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Title: COMBINATORIAL AND ALGORITHM ASPECTS OF HYPERBOLIC POLYNOMIALS

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Combinatorial and algorithmic aspects of hyperbolic polynomials

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Abstract

Let $p(x_1,...,x_n) = \sum_{(r_1,...,r_n)\in I_{n,n}} a_{(r_1,...,r_n)} \prod_{1\leq i\leq n} x_i^{r_i}$ be homogeneous polynomial of degree n in n real variables with integer nonnegative coefficients. The support of such polynomial $p(x_1,...,x_n)$ is defined as $supp(p) = \{(r_1,...,r_n) \in I_{n,n} : a_{(r_1,...,r_n)} \neq 0\}$. The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p . We study the following decision problems , which are far-reaching generalizations of the classical perfect matching problem , :

- **Problem 1 .** Consider a homogeneous polynomial $p(x_1,...,x_n)$ of degree n in n real variables with nonnegative integer coefficients given as a black box (oracle) . *Is it true that* $(1,1,..,1) \in supp(p)$ *?*
- **Problem 2 .** Consider a homogeneous polynomial $p(x_1,...,x_n)$ of degree n in n real variables with nonnegative integer coefficients given as a black box (oracle) . *Is it true that* $(1,1,..,1) \in CO(supp(p))$ *?*

We prove that for hyperbolic polynomials these two problems are equivalent and can be solved by deterministic polynomial-time oracle algorithms . This result is based on a "hyperbolic" generalization of Rado theorem .

1 Introduction and motivating examples

Let $p(x_1,...,x_n) = \sum_{(r_1,...,r_n)\in I_{n,n}} a_{(r_1,...,r_n)} \prod_{1\leq i\leq n} x_i^{r_i}$ be homogeneous polynomial of degree n in n real variables. Here $I_{k,n}$ stands for the set of vectors $r = (r_1,...,r_k)$ with nonnegative integer components and $\sum_{1\leq i\leq k} r_i = n$. We mainly study in this paper homogeneous polynomials with nonnegative integer coefficients .

Definition 1.1: The support of polynomial $p(x_1,...,x_n)$ as above is defined as $supp(p) = \{(r_1,...,r_n) \in I_{n,n} : a_{(r_1,...,r_n)} \neq 0\}$. The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p . ∎

We will study the following decision problems :

- **Problem 1 .** Consider a homogeneous polynomial $p(x_1,...,x_n)$ of degree n in n real variables with nonnegative integer coefficients given as a black box (oracle) . *Is it true that* $(1,1,..,1) \in supp(p)$ *?*
- **Problem 2 .** Consider a homogeneous polynomial $p(x_1,...,x_n)$ of degree n in n real variables with nonnegative integer coefficients given as a black box (oracle) . *Is it true that* $(1,1,..,1) \in CO(supp(p))$ *?*

Our goal is solve these decision problems using deterministic polynomial-time oracle algorithms , i.e algorithms which evaluate the given $p(x_1,...,x_n)$ at polynomial in n and $\log(p(1,1,..,1))$ number of rational vectors $(q_1,...,q_n)$; these rational vectors $(q_1,...,q_n)$ have polynomial in n and $\log(p(1,1,..,1))$ bit-wise complexity ; and the additional auxilary arithmetic computations also take polynomial in n and $\log(p(1,1,..,1))$ number of steps .
The next example explains some (well known) origins of the both problems .

Example 1.2: Consider first a multilinear polynomial $mul(x_1,...,x_n) = \prod_{1\leq i\leq n}\sum_{1\leq j\leq n} a_{i,j}x_j$, where $A = (a_{i,j} : 1 \leq i,j \leq n)$ is a square integer matrix . Then

$$p(x_1,...,x_n) = \sum_{r=(r_1,...,r_n)\in I_{n,n}} \prod_{1\leq i\leq n} \alpha_i^{r_i} per(A_r)\frac{1}{\prod_{1\leq i\leq n} r_i!}, \tag{1}$$

where A_r is a square integer matrix consisting of r_i copies of the ith column of A , $1 \leq i \leq n$; and $per(A)$ is the usual permanent of A . Notice that in this multilinear case the polynomial $p(.)$ can be evaluated in $O(N^2)$ arithmetic operations and $(1,1,...,1) \in supp(p)$ iff $per(A) \neq 0$. Therefore unless $P = NP$ there is no hope to design deterministic polynomial oracle algorithm solving Problem 1 in this case . Next consider even more general case of determinantal polynomials :

$$q(x_1,...,x_n) = \det(\sum_{1\leq i\leq n} A_i x_i),$$

where $\mathbf{A} = (A_1,...,A_n)$ is a n-tuple of positive semidefinite $n\times n$ hermitian matrices , i.e. $A_i \succeq 0$, with integer entries . Recall that the mixed discriminant

$$D(\mathbf{A}) = \frac{\partial^n}{\partial\alpha_1...\partial\alpha_n}\det(\sum_{1\leq i\leq n} A_i x_i).$$

It is well known (see , for instance , [19]) that a determinantal polynomial $q()$ can be represented as

$$q(x_1,..,x_n) = \sum_{r\in I_{n,n}} \prod_{1\leq i\leq n} x_i^{r_i} D(\mathbf{A}_r), \frac{1}{\prod_{1\leq i\leq n} r_i!} \tag{2}$$

where a n-tuple of square matrices consists of r_i copies of $A_i, 1 \leq i \leq k$. One of the equivalent formulations [28] of the classical Rado theorem states that $D(\mathbf{A}_r) > 0$ iff

$$Rank(\sum_{i\in S} A_i) \geq |S| \text{ for all } S \subset \{1,2,...,n\} \tag{3}$$

One important corollary of the Rado conditions (3) is that

$$supp(q) = CO(supp(q)) \cap I_{n,n}. \tag{4}$$

I.e. if integer vectors $r, r(1), r(2), ..., r(k) \in I(n,n)$ and

$$r = \sum_{1\leq i\leq k} a(i)r(i), a(i) \geq 0, 1 \leq i \leq k; \sum_{1\leq i\leq k} a(i),$$

and $D(\mathbf{A}_{r(i)}) > 0, 1 \le i \le k$ then also $D(\mathbf{A}_r) > 0$. Notice that in this case Problem 1 and Problem 2 are equivalent .
We can rewrite Rado conditions (3) as follows :

$$\max_{r \in supp(q)} \sum_{i \in S} r_i \ge |S| \text{ for all } S \subset \{1, 2, ..., n\} \tag{5}$$

Putting things together we get the following Fact .

