action anymore, except for a coupling between the ρ -field and the dilaton, and the only place the dilaton appears is in the generators of the $N = 2$ algebras. The dilaton contributions to the holomorphic $N = 2$ algebra are identical to the ones derived in [21]. The contributions to the anti-holomorphic $N = 2$ generators can be derived in a similar way, but since we will not need those results we will not present them here.

After inclusion of the dilaton, the action still has the local $U(1)$ invariance given in (3.3), if one also varies $\rho, \phi, \bar{\phi}$ as follows

$$\delta\phi = -\frac{1}{2}\Lambda, \quad \delta\bar{\phi} = \frac{1}{2}\Lambda, \quad \delta\rho = i\Lambda. \quad (3.32)$$

As in [21], it will be convenient to redefine ρ so that it becomes a $U(1)$ -invariant quantity, by defining

$$\rho \rightarrow \rho - i(\phi - \bar{\phi}). \quad (3.33)$$

After this redefinition, ρ itself is a chiral boson that does not couple to any of the other fields in the theory, and it can be quantized exactly. We need to do so in order to perform an expansion in α' in the theory; the field ρ does not carry a factor of α' in the action, and therefore an arbitrary number of loops for the ρ -field contributes to the theory at a given order in α' .

In the presence of an additional null current in the superconformal algebra it is in principle possible to introduce another dilaton-like field, that couples in a Fradkin-Tseytlin term to the curvature of the $U(1)$ gauge field needed to gauge the null current. We have not examined whether or not it is consistent to introduce such a field but it would be interesting to know if it is at all possible, and whether there is a corresponding state in the cohomology of the $N = 2$ string.

3.2 The internal sector

Until now we have suppressed the internal sector. One may turn on the corresponding background fields by including terms in the sigma model of the form[20]

$$S_{int} = \int d^2z \{H^+, \{H^-, \{\bar{G}, M^i \bar{\Omega}\}\}\} \quad (3.34)$$

where M^i is set of q real anti-chiral superfields and $H^\pm, \bar{G}$ are the generators of the left and right superconformal algebra in a curved background (3.4)-(3.6). Including

these terms into the action partially changes the analysis of the previous section since now the worldsheet field equations receive contributions from the internal sector and there are additional vertices to be taken into account into our 1-loop and beta function computation. One may easily verify that the new terms in the field equations contain a factor $\exp i\rho$. This factor is essential so that one does not run into problems already at tree level. The same factor, however, presents the most serious obstacle in computing the contribution of the internal sector to the one loop results. We have been using a hybrid method that treats the ρ field exactly and the rest of the fields perturbatively in α' . This seems quite difficult in the presence of the internal sector since now the ρ field does not decouple from the rest of the fields. We shall not discuss the internal sector in the remainder of this section.

3.3 The null current

So far our results were generic for any heterotic (1, 2) string, regardless of the choice of null current. In this section we analyze what further constraints we have to impose on the background in order to incorporate the null current (2.22) into the theory. The null currents, after incorporating the dilaton and the redefinition of the ρ -field, read

$$\mathcal{J}_{null}^{1/2} = e^{i\rho} e^{\phi - \bar{\phi}} u^\alpha d_\alpha, \quad \mathcal{J}_{null}^1 = u^\alpha \Pi_\alpha. \quad (3.35)$$

First, we examine the conditions imposed by requiring that the null currents be holomorphic. In fact, as in [21], we will only require the weaker condition that $\bar{\partial}\mathcal{J}$ is proportional to $\mathcal{J}$. The weaker condition guarantees that, at tree level, we can still gauge the $N = 2$ algebra together with the null current, and the theory is well-defined. Using the equations of motion for the background fields (3.8), (3.9), (3.20) and the auxiliary identity

$$\nabla \bar{\Pi}^A - \bar{\nabla} \Pi^A = -\Pi^B \bar{\Pi}^C T_{CB}{}^A \quad (3.36)$$

we find the following set of equations

$$\begin{aligned} u^\alpha T_{\underline{cb}\alpha} &= 0 \\ \nabla_{\underline{b}} u_\alpha &= 0 \\ u^\alpha \nabla_\delta T_{\underline{cb}\alpha} - (\nabla_{\underline{c}} u_\delta) T_{\underline{cb}\gamma} &= P_{\underline{cb}} u_\delta \\ T_{\underline{cb}}^{\underline{d}} \nabla_{\underline{d}} u^\alpha - u^\delta \nabla_\delta T_{\underline{cb}}^\alpha &= Q_{\underline{cb}} u^\alpha \end{aligned} \quad (3.37)$$

for some tensors $P_{\underline{cb}}$, $Q_{\underline{cb}}$. These equations are not all independent. The second equation in (3.37) implies in particular $[\nabla_{\underline{b}}, \nabla_{\underline{c}}] u_\alpha = 0$, and this implies both the third equation with $P_{\underline{bc}} = 0$, and the fourth equation, with $Q_{\underline{cb}} \sim F_{\underline{cb}}$.

3.4 Background Field Expansion

To further analyze the theory we will compute some of the OPE's between the various generators. The techniques are identical to those described in [21]. The only new

ingredient is the presence of the fermions ψ^a . For these, the background field rules read

$$\Delta\psi^a = 0, \quad (3.38)$$

$$\Delta(\nabla\psi^a) = \Pi^B y^C R_{CB}{}^{ad}\psi_d. \quad (3.39)$$

After performing the background field expansion, we will give the fermions a background expectation value by replacing $\psi^a \rightarrow \psi^a + \psi_{bg}^a$. In addition to the background field expansion presented in [21], there are now two kinds of additional terms. One kind comes from the terms containing fermions ψ_a , the other from terms containing $\bar{\Pi}^a$. The latter is no longer zero but rather satisfies the field equation given in (3.8). Loops are computed as in [21] by treating the fermions ψ^a in the same way as d^α, y_α .

