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Author(s):

Harry B. Hunt, III, D-2
Richard E. Stearns, State University of NY, Albany
Madhav V. Marathe, D-2

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Complexity and Approximability of Quantified and Stochastic Constraint Satisfaction Problems

HARRY B. HUNT III[2,3] RICHARD E. STEARNS[2] MADHAV V. MARATHE[1]

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Abstract

Let D be an arbitrary (*not* necessarily finite) nonempty set, let C be a finite set of constant symbols denoting arbitrary elements of D, and let **S** be an arbitrary finite set of finite-arity relations on D. We denote the problem of determining the satisfiability of finite conjunctions of relations in **S** applied to variables (to variables and symbols in C) by SAT(**S**) (by $\text{SAT}_C(\mathbf{S})$.) Here, we study *simultaneously* the complexity of and the existence of efficient approximation algorithms for a number of variants of the problems SAT(**S**) and $\text{SAT}_C(\mathbf{S})$, and for many different D, C, and **S**. These problem variants include decision and optimization problems, for formulas, quantified formulas stochastically-quantified formulas. We denote these problems by Q-SAT(S), MAX-Q-SAT(S), S-SAT(S), MAX-S-SAT(S) MAX-NSF-Q-SAT(**S**) and MAX-NSF-S-SAT(**S**).

The main contribution is the development of a unified predictive theory for characterizing the the complexity of these problems. Our unified approach is based on the following basic two basic concepts: (i) strongly-local replacements/reductions and (ii) relational/algebraic representability. Let $k \geq 2$. Let **S** be a finite set of finite-arity relations on Σ_k with the following condition on **S**: All finite arity relations on Σ_k can be represented as finite existentially-quantified conjunctions of relations in **S** applied to variables (to variables and constant symbols in C). Then we prove the following new results.

1. The problems SAT(**S**) and $\text{SAT}_C(\mathbf{S})$ are both **NQL**-complete and $\leq_{logn}^{bw}$-complete for **NP**.
2. The problems Q-SAT(**S**), $\text{Q-SAT}_C(\mathbf{S})$, are **PSPACE**-complete. Letting $k = 2$, the problem S-SAT(**S**) and $\text{S-SAT}_C(\mathbf{S})$ are **PSPACE**-complete.
3. $\exists\ \epsilon > 0$ for which approximating the problems MAX-Q-SAT(**S**) within ϵ times optimum is **PSPACE**-hard. Letting $k = 2$, $\exists\ \epsilon > 0$ for which approximating the problems MAX-S-SAT(**S**) within ϵ times optimum is **PSPACE**-hard.
4. $\forall\ \epsilon > 0$ the problems MAX-NSF-Q-SAT(**S**) and MAX-NSF-S-SAT(**S**), are **PSPACE**-hard to approximate within a factor of n^ϵ times optimum.

These results significantly extend the earlier results by (i) Papadimitriou [Pa85] on complexity of stochastic satisfiability, (ii) Condon, Feigenbaum, Lund and Shor [CF+93, CF+94] by identifying natural classes of **PSPACE**-hard optimization problems with provably **PSPACE**-hard ϵ-approximation problems. Moreover, most of our results hold not just for Boolean relations: most previous results were done only in the context of Boolean domains. The results also constitute as *a significant step towards obtaining a dichotomy theorems* for the problems MAX-S-SAT(**S**) and MAX-Q-SAT(**S**): a research area of recent interest [CF+93, CF+94, Cr95, KSW97, LMP99].

[1]Corresponding Author. Los Alamos National Laboratory, P. O. Box 1663, MS M997, Los Alamos NM 87545. The work is supported by the Department of Energy under Contract W-7405-ENG-36. Email: `marathe@lanl.gov`.

[2]Department of Computer Science, University at Albany - State University of New York, Albany, NY 12222. Research supported in part by NSF Grant CCR94-06611. Email: `hunt,res@cs.albany.edu`.