Fact 1.3: The following properties of determinantal polynomial $q((x_1, ..., x_n) = \det(\sum_{1 \le i \le n} A_i x_i)$ with $n \times n$ hermitian matrices $A_i \succeq 0, 1 \le i \le n$ are equivalent .

1. $(1, 1, .., 1) \notin supp(q)$.
2. $(1, 1, .., 1) \notin CO(supp(q))$.
3. There exists nonempty $S \subset \{1, 2, ..., n\}$ such that

$$\sum_{1 \le i \le n} r_i s_i < \sum_{1 \le i \le n} s_i = |S| \text{ for all} (r_1, ..., r_n) \in supp(q), \tag{6}$$

, where $(s_1, ..., s_n)$ is a characteristic function of the subset S , i.e. $s_i = 1$ if $i \in S$, and $s_i = 0$ otherwise .
Notice that if (6) holds then the distance $dist(e, CO(supp(q)))$ *from the vector* $e = (1, ..., 1)$ *to the Newton polytope* $CO(supp(q))$ *is at least* $\sqrt{\frac{n}{|S|(n-|S|)}} \ge \frac{2}{\sqrt{n}}$.

▮

We will show that for any class of polynomials satisfying Fact () there exists a deterministic polynomial-time oracle algorithm solving both Problem 1 and Problem 2 , which are , of course , equivalent in this case . Our algorithm is based on the reduction to some convex programming problem and the consequent use of the Ellipsoids method .
The next fact about determinantal polynomials , namely their hyperbolicity , is happened to be the most important .

Fact 1.4: Consider a determinantal polynomial $q((x_1, ..., x_n) = \det(\sum_{1 \le i \le n} A_i x_i)$ with $A_i \succeq 0, 1 \le i \le n$. Assume that q is not identically zero , i.e. that $B =: \sum_{1 \le i \le n} A_i \succ 0$ (the sum is strictly positive definite). For a real vector $(x_1, ..., x_n) \in R^n$ consider the following polynomial equation of degree n in one variable :

$$P(t) = q(x_1 - t, x_2 - t, ..., x_n - t) = \det(\sum_{1 \le i \le n} A_i x_i - t \sum_{1 \le i \le n} A_i) = 0. \tag{7}$$

The equation (7) has n real roots roots counting the multiplicities ; if the real vector $(x_1, ..., x_n) \in R^n$ has nonnegative entries then all roots of (7) are nonnegative real numbers . ▮

Proof: First , the matrix $A =: \sum_{1 \leq i \leq n} A_i x_i$ is hermitian . Second , $\det(A - tB) = 0$ iff $\det(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} - tI) = 0$, where $B^{-\frac{1}{2}}$ is the unique positive definite operator square root of positrive definite matrix B^{-1} . As , clearly , $B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$ is also hermitian hence its eigenvalues , which are the roots of () , are real . If $x_i \geq 0, 1 \leq i \leq n$, then the matrix $B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \succeq 0$. Therefore in this case the roots of (7) are nonnegative real numbers . ∎

The main result of this paper that this hyperbolicity , which we will describe formally in Section 1.1 , is sufficient for Fact 1.3 ; i.e. Fact 1.4 implies Fact 1.4 . ∎

1.1 Hyperbolic polynomials

The following concept of hyperbolic polynomials was originated in the theory of partial differential equations [15] .

A homogeneous polynomial $p(x), x \in R^m$ of degree n in m real varibles is is called hyperbolic in the direction $e \in R^m$ (or e- hyperbolic) if for any $x \in R^m$ the polynomial $p(x - \lambda e)$ in the one variable λ has exactly n real roots counting their multiplicities. We assume in this paper that $p(e) > 0$. Denote an ordered vector of roots of $p(x - \lambda e)$ as $\lambda(x) = (\lambda_1(x) \geq \lambda_2(x) \geq ... \lambda_n(x))$. It is well known that the product of roots is equal to $p(x)$. Call $x \in R^m$ e-positive (e-nonnegative) if $\lambda_n(x) > 0$ ($\lambda_n(x) \geq 0$). The fundamental result [15] in the theory of hyperbolic polynomials states that the set of e-nonnegative vectors is a closed convex cone. A k-tuple of vectors $(x_1, ... x_k)$ is called e-positive (e-nonnegative) if $x_i, 1 \leq i \leq k$ are e-positive (e-nonnegative).

We denote the closed convex cone of e-nonnegative vectors as $N_e(p)$, and the open convex cone of e-positive vectors as $C_e(p)$. It has been shown in [15] (see also [21]) that an e- hyperbolic polynomial p is also d- hyperbolic for all e-positive vectors $d \in C_e(p)$.

Let us fix n real vectors $x_i \in R^m, 1 \leq i \leq n$ and define the following homogeneous polynomial:

$$P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) = p(\sum_{1 \leq i \leq n} \alpha_i x_i) \tag{8}$$

Following [21] , we define the p-mixed value of an n-vector tuple $\mathbf{X} = (x_1, .., x_n)$ as

$$M_p(\mathbf{X}) =: M_p(x_1, .., x_n) = \frac{\partial^n}{\partial \alpha_1 ... \partial \alpha_n} p(\sum_{1 \leq i \leq n} \alpha_i x_i) \tag{9}$$

Equivalently, the p-mixed value $M_p(x_1, .., x_n)$ can be defined by the polarization (see [21]) :

$$M_p(x_1, .., x_n) = 2^{-n} \sum_{b_i \in \{-1,+1\}, 1 \leq i \leq n} p(\sum_{1 \leq i \leq n} b_i x_i) \prod_{1 \leq i \leq n} b_i \tag{10}$$