Using the techniques described above, we examine the tree-level OPE between G and $\mathcal{J}_{null}$. We will not consider the contributions of the dilaton to the tree-level OPE. As discussed in [21], in many cases these contributions are ambiguous and can be put equal to zero either by adding total derivatives to the action, or by modifying the background field expansion of the d -field. The tree-level OPE between G and $\mathcal{J}_{null}^1$ should have a single pole which is proportional to $\partial(\mathcal{J}_{null}^{1/2})$. A straightforward calculation shows that this is only true if

$$\nabla u_\alpha = -\nabla_\alpha u^\beta \Pi_\beta. \quad (3.40)$$

From this we deduce two new equations for u_α , namely $\nabla_\beta u_\alpha = 0$ and $\nabla_{(\alpha} u_{\beta)} = 0$. Further constraints can be found by considering the OPE between $\mathcal{J}_{null}^{1/2}$ and $\mathcal{J}_{null}^1$. That OPE contains a single pole with a coefficient proportional to $u^\alpha(\nabla_\alpha u^\beta)\Pi_\beta$. This can only vanish if $\nabla_\alpha u_\beta = 0$. Putting everything together, we have derived that $\nabla_A u_\beta = 0$, i.e. u_β should be a covariantly constant bosonic null spinor. This is the most natural generalization of the constant spinor u_β to curved superspace. By computing $[\nabla_A, \nabla_B]u_\gamma = 0$ we can now derive further consistency conditions that the background fields have to satisfy. One finds

$$\nabla_A u_\beta = 0 \implies H^{\dot{\alpha}\beta} u_\beta = T_{ab}{}^\gamma u_\gamma = 0. \quad (3.41)$$

From here on, one could in principle proceed and compute further OPE's in order to check the rest of the $N = 2$ algebra. Before doing so, it is useful to first impose as many constraints as possible on the background fields in order to simplify the calculations. In the analysis of the vertex operators we used spectral flow to argue that certain operators could be identified with each other. In particular, if one obtains the constraint $u^\alpha \nabla_\alpha M = 0$ from the null current, the identification under spectral flow can be used to choose $\nabla_\alpha M = 0$. Spectral flow acts on vertex operators by multiplication by $\exp(\omega \int^z \mathcal{J}_{null}^1)$ which in flat space reduces to $\exp(\omega u^\alpha \theta_\alpha)$. It acts on each vertex operator independently, and one can choose a different parameter ω for each choice of momentum of the vertex operator. These expressions clearly indicate the difference between the *RNS* formulation of the (1,2) strings and our new formulation. In the former the $U(1)$ current generates a gauge invariance that kills one (or two) spacetime bosonic coordinates. In our case the null current effectively kills one fermionic coordinate instead. The vertex operators contain prepotentials, whereas the action contains potentials, and it is quite hard to see precisely how spectral flow affects the various background fields, connections, etc. Our philosophy will be that if the null-current gives rise to a constraint of the form $u^\alpha(\text{something}) = 0$, then one should be able to choose $(\text{something}) = 0$ as well. One

advantage of this is that there will be no constraints that explicitly depend on u^α anymore, and as in the case of the vertex operators the null current imposes a covariant set of constraints. In particular, we will choose the constraints

$$T_{ab}{}^\gamma = H_{\alpha\beta} = 0 \quad (3.42)$$

from now on. Indeed, these equations are Lorentz covariant and do not depend on the choice of u^α . This reflects the fact that Majorana-Weyl fermions exist in $2 + 2$ dimensions and, therefore, one can effectively kill one fermionic direction without breaking the Lorentz group. By imposing the constraints (3.42) the only nonzero fields left in the theory are the dilaton and $W_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$. The supergravity algebra obtained after imposing (3.42) reads

$$\begin{aligned} \{\nabla_\alpha, \nabla_\beta\} &= 0, \\ \{\nabla_\alpha, \nabla_{\dot{\beta}}\} &= -2i\nabla_{\alpha\dot{\beta}}, \\ [\nabla_\alpha, \nabla_{\dot{b}}] &= -2iC_{\alpha\beta}W_{\dot{\beta}\dot{\gamma}}{}^{\dot{\delta}}M_{\dot{\delta}}{}^{\dot{\gamma}}, \\ [\nabla_{\dot{a}}, \nabla_{\dot{b}}] &= 0, \\ [\nabla_{\underline{a}}, \nabla_{\underline{b}}] &= C_{\alpha\beta}(W_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}\nabla_{\dot{\gamma}} + \frac{1}{24}\nabla_{(\dot{\alpha}}W_{\dot{\beta}\dot{\gamma}}{}^{\dot{\delta}})M_{\dot{\delta}}{}^{\dot{\gamma}}), \end{aligned} \quad (3.43)$$

So far, we ignored the condition that the complex structure has to be covariantly constant, and one may worry that this condition imposes further constraints on the background fields. The complex structure can be written as

$$I_{ab} = C_{\alpha\beta}\bar{I}_{\dot{\alpha}\dot{\beta}} + C_{\dot{\alpha}\dot{\beta}}I_{\alpha\beta}, \quad (3.44)$$

where $I_{\alpha\beta}$ and $I_{\dot{\alpha}\dot{\beta}}$ are symmetric. In order for I to square to the identity matrix, we need that either $I_{\alpha\beta} = 0$ or $\bar{I}_{\dot{\alpha}\dot{\beta}} = 0$. In our case, the most natural choice that is compatible with (3.43) is to take $\bar{I} = 0$. One immediately verifies that covariant constancy of $I_{\alpha\beta}$ does not lead to any further constraints on the background fields, which validates our original claim that the right-moving part of the new $N = 2$ string does not lead to any new equations.

One may wonder whether or not it is possible to relax some of the conditions given in (3.42) and still arrive at a consistent string theory. It would certainly be interesting to investigate this further. Therefore, when some of the results presented below are to our knowledge still valid after relaxing some of the conditions in (3.42), we will make a comment to this effect. We already notice, however, that the number of degrees of freedom left after imposing (3.42) is in agreement with the analysis of the vertex operators.

3.5 Further tree-level and one-loop results

As a further consistency check of this $N = 2$ string we have made a partial check of the OPE's of the holomorphic $N = 2$ algebra. After imposing (3.42), the tree-level OPE's between the null currents come out correct. Notice that due to the presence of the ρ -field in $\mathcal{J}_{null}^{1/2}$, which satisfies the OPE $e^{i\rho(z)}e^{i\rho(w)} \sim e^{2i\rho(w)}/(z-w) + \dots$, we have to keep terms

with up to two background fields in its OPE (counting ψ^a as $1/2$ background field). The tree-level diagrams for the OPE's of $G, \bar{G}$ with $G, \bar{G}$ and the null current also yield the right results. If we drop the condition $H_{\dot{\alpha}\beta} = 0$ but keep $T_{\underline{ab}\gamma} = 0$, the only vertex in the action contributing to the OPE of G with $\mathcal{J}_{null}^{1/2}$ itself is $-2i \int d^2z D_\alpha y^\beta y_\beta \bar{\Pi}^\beta H_{\dot{\beta}}^\alpha$. The result is proportional to $H_{\dot{\beta}}^\alpha u_\alpha$, which vanishes. If we also drop the constraint $\nabla_\alpha u_\beta = 0$ but keep $\nabla_{(\alpha} u_{\beta)} = 0$, the OPE of G with $\mathcal{J}_{null}^{1/2}$ can still work if we add a suitable total derivative to the action and use the identity $\nabla_{\dot{\alpha}} \nabla^\beta u_\beta \sim H_{\dot{\alpha}}^\beta u_\beta$.

