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CALCULATION OF THE NANBU-TRUBNIKOV KERNEL: IMPLICATIONS FOR NUMERICAL MODELING OF COULOMB COLLISIONS

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Abstract

We investigate the accuracy of and assumptions underlying the numerical binary Monte-Carlo collision operator due to Nanbu [K. Nanbu, Phys. Rev. **E 55** (1997)]. The numerical experiments that resulted in Nanbu's parameterized collision kernel are approximate realizations of the Coulomb-Lorentz pitch-angle scattering process, for which an analytical solution is available. It is demonstrated empirically that Nanbu's collision operator quite accurately recovers the effects of Coulomb-Lorentz pitch-angle collisions, or processes that approximate these even for very large values of the collisional time step. An investigation of the analytical solution shows that Nanbu's parameterized kernel is highly accurate for small values of the normalized collision time step, but loses some of its accuracy for larger values of the time step. Finally, a practical collision algorithm is proposed that for small-mass-ratio Coulomb collisions improves on the accuracy of Nanbu's algorithm.

- Coulomb collisions are important in many plasma systems.

- There is a long history of study of Coulomb collisions in plasmas.

    - Many important analytical results reviewed by B.A. Trubnikov, [in Reviews of Plasma Physics (M. A. Leontovich, ed., Consultants Bureau, New York, 1965), Vol. **1**, p. 105.]

- The Lorentz model, represents elastic scattering of particles off infinitely heavy (fixed) scatterers.

    - pedagogical model and a reasonable approximation that qualitatively captures the important effects of collisions in gas-dynamic and plasma systems.

    - useful for the quantitative modeling of parts of some plasma systems when time-scale separation is present, e.g., slowing down of fast electrons by a high-Z plasma, e.g., in fast-ignition inertial-confinement-fusion targets [15].

- An influential numerical binary Monte Carlo Coulomb collision algorithm was developed by T. Takizuka and H. Abe [(TA) J. Comp. Phys. **25**, 205 (1977).]
    - o particles are paired locally in space and undergo binary scattering events, which conserve particle number, energy, and momentum. The relative velocity, the magnitude of which is preserved, scatters through some angle $\Delta\theta$. The distribution of the angles $\Delta\theta$ is chosen so that for $\nu\Delta t << 1$, where ν is a mean collision rate, $\Delta\theta$ is small, and the accumulation of many collision events gives evolution in agreement with the Landau-Fokker-Planck operator for Coulomb collisions.

- This scheme was modified in K. Nanbu, [Phys. Rev. E. **55**, 4642-4652 (1997).], who aimed to develop a scheme that would accurately represent an accumulation of Coulomb collisions even for large time step values (i.e., $\nu\Delta t$ not necessarily small).

- Wang et al. [C. M. Wang, T. Lin, R. Caflisch, B. I. Cohen, A. M. Dimits, J. Comp. Phys. **227**, 4308 (2008)] performed a numerical convergence study for the methods of TA and Nanbu.

- The pointwise errors for the Nanbu method were found to be comparable to those of the TA method run at approximately half the timestep. Thus the collision aggregation was at best partially successful.

- The present work is reported in A.M. Dimits, C.M. Wang, R. Caflisch , B.I. Cohen , Y. Huang, "*Understanding the Accuracy of Nanbu's Numerical Coulomb Collision Operator,"* Journal of Computational Physics, 2009, in press.

- Here, we address: (1) What is the physics basis for the Nanbu's collision kernel? (2) How accurate is Nanbu's collision kernel and numerical collision operator over a range of time-step and mass-ratio values?

- Nanbu's collision kernel in is an empirically obtained parameterization of the kernel for the Coulomb-Lorentz pitch-angle scattering (diffusion) process, for which an analytical expression is known.

- Nanbu's collision operator quite accurately recovers the effects of such collisions, even for very large values of the collisional time step.

- For a standard temperature-anisotropy-relaxation test problem we delineate the regime of validity of Nanbu's operator. A breakdown of accuracy of Nanbu's operator as a function of time step and mass ratio occurs because energy diffusion competes with isotropization (i.e., diffusion in pitch angle). This energy diffusion can be thought of as a surrogate for other processes that may compete with isotropization in more complicated plasma systems.

ACCURACY OF THE NANBU OPERATOR FOR LONG TIME STEPS

Nanbu's collision kernel is actually derived from numerical experiments that are accurately represented by the Coulomb-Lorentz collision operator.

This operator represents diffusion with a uniform (isotropic) diffusion coefficient on a (unit) sphere of the tips of the orientation vectors.

$$\frac{\partial f_a}{\partial s} = \frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial f_a}{\partial \mu}\right] + \frac{1}{\left(1-\mu^2\right)}\frac{\partial^2 f_a}{\partial \phi^2} \ ,$$

where μ=cosθ, θ is the polar angle, ϕ is the azumuthal angle, s=t/2$\tau_s^{\alpha/\beta}$, t is the physical time, and $\tau_s^{\alpha/\beta}$ is the longitudinal slowing down time for a charged particle of species α colliding off a particle of species β:

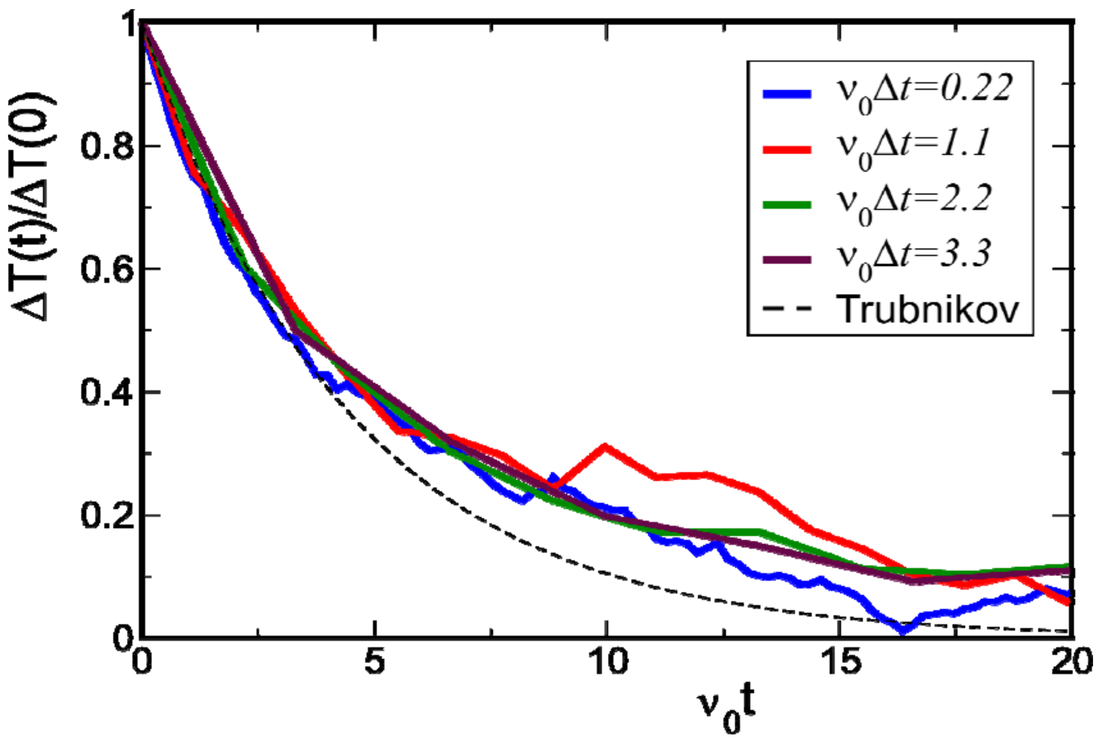
$$\tau_s^{\alpha/\beta} = \frac{v^3}{4\pi\Lambda^{\alpha/\beta} n_\beta}\left(\frac{m_\alpha}{q_\alpha q_\beta}\right)^2 \ .$$

v is the test-particle's speed, $\Lambda^{\alpha/\beta}$ is the "Coulomb logarithm," and n_α, m_α, and q_α are respectively the number density, mass, and charge of a particle of species α.