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1 Introduction and motivation

Over the past thirty years, researchers in theoretical computer science, AI, operations research, combinatorial optimization, and algorithmic algebra have studied variants of satisfiability and constraint satisfaction problems. This research was motivated by the facts that (i) such problems have wide ranging applicability in modeling real life problem and (ii) these problems are also fundamental from a complexity theory stand point providing prototypical complete problems for various complexity classes. Specific research topics areas studied in these application contexts include:

1. complexity of 3SAT and in general Boolean constraint satisfaction problems studied in [FV93, CJ+00, JCG97]

2. dichotomy type results for decision and optimization versions of Boolean constraint satisfaction problems [Cr95, KSW97, LMP99, MH+94],

3. recent research on the complexity and (non)-approximability of **PSPACE**-hard quantified and stochastic Boolean satisfiability problems [Pa94, Pa85, CF+93, CF+94, MH+94, HSM94, LMS96].

Here, we combine these lines of research and *simultaneously study* the decision, optimization and counting complexity of quantified and stochastic constraint satisfaction problems. Moreover, we do this not only for Boolean domains but for finite (≥ 2) as well as infinite domains.

We use the following notation for describing constraint satisfaction problems. Throughout this paper, unless otherwise stated explicitly, D is an arbitrary (*not* necessarily finite) nonempty set; C is a finite set of constant symbols denoting elements of D; and **S** and **T** are (usually finite) sets of finite-arity relations/algebraic on D. An **S**-*clause* (a constant free **S**-*clause*) is a relation in **S** applied to variables and constants (to variables only respectively). An **S**-clause is also sometimes referred to as a *term* or a *constraint*. An **S**-*formula* (a constant free **S**-*formula*) is a finite nonempty conjunction of **S**-*clauses* (constant free **S**-*clauses* respectively). An **S**-formula is *satisfiable*, if each individual **S**-clause is simultaneously satisfiable. An **S**-clause (S_i, R_i) with variables S_i and the relation R_i is satisfiable, if the assignment to the variables of the **S**-clause yields a tuple that belongs to R_i.

The above definitions are given when all the variables are existential variables. We now extend the above concepts to define quantified constraint satisfaction problems. A **quantified S-formula** F is of the form $F = (Q_1 x_{i_1}) \cdots (Q_m x_{i_m}) f(x_1, \ldots, x_n)$, where the following hold: (i) the variables of F are x_i for $1 \leq i \leq n$; (ii) $Q_1, \ldots, Q_m \in \{\exists, \forall\}$; and (iii) f is an **S**-formula. We say that F is *constant-free*, when the formula f is constant-free. Let us assume that F has the following structure:

$$F = \forall x_1 \exists x_2, \ldots \forall x_{n-1} \exists x_n F(x_1, \ldots, x_n)$$

i.e. all odd numbered quantifiers are universal and the even numbered quantifiers are existential[4]. Then F is satisfiable if for all values of x_1 there exists an assignment of x_2, such that for assignments to $x_3 \ldots$ such that $F(x_1, \ldots x_n)$ evaluates to true. Alternatively, the formula F is **satisfiable iff** there exists at least one proof-tree for F. Papadimitriou [Pa85] introduced Stochastic Satisfiability problems. An instance of a stochastic satisfiability problem is of the form

$$F = \mathbf{R} x_1 \exists x_2, \ldots \mathbf{R} x_{n-1} \exists x_n (E(F(x_1, \ldots, x_n) \geq 1/2).$$

Here **R** denotes a random quantifier. In other words, we ask if there is a random assignment to x_1 (with equiprobable values for 0 and 1) such that there exists an assignment to x_2 s.t. for a random assignment to x_3 $\ldots$ the boolean formula F is satisfied with an expectation of at least a 1/2. Under the notion of equiprobable values that the variables can take, this is equivalent to saying if for at least half of the leaves of a proof tree the formula evaluates to true. The universal and stochastic quantifiers can be combined the expected

[4]Sometimes we might have a sequence of existential or universal quantifiers.

value threshold can be different from 1/2 and variables need not be assigned 0 or 1 equiprobable. we will leave discussion of these extension to the full paper. See [LMP99] for details on how to incorporate these extensions as a part of a general definition.

