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Solutions to Master Equations of Quantum Brownian Motion in a General Environment with External Force

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Abstract

We revisit the model of a system made up of a Brownian quantum oscillator linearly coupled to an environment made up of many quantum oscillators at finite temperature. We show that the HPZ master equation for the reduced density matrix derived earlier [B.L. Hu, J.P. Paz, Y. Zhang, Phys. Rev. D **45**, 2843 (1992)] has incorrectly specified coefficients for the case of nonlocal dissipation. We rederive the QBM master equation, correctly specifying all coefficients, and determine the position uncertainty to be free of excessive cutoff sensitivity. Our coefficients and solutions are reduced entirely to contour integration for analytic spectra at arbitrary temperature, coupling strength, and cut-off. As an illustration we calculate the master equation coefficients and solve the master equation for ohmic coupling (with finite cutoff) and example supra-ohmic and sub-ohmic spectral densities. We determine the effect of an external force on the quantum oscillator and also show that our representation of the master equation and solutions naturally extends to a system of multiple oscillators bilinearly coupled to themselves and the bath in arbitrary fashion. This produces a formula for investigating the standard quantum limit which is central to addressing many theoretical issues in macroscopic quantum phenomena and experimental concerns related to low temperature precision measurements. We find that in a dissipative environment, all initial states settle down to a Gaussian density matrix whose covariance is determined by the thermal reservoir and whose mean is determined by the external force. We specify the thermal covariance for the spectral densities we explore.

Contents

I. INTRODUCTION

A. Historical Context and New Results

In this paper we continue the lineage of work on quantum Brownian motion (QBM) via the influence functional path-integral method of Feynman and Vernon[1] used by Caldeira and Leggett[2] to derive a master equation for a high-temperature ohmic environment, which corresponds to the Markovian regime. Following this, Caldeira, Cerdeira and Ramaswamy (CCR)[4] derived the Markovian master equation for the system with weak coupling to an ohmic bath, that was claimed to be valid at arbitrary temperature (Sec. IV C 3). At the same time Unruh and Zurek[5] derived a more complete and general master equation that incorporated a colored noise at finite temperature. Finally, and barring certain errors we will discuss, Hu, Paz and Zhang (HPZ) [12] derived a master equation for a general environment (arbitrary temperature and spectral density). Our contribution to this legacy are threefold:

1. We have completely determined the precise form of the master equation coefficients, Sec. III B, with the correct "advanced" propagator (II.24); no integro-differential equations need to be solved. Previous expressions were not correct near the initial time which is very critical due to the "jolts".

2. We have found concise and efficient solutions to the master equation, Sec. IV.

3. We have extended the theory to that of a system of multiple oscillators bilinearly coupled to themselves and the bath in arbitrary fashion and acted upon by classical forces, Sec. V. This goes far beyond previous generalizations of the theory[28] which assume specific forms of coupling.

For many calculations, physicists often invoke Markovian master equations as they are easier to solve owing to their having compact, functional representations. But one runs into trouble if one is interested in low temperatures, short times or non-ohmic baths, which likely fall under the non-Markovian regime (see Ref. [22] for a discussion). The HPZ equation was to be capable of dealing with the full range of parameters for a general environment but its form was somewhat involved. For example, the coefficients are represented by solutions to integro-differential equations and multiple integrals over Green's functions. However, we have found that for analytic spectra, when both the spectral density and dissipation kernel

are analytic to within some finite number of simple poles, our master equation coefficients and solutions are reduced to simple contour integration. In this paper we present the correct (*cf.* for instance CCR) late-time non-Markovian master equation in compact functional representation for arbitrary temperature, coupling strength (Sec. IV C 3), and even cut-off (Sec. IV C 2). Additionally we obtain the full time time dependence at times before the oscillator and reservoir have equilibrated, but later than the cutoff timescale.

Solutions to the HPZ equation have been attempted before [16] and the role of the master equation coefficients (renormalized frequency, dissipation, diffusion, and anomalous diffusion coefficients) has been discussed in the past. Much focus has been given to initial time cutoff sensitivities and their role in decoherence. We find such effects to be contingent upon nonlocal dynamics which have, as of yet, only been (correctly) available via the Langevin equation and not thoroughly explored. Here we present complete and correct solutions of the master equation for all regimes. We consider a quantum oscillator under the influence of an external classical force that is linearly coupled to a thermal reservoir of quantum oscillators. The particle begins decoupled from the reservoir and uncorrelated with it, with a short switch-on time for the coupling, which we later take to be infinitesimal. The master equation coefficients are reduced to contour integration for analytic spectra, when both the spectral density and dissipation kernel are analytic. For an ohmic spectrum, even with finite cut-off, the necessary coefficients are reduced to compact form in the Laplace domain and at late-time. We express the moderate-time (later than the cutoff timescale) behavior in an asymptotic high temperature expansion and with transcendental functions. For analytic spectra, the master equation solutions are also reduced to contour integration. The master equation is then solved explicitly for the late-time evolution and formally in general.

It is clearly shown how each coefficient enters into the solution and how the state evolves in time. In the high cut-off limit, the anomalous diffusion coefficient is actually an "anti-diffusion" term that keeps the position uncertainty finite even when the ultraviolet (UV) cut-off tends to infinity (in contrast with the momentum uncertainty). Having the solution gives us a plethora of information such as the late time thermal covariance and uncertainty function (see Sec. IV C 4). These results generalize the work of Anastopoulos and Halliwell [15], Ford and O'Connell [16], who already found the late time state to be a Gaussian, and the earlier work of Hu and Zhang [13] on the generalized uncertainty function for Gaussian states.

B. Systematic Overview

The paper is organized as follows. We begin our study in Sec. II with a quantum oscillator linearly coupled to a thermal reservoir of quantum oscillators. The particle begins both *decoupled* from and *uncorrelated* with the reservoir. The coupling is turned off before the initial time $t = 0$ and is switched on within a short timescale. For an infinitesimal switch-on time, we reduce the evolution of states and correlations to quadrature.