Associate with any vector $r = (r_1, ..., r_n) \in I_{n,n}$ an n-tuple of m-dimensional vectors $\mathbf{X}_r$ consisting of r_i copies of $x_i (1 \leq i \leq n)$. It follows, for instance from the polarization identity (10), that

$$P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) = \sum_{r \in I_{n,n}} \prod_{1 \leq i \leq n} \alpha_i^{r_i} M_p(\mathbf{X}_r) \frac{1}{\prod_{1 \leq i \leq n} r_i!} \tag{11}$$

For e-nonnegative tuple $\mathbf{X} = (x_1, .., x_n)$, define its capacity as:

$$Cap(\mathbf{X}) = \inf_{\alpha_i > 0, \prod_{1 \leq i \leq n} \alpha_i = 1} P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) \tag{12}$$

Probably the best known example of a hyperbolic polynomial is

$$P(\alpha_0, ..., \alpha_k) = Det(\sum_{0 \leq i \leq k} \alpha_i A_i) \tag{13}$$

where $A_i, 0 \leq i \leq k$ are hermitian matrices and the linear space spanned by $A_i, 0 \leq i \leq k$ contains a strictly positive definite matrix: $\sum_{0 \leq i \leq k} \beta_i A_i = B \succ 0$. This polynomial is hyperbolic in the direction $\beta = (\beta_1, ..., \beta_k)$. We can assume wlog that $B = I$ and that $\beta = (1, 0, 0, ..., 0)$. In other words, after a nonsingular linear change of variables

$$P(\alpha_0, ..., \alpha_k) = Det(\sum_{0 \leq i \leq k} \alpha_i B_i) \tag{14}$$

where the matrices $B_i, 1 \leq i \leq k$ are hermitian and $B_0 = I$.
In this case mixed forms are just mixed discriminants.
We make a substantial use of the following very recent result [22] , which is a rather direct corollary of [1] , [30] and even much older [10] .

Theorem 1.5: *Consider a homogeneous polynomial $p(y_1, y_2, y_3))$ of degree n in 3 real variables which is hyperbolic in the direction $(0, 0, 1)$. Assume that $p(0, 0, 1) = 1$. Then there exists two $n \times n$ real symmetric matrices A, B such that*

$$p(y_1, y_2, y_3)) = \det(y_1 A + y_2 B + y_3 I).$$

It has been shown in [16] that most of known facts (and some opened problems) about hyperbolic polynomials follow from Theorem 1.5 .

2 A hyperbolic analogue of the Rado theorem

Definition 2.1: Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e.Denote an ordered vector of roots of $p(x - \lambda e)$ as $\lambda(x) = (\lambda_1(x) \geq \lambda_2(x) \geq ...\lambda_n(x))$. We define the p-rank of $x \in R^m$ in direction e as $Rank_p(x) = |\{i : \lambda_i(x) \neq 0\}|$. It follows from Theorem 1.5 that the p-rank of $x \in R^m$ in any direction $d \in C_e$ is equal to the p-rank of $x \in R^m$ in direction e , which we call the p-rank of $x \in R^m$. ∎

Consider the following polynomial in one variable $D(t) = p(td + x) = \sum_{0 \leq i \leq n} c_i t^i$. It follows from the identity (11) that

$$c_n = M_p(d, .., d)(n!)^{-1} = p(d), c_{n-1} = M_p(x, d, .., d)(1!(n-1)!)^{-1}, ..., c_0 = M_p(x, .., x)(n!)^{-1} = p(x). \tag{15}$$

Let $(\lambda_1^{(d)}(x) \geq \lambda_2^{(d)}(x) \geq ... \geq \lambda_n^{(d)}(x))$ be the (real) roots of x in the e-positive direction d, i.e. the roots of the equation $p(td - x) = 0$. Define (canonical symmetric functions) :

$$S_{k,d}(x) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1}(x) \lambda_{i_2}(x) ... \lambda_{i_k}(x).$$

Then $S_{k,d}(x) = \frac{c_{n-k}}{c_n}$. Clearly if x is e-nonnegative then for any e-positive d the p-rank $Rank_p(x) = \max\{k : S_{k,d}(x) > 0\}$. The next theorem , which we prove in Appendix **A** , is the main mathematical result of this paper .

Theorem 2.2: *Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e. Let $(\mathbf{X}) = (x_1, ...x_k), x_i \in R^m$ be e-nonnegative) n-tuple of m-dimensional vectors , i.e. $x_i, 1 \leq i \leq k$ are e-nonnegative .*
Then the p-mixed form $M_p(\mathbf{X}) =: M_p(x_1, .., x_n)$ is positive iff the following generalized Rado conditions hold :

$$Rank_p(\sum_{i \in S} x_i) \geq |S| \quad \text{for all} \quad S \subset \{1, 2, ..., n\}. \tag{16}$$

Definition 2.3: Call a homogeneous polynomial $p(x), x \in R^n$ of degree n in n real variables P-hyperbolic if it is hyperbolic in direction $e = (1, 1, ...1)$ (vector of all ones) and all canonical orts $e_i, 1 \leq i \leq n$ (rows of the identity matrix I) are e-nonnegative . Call a homogeneous polynomial $q(x), x \in R^n$ of degree n in n real variables with nonnegative coefficients S-hyperbolic if there exists a P-hyperbolic polynomial p such that $supp(p) = supp(q)$. ∎

Corollary 2.4: *Let $q(x), x \in R^n$ be S-hyperbolic polynomial of degree n .*
Then $CO(supp(q)) \cap I_{n,n}$.