At one-loop level we also considered the diagrams that contribute to the OPE's of $G, \bar{G}, \mathcal{J}_{null}$ with themselves. Again, everything works out correctly. If one assumes $T_{\underline{ab}}^\gamma = 0$ but keeps $H_{\dot{\alpha}\beta}$ unconstrained, there are still many simplifications. For example, the action contains no vertices $\int d^2(\nabla y^\alpha) y^B \psi_{bg}^\varepsilon \psi_{bg}^d$ or $\int d^2 y^\alpha (\nabla y^B) \psi_{bg}^\varepsilon \psi_{bg}^d$, and the OPE's of $G, \bar{G}$ with themselves are still OK. The other one-loop diagrams are more complicated and it seems at first sight quite unlikely that there would be any chance that their sum vanishes unless we impose $H_{\dot{\alpha}\beta} = 0$.

We have not computed any further OPE's. In the case of the heterotic string we found the field equations from the OPE of T with G and T with T , and noticed that these can at the same time be derived using a conventional beta-function calculation. Here, we will assume the same relation holds, and proceed by doing a much faster beta-function calculation. The results will show that the background has to be Ricci-flat. This is the usual condition in order to have a conventional $N = 2$ algebra, strongly suggesting that the anti-holomorphic $N = 2$ algebra will also persist at one-loop. We will therefore not perform any diagrammatic analysis to check the anti-holomorphic algebra either.

3.6 Beta-function calculation

The conventional beta-function calculation can be performed along the same lines as explained in [22]. The idea is to find all UV divergent contributions to the effective action at one-loop, and to cancel the resulting conformal anomaly using the dilaton. In contrast to the case of the heterotic string discussed in [22], there are several different one-loop diagrams that contribute. The new diagrams all have one background Π^A or D^a field sticking out, and two background ψ^a 's, but not all at the same vertex. All of them can be worked out in a straightforward fashion using dimensional regularization. The only point that has to be treated with some care is the fact the some diagrams have cancelling UV and IR divergences, and in order to isolate the UV divergence we have to first subtract out the IR divergence. The results, not assuming any constraints coming from a null current, are given in appendix D. Here we will discuss the results specialized to the case where we impose in addition $T_{\underline{ab}\gamma} = H_{\dot{\alpha}\beta} = 0$.

Inserting $T_{\underline{ab}\gamma} = H_{\dot{\alpha}\beta} = 0$ into the results in appendix D yields the equations

$$\begin{aligned}
0 &= \frac{1}{2} \nabla^\varepsilon R_{\underline{ca}\underline{f}\underline{g}} + \frac{1}{2} \nabla_{\underline{a}} T_{\underline{g}\underline{f}}^{\dot{\alpha}} \nabla_{\dot{\alpha}} (\phi + \bar{\phi}) \\
&\quad + \frac{1}{2} T_{\underline{g}\underline{f}}^{\dot{\beta}} \nabla_{\underline{a}} \nabla_{\dot{\beta}} (\phi + \bar{\phi}) + \frac{1}{2} R_{\underline{da}\underline{f}\underline{g}} \nabla_{\underline{a}} (\phi + \bar{\phi}) \\
0 &= \frac{1}{2} \nabla^\varepsilon R_{\underline{c}\alpha\underline{f}\underline{g}} + \frac{1}{2} \nabla_{\dot{\alpha}} T_{\underline{g}\underline{f}}^{\dot{\beta}} \nabla_{\dot{\beta}} (\phi + \bar{\phi})
\end{aligned} \tag{3.45}$$

$$-\frac{1}{2}T_{gf}^{\dot{\beta}}\nabla_{\alpha}\nabla_{\dot{\beta}}(\phi+\bar{\phi})+\frac{1}{2}R_{d\alpha fg}\nabla_d(\phi+\bar{\phi}) \quad (3.46)$$

$$0 = T_{gf}^{\dot{\beta}}\bar{\nabla}^2(\phi+\bar{\phi}) \quad (3.47)$$

$$\begin{aligned} 0 = & \frac{1}{2}\nabla^{\epsilon}\nabla_{\epsilon}T_{gf\dot{\alpha}}+\frac{1}{2}R_{cbgf}T_{cb\dot{\alpha}}-\frac{1}{2}T_{[f|b\dot{\alpha}}T_{g]b}^{\dot{\gamma}}\nabla_{\dot{\gamma}}(\phi+\bar{\phi}) \\ & +\frac{1}{2}\nabla_dT_{gf\dot{\alpha}}\nabla_d(\phi+\bar{\phi})-T_{[f|e\dot{\alpha}}H_{g]ed}\nabla_d(\phi+\bar{\phi}) \end{aligned} \quad (3.48)$$

$$0 = \nabla_{\beta}\nabla_a(\phi+\bar{\phi}) \quad (3.49)$$

$$0 = \nabla_{\dot{\beta}}\nabla_a(\phi+\bar{\phi}). \quad (3.50)$$

Quite interestingly, using (3.49) and (3.50) one can show that all the dilaton terms in (3.45), (3.46) and (3.48) vanish. For example, (3.49) and (3.50) imply $\nabla_b\nabla_a(\phi+\bar{\phi})=0$ and by antisymmetrizing in a and b we find $T_{ab}^{\dot{\gamma}}\nabla_{\dot{\gamma}}(\phi+\bar{\phi})=0$. From here we conclude that $\nabla_{\dot{\gamma}}(\phi+\bar{\phi})=0$. Similarly, $[\nabla_{\epsilon},\nabla_{\beta}]\nabla_a(\phi+\bar{\phi})=0$ implies $R_{d\alpha bc}\nabla_d(\phi+\bar{\phi})=0$, and $[\nabla_{\epsilon},\nabla_b]\nabla_a(\phi+\bar{\phi})=0$ implies $R_{dabc}\nabla_d(\phi+\bar{\phi})=0$, etc. Equation (3.47) is now satisfied since the dilaton, in addition to being chiral, also satisfies $\nabla^2\phi+\nabla^{\alpha}\phi\nabla_{\alpha}\phi=\bar{\nabla}^2\bar{\phi}+\bar{\nabla}^{\dot{\alpha}}\bar{\phi}\bar{\nabla}_{\dot{\alpha}}\bar{\phi}=0$. These relations can be seen by examining the OPE of G with itself and also follow from the low-energy effective action presented in [21].

The precise interpretation of these statements is not entirely clear. We already noticed that $\phi+\bar{\phi}$ is a sum of a target space holomorphic and anti-holomorphic function. The results above show it is also the sum of a world-sheet holomorphic and anti-holomorphic function. The fact that the dilaton decouples from the field equations may either mean that it is always forced to be a constant, or that it plays the role of some kind of Lagrange multiplier in the theory. It would be very interesting to see if this can be related to the results of [37] who showed that in order to restore target space covariance in the $N=2$ string one has to let the dilaton transform in a non-trivial way under target space Lorentz transformations. We have not performed a detailed analysis of Lorentz anomalies in our $N=2$ string, but it is possible that the one-loop anomalies can only be canceled by assigning a non-trivial transformation rule to the dilaton. We leave this issue to a future discussion and will for the time being simply drop the dilaton from our considerations.