In Sec. III we transition from the Langevin equation to the master equation. We start with a rederivation of the master equation using our newly found results, which correct previous findings for nonlocal dissipation. In this way we are able to offer the most complete and correct formulation of the QBM master equation, wherein all coefficients are reduced to quadrature.

In Sec. IV we turn our attention to the evolution of quantum states, i.e. the master equation solutions, which we have reduced to quadrature. If both the spectral density and dissipation kernel are analytic up to some finite number of simple poles, everything is additionally reduced to contour integration. It is seen that the initial solution undergoes damped oscillations while at the same time being smoothed into a Gaussian state of thermal equilibrium. All the cumulants of the Wigner distribution are easily determined as functions of time (Sec. IV A 1). We work out example ohmic (Sec. IV C), supra-ohmic (Sec. IV D), and sub-ohmic (Sec. IV E) spectral densities. For ohmic coupling, diffusion coefficients are obtained in compact form in the Laplace frequency domain and at late time, for finite temperature, coupling strength, and even cutoff. In the time domain, the coefficients are expressed as a high temperature asymptotic expansion and with transcendental functions. In particular we can provide the late-time uncertainty function for local (Sec. IV C 4) and nonlocal dissipation.

Finally in Sec. V we trivially extend our Langevin equation, master equation, and solutions to cover a system of multiple oscillators bilinearly coupled to themselves and the bath in arbitrary fashion and acted upon by classical forces. It is seen that the external force drives the mean position and momentum while the evolution of the variance remains unchanged and determined by the reservoir.

In the last section we conclude with a list of our findings and suggestions for their applications.

II. THE LANGEVIN EQUATION

A. General Theory

The Lagrangian of a system consisting of a quantum Brownian oscillator with mass M, natural frequency Ω and coordinate x coupled with coupling constants c_n to an environment at temperature T made up of oscillators with mass m_n, natural frequency ω_n and coordinates x_n is most straightforwardly given by

$$L = \frac{1}{2}M\left(\dot{x}^2 - \Omega_{\text{bare}}^2 x^2\right) + \sum_n \frac{1}{2}m_n\left(\dot{x}_n^2 - \omega_n^2 x_n^2\right) - \sum_n c_n x x_n. \tag{II.1}$$

For reasons of calculation, we begin with the system and environment uncorrelated. Therefore we are physically motivated to also begin with the system and environment uncoupled. This will also serve to prevent certain initial time divergences that stem from the pathology of coupling without correlation.

We are then confronted with a new issue: before the coupling is turned on, the system oscillates at frequency Ω_{bare}, while after the coupling is turned on the system oscillates at frequency $\Omega_{\text{bare}} - \delta\Omega^2$, where $\delta\Omega^2$ is generally proportional to the coupling strength and the highest bath frequencies coupled to. In the high cut-off limit, the renormalized frequency is kept finite but the bare frequency then becomes infinite.

We sidestep both issues by focusing on the more well behaved and mathematically convenient Lagrangian

$$L = \frac{1}{2}M\left(\dot{x}^2 - \Omega^2 x^2\right) + \sum_n \frac{1}{2}m_n\left(\dot{x}_n^2 - \omega_n^2\left(x_n - \theta_{\text{s}}(t)\frac{c_n}{m_n\omega_n^2}x\right)^2\right), \tag{II.2}$$

where $\theta_{\text{s}}(t)$ is a switch-on function with a very short characteristic time-scale t_{s} which we will eventually take to be negligible. This switches on the interaction at the initial time in accordance with the fact that our system and environment are initially uncorrelated. The inclusion of the interaction in the square of the bath oscillator potentials is a well-known method to counter the renormalization, $\delta\Omega^2$, of the system potential. We discuss the relation to other choices of interaction in App. C.

Time-dependent coupling has been considered by Hu and Matacz [14], wherein all parameters of the system and bath oscillators and their couplings were allowed to be time-dependent. When only the system-environment coupling is time-dependent, as in our case,

the so-called dissipation and noise kernels are given respectively by

$$\mu(t,\tau) = -\int_0^\infty d\omega \sin\left[\omega(t-\tau)\right] I(\omega)\theta_{\rm s}(t)\theta_{\rm s}(\tau), \tag{II.3}$$

$$\nu(t,\tau) = +\int_0^\infty d\omega \coth\Big(\frac{\omega}{2T}\Big) \cos\left[\omega(t-\tau)\right] I(\omega)\,\theta_{\rm s}(t)\theta_{\rm s}(\tau), \tag{II.4}$$

where $I(\omega)$ is the spectral density function defined by $I(\omega) = \sum_n (c_n^2/2m_n\omega_n)\delta(\omega-\omega_n)$ and is typically taken to be ohmic $I(\omega) = (2/\pi)M\gamma_0\,\omega$, but with a cut-off regulator such that it vanishes above some high frequency scale Λ.

