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APPLICATIONS OF OPERATIONAL CALCULUS:

TRIGONOMETRIC INTERPOLATING EQUATION

FOR THE EIGHT-POINT CUBE

by

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ABSTRACT

A general method for obtaining a trigonometric-type interpolating

equation for the eight-point cubical array is illustrated. It can often

be used to reproduce a ninth datum at an arbitrary point near the

center of the array by adjusting a variable exponent. The new method

complements operational polynomial and exponential methods for the

same design.

2

1. Introduction

This paper describes a new method for interpolating eight data in a cubical array by means of the circular or hyperbolic functions. It depends on reinterpreting the identities of trigonometry as finite-difference equations by means of the shifting operator, $\exp(x)F(x)=F(x+h)$ [1]. The operational polynomial- and exponential-type equations for the eight-point array are invariant under data translation and rotation but the trigonometric-types do not have the advantage of translational invariance [2,3,4]. Literature related to the interpolation of the eight-point cube is limited in scope. This paper continues previous expositions on interpolation of data in geometric arrays by shifting-operator derived equations.

2. The eight-point cube

A recent manuscript summarizes a trigonometric-type equation for interpolating numbers arranged at the vertices of a cube [3]. It is based on the identities in Eqs. (1a) and (2a) and their operational interpretations in Eqs. (1b) and (2b), respectively. Eqs. (1b) and (2b) yield Eq. (3). The three finite-difference equations (1b), (2b), (3) apply to the 9-point rectangle in Fig. 1. A capital letter like A presently represents a location in the rectangle as well as the datum at the same place. See Fig. 1.

$$(2)\sin(x)\cos(x)\cos(x+y) - (2)\sin(x+y)\cos(x)^2 + \sin(x+y) = \sin(x-y) \tag{1a}$$

$$(F-D)(F+D)(I+A) - (I-A)(F+D)^2 + 2E^2(I-A) = 2E^2(C-G) \tag{1b}$$

3

$$(2)\sin(x)\cos(x)\cos(y-x) + (2)\cos(x)^2\sin(y-x) - \sin(y-x) = \sin(x+y) \tag{2a}$$

$$(F-D)(F+D)(G+C) + (F+D)^2(G-C) = 2E^2(G-C+I-A) \tag{2b}$$

$$E^2 = [FD(I+G-A-C)-F^2(A-G)-D^2(C-I)] / [2(I-A-C+G)] \tag{3}$$

The substitution $D=F(A+G)/(I+C)$ simplifies Eq. (3) [3]. The simplified form can be solved for F as in Eq. (4). See Fig. 1. Choose the positive solution because most laboratory data are positive numbers. Eq. (4) can be re-interpreted as a relationship in three dimensions as in Eq. (5). See Fig. 2. The notation BDIG denotes an estimate of the center point of the right-hand face of the cube. In three dimensions, a single letter represents a number at the corresponding vertex of the cube in Fig. 2. A double-letter combination like AB represents the product of two vertex-point numbers.

$$F = E(C + I)SQRT[(A + C - G - I) / ((A + C + G + I)(AC - GI))] \tag{4}$$

$$BDIG = E(B + I)SQRT[(A + B - H - I) / ((A + B + H + I)(AB - HI))] \tag{5}$$

Equation (5) has the disadvantage that it represents only five numbers in the cube. However, the cube can be rotated through a right angle to obtain a second representation of BDIG as illustrated by Eq. (6).

$$BDIG = E(G + D)SQRT[(F + G - C - D) / ((F + G + C + D)(FG - CD))] \tag{6}$$

4

The product of Eqs. (5) and (6) generate a new representation of BDIG as in Eq. (7). Eq. (8) is obtained in a similar manner or, more easily, by rotation of the cube in Fig. 2. Similar expressions for the midpoints of the remaining faces of the cube appear as Eqs. (9)-(12). See also Eqs. (29)-(37) in Ref. [3].