Without the dilaton, the following short list of equations remains

$$0 = \nabla^{\epsilon}R_{c\alpha fg} \quad (3.51)$$

$$0 = \nabla^{\epsilon}R_{\epsilon\alpha fg} \quad (3.52)$$

$$0 = \nabla^{\epsilon}\nabla_{\epsilon}T_{gf\dot{\alpha}}+R_{cbgf}T_{cb\dot{\alpha}} \quad (3.53)$$

We can work this further out by expressing everything in terms of $W_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$, using (3.43). Before doing so, notice that (3.43) in particular implies that $R_{abcd}\sim C_{\alpha\beta}C_{\gamma\delta}\nabla_{(\dot{\alpha}}W_{\dot{\beta}\dot{\gamma}\dot{\delta})}$, so that the Ricci tensor vanishes, as we expected from the presence of the ‘‘conventional’’ anti-holomorphic $N=2$ algebra.

Coming back to (3.51)-(3.53), we find that (3.51) is a direct consequence of $\{\nabla^{\beta},\nabla^{\dot{\beta}}\}W_{\dot{\beta}\dot{\gamma}\dot{\delta}}=0$, and that (3.52) follows from $[\nabla^{\beta},\bar{\nabla}^2]W_{\dot{\gamma}\dot{\delta}\dot{\alpha}}=0$. Equation (3.53)

is the only one that yields a non-trivial result, namely

$$-\frac{1}{4}\nabla^{\dot{\alpha}}(W^{\dot{\rho}}_{(\dot{\alpha}\dot{\beta}}W_{\dot{\gamma}\dot{\delta})\dot{\rho}})+2\nabla_{\dot{\alpha}}(W_{\dot{\rho}\dot{\beta}\dot{\gamma}}W^{\dot{\alpha}\dot{\rho}}_{\dot{\delta}})=0. \quad (3.54)$$

This implies finally

$$\nabla^{\dot{\gamma}}(W^{\dot{\rho}}_{\dot{\gamma}(\dot{\delta}}W_{\dot{\lambda})\dot{\alpha}\dot{\rho}})=0. \quad (3.55)$$

This equation is the only additional piece of information that we have in addition to the supergravity algebra with $W_{\alpha\beta\gamma}=H_{\underline{a}}=0$. This equation may still receive corrections from the internal sector.

4 Discussion

We have constructed in this article a "heterotic type II" $N=2$ string theory that has manifest target space supersymmetry, and we have shown by computing the beta functions and by checking the OPE's of the superconformal algebra perturbatively that its low energy theory is $N=1$ self-dual supergravity. We have, thus, solved the problem of finding a consistent quantum theory that has SD SUGRA as its low energy limit.

This new $N=2$ string was obtained by combining a conventional $N=2$ string in one sector with a Green-Schwarz-Berkovits string in the other sector. The Berkovits string has an $N=2$ superconformal invariance but it is equivalent to the conventional $N=1$ *RNS* string through a non-local field redefinition (one first embeds the $N=1$ string into an $N=2$ string and then performs a non-local field redefinition that involves the $N=1$ ghost fields). So, in this sense our new "heterotic type II" string is actually a new $(1,2)$ string. In the conventional heterotic $N=2$ strings one has to gauge a null current. Depending on the null current one finds that the target space is effectively either two dimensional or three dimensional. In our new $N=2$ string we also have to gauge a null current. We chose our null current such that it commutes with target space supersymmetry. With this choice the effective target space is still four dimensional, but a fermionic direction is effectively gauged away, leading to self-dual superspace.

Let us indicate what happens if one had chosen a fermionic spinor u^{α} instead of a bosonic one. A fermionic null spinor can always be written as λv^{α} , where λ is anticommuting and v^{α} is a bosonic spinor. In order to obtain self-dual supergravity, the following conditions are sufficient: the vertex operators and gauge invariances should not depend on λ , and λv^{α} should be covariantly constant. It is not clear to us how the theory would implement the first condition, and whether λ is a new independent object or should be expressed in terms of the world-sheet fields. Nevertheless, under these conditions all results in the paper also apply to the case of a fermionic null spinor, except that the central charge of the internal sector now has to be 12 rather than 6.

An interesting question is whether it is possible to formulate our model in *RNS* variables. One immediate problem is that space-time fermions are very difficult to deal with in the *RNS* formalism. Conversely, it will be interesting to know whether one can translate the usual heterotic $(1,2)$ string with the conventional null vector in our formalism. Obviously, this is desirable only for the cases where the conventional $(1,2)$ strings do have target space supersymmetry. One approach would be to follow the field redefinition from the *RNS* string to the Green-Schwarz-Berkovits string. This involves,

however, non-local field redefinitions, so generically the null current will be non-local. It may be, though, that in the cases where the conventional $(1, 2)$ strings do have target space supersymmetry a yet new field redefinition exists such that the null current becomes local. This question definitely deserves further investigation.

In the usual $(1, 2)$ strings the four dimensional target space may be interpreted as a $(2, 2)$ -brane moving in a 12 dimensional space-time of signature $(10, 2)$ (thus, indicating connections with F -theory). The new $N = 2$ string we constructed has as internal space an $N = 2$ SCFT with central charge $c = 6$ or $c = 12$, that typically involve 4 or 8 bosons. It seems tempting to interpret our four dimensional target space as a $(2, 2)$ -brane moving in a $(6, 2)$ or $(10, 2)$ space. Then the bosons would describe fluctuations in the transverse directions. This interpretation becomes problematic, though, as soon as one computes the spectrum. Only half of those scalars survive the null projection. So, at least at first sight, it seems unlikely that such a picture is correct⁵.

An alternative way to derive the low energy effective action is to compute scattering amplitudes. For $N = 2$ strings the most convenient way to do such calculations is to use the reformulation as $N = 4$ topological strings developed by Berkovits and Vafa [40] and subsequently used by Berkovits[41], Ooguri and Vafa [42]. It will be interesting to perform such a calculation not only in order to confirm the results we obtained using sigma model methods but, more importantly, to study the internal sector, which is hard to analyze using sigma model techniques.