Corollary 2.5: *Let $q(x), x \in R^n$ be S-hyperbolic polynomial of degree n . Then the following conditions are equivalent*

1. *$e \in CO(supp(q))$.*
2. *$e \in supp(q)$, i.e. $\frac{\partial^n}{\partial \alpha_1 ... \partial \alpha_n} q(x) > 0$.*
3. *$Cap(p) =: \inf_{\alpha_i > 0, \prod_{1 \leq i \leq n} \alpha_i = 1} q(\alpha_1, ..., \alpha_n) > 0.$*
4. *For all $\epsilon > 0$ there exists a vector $(\alpha_1, ..., \alpha_n)$ with positive entries such that the following inequality holds :*
$$\sum_{1 \leq i \leq n} |\frac{\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n)}{q(\alpha_1, ..., \alpha_n)} - 1|^2 \leq \epsilon. \tag{17}$$
5. *There exists a vector $(\alpha_1, ..., \alpha_n)$ with positive entries such that the following inequality holds :*
$$\sum_{1 \leq i \leq n} |\frac{\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n)}{q(\alpha_1, ..., \alpha_n)} - 1|^2 \leq \frac{1}{n}. \tag{18}$$
6. *For all subsets $S \subset \{1, 2, ..., n\}$ the following inequality holds :*
$$\sum_{i \in S} r_i \geq |S| \quad \text{for all} \quad (r_1, ..., r_n) \in supp(q). \tag{19}$$

(We sketch a proof in Appendix **C** .)

The following result , which we prove in Appendix **B** , is a "polynomial" generalization of Lemma 4.2 in [17] .

Proposition 2.6: *The condition (18) implies the condition (19) for all homogeneous polynomial $q(x), x \in R^n$ of degree n in n real variables with nonnegative coefficients .*

3 The ellipsoid algorithm

Consider a homogeneous polynomial $q(x), x \in R^n$ of degree n in n real variables with nonnegative integer coefficients . Associate with such q the following convex functional

$$f(y_1, ..., y_n) = \log(q(e^{y_1}, e^{y_2}, ..., e^{y_n}).$$

Proposition 3.1: *The following conditions are equivalent*

1. $e = (1, 1, .., 1) \in CO(supp(q))$.
2. $\inf_{y_1+...+y_n=0} f(y_1, ..., y_n) \geq 0$.

If $e = (1, 1, .., 1) \notin CO(supp(q))$ *then* $\inf_{y_1+...+y_n=0} f(y_1, ..., y_n) = -\infty$.
Let $dist(e, CO(supp(q))) = \Delta^{-1} > 0$ *and* $Q = \log(q(e))$. *Define* $\gamma = (Q+1)\Delta$. *Then*

$$\inf_{y_1+...+y_n=0, (|y_1|^2+...+|y_n|^2)^{\frac{1}{2}} \leq \gamma} f(y_1, ..., y_n) = \min_{y_1+...+y_n=0, |y_1|^2+...+|y_n|^2 \leq \gamma} f(y_1, ..., y_n) \leq -1. \quad (20)$$

Proof: Our proof is a strigthforward application of the concavity of the logarithm on the positive semi-axis and of Hanh-Banach separation theorem . It will be included in the full version . ∎

Proposition 3.1 suggests the following natural approach to solve Problem 2 , i.e. to decide whether $e = (1, 1, .., 1) \in CO(supp(q))$ or not :
find $\min_{y_1+...+y_n=0, |y_1|^2+...+|y_n|^2 \leq \gamma} f(y_1, ..., y_n)$ with absolute accuracy $\frac{1}{3}$. If the resulting value is greater or equal $-\frac{1}{3}$ then $e = (1, 1, .., 1) \in CO(supp(q))$; if the resulting value is less or equal $-\frac{2}{3}$ then $e = (1, 1, .., 1) \notin CO(supp(q))$. And , of course , it is natural to use the ellipsoid method . Our main tool is the following property of the ellipsoid algorithm [26]: For a prescribed accuracy $\delta > 0$, it finds a δ-minimizer of a differentiable convex function f in a ball B, that is a point $x_\delta \in B$ with $f(x_\delta) \leq \min_B f + \delta$, in no more than

$$O\left(n^2 \ln\left(\frac{2\delta + \mathrm{Var}_B(f)}{\delta}\right)\right), \qquad \mathrm{Var}_B(f) = \max_B f - \min_B f \quad (21)$$

iterations. Each iteration requires a single computation of the value and of the gradient of f at a given point, plus $O(n^2)$ elementary operations to run the algorithm itself. In our case, this is

easily seen to cost at most $O(n^2)$ oracle calls and $O(n)$ elementary arithmetic operations .
We have $n-1$ dimensional ball $B_\gamma = \{(y_1,...,y_n) : y_1 + ... + y_n = 0, |y_1|^2 + ... + |y_n|^2 \leq \gamma\}$. A straigthforward computations show that

$$Var_B(f) \leq \log(q(1,1,..,1)e^{\gamma n}) - \log(q(1,1,..,1)e^{-\gamma n}) \leq 2\gamma n.$$

Which gives $O(n^2(\ln(n) + \ln(\gamma))$ iterations of the ellipsoid method needed to solve Problem 2 , it amounts to $O(n^4(\ln(n) + \ln(\gamma))$ oracle calls . And $O(n^4(\ln(n) + \ln(\gamma))$ is polynomial in n even if γ is exponentially large ($dist(e, CO(supp(q)))$ is exponentially small). The problem is that if γ is exponentially large (and it can happened) then we need to call oracles on inputs with exponential bit-size .
Putting things together , we get the following conclusion :
If it is promised that either $e = (1,1,..,1) \in CO(supp(q))$ or $dist(e, CO(supp(q))) \geq poly(n)^{-1}$ for some fixed polynomial $poly(n)$ then Problem 1 can be solved by a deterministic polynomial-time oracle algorithm based on the ellipsoid method .
And at this point we can say nothing about Problem 1 , i.e. deciding whether $e = (1,1,..,1) \in supp(q)$ or not . Corollary 2.5 says that if q is S-hyperbolic polynomial then Problem 1 and Problem 2 are equivalent ; moreover if $e = (1,1,..,1) \notin supp(q)$ then here exists nonempty $S \subset \{1,2,...,n\}$ such that

$$\sum_{1 \leq i \leq n} r_i s_i < \sum_{1 \leq i \leq n} s_i = |S| \text{ for all} (r_1,...,r_n) \in supp(q), \tag{22}$$

, where $(s_1,...,s_n)$ is a characteristic function of the subset S , i.e. $s_i = 1$ if $i \in S$, and $s_i = 0$ otherwise .
Notice that if (22) holds then the distance $dist(e, CO(supp(q)))$ from the vector $e = (1,...,1)$ to the Newton polytope $CO(supp(q))$ is at least $\sqrt{\frac{n}{|S|(n-|S|)}} \geq \frac{2}{\sqrt{n}}$. Thus we have the next theorem .