The analytical solution for the kernel for this operator has long been known [see, e.g., Trubnikov]

Nanbu's parameterization is an excellent approximation for short time steps, and a quite good approximation for all values, with a maximum relative error in the kernel of about 8%

FOR COULOMB COLLISIONS OFF INFINITELY HEAVY (FIXED) SCATTERERS, NANBU'S COLLISION OPERATOR IS ACCURATE EVEN FOR VERY LARGE TIME STEPS.



Temperature isotropy, normalized initial value for collisional isotropization test for Lorentz collisions for four different values of the time step. Also shown is the analytical result.

EXPONENTIAL-DECAY "APPROXIMATION"

(Derived by Kagan for $m_e = m_i$; Also given in NRL Plasma Formulary, and works reasonably well for arbitrary $m_e = m_i$.)

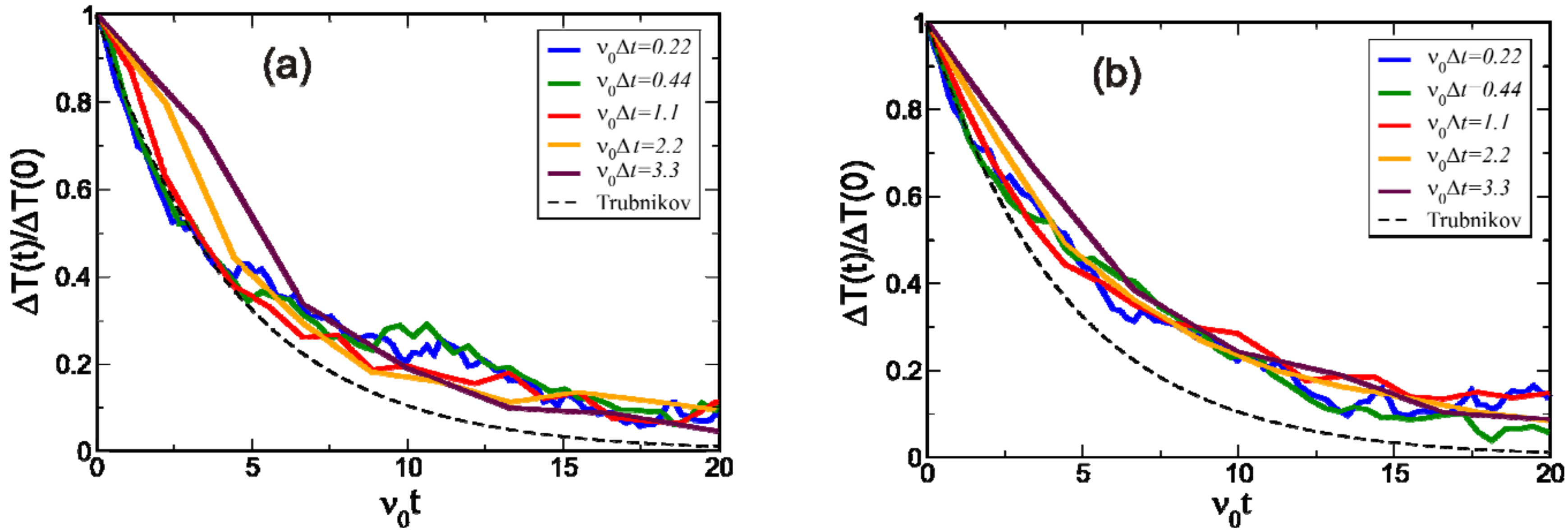
$$\frac{\Delta T(t)}{\Delta T(0)} = \exp\left(-\frac{2}{5\sqrt{\pi}}\nu_0 t\right)$$

ν_0 is a thermally averaged collisional relaxation rate:

$$\nu_0 = \frac{1}{\tau_s^{e/\beta}\left(v = v_{\mathrm{th}e}\right)} = \frac{4\pi\Lambda^{e/\beta} n_\beta e^2 q_\beta^2}{\sqrt{m_e} T_e^{3/2}},$$

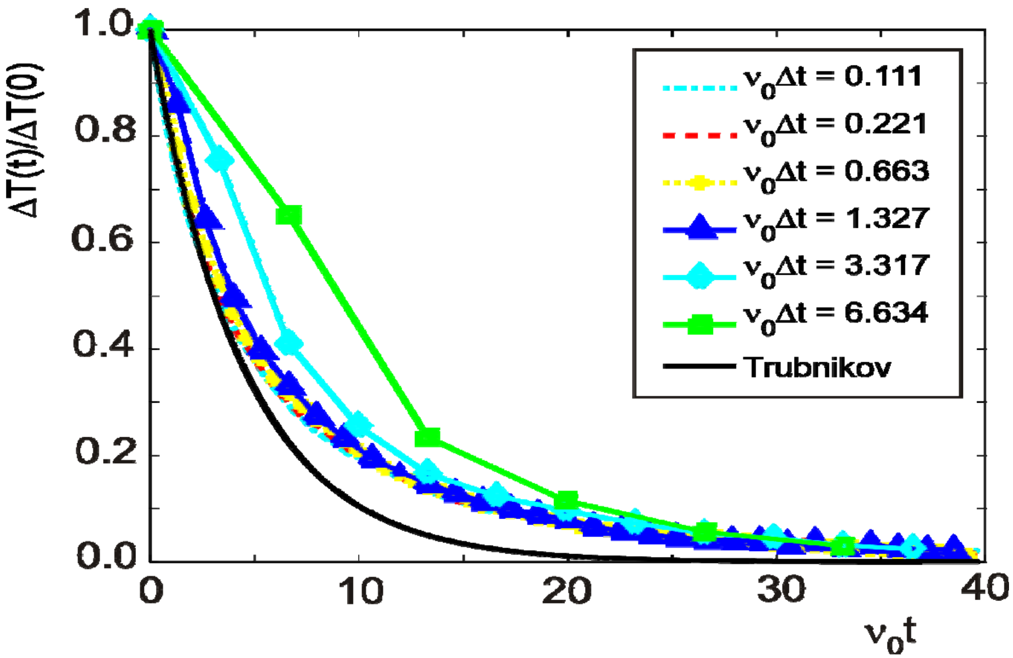
At late times ($\nu_0 t \gtrsim 4$), the simulation curves systematically depart from this exponential curve.

FOR COULOMB COLLISIONS OFF MUCH HEAVIER SCATTERERS, NANBU'S COLLISION OPERATOR IS ACCURATE EVEN FOR VERY LARGE TIME STEPS.