One may use the techniques we developed in this article to construct new $N = 1$ string theories. For instance, one could start from the type II string in the Green-Schwarz-Berkovits formalism[20] (which is the $N = (1, 1)$ model in the RNS formalism), change the signature of space time to $(2, 2)$ and introduce a null current symmetrically in both sectors. A chiral null current contributes $c = \pm 3$ to the central charge. Taking into account that the non-compact sector contributes another $c = -3$ and the $N = 2$ ghosts $c = -6$, we find that one would need an internal sector with $c = 6$ or $c = 12$. As such one could take a Calabi-Yau 2- or 4-fold. Alternatively, one may gauge (symmetrically) a full $N = 2$ null multiplet, in which case we would need a $c = 15$ internal space which may be taken to be a Calabi-Yau 5-fold. Yet more models can be constructed by asymmetric choices of the null currents. One can easily find appropriate target space supersymmetric null currents for each of these cases. All these models are difficult to construct in the RNS formalism. Altogether, we have seen many examples where the new Green-Schwarz-Berkovits techniques are very powerful. In [21] we witnessed string theory selecting a particular off-shell supergravity and in this article we obtained a new $N = 2$ string theory that has self-dual supergravity as an effective field theory. It remains to be seen what the significance of these new $N = 1$ models is. We intend to return to these issues in the future.

⁵There still exists the possibility that our model with $c = 12$ describes an $N = 1$ self-dual $(2, 2)$ -brane moving in the spacetime of 12 dimensional $N = 1$ self-dual supergravity in such a way that the self-duality effectively freezes half of the eight transverse directions. If such a scenario is correct then the reduction from 12 to 4 dimensions should be made in such a way that only an $N = 1$ supersymmetry survives. Similar ideas have been advocated by Ketov in [38]. Let us also mention that our model may be related to theories in spacetimes with signature $(11, 3)$ proposed in [39] since we also have 3 timelike coordinates (two non-compact and one compact, the ρ field).

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A Conventions

Let y^1, y^2, y^3, y^4 be real coordinates in a space of $(2, 2)$ signature,

$$ds^2 = (dy^1)^2 + (dy^2)^2 - (dy^3)^2 - (dy^4)^2 \quad (\text{A.1})$$

We define complex coordinates as follows

$$\begin{aligned} x^1 &= \frac{1}{\sqrt{2}}(y^1 + iy^2); & x^2 &= \frac{1}{\sqrt{2}}(y^3 + iy^4) \\ x^{\bar{1}} &= \frac{1}{\sqrt{2}}(y^1 - iy^2); & x^{\bar{2}} &= \frac{1}{\sqrt{2}}(y^3 - iy^4) \end{aligned} \quad (\text{A.2})$$

In these coordinates the line element is equal to

$$ds^2 = \eta_{\underline{a}\underline{b}} dx^{\underline{a}} dx^{\underline{b}} \quad (\text{A.3})$$

where the underlined indices denote both an unbar and bar index, $\underline{a} = (a, \bar{a})$, (for spinors $\underline{\alpha} = (\alpha, \dot{\alpha})$) and

$$\eta_{\underline{a}\underline{b}} = \begin{pmatrix} 0 & \eta_{a\bar{b}} \\ \eta_{\bar{a}b} & 0 \end{pmatrix} \quad (\text{A.4})$$

with

$$\eta_{a\bar{b}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.5})$$

The Dirac matrices in complex coordinates and for this specific complex structure have been worked out in [2] (appendix B in the second paper),

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \tilde{\sigma}^a & 0 \end{pmatrix}, \quad \gamma^{\bar{a}} = \begin{pmatrix} 0 & \sigma^{\bar{a}} \\ \tilde{\sigma}^{\bar{a}} & 0 \end{pmatrix}, \quad (\text{A.6})$$

where the σ matrices are given by the following matrices (σ with upper indices and $\tilde{\sigma}$ with lower indices, see (A.9))

$$\sigma^1 = -i\sqrt{2}a(-, \dot{+}), \sigma^2 = -\sqrt{2}a(-, \dot{-}), \sigma^{\bar{1}} = -i\sqrt{2}a(+, \dot{-}), \sigma^{\bar{2}} = \sqrt{2}a(+, \dot{+}) \quad (\text{A.7})$$

$$\tilde{\sigma}^1 = i\sqrt{2}a(\dot{-}, +), \tilde{\sigma}^2 = -\sqrt{2}a(\dot{+}, +), \tilde{\sigma}^{\bar{1}} = i\sqrt{2}a(\dot{+}, -), \tilde{\sigma}^{\bar{2}} = \sqrt{2}a(\dot{-}, -), \quad (\text{A.8})$$

where the $a(\alpha, \beta)$ denotes a 2×2 matrix with 1 at (α, β) and zero elsewhere. One may use these σ matrices to convert vector indices to a pair of two spinor indices. This is possible since $SO(2, 2) = SL(2, R) \otimes SL(2, R)$. Explicitly,

$$V^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} V_{\underline{a}} (\sigma^{\underline{a}})^{\alpha\dot{\alpha}}; \quad V^{\underline{a}} = \frac{1}{\sqrt{2}} (\tilde{\sigma}^{\underline{a}})_{\dot{\alpha}\alpha} V^{\alpha\dot{\alpha}}. \quad (\text{A.9})$$

In particular,

$$V^{\alpha\dot{\alpha}} = \begin{pmatrix} V_2 & -iV_1 \\ -iV_1 & -V_2 \end{pmatrix} \quad (\text{A.10})$$

One can check that the σ matrices satisfy the following relations

$$\frac{1}{2}\eta_{\underline{a}\underline{b}}(\tilde{\sigma}^{\underline{a}})_{\dot{\alpha}\alpha}(\sigma^{\underline{b}})_{\beta\dot{\beta}} = C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}}, \quad (\text{A.11})$$

$$\frac{1}{2}(\sigma^{\underline{a}})^{\alpha\dot{\alpha}}(\sigma^{\underline{b}})_{\alpha\dot{\alpha}} = \eta^{\underline{a}\underline{b}}, \quad (\text{A.12})$$

where $C_{\alpha\beta} = C_{\dot{\alpha}\dot{\beta}}$ are antisymmetric tensors with $C_{+-} = 1$. We will raise and lower spinor indices with these tensors. Our convention is the “down-hill” rule from the left to the right. These identities allow one to switch freely (with no extra factors) from one set of indices to the other. For example, one may check that

$$V^{\underline{a}}U_{\underline{a}} = V^{\alpha\dot{\alpha}}U_{\alpha\dot{\alpha}} \quad (\text{A.13})$$

From (A.10) we read off the $N = 2$ complex conjugation rules for vectors

$$\overline{V^{++}} = -V^{--}; \quad \overline{V^{+-}} = -V^{-+} \quad (\text{A.14})$$

In addition, we have

$$\overline{D_+} = -D_- \quad (\text{A.15})$$

and similar rules for the dotted indices.

