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Fingering patterns in Hele-Shaw flows are density shock wave solutions of dispersionless KdV hierarchy

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We investigate the hydrodynamics of a Hele-Shaw flow as the free boundary evolves from smooth initial conditions into a generic cusp singularity (of local geometry type $x^3 \sim y^2$), and then into a density shock wave. This novel solution preserves the integrability of the dynamics and, unlike all the weak solutions proposed previously, is not underdetermined. The evolution of the shock is such that the net vorticity remains zero, as before the critical time, and the shock can be interpreted as a singular line distribution of fluid deficit.

1. Introduction Hele-Shaw flows, introduced over a century ago [1], are known for leading to very complex patterns (FIG. 1, [2]), despite of the simplicity of the experimental set-up, FIG. 2: an almost inviscid fluid (e.g. air, occupying the domain D) is pumped in at a constant rate, displacing a very viscous fluid (e.g. oil, denoted by $\tilde{D}$), without mixing. The fluids are confined to a two-dimensional cell of height $b \to 0$, created between two horizontal glass plates. Alternative realizations [3, 4] use a 2D monolayer obtained by wetting or using granular media. As the fluids are immiscible, all the dynamics is encoded in the motion of the boundary, $\gamma(t) = \partial D$.

Neglecting the effects of surface tension, we obtain the idealized model of Hele-Shaw flow, called Laplacian growth [5–7]. Under this assumption, the equations of motion, obtained by averaging the 3-D Navier-Stokes equations over the height of the cell, give the law governing the evolution of the boundary:

$$\rho \mathbf{v}_n = -K \nabla_n p, \qquad (1)$$

where $K = \frac{b^2}{12\nu}$ is called *hydraulic conductivity*, ν is the kinematic viscosity, ρ is the density of the liquid and $p, \mathbf{v}_n, \nabla_n p$ are the exterior pressure, and the normal components of the boundary velocity and pressure gradient, respectively. Pressure solves a Dirichlet problem:

$$\Delta p = 0 \quad \text{on } \tilde{D}, \quad p_{|\gamma} = 0, \quad p_{|z\to\infty} \sim -\log|z|. \qquad (2)$$

The law (1) is named after H. Darcy, who discovered it while studying the groundflow of water in porous soil [8].

It is important to note here that the problem (1), (2) is far more fundamental than the original hydrodynamic formulation suggests. Namely, *growth* occuring at a rate ($\mathbf{v_n}$) proportional to the gradient of a harmonic function (p) and with sources/sinks at infinity, is a very wide class of processes, including electrodeposition, snowflake growth, and diffusion-limited aggregation (DLA). The common patterns observed in these phenomena are not well understood theoretically. A key element in these patterns is *branching* (or fingering, or tip-splitting), whose parameters (local geometry, frequency of occurence, etc) are essential for understanding the global properties of the cluster.

FIG. 1: Experimental Hele-Shaw fingering patterns [2]

Formally, (2) is readily solved by using the conformal map $z \to w$ of $\tilde{D}$ onto the exterior of unit circle $|w| > 1$, with the solution (here Q is the constant rate of pumping fluid, $\frac{\rho Q}{\pi} = \frac{dt_0}{dt}$, t_0 - normalized area of D and t - time):

$$p(z) = -\frac{Q}{2\pi K}\log|w(z)|, \ \rho \mathbf{v}_n = \frac{Q}{2\pi}|w'(z)|, \ z \in \tilde{D}. \quad (3)$$

We note here that, due to its analytical properties, the function $\phi(z) = -\mathrm{i}\log w(z)$ is (up to a trivial constant), the complex potential for the flow (1), such that Re $\phi \sim \psi$ (stream function), and Im $\phi \sim p$ (pressure), as in (3).

However, the simplicity of this solution is deceiving, because for an overwhelmingly large class of initial conditions, (2) turns out to be ill-defined.

2. Generic singularities of Hele-Shaw flows Since any map $w(z)$ will have singularities inside D, where $w'(z) \to \infty$, it follows that if a singularity of this type approaches the boundary γ, the local velocity will diverge. This is indeed what happens for most initial conditions for D, and it reveals a fundamental instability of Laplacian Growth, called *finite-time singularities*, because they occur for finite values of the area t_0, and therefore, of time. They were systematically studied and classified in a series of papers [9–12].

The type of singularity encountered most frequently is

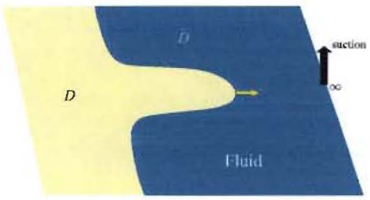

FIG. 2: Hele-Shaw flow with suction mechanism. Darker region is oil, outside is air.

a boundary cusp at (x_0, y_0), of local geometry given by two mutually prime numbers p, q:

$$(x - x_0)^p \sim (y - y_0)^q. \tag{4}$$

Among such points, the one considered generic [11] has $p = 3, q = 2$ and is referred to as the (2,3) cusp, FIG. 3.