We denote the problem of determining the satisfiability of finite conjunctions of relations in **S** or simultaneous satisfiability of systems of algebraic constraints applied to variables (applied to variables and constant symbols in C) by SAT(**S**) (by SAT$_C$(**S**).) Similarly, MAX-SAT(**S**) and MAX-SAT$_C$(**S**) denote the problems of satisfying maximum number of simultaneously satisfiable constant free **S**-clauses and **S**-clauses with constants.

The problem Q-SAT(**S**) (Q-SAT$_C$(**S**)) is the problem of determining if a constant-free (an arbitrary) quantified **S**-formula is satisfiable. The problems S-SAT(**S**) (S-SAT$_C$(**S**)) are similar except that now each $Q_i \in \{\exists, \mathbf{R}\}$. here **R** is called *random quantifier.* The problems MAX-Q-SAT(**S**), MAX-Q-SAT$_C$(**S**), MAX-S-SAT(**S**), MAX-S-SAT$_C$(**S**) are the optimization versions of these problems that aim to maximize the minimum number of simultaneously satisfiable clauses (or satisfiable with probability greater than 1/2) over all partial proof trees.

In the second class of optimization problems, we are interested in satisfying the maximum number of formulas, when instances consist of conjunction of S-formulas. Given a set of S-formulas $\phi = \{\phi_1, \cdots, \phi_n\}$ the problem MAX-NSF-SAT(S) is to find a truth assignment to the variables satisfying the maximum number of formulas in ϕ. Extensions to MAX-NSF-Q-SAT(S) and MAX-NSF-S-SAT(S) follow the above schemata and are omitted.

Important Note: *Due to space limitations, the abstract contains only a discussion of the results, overview of techniques used and significance of the results. Full proofs and detailed definitions are given in the full paper that can be obtained from the authors. Formal definitions of these problems can be found in [CF+93, CF+94, LMP99, HSM94, LMS96].*

Example 1: The generalized CNF satisfiability problems SAT(S) and SAT$_c$(S) generalize the problems 3SAT, 1-3SAT , NAE-3SAT, etc. in [GJ79]. For example, let $EO(x, y, z)$ be the ternary logical relation given by $\{(1,0,0),(0,1,0),(0,0,1)\}$. Then, the problem EXACTLY-1-IN-EX3-MONO-SAT is the same as the problem SAT({EO}). An instance of the above problem might consist of the set of variables x, y, z, w and the formula $F = EO(x,y,z) \wedge EO(x,y,w) \wedge EO(x,w,z)$. It is easy to see that F is satisfiable by setting $x = 1$ and setting all other variables to 0. Let f be an S-formula with m clauses and n_i literals in clause i, $1 \leq i \leq m$. The size of f denoted by $size(f)$ is given by $O(\sum_{i=1}^{i=m} n_i)$. Let C be a set of clauses defined over a set of variables V. We will use $F(C, V)$ to denote the formula obtained by the conjunction of clauses in C. By appropriately defining unary, binary and ternary versions of the relation EO, it is possible to define 1-3-SAT problem.

Our general results in many cases do not depend on the domain being binary or even finite. Similarly, it is possible to have finite or infinite constraint relations. This happens naturally, when we deal with algebraic constraints where constraints can be specified using algebraic (in)equations.

Example 2: Extensions of constraint satisfaction problems to quantified and stochastically quantified constraint satisfaction problems is done by allowing one to use first order quantifiers. Consider again the EO relation as defined in Example 1. An instance I of Q-SAT(EO) might look like $\forall x\ \exists y\ \forall z\ \exists w\ F$, where F is as defined above. Then it can be verified that I is not satisfiable. Moreover, $\mathbf{R}x\ \exists y\ \mathbf{R}z\ \exists w\ F$ is also not satisfiable. On the other hand $\forall x\ \exists y\ \forall z\ \exists w_1 \forall q \exists w_2\ ((EO(x, y, w_1) \wedge EO(z, y, w_2))$ is satisfiable.