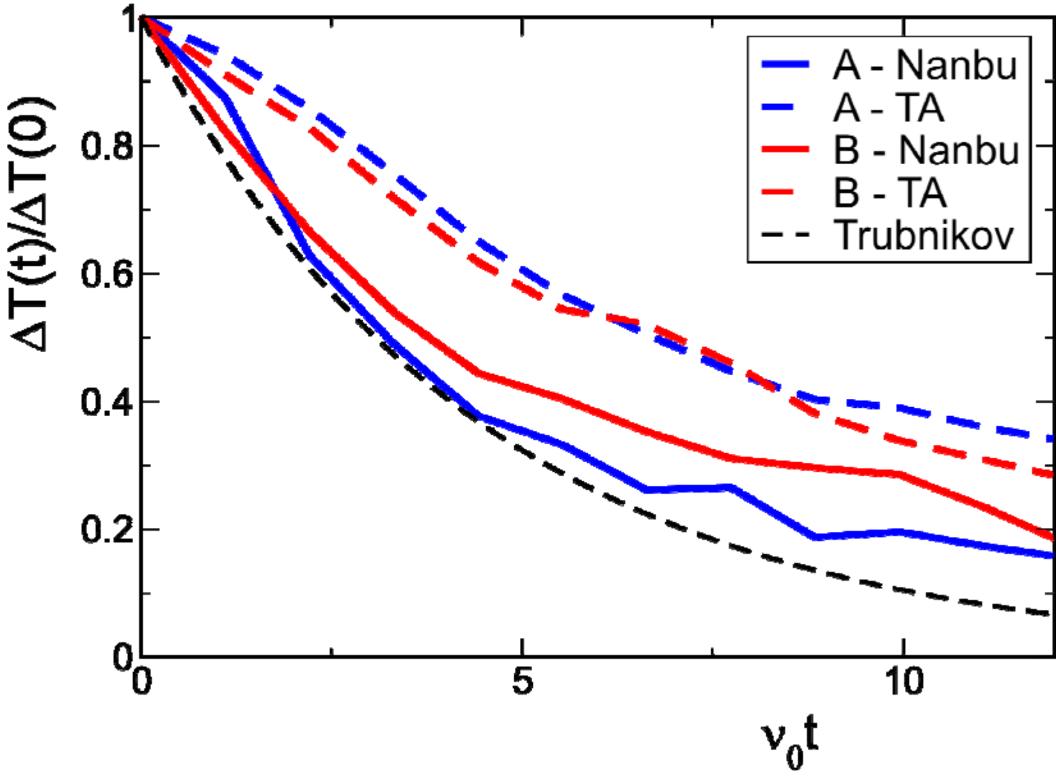
Temperature-anisotropy evolution, now with mass ratio $m_e/m_i = 10^{-4}$, for two different ensembles of 10 realizations, each with 8×10^3 electrons and the same number of ions, for $\nu_0 \Delta t$=0.22, 0.44, 1.1, 2.2 and 3.3.

FOR VERY LARGE TIMESTEPS, THERE IS A DEPARTURE FROM THE SHORT-TIME-STEP CURVES DURING THE FIRST ONE OR TWO TIME STEPS IN APPLICATION OF NANBU'S OPERATOR TO COLLISIONS OFF MUCH HEAVIER SCATTERERS.



Time-step scan for the collisional isotropization test for mass ratios $m_e/m_i=10^{-4}$, averaged over 80 realizations.

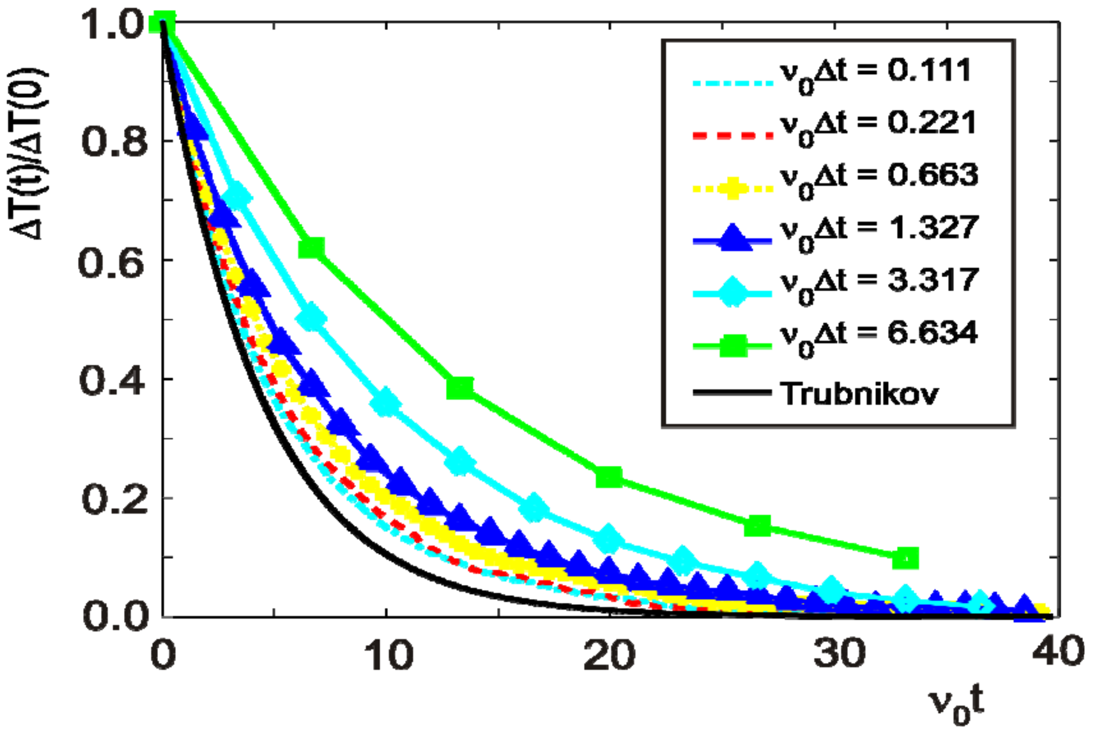
THE TAKIZUKA-ABE OPERATOR IS MUCH LESS ACCURATE THAN THE NANBU OPERATOR FOR LARGE TIMESTEPS AND SMALL MASS RATIO.



Comparison of the TA and Nanbu operators for the collisional isotropization test for $m_e/m_i=10^{-4}$, and $\nu_0\Delta t=1.1$. Cases A and B correspond to the initial loading used in the earlier figure.