Theorem 3.2: *Problem 1 and Problem 2 are equivalent for S-hyperbolic polynomials . There exists a deterministic polynomial-time oracle algorithm solving Problem 1 for a given S-hyperbolic polynomial $q(\alpha_1,...,\alpha_n)$ with integer coefficients . It requires $O(n^4(\ln(n)+\ln(\ln(q(1,1,...,1)))$ oracle calls and it bit-wise complexity (which roughly the radius of the ball B_γ) is $O(n^{\frac{1}{2}}\ln(q(1,1,...,1))$*
.

4 Hyperbolic Sinkhorn scaling

We will discuss briefly in this section another method , which is essentially a large step version of the gradient descent .

Definition 4.1: Consider an e-nonnegative tuple $\mathbf{X} = (x_1,..,x_n)$ such that the sum of its components $S(\mathbf{X}) = d = \sum_{1 \leq i \leq k} x_i$ is e-positive. Define $tr_d(x)$ as a sum of roots of the univariate polynomial equation $p(x - td) = 0$.
Define the following map (Hyperbolic Sinkhorn Scaling) acting on such tuples:

$$HS(\mathbf{X}) = \mathbf{Y} = (\frac{x_1}{tr_d(x_1)},...,\frac{x_n}{tr_d(x_n)})$$

Hyperbolic Sinkhorn Iteration (**HSI**) is a recursive procedure:

$$\mathbf{X}_{j+1} = HS(\mathbf{X}_j), j \geq 0, \ \mathbf{X}_0 \text{ is an } e\text{-nonnegative tuple with } \sum_{1 \leq i \leq k} x_i \in C_e \, .$$

We also define the doubly-stochastic defect of e-nonnegative tuples with e-positive sums as

$$DS(\mathbf{X}) = \sum_{1 \leq i \leq k} (tr_d(x_i) - 1)^2; \ \sum_{1 \leq i \leq k} x_i = d \in C_e$$

∎

We can define the map $HS(.)$ directly in terms of the P-hyperbolic polynomial

$$Q(\alpha_1, ..., \alpha_n) = P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) = p(\sum_{1 \leq i \leq n} \alpha_i x_i).$$

Indeed, if $\sum_{1 \leq i \leq n} \alpha_i x_i = d \in C_e$ then

$$tr_d(\alpha_i x_i) = \frac{\alpha_i \frac{\partial}{\partial \alpha_i} Q(\alpha_1, ..., \alpha_n)}{Q(\alpha_1, ..., \alpha_n)} \tag{23}$$

This gives the following way to redefine the map $HS(\mathbf{X})$:

$$HS(\alpha_1, ..., \alpha_n) = (\frac{Q(\alpha_1, ..., \alpha_n)}{\frac{\partial}{\partial \alpha_1} Q(\alpha_1, ..., \alpha_n)}, ..., \frac{Q(\alpha_1, ..., \alpha_n)}{\frac{\partial}{\partial \alpha_n} Q(\alpha_1, ..., \alpha_n)}); \alpha_i > 0, 1 \leq i \leq n.$$

And correspondingly the doubly-stochastic defect of $(\alpha_1, ..., \alpha_n)$ is equal to

$$\sum_{1 \leq i \leq n} |\frac{\alpha_i \frac{\partial}{\partial \alpha_i} Q(\alpha_1, ..., \alpha_n)}{Q(\alpha_1, ..., \alpha_n)} - 1|^2,$$

the same as the left side of (18) . Notice that $\sum_{1 \leq i \leq n} tr_d(x_i) = n$ by the Euler's identity .

Example 4.2: Consider the following hyperbolic polynomial in n variables: $p(z_1, ..., z_n) = \prod_{1 \leq i \leq n} z_i$. It is e- hyperbolic for $e = (1, 1, ..., 1)$. And N_e is a nonnegative orthant, C_e is a positive orthant. An e-nonnegative tuple $\mathbf{X} = (x_1, .., x_n)$ can be represented by an $n \times n$ matrix $A_{\mathbf{X}}$ with nonnegative entries: the ith column of A is a vector $x_i \in R^n$. If $Z = (z_1, ..., z_n) \in R^n$ and $d = (d_1, ..., d_n) \in R^n; z_i > 0, 1 \leq i \leq n$, then $tr_d(Z) = \sum_{1 \leq i \leq n} \frac{z_i}{d_i}$. Recall that for a square matrix $A = \{a_{ij} : 1 \leq i, j \leq N\}$ row scaling is defined as

$$R(A) = \{\frac{a_{ij}}{\sum_j a_{ij}}\},$$

column scaling as $C(A) = \{\frac{a_{ij}}{\sum_i a_{ij}}\}$ assuming that all denominators are nonzero. The iterative process $...CRCR(A)$ is called *Sinkhorn's iterative scaling* (SI). In terms of the matrix $A_{\mathbf{X}}$ the map $HS(\mathbf{X})$ can be realized as follows:

$$A_{HS(\mathbf{X})} = C(R(A_{\mathbf{X}}))$$

So, the map $HS(\mathbf{X})$ is indeed a (rather far-reaching) generalization of Sinkhorn's scaling. Other generalizations (not all hyperbolic) can be found in [20], [3], [2]. ∎