2 Summary of results

As mentioned earlier, the focus of this paper is to derive unified technique for characterizing the computational complexity and efficient approximability of quantified and stochastic satisfiability problems. For most part, we concentrate on the quantified and stochastic versions of the problems; the results for unquantified versions are derived in-situ. Specific results obtained in this paper are summarized in Figure 1. The general contributions of this paper include the following.

(1) We formalize an infinite class of quantified and stochastic constraint satisfaction problems. The type of problems studied include: decision, counting and optimization versions of these problems. Furthermore, combining these with the recent ideas of Littman et. al. we can define more general variants of the problem in which we vary the quantifiers and their semantics. We suspect that these infinite classes of problems will play a role similar to that already played by their unquantified counterparts in the earlier development of complexity theory. Of special note is the formalism of optimization and counting versions of these problems: these problems have not be been defined and studied in the literature prior to this paper. Recently there has been interest in studying the approximability of **PSPACE**-hard optimization problems:

(2) We formalize two simple yet important concepts: *local replacements/reductions* and *relational representability*. We derive the basic complexity theoretic properties related to these concepts. Using these concepts, we propose unified methods for characterizing simultaneously, the decision, optimization, approximate optimization and counting complexity of quantified and stochastic constraint satisfaction problems.

(3) We derive very general sufficient conditions and generic reductions that simultaneously show that the decision and the approximate optimization problems are hard for their respective complexity classes. There has been a recent interest in studying the approximability of PSPACE-hard optimization problems. Our general results yield an infinite set of maximization versions of stochastic and quantified constrained satisfaction problems that are **PSPACE**-hard to approximate beyond a certain fixed constant and another infinite set that are **PSPACE**-hard to approximate for any n^ϵ, $\epsilon > 0$. Since the influential paper by Papadimitriou and Yannakakis on **MAX SNP**, there has been interest in finding logical/algebraic characterization of **NP**-hard optimization problems that are hard to approximate within different factors. The results for MAX-Q-SAT(S), MAX-S-SAT(S) MAX-NSF-Q-SAT(S) and MAX-NSF-S-SAT(S) provide similar algebraic characterizations of quantified and stochastic **PSPACE**-hard optimization problems.

We now discuss some of the specific results obtained in this paper and simultaneously contrast them with known results from the literature. These results are summarized in Figure 1. In order to allow for an easy comparison between the results obtained here and the results obtained earlier by other researchers, we summarize both the results in the Figure. Moreover, previous results and our results are in 1-1 correspondence in terms of the numbering used. So for instance, 3(b) in *Part 1*, summarizes the earlier result on non-approximability of MAX-Q-3SAT, our result is given as 3(b) in *Part 2*.

Much of this discussion, but by no means all, is limited to finite sets D, since all *hardness* results given here are *tight* when D is finite. Almost all resulting reductions are *local*. Thus, they are $O(n \cdot logn)$ time-, linear size-, and $O(logn)$ space-bounded.

2.1 Discussion and Significance

We discuss some of the above specific results in some detail. Note that some of the results that follow as corollaries of our general theorems have also been obtained previously by us or other researchers. Our purpose here is to demonstrate the effectiveness of the unified approach and to show that general results presented contain much of the earlier results as subsets of the general results. Moreover, the unified approach yields a large collection of new results that are reported for the first time in the literature. We make the