The $(2, 5)$ cusp was also studied in detail [12, 13] and it was shown that its evolution may be smoothly continued beyond the critical time t_c (the moment when a cusp is formed). However, for the (2,3) cusp, a naive continuation of the classical solution is not possible. In the remainder of the paper, we will show that such a continuation can be achieved in a *weak sense*, and that its physical realization is a shock wave.

It is important to note here that there have been many attempts to find the proper way to regularize these singularities [9, 11, 14]. However, due to the singular-perturbation nature of the problem, every method proposed led to a very different physical prediction, none could explain the actual patterns observed experimentally [2, 4], and – most importantly – they were all under-determined and required extraneous information to fix a solution. Consequently, the problem stayed open for decades, despite its venerable age.

The point missed in these studies is the fact that the dynamics described by (2) has an important feature: it is integrable. This fact was established long ago [5–7], and was recently explored to the fullest extent [13, 15–19]. In this paper, we show that this important feature provides a well-defined weak solution.

3. Alternative hydrodynamic formulation In the following, we recall an alternative description of the problem, using the *height* function y, introduced in [17]. There, we showed that the dynamics of Hele-Shaw flows can be restated using the local geometry of the boundary: expanding the conformal map $z(w, t)$ around the location of the cusp z_c, as a function of time (relative to the critical time, t_c), and angle (choosing z_c real), $w = e^{i\phi}$, we have the approximation (up to an overall scale) $z(w) - z_c = x(\phi) + iy(\phi)$, where

$$x(\phi) \approx \frac{3}{2}[e(t) - \phi^2], \quad y(\phi) \approx \frac{\phi^3}{2} - \frac{3}{4}e(t)\phi, \tag{5}$$

and $e(t)$ is a critical point of the map, $z'(e) = 0$:

$$e(t) = -\frac{2}{3}\left(1 - \frac{t}{t_c}\right)^{1/2}. \tag{6}$$

Rescaling the local coordinates as $X = \frac{2x}{3}$, $Y = 4iy$, we have the local approximation in *elliptic* form:

$$Y^2 = 4(X - e)(X + e/2)^2. \tag{7}$$

The choice of rescaling is such that Y is real for $X \in [e, -e/2]$ - the region of interest.

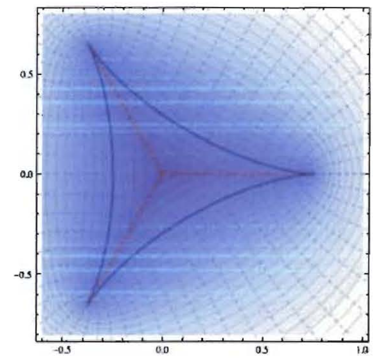

FIG. 3: Symmetric hypertrochoid evolving under Darcy law reaches the (2,3)-cusp singularities when all three critical points (red dots) hit the boundary at the same time. The shaded contour lines are the equi-pressure lines. The dashed lines are the stream lines. The red dashed lines are branch cuts -*a skeleton*.

At the critical point $t = t_c, e = 0$ the curves degenerates further into a (2,3) cusp: $Y^2 \sim X^3$. As shown in [17], the KdV-type integrable hierarchy describing the evolution of this curve is given through a set of equations; the first of them (correspoding to the standard KdV equation for $e(t)$) is

$$\partial_t Y = -i\partial_z \phi. \tag{8}$$

Since ϕ is the complex potential of the flow, we conclude that the complex velocity field $v = v_x - iv_y$ (related to $\nabla\phi$ by Darcy law) is encoded in $\dot{Y} = \partial_t Y$:

$$\mathrm{Re}\,(\dot{Y}\mathrm{d}z) = \mathrm{d}p, \quad \mathrm{Im}\,(\dot{Y}\mathrm{d}z) = -\mathrm{d}\psi, \tag{9}$$

everywhere Y, p, ψ are differentiable.

This fact is very important for formulating the physical weak solution, and cannot be stressed enough. In the form (1), (2) the problem is ill-defined and typically leads to a (2,3) cusp. However, it may turn out to be tractable, once we express it through the evolution of the height function. This is the approach that we use here: characterize the Hele-Shaw flow through $Y(z, t)$.