Our conventions for covariant derivatives, torsion, curvature, etc. are as follows

$$\nabla_A = E_A^M \partial_M + \omega_{A\beta}^{\gamma} M_{\gamma}^{\beta} + \omega_{A\dot{\beta}}^{\dot{\gamma}} M_{\dot{\gamma}}^{\dot{\beta}} + \Gamma_A Y, \quad (\text{A.16})$$

$$[\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + R_{AB\gamma}^{\delta} M_{\delta}^{\gamma} + R_{AB\dot{\gamma}}^{\dot{\delta}} M_{\dot{\delta}}^{\dot{\gamma}} + F_{AB} Y, \quad (\text{A.17})$$

where E_A^M , $\omega_{A\beta}^{\gamma}$ and Γ_A are the vielbein, the spin connection and the $U(1)$ connection, respectively. T_{AB}^C , $R_{AB\gamma}^{\delta}$ and F_{AB} are the torsion, the curvature tensor and the $U(1)$ curvature, respectively. M_{δ}^{γ} are the generators of the Lorentz group.

B Analysis of VO's in components

We show in this appendix, using a detailed component analysis, that the null current may be used to eliminate the chiral part of both the spacetime and the internal sector vertex operators. We start from the latter. By hitting equation (2.29) with $D_{\dot{\alpha}}$ and going over to momentum space one gets

$$u^{\alpha} p_{\alpha\dot{\alpha}} M_1 = 0 \quad (\text{B.1})$$

The general solution of this equation that does not set M_1 equal to zero is

$$p_{\alpha\dot{\alpha}} = p' u_{\alpha} v_{\dot{\alpha}} \quad (\text{B.2})$$

where p' is arbitrary and $v_{\dot{\alpha}}$ is a second commuting spinor. Clearly, this immediately implies that $p^2 = 0$.

The component expansion of M_1 is of the form

$$M_1 = a(x^-) + \theta_{\alpha}\xi^{\alpha}(x^-) + \theta^2 b(x^-) \quad (\text{B.3})$$

where $x_{\alpha\dot{\alpha}}^- = x_{\alpha\dot{\alpha}} - i\theta_{\alpha}\theta_{\dot{\alpha}}$ (In our conventions $D_{\alpha} = \partial_{\alpha} - i\theta^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}$, $D_{\dot{\alpha}} = \partial_{\dot{\alpha}} - i\theta^{\alpha}\partial_{\alpha\dot{\alpha}}$). Equation (2.26) implies the usual field equations for the component fields

$$\square a = \partial_{\alpha\dot{\alpha}}\xi^{\alpha} = b = 0 \quad (\text{B.4})$$

In addition (2.29) yields

$$u^{\alpha}\xi_{\alpha} = 0, \quad (\text{B.5})$$

which implies

$$\xi^{\alpha} = \xi u^{\alpha}, \quad (\text{B.6})$$

where ξ is arbitrary anticommuting variable. So the on-shell multiplet reads

$$M_1 = a(x^-) + \theta_{\alpha}u^{\alpha}\xi(x^-) \quad (\text{B.7})$$

We will now show that the gauge invariance in (2.30) is just enough to remove these degrees of freedom. Indeed, starting from a general superfield Y_1 and imposing the condition that the $u^{\alpha}D_{\alpha}Y_1$ is chiral one obtains

$$\delta M_1 = u^{\alpha}\psi_{\alpha} + \theta^{\alpha}\theta^{\dot{\alpha}}(-i\partial_{\alpha\dot{\alpha}}u^{\beta}\psi_{\beta}) + u^{\alpha}\theta_{\alpha}T \quad (\text{B.8})$$

where we have also used the on-shell condition (B.2). ψ_{α} is an arbitrary commuting spinor and T is an arbitrary anticommuting variable. Clearly, these on-shell gauge transformations eliminate completely M_1 .

Let us now consider the 4D vertex operator. The component expansion of the gauge invariant field strengths are as follows

$$\begin{aligned} W_{\alpha} &= \lambda_{\alpha} + \theta_{\alpha}D + \theta^{\beta}f_{\beta\alpha} - i\theta^2\partial_{\alpha}{}^{\dot{\alpha}}\lambda_{\dot{\alpha}} \\ W_{\dot{\alpha}} &= \lambda_{\dot{\alpha}} + \theta_{\dot{\alpha}}D + \theta^{\beta}f_{\beta\dot{\alpha}} - i\bar{\theta}^2\partial^{\alpha}{}_{\dot{\alpha}}\lambda_{\alpha} \end{aligned} \quad (\text{B.9})$$

where the same field D appears in both expansions due to the identity $D^{\alpha}W_{\alpha} = D^{\dot{\alpha}}W_{\dot{\alpha}}$. Equations (2.33) read

$$\partial_{\alpha}{}^{\dot{\alpha}}\lambda_{\dot{\alpha}} = \partial^{\alpha}{}_{\dot{\alpha}}\lambda_{\alpha} = 0, \quad (\text{B.10})$$

$$\partial_{\alpha\dot{\alpha}}D = \partial_{\alpha\dot{\alpha}}f^{\alpha}{}_{\beta} = \partial_{\alpha\dot{\alpha}}f^{\dot{\alpha}}{}_{\beta} = 0 \quad (\text{B.11})$$

Equation (2.34) implies (on-shell)

$$u^{\alpha}\lambda_{\alpha} + u^{\alpha}\theta_{\alpha}D + u^{\alpha}\theta^{\beta}f_{\alpha\beta} = 0 \quad (\text{B.12})$$

This equation is then solved by

$$\lambda^{\alpha} = \lambda u^{\alpha}; \quad D = 0; \quad f_{\alpha\beta} = f u_{(\alpha}u_{\beta)} \quad (\text{B.13})$$

where λ and f and on-shell fermionic and bosonic components. So W_α on-shell reads

$$W_\alpha = \lambda(x^-)u_\alpha + \theta^\alpha f(x^-)u_{(\alpha}u_{\beta)}. \quad (\text{B.14})$$

It is now easy to show that the on-shell gauge invariance in (2.36) removes these degrees of freedom. Indeed, working out (2.36) in components yields

$$\delta W_\alpha = \frac{1}{2}u_\alpha(F + \theta^\beta \partial_{\beta\dot{\beta}}\chi^{\dot{\beta}} - i\theta^\beta \theta^{\dot{\beta}} \partial_{\beta\dot{\beta}}F) \quad (\text{B.15})$$

which can clearly remove λ and f (in momentum space the θ^β term is proportional to $u_\alpha u_\beta v_{\dot{\alpha}} \chi^{\dot{\alpha}}$).

C Conservation of the supersymmetry currents.