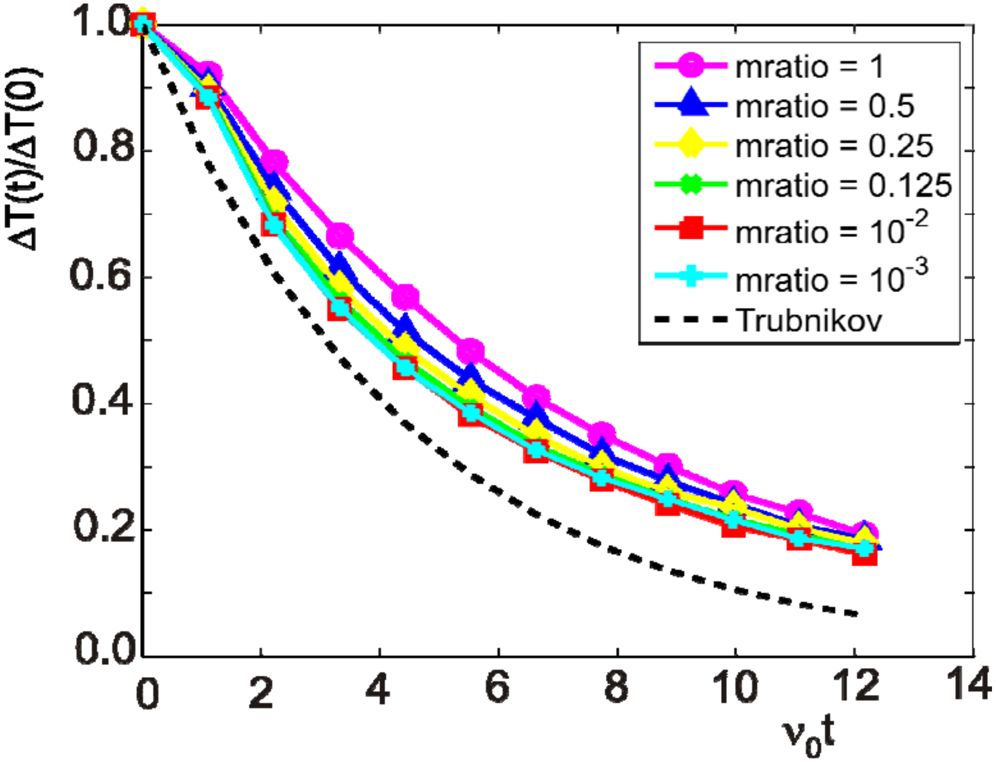
FOR m_e/m_i =1, SIMULATIONS USING NANBU'S OPERATOR CONVERGE FOR SMALL TIME STEP AND AGREE WITH THE EXPONENTIAL FORMULA.

The agreement degrades for larger timesteps, as expected



Time-step scan in the collisional isotropization test for m_e/m_i=1, averaged over 80 realizations of the initial loading, for $\nu_0\Delta t$ ranging from 0.11 to 6.6.

FOR LARGE TIME STEPS, THE AGREEMENT IS BEST FOR VERY SMALL VALUES OF THE MASS RATIO, AND DEGRADES AS THE MASS RATIO INCREASES.



Results for the collisional isotropization test for mass ratios m_e/m_i ranging from 10^{-3} to 1, with $\nu_0\Delta t$=1.1.

Analytical Solution for the Coulomb-Lorentz Collision Kernel, and Comparison With Nanbu's Parameterization

Nanbu's kernel is based on parameterization of numerical realizations of the Coulomb-Lorentz collision process.

The analytical solution for the kernel is [see, e.g., Trubnikov]

$$f_a(\theta,t) = f\left(\mu,s\right) = \frac{1}{2\pi}\sum_{l=0}^{\infty}(l+\tfrac{1}{2})P_l(\mu)\exp\left[-l(l+1)s\right],$$

where μ=cosθ. The normalization is

$$2\pi\int_{-1}^{1} d\mu \; f(\mu,s) = 1 \, .$$

Nanbu's normalized time parameter $s = \left\langle \theta_1^2 \right\rangle N/2$ is related to Trubnikov's normalized time s by $s = t/(2\tau_s)$

Nanbu observed that his simulations were quite accurately represented by linear approximations to the dependence of log[$f(\mu)$] on μ .

He also found empirically that

$$\left\langle \sin^2 \theta / 2 \right\rangle = \frac{1}{2}\left(1 - \langle \mu \rangle\right) = \frac{1}{2}\left(1 - e^{-s}\right).$$

This can be recovered easily from the above analytical solution. It follows that

$$N\left(\mu, s\right) = \frac{A(s)}{4\pi \sinh A(s)} \exp\left[A(s)\mu\right],$$

where A satisfies

$$\coth A(s) - 1/A(s) = e^{-s}$$

Another motivation for Nanbu's form can be seen from the early-time form of the Coulomb-Lorentz kernel, which results from diffusion on a unit sphere, with constant, isotropic diffusion coefficient, of the initial condition

$$f_a(\theta, t=0) = \delta(1-\mu) \Big/ \left(2\pi\right)$$

For short times, this solution is sufficiently localized to not sense the curvature of the sphere (i.e., $\mu \simeq 1$, $\theta \simeq 0$), and the diffusion equation becomes

$$\frac{\partial f_a}{\partial s} \simeq \frac{1}{\theta}\frac{\partial}{\partial \theta}\left[\theta \frac{\partial f_a}{\partial \theta}\right],$$

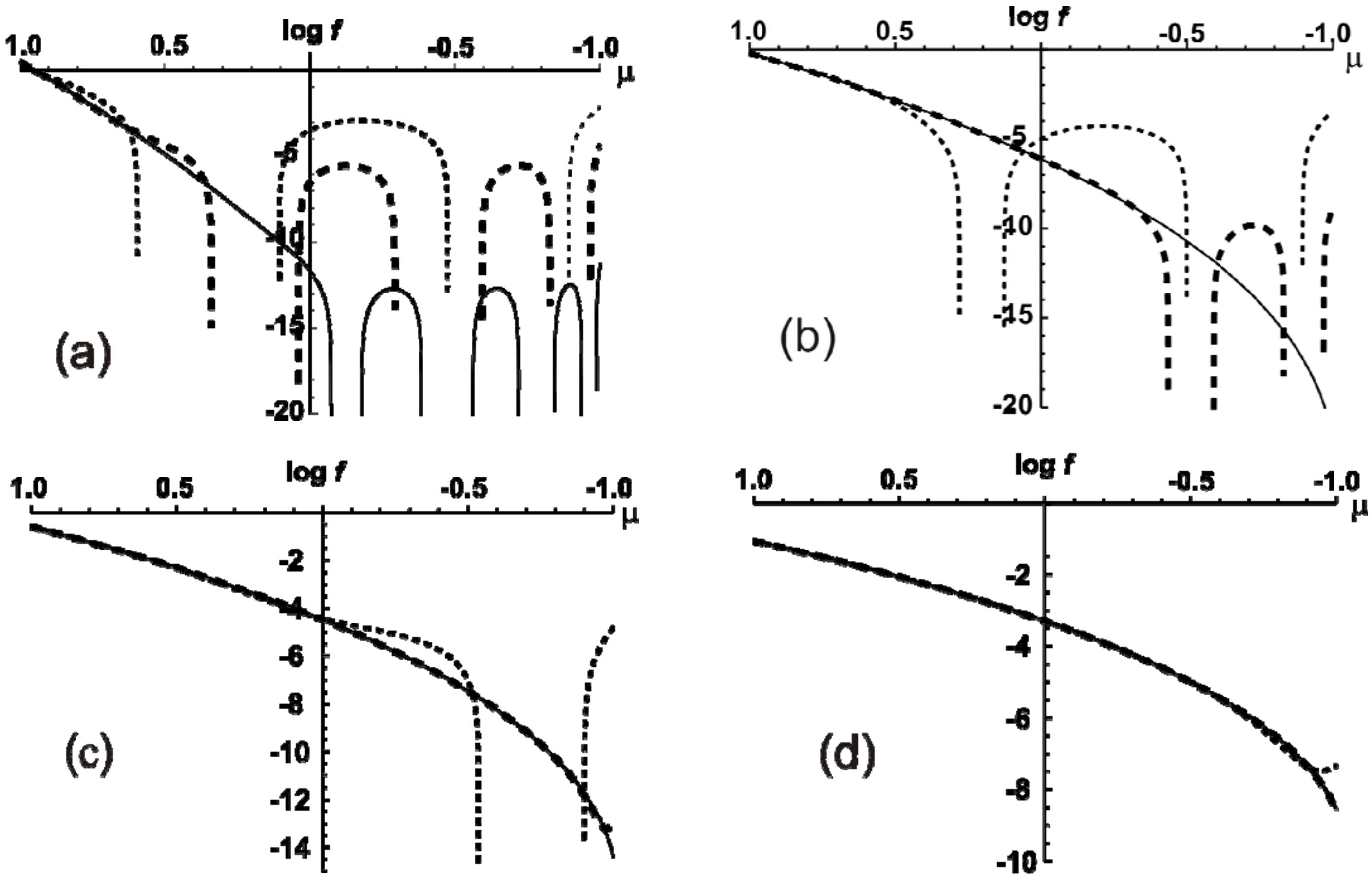
This is the equation for diffusion on a plane with axial symmetry and uniform diffusion coefficient. The kernel for this equation is