The following result , proved in [16] , allows to use (**HSI**) to solve Problem 1 for P-hyperbolic polynomials q in the same way as it was done for the perfect matching problem in [20] , [17] ; and for the Edmonds' problem in [3] . The corresponding complexity is $O(n\log(q(e)))$ iterations of (**HSI**) , which can be done in $O(n^3\log(q(e)))$ oracle calls . The algorithm works in the following way :
Run $K = O(n\log(q(e)))$ *Hyperbolic Sinkhorn Iterations* $\mathbf{X}_{j+1} = HS(\mathbf{X}_j)$ *; if* $DS(\mathbf{X}_i) \leq \frac{1}{n}$ *for some* $i \leq K$ *then the p-mixed form* $M_p(\mathbf{X}_0) > 0$ *, and* $M_p(\mathbf{X}_0) = 0$ *otherwise .*

Proposition 4.3: *Let* $y_i = \frac{x_i}{tr_d(x_1)}$ *, where* x_i *is e-nonnegative ,* $1 \leq i \leq n$ *, and* $d = \sum_{1\leq i\leq n} x_i$ *is e- positive . Then (clearly)* $w = \sum_{1\leq i\leq n} y_i$ *is e- positive . Let positive real numbers* $\lambda_1 \geq \ldots \geq \lambda_n$ *be the roots of the equation* $p(w - td) = 0$*. Then* $\sum_{1\leq i\leq n} \lambda_i = n$ *and thus* $p(w) = p(d)\prod_{1\leq i\leq n} \lambda_i \leq p(d)$ *.*
In terms of the corresponding P-hyperbolic polynomial Q , the following inequality holds :

$$Q\left(\left(\frac{\partial}{\partial\alpha_1}Q(\alpha_1,...,\alpha_n)\right)^{-1},...,\left(\frac{\partial}{\partial\alpha_n}Q(\alpha_1,...,\alpha_n)\right)^{-1}\right) \leq Q(\alpha_1,...,\alpha_n)^{-(n-1)}; \alpha_i > 0. \quad (24)$$

5 Conclusion and Acknowledgments

Univariate polynomials with real roots appear quite often in modern combinatorics , especially in the context of integer polytopes . We discovered in this paper rather unexpected and very likely far-reaching connections between hyperbolic polynomials and many classical combinatorial and algorithmic problems . There are still several open problems . The most interesting is a hyperbolic generalization of the van der Waerden conjecture for permanents of doubly stochastic matrices .

Question 5.1: Define the van der Waerden constant of a hyperbolic in direction e polynomial $p(y_1,...,y_m)$ of degree n in m real variables as

$$VDW(p) = \inf \frac{M_p(x_1,..,x_n)}{Cap(x_1,..,x_n)}$$

where the infimum is taken over the set of tuples $(x_1,..,x_n)$ of e-positive vectors. It is easy to see that $VDW(p) \leq \frac{n!}{n^n}$. Is $VDW(p) = \frac{n!}{n^n}$? Is it positive ? ∎

For a hyperbolic in direction $(1,1,..,1)$ polynomial $Mul(y_1,...,y_n) = y_1y_2...y_n$ this question is equivalent to the famous van der Waerden conjecture for permanents of doubly stochastic matrices , proved in [12] , [13] . For a hyperbolic in direction I polynomial $\det(X)$, X is $n\times n$ hermitian matrix , it is equivalent to Bapat's conjecture [5] (it was also hinted in [12]) , proved by the author in [18] , [29] .

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A Proof of the (main) Theorem 2.2

Before proving Theorem 2.2 , we will recall some basic properties of p-mixed forms and prove a few auxillary results . The following fact was proved in [21]

Fact A.1: Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e. Then the following properties hold .

1. The p-mixed form $M_p(x_1, .., x_n)$ is linear in each $x_i, 1 \leq i \leq n$.
2. If $x_1, x_2, .., x_{n-1}$ are e-nonnegative then the linear functional $l(x) = M_p(x_1, .., x_{n-1}, x)$ is nonnegative on the closed cone N_e of e-nonnegative vectors .
3. If the tuples $(x_1, .., x_n), (y_1, .., y_n), (x_1 - y_1, .., x_n - y_n)$ are e-nonnegative then
$$0 \leq M_p(y_1, .., y_n) \leq M_p(x_1, .., x_n).$$
4. Fix e-positive vector d and consider the following homogeneous polynomial $p_d(x), x \in R^m$ of degree $n-1$ in m real variables : $p_d(x) =: M_p(x, x, ..., x, d)$. Then p_d is hyperbolic in any e-positive direction $v \in C_e(p)$. If $g \in C_e(p)$ (e-positive respect to the polynomial p) then also $q \in C_v(p_d)$ for all $v \in C_e(p)$.

∎

The next fact is well known .

Fact A.2: Consider a sequence of univariate polynomials of the same degree n : $P_k(t) = \sum_{0 \leq i \leq n} a_{i,k} t^i$. suppose that $\lim_{k \to \infty} a_{i,k} = a_i, 0 \leq i \leq n$ and $a_n \neq 0$.
Define $P(t) = \sum_{0 \leq i \leq n} a_i t^i$. Then roots of P_k converge to roots of P . In particular if roots of all polynomials P_k are real then also roots of P are real ; if roots of all polynomials P_k are real nonnegative numbers then also also roots of P are real nonnegative numbers . ∎

The following corollary of Theorem 1.5 plays crucial role in our proof of Theorem 2.2 .

Corollary A.3:

1. *Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e. Let x_1, x_2, x_3 be three e-nonnegative vectors and $d = x_1 + x_2 + x_3$ is e-positive . Assume wlog that $p(x_1 + x_2 + x_3) = 1$. Then there exists three symmetric positive semidefinite matrices A, B, C such that $p(a_1 x_1 + a_2 x_2 + a_3 x_3) = \det(a_1 A + a_2 B + a_3 C$ for all real a_1, a_2, a_3. Additionally , the roots of $a_1 x_1 + a_2 x_2 + a_3 x_3$ in the direction d , i.e. the roots of the equation $p(a_1 x_1 + a_2 x_2 + a_3 x_3 - td) = 0$, coincide with the eigenvalues of $a_1 A + a_2 B + a_3 C$.*
2. *Theorem 2.2 is true for e-nonnegative tuples $(\mathbf{X}) = (x_1, ... x_n), x_i \in R^m$ consisting of at most three distinct components , i.e the cardinality of the set $\{x_1, ... x_n\}$ is at most three .*