In this appendix we analyze the equations that follow by requiring conservation of the supersymmetry currents, $\partial H^\pm = 0$. The terms linear in the fermion field ψ^a have been analyzed in the main text. The terms cubic in the fermions yield the following equations

$$(\eta_{a\bar{e}} \pm I_{a\bar{e}})(R_{\bar{e}dbc} + 2H_{f[eb}H_{c]df} + 2\nabla_{\underline{d}}H_{\bar{c}be} - 2\nabla_{\bar{e}}H_{cb\bar{d}}) - \\ -6H_{\bar{a}de}C_{\bar{e}bc}^\pm + \nabla_{\underline{d}}C_{\bar{c}ba}^\pm + \text{permutations with signs in } \underline{a}, \underline{b}, \underline{c} = 0 \quad (\text{C.1})$$

$$(\eta_{a\bar{e}} \pm I_{a\bar{e}})(R_{\bar{e}dbc} + 2\nabla_{\underline{d}}H_{\bar{c}be}) + \\ + \nabla_{\bar{d}}C_{\bar{c}ba}^\pm + \text{permutations with signs in } \underline{a}, \underline{b}, \underline{c} = 0, \quad (\text{C.2})$$

$$(\eta_{a\bar{e}} \pm I_{a\bar{e}})(\nabla_{\bar{e}}T_{cb\bar{d}} - 4T_{\bar{b}f\bar{d}}H_{\bar{c}f\bar{e}}) - \\ -3T_{\bar{a}e\bar{d}}C_{\bar{e}bc}^\pm + \text{permutations with signs in } \underline{a}, \underline{b}, \underline{c} = 0. \quad (\text{C.3})$$

We start our analysis with (C.2). By using the identity

$$R_{\underline{a}[bcd]} = -8\nabla_{\underline{a}}H_{bcd} \quad (\text{C.4})$$

one finds that the complex structure independent part of $C_{\bar{c}ba}^\pm$ is equal to $(-2/3)H_{\bar{c}ba}$. To obtain the complex structure dependent part of $C_{\bar{c}ba}^\pm$ we observe that the term containing I and the curvature R in (C.2) can be rewritten as a linear combination of $I_{a\bar{e}}R_{\bar{d}[ebc]}$ and $[\nabla_{\bar{d}}, \nabla_{\bar{e}}]I_{ab}$. The first piece can be rewritten using the cyclic identity (C.4), the second piece using (3.24) and (3.25). After some straightforward algebra one obtains that the complex structure dependent part of $C_{\bar{c}ba}^\pm$ is equal to $\pm I_{\underline{d}[\bar{c}}H_{ba]\bar{d}}/3$. Putting these results together we get that equation (C.2) is solved by

$$C_{\bar{c}ba}^\pm = \frac{1}{3}(-2H_{\bar{c}ba} \pm I_{\underline{d}[\bar{c}}H_{ba]\bar{d}}) \quad (\text{C.5})$$

An alternative route to obtain this result is to first use (3.21) to eliminate the curvature from (C.2) and then use the following identities involving torsions

$$T_{\bar{c}b\bar{a}}C_{\delta\alpha} + \text{cyclic in } \underline{a}, \underline{b}, \underline{c} = -i\nabla_{\bar{\delta}}H_{\bar{b}ca} \quad (\text{C.6})$$

$$C_{\alpha\bar{\delta}}T_{\bar{e}c\bar{a}}I_{\bar{e}b} + T_{\bar{a}c\bar{e}}C_{\delta\bar{e}}I_{\bar{e}b} - \underline{a} \leftrightarrow \underline{b} = -i\nabla_{\bar{\delta}}(I_{\underline{d}[\bar{a}}H_{\bar{b}]\bar{c}d}) \quad (\text{C.7})$$

The latter is obtained by differentiating (3.25) by ∇_δ and using

$$\begin{aligned} T_{\delta\dot{\alpha},\underline{g},\dot{\epsilon}}C_{\alpha\epsilon}I_{\underline{e}\underline{b}} - \underline{a} \leftrightarrow \underline{b} &= (C_{\alpha\delta}T_{\underline{e}\underline{c}\dot{\alpha}}I_{\underline{e}\underline{b}} + T_{\underline{a}\underline{c}\dot{\epsilon}}C_{\delta\epsilon}I_{\underline{e}\underline{b}} + i\nabla_\gamma H_{\gamma(\delta}I_{\alpha)\dot{\alpha},\underline{b}} - \underline{a} \leftrightarrow \underline{b}) \\ &\quad - 4i\nabla_\gamma H_{\gamma\delta}I_{\underline{a}\underline{b}}, \end{aligned} \quad (C.8)$$

which, in turn, can be obtained by using (3.30).

Equations (C.1) and (C.3) should now automatically be satisfied. The complex structure independent part of (C.3) reads

$$\nabla_{[\underline{a}}T_{\underline{b}\underline{c}]}^{\underline{\delta}} - 2H_{\underline{e}[\underline{a}\underline{b}}T_{\underline{c}]\underline{e}}^{\underline{\delta}} \quad (C.9)$$

which is equal to zero by using the Bianchi identities. The complex structure dependent part is equal to

$$I_{[\underline{a}]}^{\underline{e}}\nabla_{\underline{e}}T_{\underline{b}\underline{c}]}^{\underline{\delta}} - 2I_{\underline{e}\underline{f}}H_{[\underline{b}\underline{c}]}^{\underline{e}}T_{\underline{a}]\underline{f}\underline{\delta}}. \quad (C.10)$$

To show that this is equal to zero one first rewrites the derivative of the torsion using (C.9). Next one can “partially integrate” the derivative in these terms using (3.26), after which one can use (3.25) to work things out and obtain zero.

It remains to demonstrate (C.1). This is most easily done using the cyclic identity

$$R_{\underline{a}[\underline{b}\underline{c}\underline{d}]} = -4\nabla_{\underline{a}}H_{\underline{b}\underline{c}\underline{d}} \quad (C.11)$$

which one can prove using the explicit form of R . It can be used to rewrite the complex structure independent part of (C.1) as a linear combination of $\nabla_{[\underline{a}}H_{\underline{b}\underline{c}\underline{d}]}$ and $H_{\underline{e}[\underline{a}\underline{b}}H_{\underline{c}\underline{d}]\underline{e}}$. The second of these terms vanishes identically, due to lack of indices in dimension 4, and the first term is then zero by virtue of the Bianchi identity for H . The complex structure dependent part of (C.1) can be analyzed as follows. The term $I_{\underline{a}\underline{e}}R_{\underline{e}\underline{d}\underline{b}\underline{c}}$ can be rewritten as a linear combination of $I_{\underline{a}\underline{e}}R_{\underline{d}[\underline{e}\underline{b}\underline{c}]}$ and $[\nabla_{\underline{a}}, \nabla_{\underline{c}}]I_{\underline{b}]\underline{a}}$. The first term can be manipulated using (C.11), the second using (3.25). After some manipulations we then end up with a term proportional to $I_{\underline{a}\underline{e}}\nabla_{\underline{e}}H_{\underline{b}\underline{c}\underline{d}}$ and one proportional to $I_{\underline{a}\underline{f}}H_{\underline{e}[\underline{f}\underline{b}}H_{\underline{c}\underline{d}]\underline{e}}$, and both vanish for the reasons mentioned above.