$$f_a\left(\theta, t\right) = \frac{1}{2\pi s}\exp\left(-\frac{\theta^2}{2s}\right) \quad \forall s > 0$$

Nanbu's Kernel reduces to this for small s, for which $A(s) \simeq s^{-1}$.

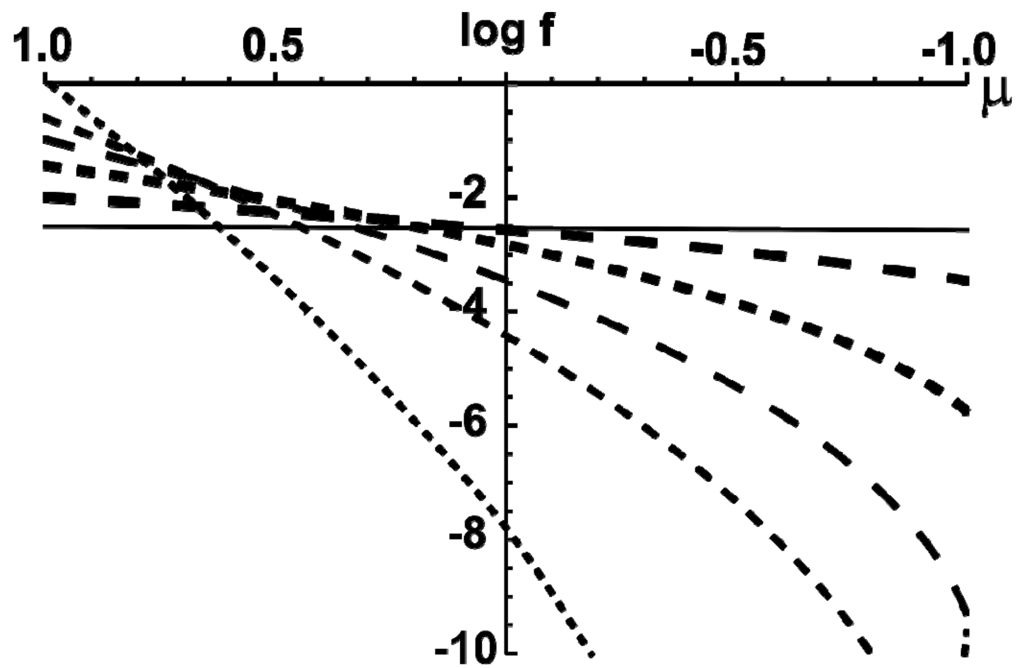
We have coded both the analytical and Nanbu kernels in Mathematica.

8 TERMS ARE SUFFICIENT FOR GOOD ACCURACY OF ANY INTEGRALS INVOLVING $f(\mu)$ FOR $s \geq 0.2$.



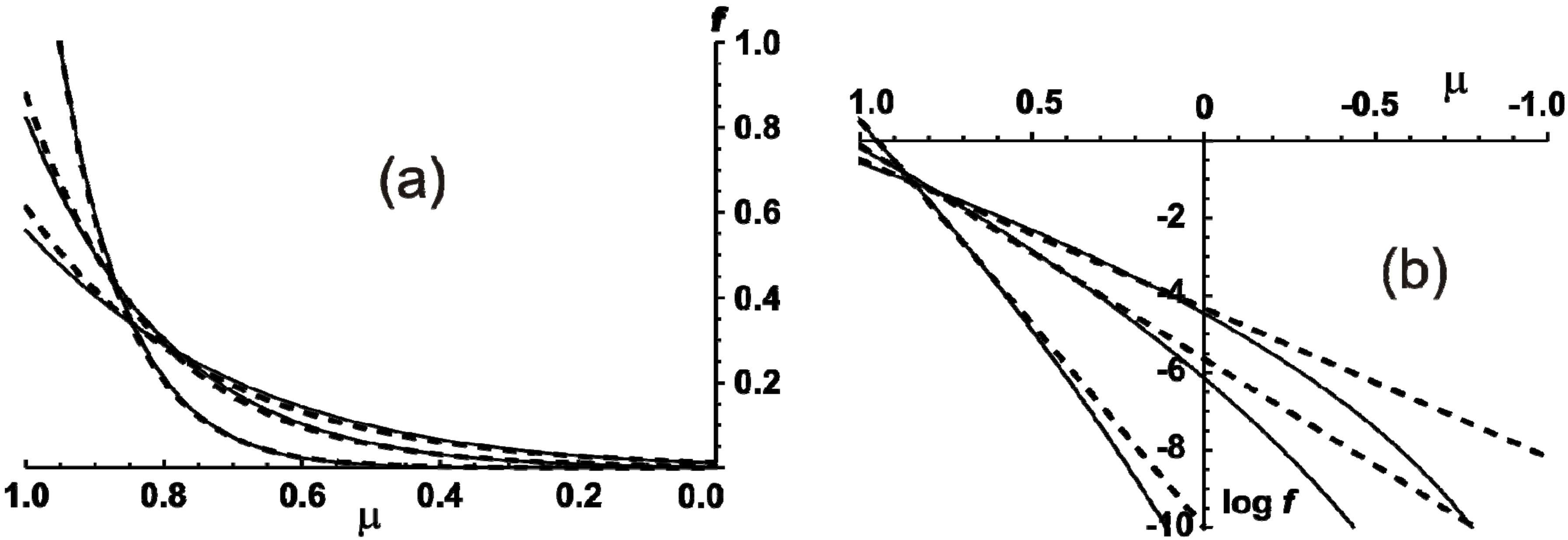
Comparison of log$f(\mu)$,using 4 terms (small dashed), 8 terms (thick dashed), and 14 terms (solid) for (a) s=0.1, (b) s=0.2, (c) s=0.3, and (d) s=0.5.