The quantum Langevin equation is then given by $(L\cdot x)(t) = \xi(t)$. The noise kernel is the covariance of the Gaussian stochastic process $\xi(t)$ with $\langle\xi(t)\rangle_\xi = 0$ and $\langle\xi(t)\xi(\tau)\rangle_\xi = \nu(t,\tau)$, while the dissipation kernel is contained within the Langevin operator

$$(L\cdot x)(\tau) = M\ddot{x}(\tau) + M\Omega^2 x(\tau) + 2\int_0^\tau d\tau'\,\mu(\tau,\tau')\,x(\tau') + M\delta\Omega^2\,\theta_{\rm s}^2(\tau)x(\tau). \tag{II.5}$$

The Langevin operator can also be expressed in terms of the phenomenological *damping kernel*

$$\gamma(t,\tau) = \frac{1}{M}\int_0^\infty d\omega\, \cos[\omega(t-\tau)]\frac{I(\omega)}{\omega}\theta_{\rm s}(t)\theta_{\rm s}(\tau), \tag{II.6}$$

which we cover more thoroughly in App. C 1. The damping kernel is positive definite like the noise kernel and it provides a cancelation of the frequency renormalization while introducing a kick. In the limit of an infinitesimal switch-on time, the kick is trivial to calculate and the Langevin operator can be expressed:

$$(L\cdot x)(\tau) = M\left(\ddot{x}(\tau) + 2\int_0^\tau d\tau'\,\gamma(\tau,\tau')\dot{x}(\tau') + \Omega^2 x(\tau)\right) + 2M\gamma(t)x(0), \tag{II.7}$$

and one can see that we have a quadratic eigenvalue problem with all positive definite kernels. Therefore, in the short switch-on time limit, the system behavior is indeed dissipative.

B. Langevin Equation Solutions

As our Langevin equation can be written

$$L\cdot x = M\left(\ddot{x} + 2\,\gamma * \dot{x} + \Omega^2 x\right) + 2Mx_0\gamma = \xi, \tag{II.8}$$

where $*$ denotes the Laplace convolution [*i.e.*, $(A*B)(t) = \int_0^t d\tau A(t-\tau)B(\tau)$] and x_0 is the initial condition at $t=0$, it is then best to perform a Laplace transform

$$\hat{f}(s) = \mathcal{L}\{f\}(s) = \int_0^\infty dt\, e^{-st} f(t), \tag{II.9}$$

under which the equation becomes purely algebraic. The Laplace transform of Eq. (II.8) is given by

$$M\left(s^2 + 2s\hat{\gamma}(s) + \Omega^2\right)\hat{x}(s) \;=\; M\left(sx_0 + \dot{x}_0\right) + \hat{\xi}(s), \tag{II.10}$$

whose solution is

$$\hat{x}(s) \;=\; M\left(sx_0 + \dot{x}_0\right)\hat{G}(s) + \hat{G}(s)\hat{\xi}(s), \tag{II.11}$$

$$\hat{G}(s) \;=\; \frac{\frac{1}{M}}{s^2 + 2s\hat{\gamma}(s) + \Omega^2}, \tag{II.12}$$

where terms proportional to the initial conditions x_0 and $\dot{x}_0$ correspond to the homogeneous solution while the noise term corresponds to the driven solution.

$G(t)$ satisfies the initial boundary conditions $G(0) = 0$, $\dot{G}(0) = \frac{1}{M}$ and fully determines the retarded Green function or propagator. In the time domain, the solution can be expressed

$$x(t) \;=\; M\left(x_0\dot{G}(t) + \dot{x}_0 G(t)\right) + (G * \xi)(t), \tag{II.13}$$

1. *Analytic Spectra*

For an ohmic environment in the infinite cut-off limit one has $\hat{\gamma}(s) = \gamma_0$. More accurately, for nonlocal dissipation $\hat{\gamma}(s)$ should vanish at high frequency. If $\hat{\gamma}(s)$ is analytic, the inverse Laplace transform of $\hat{G}(s)$ is an easy to perform contour integral. Given the expression (II.6) of the damping kernel, one can compute its Laplace transform:

$$\hat{\gamma}(s) = \frac{1}{M}\int_0^\infty d\omega\, \frac{I(\omega)}{\omega}\frac{s}{\omega^2 + s^2}. \tag{II.14}$$

If take the odd extension of the spectral density for negative frequencies, i.e. $I(-|\omega|) = -I(|\omega|)$, then we have the contour integral

$$\hat{\gamma}(s) \;=\; \frac{1}{2M}\int_{-\infty}^{+\infty} d\omega\, \frac{I(\omega)}{\omega}\frac{s}{\omega^2 + s^2}, \tag{II.15}$$

which can easily be evaluated if the odd extension of $I(\omega)$ is analytic, i.e. for $I(\omega) \sim \omega$ but not $I(\omega) \sim \omega^2$.

This is still less than ideal as the difficulty of solving the Langevin equation is determined more directly by the nature of the damping kernel. One would rather first make their choice of damping kernel, preferably in the Laplace domain, as opposed to deriving it from the

spectral density. But the spectral density is still required for noise integrals, so we need the inverse relationship. Eq. (II.6) implies a simple relation between the spectral density and the Fourier transform of the damping kernel: $I(\omega) = \frac{M}{\pi}\omega\tilde{\gamma}(\omega)$ and using Eq. (B.13) applied to $\tilde{\gamma}(\omega)$ we then get the following result for $I(\omega)$ in terms of the Laplace transform of the damping kernel:

$$I(\omega) = \frac{2}{\pi} M\omega \lim_{\epsilon \to 0} \mathrm{Re}[\hat{\gamma}(\epsilon + \imath\omega)]\,. \tag{II.16}$$

From this we see that analytic damping kernels (possibly up to a finite number of simple poles, but without branch cuts) result in spectral densities which are odd analytic functions. On the other hand, we just found that odd analytic spectral densities (up to a finite number of simple poles) make it possible to compute the damping kernel via contour integration through Eq. (II.15). We will thus refer to this class of analytic spectral densities and corresponding analytic damping kernels as *analytic spectra.*

Moreover, as we will see in later sections, given that Bromwich's formula for the inverse Laplace transform can be computed as a contour integral, with the choice of these odd analytic spectral densities (or, equivalently, analytic damping kernels) all important quantities will be calculable via contour integration.