D Beta-functions without the null current

For completeness, we present here the full result of the beta-function calculation. If we were to study the (1,2)-string with a different choice of null current, one would first determine the additional constraints it implies and subsequently insert them in the expressions below in order to determine the field equations for that string theory.

$$\begin{aligned} 0 &= R_{\underline{c}\underline{b}\underline{g}\underline{f}}H_{\underline{c}\underline{a}\underline{b}} - T_{\underline{c}\underline{b}}^{\underline{a}}T_{\underline{g}\underline{f}\underline{a}}H_{\underline{c}\underline{a}\underline{b}} + 2\nabla_{\underline{c}}T_{\underline{b}\underline{g}\underline{f}}H_{\underline{c}\underline{a}\underline{b}} \\ &\quad + 4iT_{\underline{c}\underline{f}\underline{a}}T_{\underline{c}\underline{g}\underline{\alpha}} - 2T_{\underline{c}\underline{f}\underline{b}}T_{\underline{c}\underline{g}\underline{d}}H_{\underline{b}\underline{a}\underline{d}} - 2T_{\underline{c}\underline{f}\underline{b}}T_{\underline{d}\underline{g}\underline{b}}T_{\underline{d}\underline{c}\underline{a}} \\ &\quad - 4iT_{\underline{c}\underline{g}\underline{\alpha}}T_{\underline{c}\underline{f}\underline{\alpha}} + 2T_{\underline{c}\underline{g}\underline{b}}T_{\underline{c}\underline{f}\underline{d}}H_{\underline{b}\underline{a}\underline{d}} + 2T_{\underline{c}\underline{g}\underline{b}}T_{\underline{d}\underline{f}\underline{b}}T_{\underline{d}\underline{c}\underline{a}} \\ &\quad - \frac{1}{4}T_{\underline{g}\underline{f}}^{\underline{\beta}}R_{\underline{\beta}\underline{c}\underline{a}}^{\underline{c}} + \frac{i}{2}T_{\underline{g}\underline{f}}^{\underline{\alpha}}T_{\underline{\alpha}\underline{\beta},\underline{a}}^{\underline{\beta}} + \frac{i}{2}T_{\underline{g}\underline{f}}^{\underline{\alpha}}T_{\underline{\alpha}\underline{\beta},\underline{a}}^{\underline{\beta}} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}T_{\underline{ca}}{}^B R_{\underline{cBfg}} + \frac{1}{2}\nabla^\varepsilon R_{\underline{c}a f g} - \frac{1}{2}R_{\underline{ac}}{}^b T_{\underline{g}f b} \\
& + \frac{1}{2}T_{\underline{eab}}T_{\underline{ebc}}T_{\underline{gfa}} - \frac{1}{2}\nabla^\varepsilon T_{\underline{ca}}{}^b T_{\underline{gfb}} - T_{\underline{cab}}\nabla_\varepsilon T_{\underline{gfb}} \\
& + \frac{1}{2}\nabla^\varepsilon \nabla_\varepsilon T_{\underline{gfa}} - \frac{1}{2}T_{\underline{ba}[f}T_{\underline{g]b}}{}^\alpha \nabla_\alpha(\phi + \bar{\phi}) + \frac{1}{2}\nabla_\alpha T_{\underline{g}f}{}^\alpha \nabla_\alpha(\phi + \bar{\phi}) \\
& + \frac{1}{2}T_{\underline{g}f}{}^\alpha \nabla_\alpha \nabla_\alpha(\phi + \bar{\phi}) + \frac{1}{2}R_{\underline{d}a f g} \nabla_d(\phi + \bar{\phi}) + H_{\underline{e[d}f}H_{\underline{g]ae}} \nabla_d(\phi + \bar{\phi}) \\
& + \nabla_\alpha H_{\underline{g}f d} \nabla_d(\phi + \bar{\phi}) - \nabla_d H_{\underline{g}f a} \nabla_d(\phi + \bar{\phi}) \\
0 = & \frac{1}{2}R_{\underline{cbgf}}T_{\underline{cb}\alpha} + \frac{1}{2}T_{\underline{cb}}{}^a T_{\underline{gfa}}T_{\underline{cb}\alpha} - \nabla_\varepsilon T_{\underline{bgf}}T_{\underline{cb}\alpha} \\
& + T_{\underline{ca}[f}T_{\underline{g]cd}}T_{\underline{ad}\alpha} + 2T_{\underline{ac}[f}T_{\underline{g]dc}}T_{\underline{da}\alpha} + \frac{1}{2}\nabla^\varepsilon \nabla_\varepsilon T_{\underline{gfa}} \\
& - \frac{1}{2}T_{\underline{fb}\alpha}T_{\underline{gb}}{}^\delta \nabla_\delta(\phi + \bar{\phi}) + \frac{1}{2}\nabla_d T_{\underline{gfa}} \nabla_d(\phi + \bar{\phi}) - T_{\underline{fe}\alpha}H_{\underline{ged}} \nabla_d(\phi + \bar{\phi}) \\
& + \frac{1}{2}T_{\underline{gb}\alpha}T_{\underline{fb}}{}^\delta \nabla_\delta(\phi + \bar{\phi}) + T_{\underline{ge}\alpha}H_{\underline{fed}} \nabla_d(\phi + \bar{\phi}) \\
0 = & \frac{1}{2}\nabla^\varepsilon R_{\underline{c}a f g} - \frac{1}{2}R_{\underline{ac}b}T_{\underline{gfb}} + \frac{1}{2}\nabla_\alpha T_{\underline{g}f}{}^\delta \nabla_\delta(\phi + \bar{\phi}) \\
& - \frac{1}{2}T_{\underline{g}f}{}^\delta \nabla_\alpha \nabla_\delta(\phi + \bar{\phi}) + \frac{1}{2}R_{\underline{d}a f g} \nabla_d(\phi + \bar{\phi}) + \nabla_\alpha H_{\underline{g}f d} \nabla_d(\phi + \bar{\phi}) \\
0 = & 2\nabla_\alpha H_{\dot{\alpha}\beta} - \nabla_\beta \nabla_\alpha(\phi + \bar{\phi}) \\
0 = & 2\nabla_{\dot{\alpha}} H_{\dot{\beta}\alpha} + \nabla_{\dot{\beta}} \nabla_\alpha(\phi + \bar{\phi})
\end{aligned} \tag{D.1}$$

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