Part 1: Summary of related results applicable to this paper

1. [Sc78, CES85, MS81]:The problems 3SAT and 3-COLORABLE GRAPH are **NQL**-complete. The problems EX-3SAT, EXACTLY1-EX3MONOSAT, NAE-EX3SAT, GOLD'S-MONOTONE-3SAT are $\leq^{bw}_{logn}$-complete for **NP**.
2. [Sc78, Pa85]:The problems SAT(**S**) and SAT$_C$(**S**) are **NP**-complete and the problems Q-SAT(**S**) and Q-SAT$_C$(**S**) are **PSPACE**-complete, for all finite sets **S** of finite-arity Boolean relations such that Rep_C(**S**)=BOOLEAN-RELATIONS[a]. The problem S-3SAT is **PSPACE**-complete.
3. (a) [ALM+98, PY91]: The problems MAX-3SAT and MAX-NAE-3SAT are **MAX SNP**-complete. Consequently, there exists $\epsilon > 0$ for which approximating these two problems within ϵ times optimum is **NP**-hard.
   (b) [CF+93]: $\exists\ \epsilon > 0$ for which approximating the problem MAX-Q-3SAT within ϵ times optimum is **PSPACE**-hard.
   (c) [CF+94]: $\exists\ \epsilon > 0$ for which approximating the problem MAX-S-3SAT within ϵ times optimum is **PSPACE**-hard.
4. (a) [PR93]:The problem MAX-NSF-3SAT is **MAX** Π_1-complete. Consequently for all $\epsilon > 0$ approximating this problem within ϵ times optimum is **NP**-hard.
   (b) [CF+93]:For all $\epsilon > 0$ approximating the problem MAX-Q-FORMULA-3SAT within ϵ times optimum is **PSPACE**-hard.

Part 2: Summary of the results obtained in this paper

Let $k \geq 2$. Let **S** be a finite set of finite-arity relations on Σ_k such that Rep(**S**) $= \mathbf{\Sigma_k - RELATIONS}$.[b] Then the following hold:

1. The problems SAT(**S**) and SAT$_C$(**S**) are both **NQL**-complete and $\leq^{bw}_{logn}$-complete for **NP**.
2. The problems Q-SAT(**S**), Q-SAT$_C$(**S**), are **PSPACE**-complete. Letting $k = 2$, the problem S-SAT(**S**) and S-SAT$_C$(**S**) are **PSPACE**-complete.
3. Let $k \geq 2$. Let **S** be any finite set of finite-arity relations on Σ_k such that Rep(**S**)=Σ_k-RELATIONS. Then, the following hold:
   (a) The problem MAX-SAT(**S**) is **MAX SNP**-complete. Consequently, there exists $\epsilon > 0$ such that approximating the problem within ϵ times optimum is **NP**-hard.
   (b) $\exists\ \epsilon > 0$ for which approximating the problems MAX-Q-SAT(**S**) within ϵ times optimum is **PSPACE**-hard.
   (c) Letting $k = 2$, $\exists\ \epsilon > 0$ for which approximating the problems MAX-S-SAT(**S**) within ϵ times optimum is **PSPACE**-hard.
4. Let S and T be finite sets of finite-arity relations on an arbitrary nonempty set D. Let $\epsilon > 0$. Then, the following hold:
   (a) The problem SAT(**S**) is $O(n \cdot logn)$ time-, linear size-, and $O(logn)$ space-bounded reducible to the problem of approximating the problem MAX-NSF-SAT(**S**) within a factor of ϵ times optimum. Therefore whenever the problem SAT(**S**) is **NP**-hard, approximating the problem MAX-NSF-SAT(**S**) within ϵ times optimum is **NP**-hard.
   (b) The problems Q-SAT(**S**), Q-SAT(S) are $O(n \cdot logn)$ time-, linear size-, and $O(logn)$ space-bounded reducible to the problems of approximating the problems MAX-NSF-Q-SAT(**S**), MAX-NSF-S-SAT(**S**), respectively, within a factor of n^{ϵ} times optimum.

[a] This is the terminology used in [Sc78] to say that we can represent all finite arity-boolean relations.

[b] Like in Boolean case, this means that all finite arity relations on Σ_k can be equivalently represented as finite existentially-quantified conjunctions of relations in **S** applied to variables (to variables and constant symbols in C).

Figure 1: Summary of results for constrained satisfaction problems. Note that a few of the results have been obtained earlier. The purpose here is to show the unified use of our techniques.

following additional observations about the results summarized above.