$$BDIG = E[(B+I)(D+G)Q_1]^{(1/2)} \tag{7}$$

$$ACHF = E[(C+F)(A+H)Q_1]^{(1/2)} \tag{8}$$

$$ABGF = E[(A+G)(B+F)Q_3]^{(1/2)} \tag{9}$$

$$CDIH = E[(D+H)(C+I)Q_3]^{(1/2)} \tag{10}$$

$$ABDC = E[(B+C)(A+D)Q_5]^{(1/2)} \tag{11}$$

$$FGIH = E[(F+I)(G+H)Q_5]^{(1/2)} \tag{12}$$

$$Q_1 = ((H+I-A-B)(F+G-C-D) / ((A+B+H+I)(HI-AB)(F+G+C+D)(FG-CD)))^{(1/2)} \tag{13}$$

$$Q_3 = ((F+H-B-D)(G+I-A-C) / ((B+D+F+H)(FH-BD)(A+C+G+I)(GI-AC)))^{(1/2)} \tag{14}$$

$$Q_5 = ((C+H-B-G)(A+F-D-I) / ((B+G+C+H)(CH-BG)(D+I+A+F)(AF-DI)))^{(1/2)} \tag{15}$$

The scale of the three-dimensional (x,y,z) coordinate system in Fig. 2 is (–1 .. 1). The effect of a change in a particular direction is reflected in the coefficient of the corresponding parameter in the function representing the cube. The hyperbolic cosines of the x-, y-, and z- coefficients are (BDIG + ACHF)/(2E), (CDIH + ABGF)/(2E), and (FGIH + ABDC)/(2E), respectively. The letter E disappears from the three ratios.

In principle, many trigonometric-type equations for the eight-point cube could be generated by seeking replacements for Eqs. (1a) and (2a). In practice, a variety can be generated more easily. Define new terms called T_1 and T_2. A new series of six terms denoted by W_x are ratios of T_1 and T_2 as in Eqs. (16)-(18). Only W_1 is defined by T_1 and T_2 as they are presently written. NN in the exponent (NN/4) remains to be assigned.

$$T_1 = (AH+CF-IB-DG)^2(I+B+D+G)^4$$

$$/ [(4(F+I+A+C+G+H+D+B)^2(F-I+A+C-G+H-D-B)^2)] \tag{16}$$

$$T_2 = (I+B)^2(D+G)^2(AH-IB)(CF-DG)$$

$$/ [(F+C+G+D)(F+C-G-D)(A-B+H-I)(A+B+H+I)] \tag{17}$$

$$W_1 = (T_1/T_2)^{(NN/4)} \tag{18}$$

In the preceding expressions, a double-letter combination (except NN) is the product of two numbers at the corresponding vertices of the cube. See Fig. 2. Rearrange the letters in W_1 in order to define W_2. Where the letter A occurs in W_1, change it to D, where the letter B occurs in W_1, change it to C, and so on according to Eqs. (19)-(24). That is, the sequence (A,B,C,D,F,G,H,I) in W_1 becomes the new sequence (D,C,B,A, I,H,G,F) in W_2. Use the correspondences in Eqs. (19)-(24) to define W_1 .. W_6.

$$W_1: (A,B,C,D,F,G,H,I) \rightarrow (A,B,C,D,F,G,H,I) \tag{19}$$

$$W_2: (A,B,C,D,F,G,H,I) \rightarrow (D,C,B,A,I,H,G,F) \tag{20}$$

$$W_3: (A,B,C,D,F,G,H,I) \rightarrow (C,A,D,B,H,F,I,G) \tag{21}$$

W_4: (A,B,C,D,F,G,H,I) → (B,D,A,C,G,I,F,H)   (22)

W_5: (A,B,C,D,F,G,H,I) → (H,C,I,D,F,A,G,B)   (23)

W_6: (A,B,C,D,F,G,H,I) → (D,I,C,H,B,G,A,F)   (24)

Eqs. (25)-(30) are new expressions for the center points of the cube faces in Fig. 2. They are more versatile than the forms as in Eqs. (7)-(12). The letter E vanishes (see above) on evaluating the arccosh function making the procedure an 8-point method.

$$BDIG = (W_1)E[(B+I)(D+G)Q_1]^{(1/2)} \qquad (25)$$

$$ACHF = (W_2)E[(C+F)(A+H)Q_1]^{(1/2)} \qquad (26)$$

$$ABGF = (W_3)E[(A+G)(B+F)Q_3]^{(1/2)} \qquad (27)$$

$$CDIH = (W_4)E[(D+H)(C+I)Q_3]^{(1/2)} \qquad (28)$$

$$ABDC = (W_5)E[(B+C)(A+D)Q_5]^{(1/2)} \qquad (29)$$

$$FGIH = (W_6)E[(F+I)(G+H)Q_5]^{(1/2)} \qquad (30)$$

The left-hand members of Eqs. (25)-(30) are used to generate new expressions for the numerical coefficients of the x-, y-, and z-parameters in the interpolating equation. They are listed as Eqs. (31)-(33), respectively. If the argument of an arccosh function is less than unity, replace it with the arccos notation so that p, q, and r are real numbers. Then replace sinh and cosh by sine and cosine, respectively, in the interpolating equation. Choose another method if the arguments of the arcos or arccosh functions are complex numbers that do not represent artifacts of calculation precision.

