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Title:	A Proof of the Log-Concavity Conjecture Related to the Computation of the Ergodic Capacity of MIMO Channels
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A proof of the log-concavity conjecture related to the computation of the ergodic capacity of MIMO channels

Leonid Gurvits *

September 21, 2009

Abstract

An upper bound on the ergodic capacity of **MIMO** channels was introduced recently in [1]. This upper bound amounts to the maximization on the simplex of some multilinear polynomial $p(\lambda_1, ..., \lambda_n)$ with non-negative coefficients. In general, such maximizations problems are **NP-HARD**. But if say, the functional $\log(p)$ is concave on the simplex and can be efficiently evaluated, then the maximization can also be done efficiently. Such log-concavity was conjectured in [1]. We give in this paper self-contained proof of the conjecture, based on the theory of **H-Stable** polynomials.

1 The conjecture

Let B be $M \times M$ matrix. Recall the definition of the **permanent** :

$$Per(B) = \sum_{\sigma \in S_M} \prod_{1 \leq i \leq M} A(i, \sigma(i)).$$

The following Conjecture was posed in [1].

Conjecture 1.1: Let A be $M \times N, M < N$ matrix with non-negative entries. We denote as A_S a submatrix

$$A_S = \{A(i,j) : 1 \leq i \leq m; j \in S \subset \{1, ..., N\}.$$

Define the following multi-linear polynomial with non-negative coefficients

$$F_A(\lambda_1, ..., \lambda_N) = \sum_{|S|=M, S \subset \{1,...,N\}} Per(A_S) \prod_{j \in S} \lambda_j. \tag{1}$$

Then the functional $\log(F_A)$ is concave on $R_+^N = \{(\lambda_1, ..., \lambda_N) : \lambda_j \geq 0, 1 \leq j \leq N\}$. ∎

We present in this paper a proof of Conjecture(1.1). Actually we prove that the polynomial F_A is either zero or **H-Stable**.

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2 H-Stable polynomials

To make this note self-contained, we present in this section proofs of a few necessary results. The reader may consult [5] and [3] for the further reading and references.

We denote as $Hom_+(m,n)$ a convex closed cone of homogeneous polynomials with non-negative coefficients of degree n in m variables and with non-negative coefficients; as R^m_+ a convex closed cone of non-negative vectors in R^m and as R^m_{++} a convex open cone of positive vectors in R^m.

Definition 2.1: A homogeneous polynomial $p \in Hom_+(m,n)$ is called **H-Stable** if

$$|p(z_1,...,z_m)| > 0; Re(z_i) > 0, 1 \le i \le m;$$

is called **H-SStable** if $|p(z_1,...,z_m)| > 0$ provided that
$Re(z_i) \ge 0, 1 \le i \le m$ and $0 < \sum_{1 \le m \le m} Re(z_i)$.
∎

Example 2.2: Consider a bivariate homogeneous polynomial $p \in Hom_+(2,n)$, $p(z_1,z_2) = (z_2)^n P(\frac{z_1}{z_2})$, where P is some univariate polynomial. Then p is **H-Stable** iff the roots of P are non-positive real numbers. This assertion is just a rephrasing of the next set equality:

$$\mathbf{C} - \{\frac{z_1}{z_2} : Re(z_1), Re(z_2) > 0\} = \{x \in R : x \le 0\}.$$

In other words

$$P(t) = a \prod_{1 \le i \le k \le n} (t + a_i) : a_i \ge 0, 1 \le i \le k; a > 0.$$

Which gives the following expression for the bivariate homogeneous polynomial p:

$$p(z_1,z_2) = a z_2^{n-k} \prod_{1 \le i \le k \le n} (z_1 + a_i z_2)$$

∎

Fact 2.3: Let $p \in Hom_+(m,n)$ be **H-Stable**. Then $\log(p)$ is concave on R^m_+.

Proof: Consider two vectors $X, Y \in R^m_+$ such that their sum $X + Y \in R^m_+$ has all positive coordinates. It is sufficient to prove that the bivariate homogeneous polynomial $q \in Hom_+(2,n)$

$$q(t,s) = p(tX + sY),$$

is log-concave on R^2_+. Clearly, the polynomial q is **H-Stable**. Therefore, using Example(2.2), we get that

$$\log(q(t,s)) = \log(a) + (n-k)\log(s) + \sum_{1 \le i \le k \le n} \log(t + a_i s) : a_i \ge 0, 1 \le i \le k; a > 0.$$

The log-concavity of q follows now from the concavity of the logarithm on $[0,\infty)$. ∎

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The log-concavity of q follows now from the concavity of the logarithm on $[0,\infty)$. ▮