2. *Phase Space Representation*

In terms of phase space coordinates, $\mathbf{q} = (x, p)$, our Langevin equation can be written

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & + & \frac{1}{M}p \\ -M\Omega^2 x & + & -2\,\gamma * p \end{bmatrix} + \begin{bmatrix} 0 \\ \xi \end{bmatrix}, \tag{II.17}$$

or more simply $\dot{\mathbf{q}} = \mathbf{L}\cdot\mathbf{q} + \boldsymbol{\xi}$. In Laplace space we then have the solutions

$$\hat{\mathbf{q}}(s) = \hat{\mathbf{\Phi}}(s)\,\mathbf{q}_0 + \hat{\mathbf{\Phi}}(s)\,\hat{\boldsymbol{\xi}}(s), \tag{II.18}$$

$$\hat{\mathbf{\Phi}}(s) = \begin{bmatrix} s + 2\hat{\gamma}(s) & \frac{1}{M} \\ -M\Omega^2 & s \end{bmatrix} M\hat{G}(s), \tag{II.19}$$

where $\hat{G}(s)$ is the same propagator derived in the position representation and given by Eq. (II.12). In the time domain we can then represent the initial value solutions

$$\mathbf{q}(t) = \mathbf{\Phi}(t)\,\mathbf{q}_0 + (\mathbf{\Phi} * \boldsymbol{\xi})(t), \tag{II.20}$$

$$\mathbf{\Phi}(t) = M \begin{bmatrix} \dot{G}(t) + 2(\gamma * G)(t) & \frac{1}{M}G(t) \\ -M\Omega^2 G(t) & \dot{G}(t) \end{bmatrix}, \tag{II.21}$$

For an earlier time τ, the solution $\mathbf{q}(\tau)$ will take an analogous form and the two expressions can be used to find means to propagate from τ to t

$$\begin{aligned} \mathbf{q}(t) &= \mathbf{\Phi}(t,\tau)\,\mathbf{q}(\tau) + \int_\tau^t d\tau'\,\mathbf{\Phi}(t-\tau')\,\boldsymbol{\xi}(\tau') \\ &\quad + \int_0^\tau d\tau'\,[\mathbf{\Phi}(t-\tau') - \mathbf{\Phi}(t,\tau)\,\mathbf{\Phi}(\tau-\tau')]\,\boldsymbol{\xi}(\tau'), \end{aligned} \tag{II.22}$$

$$\mathbf{\Phi}(t,\tau) \equiv \mathbf{\Phi}(t)\,\mathbf{\Phi}^{-1}(\tau), \tag{II.23}$$

and note that $\mathbf{\Phi}(t,\tau) \neq \mathbf{\Phi}(t-\tau)$ for nonlocal dissipation where one is not working with exponential matrices. Therefore, the last term in the solution will only vanish for local dissipation.

The last step is then to solve for $\mathbf{q}(\tau)$ in terms of the final conditions $\mathbf{q}(t)$.

$$\begin{aligned} \mathbf{q}(\tau) &= \mathbf{\Phi}(\tau,t)\,\mathbf{q}(t) + \int_t^\tau d\tau'\,\mathbf{\Phi}(\tau,t)\,\mathbf{\Phi}(t-\tau')\,\boldsymbol{\xi}(\tau') \\ &\quad - \int_0^\tau d\tau'\,[\mathbf{\Phi}(\tau,t)\,\mathbf{\Phi}(t-\tau') - \mathbf{\Phi}(\tau-\tau')]\,\boldsymbol{\xi}(\tau'), \end{aligned} \tag{II.24}$$

and again, the last term will only vanish for local dissipation. This is not what one would expect from working with local differential equations, and in fact previous derivations of the HPZ master equation[12, 19] completely neglect this term and report so-called advanced propagators which are only correct in the limit of local dissipation. To our knowledge, this is the first derivation of the correct final propagator and therefore with this we have the first correct derivation of the QBM master equation wherein all quantities are completely specified.

C. Evolution of States

Using the expression for the Wigner function as a double average (over initial conditions and realizations of the stochastic source)[19], its characteristic function (i.e. its Fourier

transform) is given by

$$\begin{aligned}
\mathcal{W}_{\rm r}(t,\mathbf{k}) &= \int d\mathbf{q}\, e^{-i\mathbf{k}^{\rm T}\mathbf{q}} \left\langle \langle \delta(\mathbf{q}-\mathbf{q}(t)) \rangle_{\mathbf{q}_0} \right\rangle_{\boldsymbol{\xi}} && \text{(II.25)} \\
&= \left\langle \langle e^{-i\mathbf{k}^{\rm T}\mathbf{q}(t)} \rangle_{\mathbf{q}_0} \right\rangle_{\boldsymbol{\xi}} \\
&= \left\langle e^{-i\mathbf{k}^{\rm T}\boldsymbol{\Phi}(t)\mathbf{q}_0} \right\rangle_{\mathbf{q}_0} \left\langle e^{-i\mathbf{k}^{\rm T}\boldsymbol{\Phi}(t-\tau)\cdot\boldsymbol{\xi}(\tau)} \right\rangle_{\boldsymbol{\xi}} && \text{(II.26)} \\
&= \mathcal{W}_{\rm r}(0, \boldsymbol{\Phi}^{\rm T}(t)\mathbf{k})\, e^{-\frac{1}{2}\mathbf{k}^{\rm T}\boldsymbol{\sigma}_T(t)\mathbf{k}}, && \text{(II.27)}
\end{aligned}$$

where the thermal covariance is given by

$$\boldsymbol{\sigma}_T(t) = \int_0^t d\tau \int_0^t d\tau'\, \boldsymbol{\Phi}(t-\tau)\, \boldsymbol{\nu}(\tau,\tau')\, \boldsymbol{\Phi}^{\rm T}(t-\tau'), \tag{II.28}$$

In the third equality we used the initial value solution (II.20) for $\mathbf{q}(t)$ and in the last step we took into account the definition of the noise average

$$\langle \cdots \rangle_{\boldsymbol{\xi}} = \frac{1}{\sqrt{2\pi \det(\boldsymbol{\nu})}} \int \mathcal{D}\boldsymbol{\xi} \cdots e^{-\frac{1}{2}\boldsymbol{\xi}\cdot\boldsymbol{\nu}^{-1}\cdot\boldsymbol{\xi}} \tag{II.29}$$

and completed the square to calculate the Gaussian functional integral. For our Lagrangian, the stochastic force $\boldsymbol{\xi}$ only has momentum component and therefore its covariance $\boldsymbol{\nu}$ only has momentum-momentum component. If the bath were to be linearly coupled to system momenta, that would introduce terms in the noise and damping kernels which operate strictly between system positions.