First, note that several simple but fundamental properties of our model, that generalize those of previous models such as the *generalized CNF satisfiability problems*, the *constrained satisfiability problems*, and the classes of graphical problems *ECC* and *LCC* of [Sc78, FV93, JCG97, CJ+00, JCG97], respectively.

1. Most of our constructions hold, for domains D of arbitrary *not* necessarily finite cardinality. Moreover, they hold for problems expressed in terms of fairly arbitrary sets of algebraically-expressed constraints **S** on D. In particular, these sets of constraints also need not be finite.

2. Most of our constructions use the Boolean operator *and*, only in the sense of *simultaneously satisfiable* over the domain D and given set of constraints from **S**.

3. All of our constructions are explicitly expressed as *strongly-local graph /hypergraph replacements*. This guarantees their extensibility.

Second, the problems MAX-Q-SAT(S) and MAX-S-SAT(S) are **PSPACE**-hard (as opposed to **NP**-hard) to approximate beyond a fixed constant (a separate constant for each problem). Moreover MAX-NSF-Q-SAT(S) and MAX-NSF-S-SAT(S) are **PSPACE**-hard within any n^{ϵ} factor. Thus our results provide natural algebraic classes of optimization problems that can be potentially used for proving non-approximability of **PSPACE**-hard optimization problems. The un-quantified version of these problems have been used in the past to derive a number of non-approximability results. Similar results can be now obtained in a game theoretic setting. For e.g. it is possible to define a game theoretic version of the MAX-CUT problem: our results how that it is **PSPACE**-hard to approximate.

Third, except for results in [FV93, JCG97] on when the problems SAT(S) are polynomially solvable and the well-known results that, the problems k-COLORABLE-GRAPH and MAX-k-COLORABLE-GRAPH are **NP**- and **MAX SNP**-complete, respectively, very few general hardness results were known previously for sets of relations on sets D such that $3 \leq |D| < \infty$.

Our results extend earlier results and/or answer open problems in (i)Condon, Feigenbaum, Lund and Shor [CF+93, CF+94] to identify natural classes of **PSPACE**-hard optimization problems with provably **PSPACE**-hard ϵ-approximation problems, (ii) work of Papadimitriou [Pa85] on stochastic satisfiability problems (where only S-3SAT was considered) and (iii) Schaefer [Sc78] on quantified generalized satisfiability problems extending it to non-Boolean domain and providing tighter reductions). Progress is made on the approximability of the problems MAX-S-SAT(S) and MAX-Q-SAT(S): *a significant step towards obtaining a dichotomy theorems* for these problems. recently there has been substantial interest in obtaining dichotomy results for decision, optimization and counting versions of satisfiability problems. [CF+93, CF+94, Cr95, KSW97, LMP99]. While (non)-approximability of **NP**-hard optimization problems has received a lot of attention over the recent years, approximability of **PSPACE**-hard optimization problems has only been studied by us [HSM94, MH+94] for quantified and succinctly specified problems, by Condon, Feigenbaum, Lund and Shor [CF+93, CF+94] for quantified and stochastic satisfiability problems and by Lincoln, Mitchell and Scederov in the context of linear logic [LMS96].

3 Overall technique

Our methodology is based upon the following two *simple yet powerful* concepts.

1. Relational Representability: As the name suggests, letting **S** and **T** be sets of relations/algebraic constraints on a common domain D, the intuitive concept that the relations in **S** are *expressible* (or extending the terminology from [Sc78] are *representable*) by finite conjunctions of the relations in **T**. This is formalized in Definition 3.1 below:

Definition 3.1 *1. We denote the set of all finite-arity relations on a non-empty set D logically equivalent to finite existentially-quantified conjunctions of relations/algebraic constraints in* **S** *applied to variables* *(to variables and constant symbols in C) by Rep*($\mathbf{S}$) *(by* $Rep_C(\mathbf{S})$.*)*