Remark 2.4: Since the polynomial p is homogeneous of degree n hence, by the standard argument, the function $p^{\frac{1}{n}}$ is concave on R_+^m as well. ∎

Fact 2.5: Let $p \in Hom_+(m,n)$ be **H-Stable** and $x_i \geq 0, 1 \leq i \leq m$ then the following inequality holds

$$|p(x_1 + iy_1, ..., x_m + iy_m)| \geq p(x_1, ..., x_m) \tag{2}$$

Proof: Consider without loss of generality the positive case $x_i > 0, 1 \leq i \leq m$. Then there exists a positive real number $\mu > 0$ such that $y_i + \mu x_i > 0, 1 \leq i \leq m$. It follows from Example(2.2) that for all complex numbers $z \in C$

$$p(zx_1 + (y_1 + \mu x_1), ..., x_m + z(y_m + \mu x_m) = p(x_1, ..., x_m) \prod_{1 \leq i \leq n} (z + a_i); a_i > 0, 1 \leq i \leq m.$$

Thus

$$p(zx_1 + y_1, ..., zx_m + y_m) = p(x_1, ..., x_m) \prod_{1 \leq i \leq n} (z + a_i - \mu)$$

We get, using the homogeniuty of the polynomial p, that

$$p(x_1 + iy_1, ..., x_m + iy_m) = p(x_1, ..., x_m) \prod_{1 \leq j \leq n} (1 + i(a_j - \mu)).$$

As $|\prod_{1 \leq j \leq n}(1 + i(a_j - \mu))| \geq 1$ this proves that the inequality (2) holds. ∎

Corollary 2.6: *A nonzero polynomial $p \in Hom_+(m,n)$ is* **H-Stable** *if and only the inequlity (2) holds.*

Corollary 2.7: *Let $p_i \in Hom_+(m,n)$ be a sequence of* **H-Stable** *polynomials and $p = \lim_{i \to \infty} p_i$. Then p is either zero or* **H-Stable**.

Some readers might recognize Corollary (2.7) as a particular case of A. Hurwitz's theorem on limits of sequences of nowhere zero analytical functions. Our proof below is elementary.

Proof: Suppose that p is not zero. Since $p \in Hom_+(m,n)$ hence $p(x_1, \ldots, x_m) > 0$ if $x_j > 0 :$ $1 \leq j \leq m$. As the polynomials p_i are **H-Stable** therefore $|p_i(Z)| \geq |p_i(Re(Z))| : Re(Z) \in R_{++}^m$. Taking the limits we get that $|p(Z)| \geq |p(Re(Z))| > 0 : Re(Z) \in R_{++}^m$, which means that p is **H-Stable**. ∎

We need the following simple yet crucial result.

Proposition 2.8: *Let $p \in Hom_+(m,n)$ be* **H-Stable**. *Then the polynomial $p_{(1)} \in Hom_+(m-1, n-1)$,*

$$p_{(1)}(x_2, ..., x_m) =: \frac{\partial}{\partial x_1} p(0, x_2, ..., x_m),$$

is either zero or **H-Stable**.

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Fact 2.5: Let $p \in Hom_{+}(m,n)$ be **H-Stable** and $x_i \geq 0, 1 \leq i \leq m$ then the following inequality holds

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$$p(zx_1 + (y_1 + \mu x_1), ..., x_m + z(y_m + \mu x_m) = p(x_1, ..., x_m) \prod_{1 \leq i \leq n} (z + a_i); a_i > 0, 1 \leq i \leq m.$$

Thus

$$p(zx_1 + y_1, ..., zx_m + y_m) = p(x_1, ..., x_m) \prod_{1 \leq i \leq n} (z + a_i - \mu)$$

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$$p_{(1)}(x_2, ..., x_m) =: \frac{\partial}{\partial x_1} p(0, x_2, ..., x_m),$$

is either zero or **H-Stable**.

Proof: Fix complex numbers $z_i, 2 \leq i \leq m$ and define the following univariate polynomial

$$R(t) = p(t, z_2, ..., z_m).$$

It follows that $R'(0) = p_{(1)}(z_2, ..., z_m)$. We consider two cases.
First case: the polynomial $p \in Hom_+(m, n)$ is **H-SStable**. In this case the polynomial $p_{(1)} \in Hom_+(m-1, n-1)$ is **H-SStable** as well. Indeed, in this case if the real parts $Re(z_i) \geq 0, 2 \leq i \leq m$ and $\sum_{2 \leq i \leq m} RE(z_i) > 0$ then all the roots $v_1, ..., v_{n-1}$ of the univariate polynomial R have strictly negative real parts:

$$R(t) = h \prod_{2 \leq i \leq n-1} (t - v_i), 0 \neq h \in C.$$

Therefore

$$p_{(1)}(z_2, ..., z_m) = R'(0) = h(-1)^{n-2} (\prod_{2 \leq i \leq n-1} v_i)(\sum_{2 \leq i \leq n-1} (v_i)^{-1}) \neq 0$$

as the real part

$$Re(\sum_{2 \leq i \leq n-1} (v_i)^{-1}) = \sum_{2 \leq i \leq n-1} \frac{Re(v_i)}{|v_i|^2} > 0.$$