The form of the solution is rather simple: all initial cumulants of the Wigner function undergo damped oscillations (for the underdamped case) while a thermal covariance arises. We will discuss the solutions more thoroughly in Sec. IV.

D. General Correlations

Utilizing our initial value solution to the Langevin equation, Eq. (II.20), it is a straightforward application to calculate multiple time correlations between system observables. The first nontrivial correlation is the two-time correlation, which we find to agree with that of our master equation solution, Eq. (II.28), in the coincidence limit.

$$\begin{aligned}
\left\langle \mathbf{q}\,(t_1)\, \mathbf{q}^{\rm T}\,(t_2) \right\rangle &= \boldsymbol{\Phi}\,(t_1)\, \boldsymbol{\sigma}_0\, \boldsymbol{\Phi}^{\rm T}(t_2) + \boldsymbol{\sigma}_T\,(t_1,t_2)\,, && \text{(II.30)} \\
\boldsymbol{\sigma}_T\,(t_1,t_2) &= \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2\, \boldsymbol{\Phi}\,(t_1-\tau_1)\, \boldsymbol{\nu}(\tau_1,\tau_2)\, \boldsymbol{\Phi}^{\rm T}(t_2-\tau_2)\,, && \text{(II.31)}
\end{aligned}$$

where we have used some basic facts of our Gaussian noise, namely $\langle \boldsymbol{\xi}(t) \rangle_{\boldsymbol{\xi}} = 0$ and $\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}(\tau) \rangle_{\boldsymbol{\xi}} = \boldsymbol{\nu}(t,\tau)$.

Higher order correlations can be calculated in a similar manner, but we know from the form of our solution in Eq. (II.27) that for single-time correlations, no cumulant needs to be computed other than the thermal covariance.

III. MASTER EQUATION

A. General Theory

Relying upon the same fundamental theory as given in Sec. II A, the HPZ master equation for the reduced density matrix $\rho_{\rm r}$ and the reduced Wigner function are given respectively by

$$\frac{\partial}{\partial t}\rho_{\rm r} = -\imath\,[H_{\rm R}, \rho_{\rm r}] - \imath\Gamma\,[x, \{p, \rho_{\rm r}\}] - MD_{pp}\,[x, [x, \rho_{\rm r}]] - D_{xp}\,[x, [p, \rho_{\rm r}]]\,, \tag{III.1}$$

$$\frac{\partial}{\partial t}W_{\rm r} = \{H_{\rm R}, W_{\rm r}\} + 2\Gamma\frac{\partial}{\partial p}(pW_{\rm r}) + MD_{pp}\frac{\partial^2}{\partial p^2}W_{\rm r} - D_{xp}\frac{\partial^2}{\partial x \partial p}W_{\rm r}, \tag{III.2}$$

where $H_{\rm R}$ corresponds to the system Hamiltonian with Ω^2 replaced by a time-dependent frequency $\Omega_{\rm R}^2(t) \sim \Omega^2$ whose detailed form, together with that of the dissipation function $\Gamma(t)$ and the diffusion functions $D_{xp}(t)$ and $D_{pp}(t)$, can be found in [12]. However we have found that the previous formulations are only truly correct for local dissipation.

B. Derivation of the Master Equation

Following the derivation of Calzetta, Roura, and Verdaguer [19] we will now derive the master equation in the language of phase space vectors and matrix operators. This formalism has already allowed us to derive the correct final propagator in Sec. II B 2, and with this we will have the most complete derivation of the QBM master equation to date. We begin from the stochastic representation of the Wigner function.

$$W_{\rm r}(\mathbf{q};t) = \Big\langle \langle \delta(\mathbf{q}(t) - \mathbf{q}) \rangle_{\boldsymbol{\xi}} \Big\rangle_{\mathbf{q}_0}, \tag{III.3}$$

$$\frac{\partial}{\partial t}W_{\rm r}(\mathbf{q};t) = -\boldsymbol{\nabla}_{\mathbf{q}}^{\rm T} \Big\langle \langle \dot{\mathbf{q}}(t)\, \delta(\mathbf{q}(t) - \mathbf{q}) \rangle_{\boldsymbol{\xi}} \Big\rangle_{\mathbf{q}_0}. \tag{III.4}$$

One can then express the time derivative in terms of the Langevin equation $\dot{\mathbf{q}} = \mathbf{L}\cdot\mathbf{q} + \boldsymbol{\xi}$ and the master equation can be written

$$\frac{\partial}{\partial t} W_{\mathrm{r}}(\mathbf{q};t) \;=\; -\boldsymbol{\nabla}_{\mathbf{q}}^{\mathrm{T}} \left\langle \left\langle \left(\int_0^t d\tau\, \mathbf{L}(t,\tau)\, \mathbf{q}(\tau) + \boldsymbol{\xi}(t) \right) \delta(\mathbf{q}(t)-\mathbf{q}) \right\rangle_{\boldsymbol{\xi}} \right\rangle_{\mathbf{q}_0} . \tag{III.5}$$

Using our final propagator (II.24) we can now express $\mathbf{q}(\tau)$ in terms of the final $\mathbf{q}$ which appears in the Wigner function by way of the delta function.