$$p = \text{arccosh}((\text{BDIG}+\text{ACHF})/(2E)) \qquad (31)$$

$$q = \text{arccosh}((\text{CDIH}+\text{ABGF})/(2E)) \qquad (32)$$

$$r = \text{arccosh}((\text{FGIH}+\text{ABDC})/(2E)) \qquad (33)$$

Use Eq. (28) in Ref. [3] to determine the numerical interpolating equation for the eight-point cube in Fig. 2. Note that p, q, and r in Eqs. (31)-(33) replace p, q, and r in the cited Eq. (28). The numerical coefficients denoted J, K, M, N, S, T, U, V are explained in the text following the cited Eq. (28). The same coefficients also apply in the present method. All the W coefficients are unity when positive trilinear numbers at vertices A-I in Fig. 2 are arguments of functions such as 2^x, $\sin(10x^\circ)$, $\sinh(x/4)$.

For example, let trial data be 1^3, 2^3, 3^3, 4^3 at vertices A-D, and 6^3, 7^3, 8^3, and 9^3 be the data at vertices F-I, respectively, in Fig. 2. Eq. (38) in Ref. [3] results when the exponent NN is zero. Note that the cited Eq. (38) does not reproduce the center point datum ($R=5^3=125$) when (x, y, z) are (0, 0, 0), respectively. Instead, it yields R=116.4, nearly. A merit of the present method lies in the adjustable nature of exponent NN.

The introduction of exponent NN and six parameters denoted W yields a new interpolating equation that has more flexibility than Eq. (28) in Ref. [3]. The exponent NN can often be used to satisfy an arbitrary criterion. For example, let NN be

assigned a new value: NN=(–0.6142). The new interpolating equation, for the same trial

data, is equation is Eq. (34). At (x,y,z)=(0,0,0) it renders R=125.0, nearly. In this case,

Eq. (34) applies to the nine-point array in Fig. 2. The arbitrarily introduced criterion

(that E should be close to 5^3=125) was satisfied by means of generating a new

interpolating equation by changing the value of exponent NN. The value of NN is

adjusted until the prediction of Eq. (34) at (x,y,z) = (0,0,0) is satisfactorily close to the

datum at E.

$$
\begin{aligned}
R = {}&(125.0)\cos(0.2747x)\cosh(0.3673y)\cosh(1.224z) + \\
&(90.56)\sin(0.2747x)\cosh(0.3673y)\cosh(1.224z) + \\
&(142.9)\cos(0.2747x)\sinh(0.3673y)\cosh(1.224z) + \\
&(133.0)\cos(0.2747x)\cosh(0.3673y)\sinh(1.224z) + \\
&(79.65)\sin(0.2747x)\sinh(0.3673y)\cosh(1.224z) + \\
&(83.27)\sin(0.2747x)\cosh(0.3673y)\sinh(1.224z) + \\
&(133.5)\cos(0.2747x)\sinh(0.3673y)\sinh(1.224z) + \\
&(47.36)\sin(0.2747x)\sinh(0.3673y)\sinh(1.224z)
\end{aligned}
\tag{34}
$$

Suppose it is desired to generate an equation that yields R=128.0 when

(x,y,z)=(0,0,0.1). The proper value of exponent NN is again found by trial and error. It

is NN=(0.2968), nearly. The illustrated method is versatile and easy to apply. The

present assignments for T_1, T_2, and W_1 .. W_6 are not the only possible choices.