THE ANALYTICAL KERNEL SHOWS GOOD AGREEMENT WITH NANBU'S KERNEL



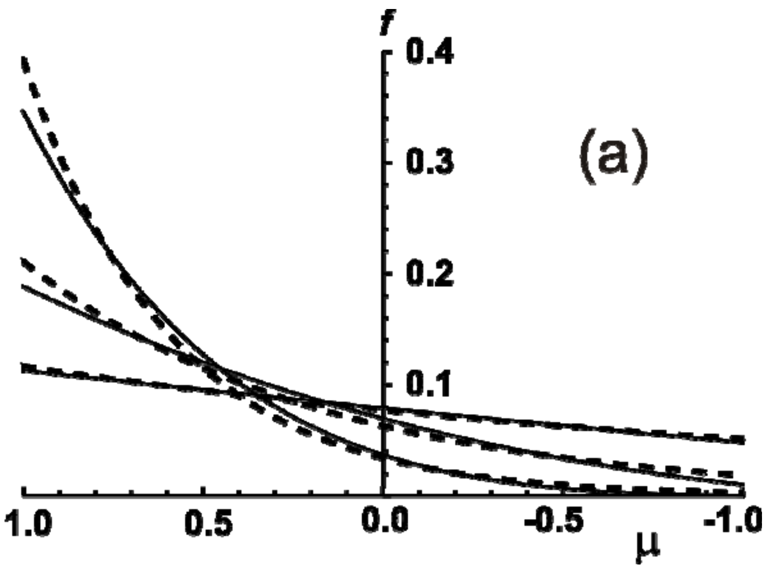
log *f* from analytical solution at times (*s*=0.153, 0.305, 0.458, 0.763, 1.53, 4.58) corresponding to those of Fig. 2 in Nanbu's paper.

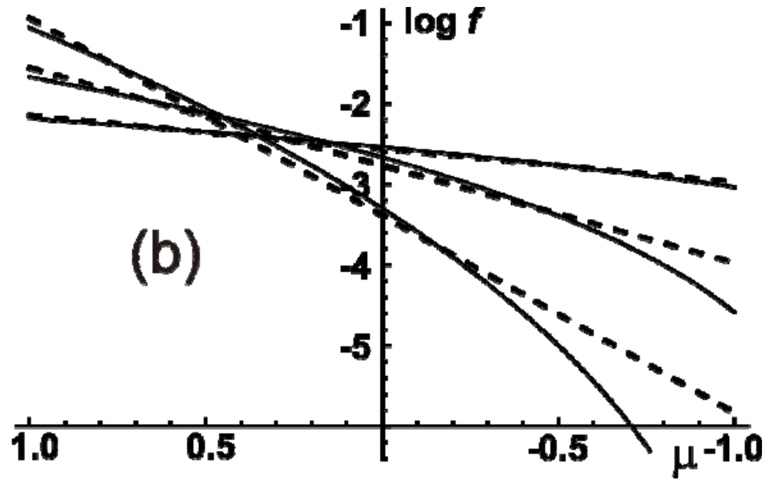
FOR SMALL VALUES OF S, NANBU'S KERNEL GIVES AN EXCELLENT APPROXIMATION TO THE ANALYTICAL ONE, AS EXPECTED FROM ANALYTICAL ARGUMENTS.



Comparison for (a) *f* and (b) log(*f*), between the Nanbu's Kernel (dashed curves) and analytical kernel for s=0.1, 0.2, and 0.3.

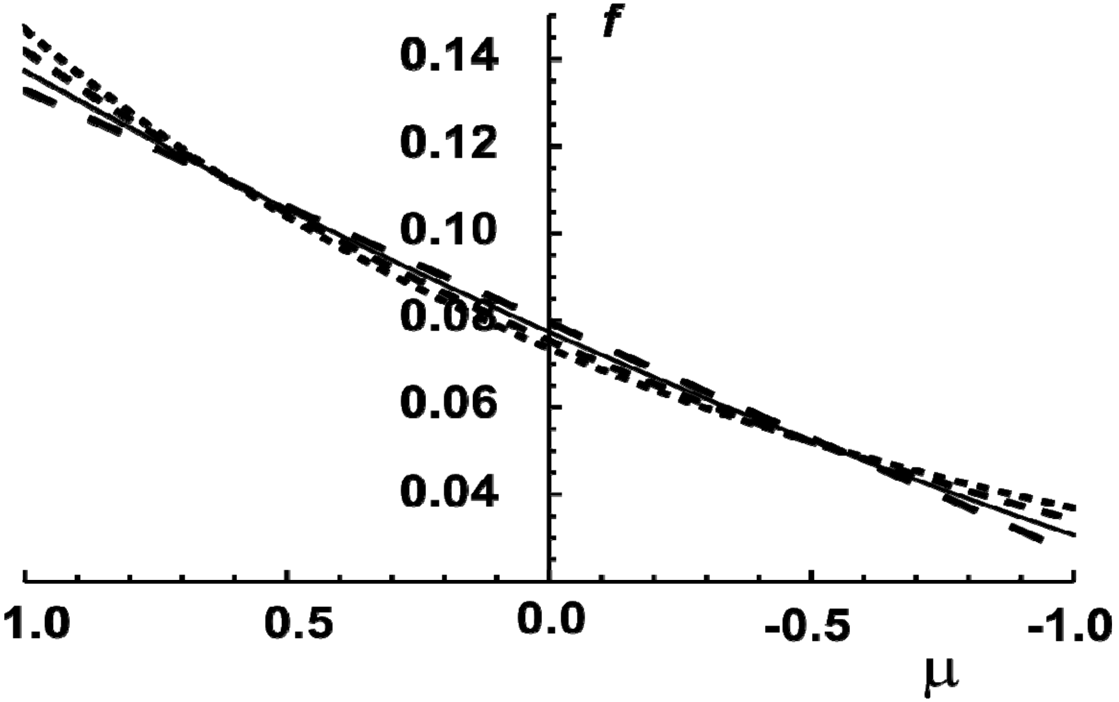
FOR LARGER VALUES OF *S*, NANBU'S PARAMETERIZATION IS A REASONABLY GOOD APPROXIMATION TO THE ANALYTICAL ONE, BUT NOT AS ACCURATE AS FOR SMALLER *S*.





Comparison for (a) *f* and (b) log(*f*), between the Nanbu kernel (dashed curves) and analytical kernel (solid curves) for s=0.5, 1.0, and 2.0

AT LARGE *s*, NANBU'S KERNEL HAS MORE CURVATURE THAN THE ANALYTICAL SOLUTION, WHICH IS QUITE ACCURATELY APPROXIMATED BY ITS FIRST TWO (LINEAR IN μ) TERMS.



Comparison for *s*=1.5 between Nanbu's Kernel [small-dashed curve], and the analytical kernel with 2 terms kept (long-dashed curve), with 16 terms kept (solid curve), and a convolution of two Nanbu Kernels each with s=0.75 (medium dashed curve).

Nanbu's kernel is a slight overestimate $\mu \simeq \pm 1$, and an underestimate near $\mu \simeq 0$. for values of $s \simeq 1$-2, the application of an operator based on this formula to isotropization by Lorentz collisions will slightly underpredict isotropization rates.

ANALYTICAL SOLUTION FOR THE RELAXATION OF TEMPERATURE ANISOTROPY BY COULOMB-LORENTZ COLLISIONS

Relaxation of a small temperature anisotropy by like-particle Coulomb collisions: V. I. Kogan [in Plasma Physics and the Problem of Controlled Thermonuclear reactions (Pergamon Press, New York, 1961), Vol. **1**, p. 153.]; also reviewed by Trubnikov.