$$\mathbf{q}(\tau) \;=\; \boldsymbol{\Phi}(\tau,t)\,\mathbf{q}(t) + \int_0^t d\tau\, \boldsymbol{\Phi}_f\,(\tau,\tau')\; \boldsymbol{\xi}(\tau'), \tag{III.6}$$

$$\boldsymbol{\Phi}_f\,(\tau,\tau') \;=\; -\boldsymbol{\Phi}(\tau,t)\,\boldsymbol{\Phi}(t-\tau') + \theta(\tau-\tau')\;\boldsymbol{\Phi}(\tau-\tau'). \tag{III.7}$$

We are left with a homogeneous term and two noisy terms.

$$\begin{aligned}
\frac{\partial}{\partial t} W(\mathbf{q};t) \;=\; & -\boldsymbol{\nabla}_{\mathbf{q}}^{\mathrm{T}} \int_0^t d\tau\, \mathbf{L}(t,\tau)\, \boldsymbol{\Phi}(\tau,t)\, \mathbf{q}\, W(\mathbf{q};t) \\
& -\boldsymbol{\nabla}_{\mathbf{q}}^{\mathrm{T}} \left\langle \left\langle \int_0^t d\tau \int_0^t d\tau'\, \mathbf{L}(t,\tau)\boldsymbol{\Phi}_f(\tau,\tau')\; \boldsymbol{\xi}(\tau')\, \delta(\mathbf{q}(t)-\mathbf{q}) \right\rangle_{\boldsymbol{\xi}} \right\rangle_{\mathbf{q}_0} \\
& -\boldsymbol{\nabla}_{\mathbf{q}}^{\mathrm{T}} \left\langle \langle \boldsymbol{\xi}(t)\, \delta(\mathbf{q}(t)-\mathbf{q}) \rangle_{\boldsymbol{\xi}} \right\rangle_{\mathbf{q}_0} .
\end{aligned} \tag{III.8}$$

The noise averages can be evaluated with the help of Novikov's formula.

$$\langle \boldsymbol{\xi}(\tau')\, \delta(\mathbf{q}(t)-\mathbf{q}) \rangle_{\boldsymbol{\xi}} \;=\; -\int_0^t d\tau''\, \boldsymbol{\nu}(\tau',\tau'') \left\langle \left[\frac{\delta \mathbf{q}(t)}{\delta \boldsymbol{\xi}(\tau'')} \right] \boldsymbol{\nabla}_{\mathbf{q}}\, \delta(\mathbf{q}(t)-\mathbf{q}) \right\rangle_{\boldsymbol{\xi}} , \tag{III.9}$$

where the required functional Jacobian can easily be extracted from the Langevin equation

$$\left[\frac{\delta \mathbf{q}(t)}{\delta \boldsymbol{\xi}(\tau'')} \right] \;=\; \boldsymbol{\Phi}^{\mathrm{T}}(t-\tau'') \tag{III.10}$$

And finally we have completely determined the master equation.

$$\frac{\partial}{\partial t} W_{\mathrm{r}}(\mathbf{q};t) \;=\; \left\{ \boldsymbol{\nabla}_{\mathbf{q}}^{\mathrm{T}}\, \mathbf{H}(t)\, \mathbf{q} + \boldsymbol{\nabla}_{\mathbf{q}}^{\mathrm{T}}\, \mathbf{D}(t)\, \boldsymbol{\nabla}_{\mathbf{q}} \right\} W_{\mathrm{r}}(\mathbf{q};t), \tag{III.11}$$

with the psuedo-Hamiltonian and diffusion matrices

$$\mathbf{H}(t) \;\equiv\; -\int_0^t d\tau\, \mathbf{L}(t,\tau)\, \boldsymbol{\Phi}(\tau,t), \tag{III.12}$$

$$\begin{aligned}
\mathbf{D}(t) \;\equiv\; & \int_0^t d\tau\, \boldsymbol{\nu}(t,\tau)\; \boldsymbol{\Phi}^{\mathrm{T}}(t-\tau) \\
& + \int_0^t d\tau \int_0^t d\tau' \int_0^t d\tau''\, \mathbf{L}(t,\tau)\, \boldsymbol{\Phi}_f\,(\tau,\tau')\; \boldsymbol{\nu}(\tau',\tau'')\; \boldsymbol{\Phi}^{\mathrm{T}}(t-\tau'') .
\end{aligned} \tag{III.13}$$

It should be noted that the diffusion matrix is not expressed in symmetric form here, but is certainly symmetrized by the master equation itself. Upon symmetrization, these matrix operators relate to the conventional representation as

$$\mathbf{H}(t) = \begin{bmatrix} 0 & -\frac{1}{M} \\ M\Omega_{\mathrm{R}}^2(t) & 2\Gamma(t) \end{bmatrix} \qquad \mathbf{D}(t) = \begin{bmatrix} 0 & -\frac{1}{2}D_{xp}(t) \\ -\frac{1}{2}D_{xp}(t) & MD_{pp}(t) \end{bmatrix}. \tag{III.14}$$

The coefficients, as expressed here, are in an analogous form with previous derivations, but this is not the simplest representation of the master equation coefficients. We will now proceed to integrate out all of the Langevin operators.

1. Simplification of the Master Equation Coefficients

Let us expand the psuedo-Hamiltonian operator.

$$\mathbf{H}(t) \;=\; -(\mathbf{L}\cdot\mathbf{\Phi})(t)\,\mathbf{\Phi}^{-1}(t), \tag{III.15}$$

$$\mathbf{H}(t) \;=\; -\dot{\mathbf{\Phi}}(t)\,\mathbf{\Phi}^{-1}(t). \tag{III.16}$$

This new definition of $\mathbf{H}$ immediately reveals the connection between the local and nonlocal dynamics as the nonlocal propagator satisfies the local equation

$$\dot{\mathbf{\Phi}}(t) \;=\; -\mathbf{H}(t)\,\mathbf{\Phi}(t), \tag{III.17}$$

where for local dissipation one would simply have $\mathbf{\Phi}(t) = e^{-t\,\mathbf{H}}$.