Proof:

1. Consider the following homogeneous polynomial $L(b_1,b_2,b_3) = P(b_1x_1 + b_2x_2 + b_3(x_1 + x_2+x_3))$ of degree n in 3 real variables . It follows from Theorem 1.5 that there exists two real symmetric matrices A and B such that $L(b_1,b_2,b_3) = \det(b_1A+b_2B+b_3I)$. It follows that they both positive semidefinite , and $C = I - A - B$ is also positive semidefinite . Take a real linear combination $z = a_1x_1 + a_2x_2 + a_3x_3$. Then

$$p(z-t(x_1+x_2+x_3)) = \det((a_1-a_3)A+(a_2-a_3)B+a_3I-tI) = \det(a_1A+a_2B+a_3C-tI).$$

This proves that $p(a_1x_1 + a_2x_2 + a_3x_3) = \det(a_1A + a_2B + a_3C)$ for all real a_1, a_2, a_3 by putting $t = 0$. And it also proves the "eigenvalues " statement .

2. Consider e-nonnegative tuple $(\mathbf{X})$ consisting of r_i copies of x_i , $1 \leq i \leq 3$; $r_1+r_2+r_3 = n$. Assume that $d = x_1 + x_2 + x_3$ is e-positive (if it is not then $M_p(\mathbf{X}) = 0$ by a simple argument based on the monotonicity of p-mixed forms). It follows from the polarization formula (10) , that

$$M_p(\mathbf{X}) = \sum_{1\leq i\leq k<\infty} d_i p(t_{1,i}x_1 + t_{2,i}x_2 + t_{3,i}x_3),$$

and this formula is universal , i.e. holds for all homogeneous polynomial of degree n , in particular for $\det(X)$, X is $n \times n$ symmetric matrix . Therefore , using the first part of this Corollary we get that the p-mixed form $M_p(\mathbf{X}) = D(\mathbf{A})$, where the matrix tuple $\mathbf{A}$ consists of r_1 copies of A , r_2 copies of B and r_3 copies of C and $D(\mathbf{A})$ is the mixed discriminant . Using Rado theorem for mixed discriminants we get that $D(\mathbf{A}) > 0$ iff

$$Rank(\sum_{i\in S} A_i) \geq \sum_{i\in S} r_i \text{ for all } S \subset \{1,2,3\}.$$

But from the first part we get that $Rank(\sum_{i\in S} A_i)$ is equal to p-rank $Rank_p(sum_{i\in S}x_i)$ of $\sum_{i\in S} x_i$ for all $S \subset \{1,2,3\}$.

∎

Proposition A.4: *Consider similarly to part 4 of Fact A.1 the polynomial $p_d(x) =: M_p(x,x,...,x,d)$ where d is e-nonnegative and $Rank_p(d) \geq 1$. Then p_d is hyperbolic in any direction $z \in N_e(p)$ which is e-nonnegative and satisfies the following inequalities :*

$$Rank_p(z) \geq n-1; \; Rank_p(z+d) = n. \tag{25}$$

Also , if $y \in N_e(p)$ is e-nonnegative then also $y \in N_z(p_d)$, i.e. is z-nonnegative respect to the polynomial p_d.

Proof: Let $z \in N_e(p)$ be e-nonnegative satisfying (25) . Consider univariate polynomial $P(t) = M_p(tz+x, tz+x, ..., tz+x, d)$. Then $P(t) = \sum_{0\leq i\leq n-1} a_i t^i$ and $a_{n-1} = M_p(z,z,...,z,d)$. It follows from Corollary A.3 that $a_{n-1} > 0$. Consider now a sequence of univariate polynomials $P_k(t) = M_p(tz_k + x, tz_k + x, ..., tz_k + x, d_k)$. Where z_k, d_k are e-positive and $\lim_{k\to\infty} z_k = z$, $\lim_{k\to\infty} d_k = d$. Then the coefficients of polynomials P_k converge to the coefficients of the polynomial P . It follows from part 4 of Fact A.1 that the roots of P_k are real . Since $a_{n-1} > 0$ hence using Fact A.2 we get that the roots of P are also real . This exactly means that the polynomial p_d is hyperbolic in direction z . The d-nonnegativity statement follows from the nonnegativity part of Fact A.2 . ▮

We are ready now to present our proof of Theorem 2.2 .

Proof: (Proof of Theorem 2.2) . The "only if" part is simple . Indeed supposed that there exists a subset $S \subset \{1, 2, ..., n\}$ such that $Rank_p(\sum_{i \in S} x_i) < |S|$, i.e. using the identities (15) $M_p(k, k, ...k, d, .., d) = 0$, where $k = \sum_{i \in S} x_i$, $d \in C_e(p)$ is e-positive and the n-tuple $(k, k, ...k, d, .., d)$ consists of $|S|$ copies of $k = \sum_{i \in S} x_i$. Let d be any e-positive positive vector such that $d - x_i$ is e-nonnegative , $1 \leq i \leq n$. Using the monotonicity of p-mixed forms we get that

$$M_p(x_1, ..., x_n) \leq M_p(k, k, ...k, d, .., d) = 0.$$

Our proof of the "if" part is by induction in the degree n . Suppose that the generalized Rado conditions (16) hold . Then at least $Rank_p(x_n) \geq 1$. Consider the following homogeneous polynomial of degree $n - 1$:

$$p_d(x) = M_p(x, x, ..., x, d), \ d = x_n.$$

We get from Proposition a.4 the following assertion :
The polynomial $p_d(x)$ is hyperbolic in direction $z = \sum_{1 \leq i \leq n-1} x_i$ and the vectors $x_i \in N_z(p_d), 1 \leq i \leq n - 1$, i.e. are z-nonnegative respect to the polynomial p_d.
Indeed , it follows from the generalized Rado conditions (16) that $Rank_p(z) \geq n - 1$ and $Rank_p(z + d) = Rank_p(\sum_{1 \leq i \leq n} x_i) = n$.