Second case: the polynomial $p \in Hom_+(m, n)$ is **H-Stable** but not **H-SStable**. We need to approximate p by a sequence of **H-SStable** polynomials. Here is one natural approach: let A be any $m \times m$ matrix with positive entries. Define the following polynomials:

$$p_{I+\epsilon A}(Z) =: p\left((I + \epsilon A)Z\right), Z \in C^m.$$

Clearly, the for all $\epsilon > 0$ the polynomials $p_{I+\epsilon A} \in Hom_+(m, n)$ and are **H-SStable**.
It follows that polynomials $\frac{\partial}{\partial x_1} p_{I+\epsilon A}(0, x_2, ..., x_m)$ are **H-SStable** as well. Note that

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial x_1} p_{I+\epsilon A}(0, z_2, ..., z_m) = p_{(1)}(z_2, ..., z_m).$$

Using Corollary(2.7) we get that the polynomial $p_{(1)}$ is either **H-Stable** or zero. ∎

3 Proof of the conjecture

Proof: We will need a few auxillary polynomials:

$$P(x_1, ..., x_M; \lambda_1, ..., \lambda_N) = \prod_{1 \leq j \leq N} (\lambda_j + \sum_{1 \leq i \leq m} A(i, j) x_i). \quad (3)$$

Clearly, the polynomial $P \in Hom_+(M + N, N)$ is **H-Stable** if the entries of the matrix A are non-negative. Applying Proposition(2.8) inductively, we get that the following polynomial

$$R(\lambda_1, ..., \lambda_N) = \frac{\partial^m}{\partial x_1 ... \partial x_m} P(X = 0; \lambda_1, ..., \lambda_N) \quad (4)$$

Proof: Fix complex numbers $z_i, 2 \leq i \leq m$ and define the following univariate polynomial

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It follows that $R'(0) = p_{(1)}(z_2, ..., z_m)$. We consider two cases.
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is either zero or **H-Stable** as well. It is easy to see that

$$R(\lambda_1, ..., \lambda_N) = \sum_{|S|=M, S\subset\{1,...,N\}} Per(A_S) \prod_{j\in\bar{S}} \lambda_j, \tag{5}$$

where $\bar{S} = \{1, ..., N\} - S$ is the compliment of the set S.
Now everything is ready for the punch line: the **multilinear homogeneous polynomial**, defined in (1),

$$F_A(\lambda_1, ..., \lambda_N) = (\prod_{1\leq i\leq N} \lambda_i) R((\lambda_1)^{-1}, ..., (\lambda_N)^{-1}). \tag{6}$$

Recall that the real part $Re(z^{-1}) = \frac{Re(z)}{|z|^2}$ for all non-zero complex numbers $z \in C$. Therefore, if the real parts $Re(\lambda_i) > 0, 1 \leq i \leq n$ then the same is true for the inverses:

$$Re((\lambda_i)^{-1}) > 0, 1 \leq i \leq n.$$

This proves that the polynomial F_A is either zero or **H-Stable**. The log-concavity follows from Fact(2.3). ∎

4 Conclusion

The reader should not be deceived by the simplicity of our proof: very similar arguments are behind the breakthrough results in [5], [4], [6]. The reader is advised to read very nice exposition in [3].
Conjecture (1.1) is actually a very profound question. Had it been asked and properly answered in 1960-70s, then the theory of permanents (and of related things like mixed discriminants and mixed volumes [6]) could have been very different now.
Though the "permanental" part in [1] is fairly standard(the authors essentially rediscovered so called Godsil-Gutman Formula [8]) it is quite amazing how naturally the permanent enters the story. Switching the expectation and the logarithm can be eventful indeed.
The log-concavity comes up really handily in the optimizational context of [1]. The thing is that maximization on the simplex of $\sum_{1\leq 1\leq j\leq N} b(i,j)x_i x_j$ is **NP-COMPLETE** even when $b(i,j) \in \{0,1\}, 1 \leq 1 \leq j \leq N$.
Our proof is yet another example on when the best answer to a question posed in the real numbers domain lies in the complex numbers domain. Yet, we don't exlude a possibility of a direct "monstrous" proof.

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