Now let us do the same for the diffusion matrix. Expanding out the final propagator we will need to simplify the following integral

$$\int_0^t d\tau\,\mathbf{L}(t,\tau)\,\mathbf{\Phi}(\tau-\tau')\,\theta(\tau-\tau') \;=\; \int_{\tau'}^t d\tau\,\mathbf{L}(t-\tau)\,\mathbf{\Phi}(\tau-\tau'), \tag{III.18}$$

where we have made use of the stationary property of the dissipation kernel. With a simple change of variables we have

$$\int_0^t d\tau\,\mathbf{L}(t,\tau)\,\mathbf{\Phi}(\tau-\tau')\,\theta(\tau-\tau') \;=\; \int_0^{t-\tau'} d\tau\,\mathbf{L}(t-\tau'-\tau)\,\mathbf{\Phi}(\tau) \tag{III.19}$$

$$\;=\; \dot{\mathbf{\Phi}}(t-\tau'). \tag{III.20}$$

With some simplification we can now express the diffusion operator

$$\begin{aligned}\mathbf{D}(t) \;=\; &\int_0^t d\tau\,\boldsymbol{\nu}(t,\tau)\;\mathbf{\Phi}^{\mathrm{T}}(t-\tau) \\ &+\int_0^t d\tau\int_0^t d\tau'\left\{\left[\frac{d}{dt}+\mathbf{H}(t)\right]\mathbf{\Phi}(t-\tau)\right\}\boldsymbol{\nu}(\tau,\tau')\;\mathbf{\Phi}^{\mathrm{T}}(t-\tau'),\end{aligned} \tag{III.21}$$

and one can clearly see how the second term vanishes for local dissipation when the transition matrix is the exponential matrix $e^{-t\,\mathbf{H}}$.

Our simplified representation of the master equation is then

$$\frac{\partial}{\partial t}W_{\rm r}(\mathbf{q};t) = \left\{\boldsymbol{\nabla}_{\mathbf{q}}^{\rm T}\,\mathbf{H}(t)\,\mathbf{q} + \boldsymbol{\nabla}_{\mathbf{q}}^{\rm T}\,\mathbf{D}(t)\,\boldsymbol{\nabla}_{\mathbf{q}}\right\}W_{\rm r}(\mathbf{q};t), \tag{III.22}$$

$$\mathbf{H}(t) = -\dot{\boldsymbol{\Phi}}(t)\,\boldsymbol{\Phi}^{-1}(t), \tag{III.23}$$

$$\begin{aligned}\mathbf{D}(t) &= \int_0^t d\tau\,\boldsymbol{\nu}(t,\tau)\;\boldsymbol{\Phi}^{\rm T}(t-\tau) \\ &\quad + \int_0^t d\tau\int_0^t d\tau'\left\{\left[\frac{d}{dt}+\mathbf{H}(t)\right]\boldsymbol{\Phi}(t-\tau)\right\}\boldsymbol{\nu}(\tau,\tau')\;\boldsymbol{\Phi}^{\rm T}(t-\tau')\,,\end{aligned} \tag{III.24}$$

which contains fewer integrals than the conventional representation and is completely determined.

It is important here to note that the nonlocal contributions to the diffusion coefficients are critical at early times when evolution is strictly nonlocal. These terms are complicated (nonlinear) and so at early times the Langevin equation is much easier to work with. At late times the master equation perspective becomes more useful as the nonlocal terms vanish and the master equation coefficients asymptote.

C. Master Equation Solutions

Here we wish to show that the master equation itself can be solved to produce the same solution as derived in Sec. II C. We consider the general master equation

$$\frac{\partial}{\partial t}W_{\rm r} = \left(\boldsymbol{\nabla}_{\mathbf{q}}^{\rm T}\,\mathbf{D}(t)\,\boldsymbol{\nabla}_{\mathbf{q}} + \boldsymbol{\nabla}_{\mathbf{q}}^{\rm T}\,\mathbf{H}(t)\,\mathbf{q}\right)W_{\rm r}\,, \tag{III.25}$$

where the Langevin equation has given us the propagator $\boldsymbol{\Phi}(t)$ that satisfies $\dot{\boldsymbol{\Phi}}(t) = -\mathbf{H}(t)\,\boldsymbol{\Phi}(t)$.

This is a hyperbolic second order partial differential equation (PDE). The equation is not separable in time, nor phase-space. The nature of the PDE suggests taking a Fourier transform of the phase-space variables as the derivatives are of higher order than the algebraic parameters. Furthermore, not only does a Fourier transform reduce the PDE to first order, but the computation of expectation values also becomes trivial since we are then working with the characteristic function of the distribution.

The Fourier transform is defined as

$$\mathcal{F}\{f\}(\mathbf{k}) = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dp\,e^{-\imath\mathbf{k}\cdot\mathbf{q}}f(\mathbf{q}), \tag{III.26}$$

and it exhibits the usual properties:

$$\imath^n \frac{\partial^n \mathcal{F}\{f\}}{\partial k_j^n}(\mathbf{0}) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp\, q_j^n f(\mathbf{q}). \tag{III.27}$$

The master equation then becomes equivalently

$$\left(\frac{\partial}{\partial t} + \mathbf{k}^{\mathrm{T}}\mathbf{H}\boldsymbol{\nabla}_{\mathbf{k}}\right) \mathcal{W}_{\mathrm{r}} = -\mathbf{k}^{\mathrm{T}}\mathbf{D}\mathbf{k}\, \mathcal{W}_{\mathrm{r}}. \tag{III.28}$$

where $\mathcal{W}_{\mathrm{r}} = \mathcal{F}\{W_{\mathrm{r}}\}$ and the normalization of $W_{\mathrm{r}}(t, \mathbf{q})$ implies $\mathcal{W}_{\mathrm{r}}(t, \mathbf{0}) = 1$.