2. *We say that a relation S is, and a set of relations* $\mathbf{S}$ *are, representable (constant-free representable) by a set of relations/algebraic constraints* $\mathbf{T}$ *if and only if* $S \in Rep_C(\mathbf{T})$ *(*$S \in Rep(\mathbf{T})$*) and* $\mathbf{S} \subset Rep_C(\mathbf{T})$ *(*$\mathbf{S} \subset Rep(\mathbf{T})$*, respectively.)*

Note: *Throughout this paper* $Rep(\mathbf{S})$ *denotes the set of relations expressible by* constant-free **S**-*formulas; and* $Rep_C(\mathbf{S})$ *denotes the set of relations expressible by* **S**-*formulas with constants from* C.

Variants of the concepts of Definition 3.1 on the *relative representability* of ordered-pairs $(\mathbf{S}, \mathbf{T})$ of sets of *relations*, henceforth denoted collectively by *relational representability*, are well known, especially in mathematical logic. Previously in complexity theory, *relational representability* as used here and the individual constraint satisfaction problems studied have usually been restricted to finite sets **S** of finite-arity relations on *finite* sets D, generally the set $\{0, 1\}$. Additionally, their uses are generally restricted to formulas or (occasionally also to quantified formulas), [Ho97, CES85, GJ79, JCG97, Sc78]. In contrast, our results apply with the exception of the problems S-SAT(**S**) to *both* finite and infinite domains and sets of relations/constraints.

For any set D and finite sets of finite-arity relations **S** and **T** on D, if $\mathbf{S} \subset Rep(\mathbf{T})$ (or $\mathbf{S} \subset Rep_C(\mathbf{T})$), then

1. the problem SAT(**S**) is *1-strongly local* reducible to the problem SAT(**T**) (or $\text{SAT}_C(\mathbf{T})$),
2. the problem Q-SAT(**S**) is efficiently reducible to the problem Q-SAT(**T**) (or Q-$\text{SAT}_C(\mathbf{T})$), and
3. (when D is finite) the problem S-SAT(**S**) is efficiently reducible to the problem S-SAT(**T**) (or S-$\text{SAT}_C(\mathbf{T})$).
4. Moreover often, the reductions of items 1-3 can also be used to relate the relative complexities of the associated MAX- problems.

Figure 2: *Meta-Result 2.* Relational Representability and Strongly-Local Reductions.

2. Local Replacements: Let $k \geq 1$. The second basic component of our methodology consists of the formalization and systematic investigation of the properties of the classes of *k-strongly-local* and *k-strongly-local-enforcer replacements* and *reductions*, especially with respect to constraint satisfaction problems. The basic idea of local reductions is not new and can be traced back to [GJ79] for decision problems, and recently in [HSM94, KSW97, Cr95] for optimization problems. The new contribution of this and companion papers is to formalize the complexity theory properties of such reductions. *In contrast, previous researchers, e.g. [GJ79, CES85], have discussed efficient reductions by local replacement; but they have* not *gone far in formalizing, or in characterizing the complexity-theoretic properties of, their concepts.*

Let $k \geq 1$. Let D_1,D_2 be nonempty sets. Let **S** with $|\mathbf{S}| = p$ and **T** with $|\mathbf{T}| = q$ be finite nonempty sets of finite-arity relations on D_1 and D_2, respectively. We define *k-strongly-local* and *k-strongly-local-enforcer reductions* of the problem SAT(**S**) to the problem SAT(**T**) to be *k-strongly-local* and *k-strongly-local-enforcer replacements* from the set of all **S**-formulas to the set of all **T**-formulas, that are also reductions Intuitively, $\forall k$, in *k-strongly-local replacements* we have *templates*, to be treated as *macros*, with the same template for each variable and distinct templates for each **S**-clause. Details about macro expansions and the the way the variables are replaced depend very simply on the value of k. Figure 2 shows how local replacement/reductions and relational representability can be combined to obtain efficient reductions between classes of satisfiability problems.

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