An approximate parameterization extending this result to Coulomb collisions of test particles with field particles of different mass, and to large values of the temperature difference is given in the NRL Plasma Formulary (Ed., H. Huba, 2007, Naval Research Laboratory document: NRL/PU/6790-07-500, http://wwwppd.nrl.navy.mil/nrlformulary/), p33.

These both predict an exponential temperature decay with rate $\dfrac{2}{5\sqrt{\pi}}\nu_0$

The numerical results disagree with this formula for $\nu_0 t \gtrsim 4$.

MORE DETAILED EXAMINATION FOR SMALL MASS RATIO (LORENTZ SCATTERING)

Use analytical kernel, which yields a closed-form expression valid for all times.

$$\Delta T = T_{\perp} - T_{\|} = m\left(\left\langle v_{\perp}^2 / 2\right\rangle - \left\langle v_{\|}^2\right\rangle\right)$$
$$= -m\left\langle v^2 P_2(\mu)\right\rangle,$$

where $\left\langle \psi(v,\mu)\right\rangle$ represents and integral of $\psi(v,\mu)$ over the distribution function.

For an initial state with Maxwellian distributions in the parallel and transverse directions,

$$f(v,\mu,t=0) \approx F_m(v) + \delta f(v,\mu,t=0)\,,$$

where

$$F_m(v) = \frac{1}{\left(\sqrt{2\pi} v_{th}\right)^3} \exp\left(-\frac{v^2}{2v_{th}^2} \right)$$

is the equilibrium Maxwellian, and

$$\delta f(v, \mu, t = 0) = \frac{1}{3v_{th}^2} \frac{\Delta T}{T} \left(\frac{1}{2} v_\perp^2 - v_\parallel^2 \right) F_m(v) = -\frac{1}{3v_{th}^2} \frac{\Delta T}{T} v^2 P_2(\mu) F_m(v) .$$

Here, $v_{th}^2 = T/m$, T is the equilibrium temperature, and m is the particle mass.

Using the analytical kernel gives

$$\delta f(v, \mu) = -\frac{1}{3} \frac{\Delta T}{T} \frac{v^2}{v_{th}^2} P_2(\mu) F_m(v) \exp\left[-3 \frac{v_{th}^3}{v^3} \frac{t}{\tau_s(v_{th})} \right],$$

and

$$\Delta\hat{T}(t) \equiv \frac{\Delta T(t)}{\Delta T(0)} = \sqrt{\frac{2}{\pi}} \frac{1}{15 v_{th}^7} \int_0^\infty dv\, v^6 \exp\left[-\frac{1}{2}\left(\frac{v^2}{v_{th}^2} + \frac{6 v_{th}^3}{v^3} \frac{t}{\tau_s(v_{th})} \right) \right].$$

It can easily be verified that

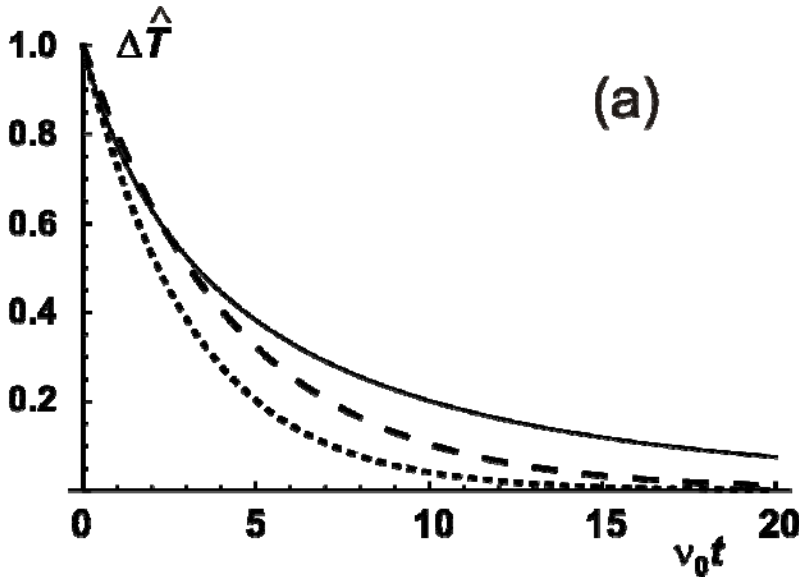
$$\Delta\hat{T}(0) \equiv \sqrt{\frac{2}{\pi}} \frac{1}{15 v_{th}^7} \int_0^\infty dv\, v^6 \exp\left[-\frac{v^2}{2 v_{th}^2} \right] = 1 .$$

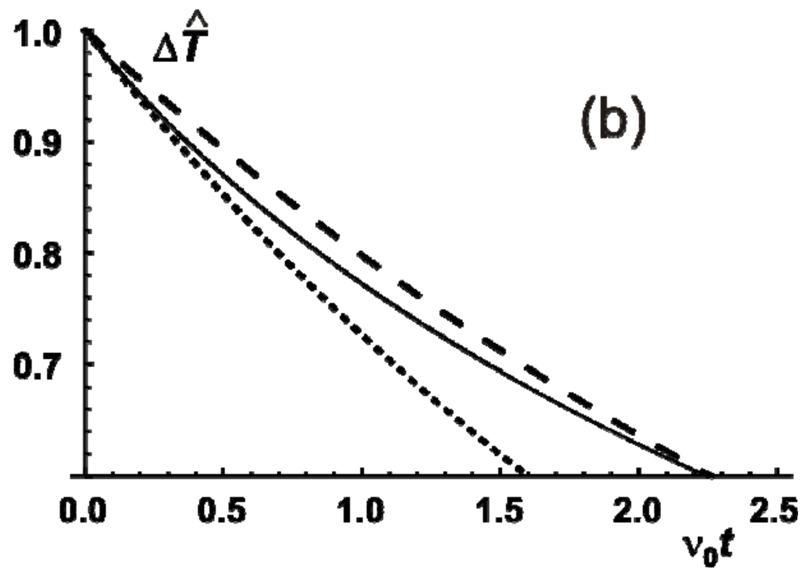
For small t/τ_s,

$$\Delta\hat{T}'(0) \equiv \left. \frac{d\Delta\hat{T}(t)}{dt} \right|_{t=0} = -\frac{1}{\tau_s(v_{th})} \sqrt{\frac{2}{\pi}} \frac{1}{5 v_{th}^4} \int_0^\infty dv\, v^3 \exp\left(-\frac{v^2}{2 v_{th}^2} \right) = -\frac{2}{5} \sqrt{\frac{2}{\pi}} \left[\tau_s(v_{th}) \right]^{-1}$$

This early-time decay rate is a factor of $\sqrt{2}$ greater than that given earlier.

THIS CAN BE BETTER UNDERSTOOD THROUGH DIRECT EVALUATION OF $\Delta\hat{T}(t)$





Comparison of the temperature-anisotropy decay vs. time predicted by analytical solution, vs. exponential decay with rates given by $\left[2\big/\left(5\sqrt{\pi}\right)\right][\tau_s(v_{th})]^{-1}$ (large-dashed curve) and $\left[2\sqrt{2}\big/\left(5\sqrt{\pi}\right)\right][\tau_s(v_{th})]^{-1}$ (small dashed curve), on two different scales. [Frame (b) shows detail of the early-time dependence.]