Next we show that the $n-1$-tuple $\mathbf{Y} = (x_1, ..., x_{n-1})$ satisfies the generalized Rado conditions :

$$Rank_{p_d}(\sum_{i \in S} x_i) \geq |S| \quad \text{for all} \quad S \subset \{1, 2, ..., n - 1\}.$$

Or equivalently , that

$$M_p(k, .., k, z, ..., z, d) > 0; k = \sum_{i \in S} x_i, z = \sum_{1 \leq i \leq n-1} x_i, d = x_n, S \subset \{1, ..., n - 1\}, \qquad (26)$$

where the n-tuple $\mathbf{T} = (k, .., k, z, ..., z, d)$ consists of $|S|$ copies of k , $n - 1 - |S|$ copies of z and one copy of d .
It is easy to see that the generalized Rado conditions for the n-tuple $\mathbf{T}$ are implied by the generalized Rado conditions for the original n-tuple $\mathbf{X} = (x_1, ..., x_{n-1}, x_n)$. Since the n-tuple $(k, .., k, z, ..., z, d)$ consists of at most three distinct components hence we can apply part 2 of Corollary A.3 . Therefore we get that indeed

$$Rank_{p_d}(\sum_{i \in S} x_i) \geq |S| \quad \text{for all} \quad S \subset \{1, 2, ..., n - 1\}. \qquad (27)$$

Thus , by induction in the degree , we get that p_d-mixed form $M_{p_d}(x_1, ..., x_{n-1}) > 0$:
the polynomial p_d of degree $n - 1$ in m real variables is z-hyperbolic . But

$M_{p_d}(x_1, ..., x_{n-1}) = \frac{\partial^{n-1}}{\partial\alpha_1 ... \partial\alpha_{n-1}} p_d(\sum_{1 \le i \le n-1} \alpha_i x_i) =$
$= \frac{\partial^{n-1}}{\partial\alpha_1 ... \partial\alpha_{n-1}} M_p(\sum_{1 \le i \le n-1} \alpha_i x_i, ..., \sum_{1 \le i \le n-1} \alpha_i x_i, x_n) = (n-1)! M_p(x_1, ..., x_n).$

We conclude that if Theorem 2.2 is true for $n-1$ then it is also true for n , and the case "$n = 1$" is trivially true . ∎

B Proof of Proposition 2.6

Proof: Assume wlog that $q(\alpha_1, ..., \alpha_n) = 1$. It follows from the Euler's identity that

$$\sum_{1 \le i \le n} \alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n) = n.$$

Let $q(\alpha_1, ..., \alpha_n) = \sum_{(r_1,...,r_n) \in supp(q)} a_{(r_1,...,r_n)} \prod_{1 \le i \le n} \alpha_i^{r_i}$.
Define (positive numbers) $b_{(r_1,...,r_n)} = a_{(r_1,...,r_n)} \prod_{1 \le i \le n} \alpha_i^{r_i}, (r_1, ..., r_n) \in supp(q)$.
Then $\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n) = \sum_{(r_1,...,r_n) \in supp(q)} r_i b_{(r_1,...,r_n)}$.

Suppose that for some subset $S \subset \{1, 2, ..., n\}, 1 \le |S| < n$ we have the inequality $\sum_{i \in S} r_i < |S|$ for all $(r_1, ..., r_n) \in supp(q)$. Then $\sum_{i \in S} \alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n) \le |S| - 1$. But the condition (18) says that $\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n) = 1 + \delta_i$ and $\sum_{1 \le i \le n} |\delta_i|^2 \le \frac{1}{n}$. By the Cauchy-Schwarz inequality , $\sum_{i \in S} |\delta_i| \le \sqrt{\frac{|S|}{n}} < 1$. Therefore ,

$$\sum_{i \in S} \alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n) \ge |S| - \sum_{i \in S} |\delta_i| > |S| - 1.$$

The last inequality gives a contradiction . ∎

C A sketch of a proof of Corollary 2.4

Proof: By Theorem 2.2 the conditions (1) and (2) are equivalent . (2) implies (3) for any homogeneous polynomial with nonnegative coefficients .
Let $\alpha_i = e^{y_i}, 1 \le i \le n; \sum_{1 \le i \le n} y_i = 0$. Consider the following convex functional

$$f(y_1, ..., y_n) = \log(q(e^{y_1}, e^{y_2}, ..., e^{y_n}).$$

Here $q(x), x \in R^n$ is a homogeneous polynomial of degree n in n real variables with nonnegative coefficients . Then

$$\frac{\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n)}{q(\alpha_1, ..., \alpha_n)} = \frac{\partial}{\partial y_i} f(y_1, ..., y_n), 1 \le i \le n.$$

Notice the condition (3) is equivalent to the following condition :

$$\inf_{y_1 + ... + y_n = 0} f(y_1, ..., y_n) = L > -\infty.$$

Consider the anti-gradient flow , i.e. the system of differential equations

$$y_i(t)' = -(\frac{\partial}{\partial y_i} f(y_1, ..., y_n) - 1), y_i(0) = 0; 1 \leq i \leq n.$$

It is well known that in this convex case the gradient flow is defined for all $t \geq 0$. Using the Euler's identity we get that

$$\frac{d}{dt} f(y_1(t), ..., y_n(t)) = -\beta(t) =: - \sum_{1 \leq i \leq n} |\frac{\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n)}{q(\alpha_1, ..., \alpha_n)} - 1|^2$$

It is easy to see that , because of the convexity of f , a nonnegative function $\beta(t)$ is non-increasing on $[0, \infty)$.
As $\inf_{y_1 + ... + y_n = 0} f(y_1, ..., y_n) = L > -\infty$ thus $\int_0^\infty \beta(t) dt < \infty$. Thus $\lim_{t \to \infty} \beta(t) = 0$. This proves the implication (3) $\to$ (4) for all homogeneous polynomials of degree n in n real variables with nonnegative coefficients .
The implication (4) $\to$ (5) is obvious . The implication (5) $\to$ (6) for general homogeneous polynomials of degree n in n real variables with nonnegative coefficients is Proposition 2.6 .
Finally , the condition (6) is equivalent to (2) . ∎