A property of the present trigonometric form is that it predicts the center point correctly whenever the data are derived from certain common functions with linear numbers as their arguments. Those functions include 2^x, $\sin(x)$, $\cos(x)$, $\sinh(x)$, and $\cosh(x)$. Eqs. (16)-(18) have the merit of preserving this property for any value of the exponent NN. Let the eight trial data be the sines of $10° .. 40°$ as A .. D and $50° .. 90°$ as F .. I in Fig. 2, respectively. The equation representing the cube of sines is Eq. (35). It has $J\sim 0.08727x$, $K\sim 0.1745y$, $L\sim 0.4363z$. Eq. (35) is maintained no matter what number is assigned to exponent NN. The described property is also maintained when the linear numbers $x=1 ... 4$, and $6 ... 9$ are assigned to 2^x in order to generate new trial data at vertices A .. D and F .. I, respectively. Equations such as Eq. (35) reproduce the original data and are invariant under data rotation by not under data translation.

$$R = (0.7660)\cos(J)\cos(K)\cos(L) + (0.6428)\cos(J)\cos(K)\sin(L) +$$
$$(0.6428)\cos(J)\sin(K)\cos(L) - (0.7660)\cos(J)\sin(K)\sin(L) +$$
$$(0.6428)\sin(J)\cos(K)\cos(L) - (0.7660)\sin(J)\cos(K)\sin(L) -$$
$$(0.7660)\sin(J)\sin(K)\cos(L) - (0.6428)\sin(J)\sin(K)\sin(L) \tag{35}$$

4. Discussion

As illustrated in this manuscript and the literature citations, the operational method represents a new approach to interpolation of data in geometric arrays. Its results are potentially useful in applications such as the estimation of curvature

coefficients and other forms of data analysis. The literature citations refer solely to shifting-operator methods because they are so new, because there are few alternative, familiar approaches that are so versatile and easy to apply, and because shifting-operator methodology is a new way to study problems in applied geometry.

The algorithm described in this paper may seem to be tedious to prepare. To lessen the tedium, and illustrate the method, the author can supply a Maple® worksheet that executes all of the calculations and delivers the trigonometric interpolating equation [6]. The user must supply the eight data in cubical array and choose a value for NN. Another way to adjust the algorithm to accommodate a ninth datum is by means of a translating term. That method has been described in Ref. [7].

Table 1 illustrates the sums of squares of deviations of four equations for interpolating the eight-point cube: (1) the trilinear equation, (2) the quadratic equation appearing as Eq. (10) in Ref. [3], the cubic polynomial equation appearing as Eq. (1) in Ref. [5], and (4) the trigonometric equation derived as illustrated in this paper. The trigonometric equation uses NN=0. The trial data are generated by simple, monotonic functions applied to the integers 1 .. 4, and 6 .. 9 as A .. D and F .. G, respectively, as in Fig. 2. As can be observed by the tabulated results, the operational equations often have lower sums of squares of deviations than the trilinear equation.

Table 1. Approximate sums of squares of deviations of four modeling surfaces from typical trial surfaces. The equations are based on different approaches to the eight-point cube. The data are generated by applying the listed functions to the integers 1 .. 4 and 6 .. 9 at vertices A .. D and F .. I, respectively, as in Fig. 2. The coefficient NN in the trigonometric equation is taken as NN=0 [3,5].

Function*	Trilinear equation	Quadratic equation	Cubic equation	Trigonometric (this paper)
M^2	229	0	0	17.6
M^3	52600	1200	0	3250
2^M	49600	10800	2540	0
$\sinh(M/2)$	265	36.7	7.12	0
$\tan(9M^\circ)$	4.89	1.55	0.418	0.443
$\cosh(M/2)$	270	36.0	7.24	0
$\cosh(M/2) + M$	270	36.0	7.24	1.47
$(M)\cosh(M/2)$	28700	5180	1120	45.4
$(5)\sin(10M^\circ) + \cos(10M^\circ)$	0.992	0.00670	0.00138	0

*$M = (5+x/2+y+5z/2)$

G H I

D E F

A B C
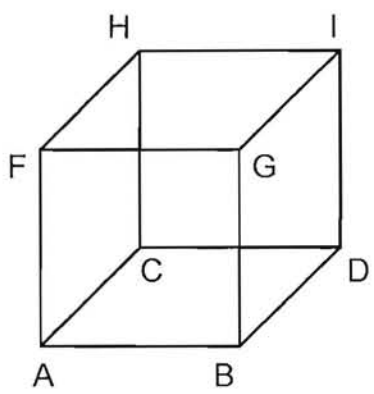
Fig. 1. The nine-point rectangle.

Fig. 2. The eight-point cube.

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