4. Hydrodynamics from the height function Starting from the complex velocity field v, we derive the identity $2\bar{\partial}v = \nabla\cdot\mathbf{v} + i\nabla\times\mathbf{v}$, where $\nabla\times\mathbf{v} = \partial_y v_x - \partial_x v_y$ is the vorticity field, and $\nabla\cdot\mathbf{v} = \partial_x v_x + \partial_y v_y$ is the divergence of velocity field. Using (8), and integrating over a closed loop ∂B bounding the domain B, we obtain:

$$\mathrm{Re}\oint_{\partial B}\dot{Y}\mathrm{d}z = \iint_B \nabla\times\rho\mathbf{v}\mathrm{d}x\mathrm{d}y = \oint \mathrm{d}p \quad (10)$$

$$\mathrm{Im}\oint_{\partial B}\dot{Y}\mathrm{d}z = \iint_B \nabla\cdot\rho\mathbf{v}\,\mathrm{d}x\mathrm{d}y = -\oint \mathrm{d}\psi. \quad (11)$$

We also note (7) that the critical points of the conformal map $e(t)$ are critical points of $Y(z)$. More precisely, they are branch points for $Y(z)$, and must be connected by branch cuts along which the function $Y(z)$ will have a finite discontinuity (jump). They are depicted in FIG. 3.

Therefore, since the leftmost contour integrals in (11, 10) reduce to line integrals along the branch cuts of $Y(z)$, we obtain the following equivalent representation of the flow $\mathbf{v}$: the *sources* of the field can be identified with the support of the jump of $\dot{Y}$, and its *strength* becomes the magnitude of the jump itself, $\sigma|\mathrm{d}z| = \mathrm{disc}(\mathrm{i}\dot{Y}\mathrm{d}z)$ for any infinitesimal arc $\mathrm{d}z$. Equivalently, we may say that the droplet of *uniform* density in D can be replaced by a *non-uniform, singular* line density σ, supported on the cuts Γ. In particular, the Newtonian potentials created by these two distributions outside D are identical.

Before the boundary singularity forms, this alternative formulation does not lead to any new physical conclusions. After the critical time t_c, however, the situation is very different, especially because at $t = t_c$, the endpoint of the cut Γ touches the boundary, and evolves outside D for higher times.

We propose to identify the evolution of the droplet D after a critical time t_c with the evolution of the branch cuts $\Gamma(t)$ *outside* of D, together with the evolution of the rest of the droplet. The branch cuts found outside of D have the physical interpretation of *density shock waves.*

In short, we will replace the (uniform) density of oil by

$$\rho(\mathbf{x},t) = \rho_0 - \delta(\mathbf{x};\Gamma)\sigma(\mathbf{x},t) \quad (12)$$

where the line density $\sigma(\mathbf{x}, t > t)_c$ represents the density of $-\mathrm{i}Y(X)$ on the cut Γ (here $\delta(\mathbf{x};\Gamma)$ is the delta-function on the cut).

Once this choice for the weak solution is made, all that remains is to specify how to choose the branch cuts beyond the critical time. As we show in the next section, this choice is uniquely determined by the physical interpretation of the integrability condition: *zero-vorticity*.

5. The physical prescription for weak solutions Integrability of the Laplacian growth can be (and has been) formulated in many ways; for our purposes, the most relevant here is the statement that the density of the height function remains (after the critical time) *real* and *positive*, as it was before t_c. We refer the reader to [20] for a comprehensive discussion on these conditions.

Here, we note only that these requirements guarantee the zero-vorticity condition for the flow after t_c. From the first equation in (10), we see that this condition yields $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\oint Y\mathrm{d}z = 0$ for all closed cycles in the fluid. The condition that the line densities are real

$$\sigma(z) = \text{real}, \quad z\in\Gamma, \quad (13)$$

satisfies this requirement, and furthermore leads to a stronger condition

$$\mathrm{Re}\oint Y(z)\mathrm{d}z = 0, \quad \text{all fluid cycles.} \quad (14)$$

The physical interpretation of the positivity condition is that shocks represent *fluid deficit*. As a result, the shock moves toward the direction of higher pressure. Let us now use this prescription to find the velocity of the shock. For that purpose, we denote by $\mathbf{n}, \ell$ the normal and tangent unit vectors at a point on the shock. Now consider the integral $\oint_{\partial B} Y(z)\mathrm{d}z$ over a loop which intersects the shock at two points $z_{1,2}$. This integral stays purely imaginary at all times:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\oint Y(z)\mathrm{d}z = 0. \quad (15)$$

Letting the contour now shrink to an infinitesimal loop, $z_{1,2}\to z$, we obtain the differential form of (15) as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\,[\mathrm{disc}\,Y\ \mathrm{d}z]_\Gamma = 0. \quad (16)$$

We denote the velocity of the shock front (normal to the instantaneous curve Γ), by $\mathbf{V}_\perp$, directed along the vector $\mathbf{n}$. Then the total time derivative (16) becomes

$$\mathrm{Re}\,[\mathrm{disc}\,\dot{Y}\ \mathrm{d}z + \nabla_\parallel(\mathrm{disc}\,S\cdot\mathbf{V}_\perp)|\mathrm{d}z|]_\Gamma = 0, \quad (17)$$

where $\nabla_\parallel$ represents the derivative along the direction tangent to the front.