IMPROVEMENT OF NANBU'S COLLISION OPERATOR FOR COULOMB-LORENTZ COLLISIONS

An improvemement to Nanbu's operator for Lorentz (small mass-ratio) collisions can be made based on the above equations.

An accurate approximation to the kernel for the Lorentz collision operator can be obtained by using a matched expression in which Nanbu's form, is used for small $s \leqslant s_0$ and the analytical solution with a finite number of terms is used for $s > s_0$.

An accurate combination is given by s_0=0.1, and 14 terms in the analytical expression.

$$f_a(\mu, s) = \begin{cases} \dfrac{A(s)}{4\pi \sinh A(s)} \exp\left[A(s)\mu\right], & \text{for} \quad s \leq 0.1, \\ \dfrac{1}{2\pi} \displaystyle\sum_{l=0}^{13} (l + \tfrac{1}{2}) P_l(\mu) \exp\left[-l(l+1)s\right], & \text{for} \quad 0.1 < s \leq 4.0, \\ \dfrac{1}{2\pi} \displaystyle\sum_{l=0}^{1} (l + \tfrac{1}{2}) P_l(\mu) \exp\left[-l(l+1)s\right], & \text{for} \quad s > 4.0, \end{cases}$$

A more general version is:

$$f_a(\mu, s) = \begin{cases} \dfrac{A(s)}{4\pi \sinh A(s)} \exp\left[A(s)\mu\right], & \text{for} \quad s \le s_0, \\ \dfrac{1}{2\pi} \displaystyle\sum_{l=0}^{m(s)} (l + \tfrac{1}{2}) P_l(\mu) \exp\left[-l(l+1)s\right], & \text{for} \quad s > s_0. \end{cases}$$

Now, compute the following indefinite integral, which lies between 0 and 1.

$$F(\mu, s) = 2\pi \int_{-1}^{\mu} d\xi \ f(\xi, s) ,$$

$$F(\mu, s) = \begin{cases} \dfrac{1}{2\sinh A(s)}\Big(\exp\big[A(s)\mu\big] - \exp\big[-A(s)\big]\Big), & s \le s_0, \\ \dfrac{1}{2}\sum_{l=0}^{m(s)}\Big\{\big[P_{l+1}(\mu) - P_{l-1}(\mu)\big] - \big[P_{l+1}(-1) - P_{l-1}(-1)\big]\Big\}\exp\big[-l(l+1)s\big], & s > s_0. \end{cases}$$

The following identity has been used, with the convention $P_{-1}(\mu)=0$.

$$\int_{-1}^{\mu} d\xi\, P_{l+1}(\xi) = \frac{1}{1+2l}\Big(\big[P_{l+1}(\mu) - P_{l-1}(\mu)\big] - \big[P_{l+1}(-1) - P_{l-1}(-1)\big]\Big)$$

Then, numerically invert $F(\mu,s)$ to obtain

$$F_I(F,s) \equiv F^{-1}(F,s) : \big[0,1\big]\times\big[0,\infty\big) \to \big[-1,1\big]\times\big[0,\infty\big)$$

on a mesh in the $(F,s) \in [0,1]\times[0,\infty)$ plane.

F_I is the function needed for sampling the kernel using (pseudo)random numbers uniformly distributed in the interval [0,1]. The resulting table of values of $F_I(F,s)$ needs to be computed only once, given a choice of (F,s) mesh.

A small preprocessor program can be used to generate this and store this table before the time advance is begun. In practice, it may be most expedient to compute and use the table directly only for intermediate values of s, $0.1 \leqslant s \leqslant 4.0$.

For $s \leqslant 0.1$, an analytical inversion can be used exactly as for Nanbu's algorithm.

For $s \geqslant 4.0$, a perturbative analytical calculation of the inversion can be used, based on the dominance of the first (l=0) term in the sum. Keeping only the l=0 term gives the leading order solution

$$\mu(F,s) = F^I(F,s) \approx F - 1 .$$

Iterating gives:

$$\mu(F,s) = F^{I}(F,s)$$
$$\approx F - 1 - \frac{1}{2}\sum_{l=1}^{m(s)} \left\{ \left[P_{l+1}(F-1) - P_{l-1}(F-1) \right] - \left[P_{l+1}(-1) - P_{l-1}(-1) \right] \right\} \times$$
$$\exp\left[-l(l+1)s \right]$$

where only a small number of terms is needed.

At any given time step, for each particle pair l with a s value s_l, use a standard pseudorandom number generator to generate a number r.

Then, given this (r, s_l), use either interpolation (if $0.1 \leqslant s_l \leqslant 4.0$) or the analytical results for $F_I(F,s)$ (for $s_l \leqslant 0.1$ $s_l \geqslant 4.0$) to find an approximation to $F_I(r, s_l) = \mu$. This value of μ represents the cosine of the angle of the relative velocity vector of the pair with respect to the pre-collision direction.

For large time steps, the operator just described will produce a more accurate approximation to the Coulomb-Lorentz (or very small mass-ratio interspecies scattering process) than Nanbu's original operator.

In applications the (Lorentz) pitch-angle scattering process takes place simultaneously with energy evolution due to scattering (if the mass ratio is not very small) or other processes (such as acceleration by collective or macroscopic electric or magnetic fields). An accurate representation of such processes may still require small frequent collisional time sub steps interspersed with sub steps that advance the effects of the other competing processes.

CONVOLUTION OF TWO NANBU KERNELS

The simplicity of the Nanbu Kernel enables a straightforward evaluation of the convolution of two Nanbu Kernels in closed form as a one-dimensional integral, which is easily evaluated numerically.

$$N_2\left(\bullet, s_1, s_2\right) = N\left(\bullet, s_1\right) * N\left(\bullet, s_2\right),$$

Denote the two reference points and one integration point on the unit sphere respectively as *O*, *P* and *P'*, given by the unit vectors and polar coordinates $\hat{\boldsymbol{o}}$:(θ=0), $\hat{\boldsymbol{p}}$:(θ,ϕ), and $\hat{\boldsymbol{p}}'$:(θ',ϕ'). Then

$$N_2\left(\mu, s_1, s_2\right) = \int dS_{P'}\, N\left(\mu', s_1\right) N\left(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}', s_2\right),$$

where $dS_{P'} = 2\pi d\mu' d\phi'$ is an area element for integration over points *P'* on the unit sphere, and $\mu' = \cos\theta'$.

Using

$$\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}' = \sin\theta \sin\theta' \cos(\phi - \phi') + \cos\theta \cos\theta',$$

and integrating over ϕ', and using $\oint d\phi' \exp(\alpha \cos\phi') = 2\pi I_0(\alpha)$, where I_0 denotes the modified Bessel function of order zero, gives

$$N_2\left(\mu, s_1, s_2\right) = 2\pi C(A_1)C(A_2)\int_{-1}^{1} d\mu' \exp[(A_1 + A_2\mu)\mu'] I_0\left(A_2\sqrt{(1-\mu^2)(1-\mu'^2)}\right)$$

where

$$C(A_i) = \frac{A(s_i)}{4\pi \sinh A(s_i)} \cdot$$

These equations have been coded in Mathematica to yield the medium-dashed curve in the next-to-last figure.