For the first term, we use a kinematic identity $\mathrm{Re}\,[\mathrm{disc}\,\dot{Y}\ \mathrm{d}z] = -\mathrm{disc}\ v_\parallel|\mathrm{d}z|$, valid on both sides of the shock. From the reality of σ, it follows that the second term in (17) is purely real, and equals $-\nabla_\parallel(\sigma\mathbf{V}_\perp)|\mathrm{d}z|$.

Together, it yields to the condition

$$\nabla_\parallel J_\perp + \mathrm{disc}\ j_\parallel = 0, \quad J_\perp = \sigma\mathbf{V}_\perp, \quad j_\parallel = \rho_0 v_\parallel. \quad (18)$$

The first term in this equation represents the transport of mass due to motion of the shock (normal to the shock itself), while the second is the vorticity of the surrounding fluid flow. Thus, condition that σ is real means zero-vorticity condition for the fluid.

Using Darcy law we replace the fluid velocity in (18) by $-\nabla_\parallel p$, and integrate (18) along the cut. We obtain the condition

$$\sigma\mathbf{V}_\perp = (\mathrm{disc}\ p)\ \mathbf{n} \quad (19)$$

(the constant of integration is fixed from the assumption that both the line density and discontinuity of pressure

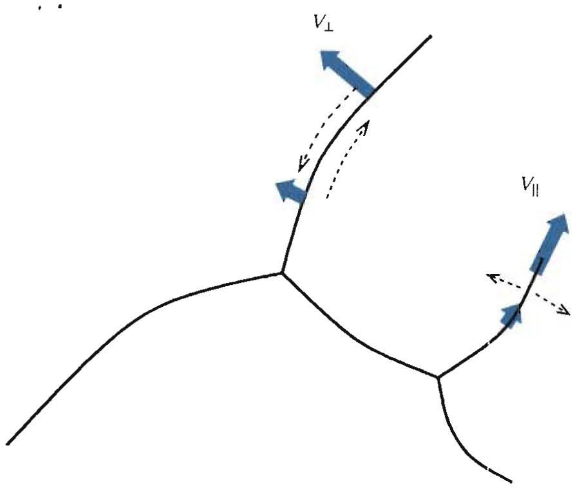

FIG. 4: In this figure, we show the normal velocity of the shock by the thick arrows and the flow of fluid around the shock by dashed arrows.

vanish at endpoints of the cut). The fact that $\sigma > 0$ is positive means that a shock moves toward the direction of larger pressure.

Furthermore, calculating the discontinuities on both sides of the Darcy law, we obtain

$$\dot{\sigma} = -\text{disc}\ \nabla_{\|}\psi\ . \tag{20}$$

This formula has the following interpretation. Let us assume the shock filled with some material (say, an inviscid fluid in the Hele-Shaw cell), of line density σ. Then the continuity condition $\dot{\sigma}+\nabla_{\|}(\sigma \mathbf{V}_{\|}) = 0$, tells that this material is moving along the shock with velocity $\mathbf{V}_{\|}$. Combining this with (20) and integrating along the shock, we obtain a counterpart of (18): divergency of the sliding current along a shock is the discontinuity of current of fluid normal to the shock (see Figure 4):

$$\nabla_{\|}J_{\|} - \text{disc}\ j_{\perp} = 0, \quad J_{\|} = \sigma \mathbf{V}_{\|}, \quad j_{\perp} = \rho_0 v_{\perp}. \tag{21}$$

Integrating this formula along a shock gives:

$$\sigma \mathbf{V}_{\|} = (\text{disc}\,\psi)\,\boldsymbol{\ell}. \tag{22}$$

Equations (19), (22) completely specify the motion of the shock.

6. Conclusions In this Letter, we have proposed a weak solution for the finite-time singularities of Laplacian growth, and at the same time, a new method for dealing with such critical events in this class of processes. The prescription we gave allows to continue the dynamics beyond the critical time, for the price of introducing desnity wave shocks. The dynamis of the shocks is integrable, and related to the KdV hierarchy. In future publications, we will explore the detailed geometric properties of the shock dynamics and indicate how they relate to known patterns observed in processes like diffusion-limited aggregation.

We note that very simple requirements for the dynamics of the shocks (reality and positivity of the density) led to a prescription of the shock dynamics which is reminiscent of the known Rankine-Hugoniot conditions for shocks in compressible Eurler flows [21]. This is not surprising, since the nature of our weak solution is that of compressibility shocks (12).

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