From Eq. (III.28) it is clear that if the master equation coefficients asymptote to constant values, then we will have a stationary Gaussian solution in the late-time limit given by

$$\mathcal{W}_T^{\infty} = e^{-\frac{1}{2}\mathbf{k}^{\mathrm{T}}\boldsymbol{\sigma}_T^{\infty}\mathbf{k}}, \tag{III.29}$$

with $\boldsymbol{\sigma}_T^{\infty}$ uniquely determined by the Lyapunov equation

$$\mathbf{H}_{\infty}\,\boldsymbol{\sigma}_T^{\infty} + \boldsymbol{\sigma}_T^{\infty}\,\mathbf{H}_{\infty}^{\mathrm{T}} = 2\mathbf{D}_{\infty}\,. \tag{III.30}$$

As we will show below, this corresponds to the thermal state of our system.

1. *Method of Characteristic Curves*

The method of characteristic curves involves looking for parameterized curves in the domain $(t, \mathbf{k})$ along which the first order PDE becomes a set of first order ODEs. For each one of those curves we have

$$\mathcal{W}_{\mathrm{r}}[t, \mathbf{k}] \;=\; \mathcal{W}_{\mathrm{r}}[t(\tau), \mathbf{k}(\tau)]\,, \tag{III.31}$$

$$\frac{d}{d\tau}\mathcal{W}_{\mathrm{r}} \;=\; \frac{dt}{d\tau}\frac{\partial}{\partial t}\mathcal{W}_{\mathrm{r}} + \frac{d\mathbf{k}^{\mathrm{T}}}{d\tau}\,\boldsymbol{\nabla}_{\mathbf{k}}\mathcal{W}_{\mathrm{r}}\,, \tag{III.32}$$

Next, we attempt to match the right-hand side of Eq. (III.32) to the left-hand side of Eq. (III.28). This results in a system of ODEs in the parameter τ. We will look for curves that synchronize with the initial time so that $t(0) = 0$, $\mathbf{k}(0) = \mathbf{k}_0$. The time solution is simple:

$$\frac{dt}{d\tau} = 1 \;\Rightarrow\; t(\tau) = \tau\,. \tag{III.33}$$

On the other hand, the solution for the Fourier transform of the phase-space variables is a bit more involved:

$$\frac{d}{d\tau}\mathbf{k}^{\mathrm{T}}(\tau) = +\mathbf{k}^{\mathrm{T}}(\tau)\mathbf{H}(\tau). \tag{III.34}$$

For local dissipation, the solution here is simply $\mathbf{k}^{\mathrm{T}}(\tau) = \mathbf{k}_0^{\mathrm{T}} e^{+\tau\,\mathbf{H}}$, and so the propagator for $\mathbf{k}^{\mathrm{T}}$ happens to be $\mathbf{\Phi}^{-1}(\tau)$. This is also true in general as the propagator for the characteristic curves $\mathbf{\Phi}_k^{\mathrm{T}}(\tau)$ must satisfy the equation

$$\frac{d}{d\tau}\mathbf{\Phi}_k^{\mathrm{T}}(\tau) \;=\; -\mathbf{\Phi}_k^{\mathrm{T}}(\tau)\,\dot{\mathbf{\Phi}}(t)\,\mathbf{\Phi}^{-1}(t), \tag{III.35}$$

where we have simply specified $\mathbf{H}$. This is equivalent to the relation

$$\frac{d}{d\tau}\left(\mathbf{\Phi}_k^{\mathrm{T}}(\tau)\,\mathbf{\Phi}(\tau)\right) \;=\; 0, \tag{III.36}$$

which necessarily implies an inverse relation between the two propagators.

We now have the rules for transforming back and forth between the domain coordinates $(t, \mathbf{k})$ and the characteristic curve coordinates $(\tau, \mathbf{k}_0)$. $\mathbf{k}_0$ does not change along the characteristic curve, but for a given τ it uniquely specifies a particular curve. Using these results, we can immediately apply the method of characteristic curves to solving Eq. (III.28) as follows:

$$\frac{d}{d\tau}\mathcal{W}_{\mathrm{r}}[t(\tau), \mathbf{k}(\tau)] \;=\; -\mathbf{k}^{\mathrm{T}}\mathbf{D}(t)\mathbf{k}\,\mathcal{W}_{\mathrm{r}}[t(\tau), \mathbf{k}(\tau)]\,, \tag{III.37}$$

$$\frac{d}{d\tau}\mathcal{W}_{\mathrm{r}}[t(\tau), \mathbf{\Phi}_k(\tau)\mathbf{k}_0] \;=\; -\Big(\mathbf{k}_0^{\mathrm{T}}\mathbf{\Phi}_k^{\mathrm{T}}(\tau)\,\mathbf{D}(\tau)\,\mathbf{\Phi}_k(\tau)\mathbf{k}_0\Big)\,\mathcal{W}_{\mathrm{r}}[t(\tau), \mathbf{\Phi}_k(\tau)\mathbf{k}_0]\,. \tag{III.38}$$

The last equation is a linear ODE whose solution can be easily found to be

$$\mathcal{W}_{\mathrm{r}}[\tau, \mathbf{\Phi}_k(\tau)\mathbf{k}_0] = \mathcal{W}_{\mathrm{r}}[0, \mathbf{k}_0]\,e^{-\int_0^\tau d\tau'\left(\mathbf{k}_0^{\mathrm{T}}\mathbf{\Phi}_k^{\mathrm{T}}(\tau')\,\mathbf{D}(\tau')\,\mathbf{\Phi}_k(\tau')\mathbf{k}_0\right)}, \tag{III.39}$$

where $\mathcal{W}_{\mathrm{r}}[0, \mathbf{k}_0]$ is the initial characteristic function at $t = 0$. We can now express the solution back in terms of $\mathbf{k}$ and $\mathbf{\Phi}$ to get the final result

$$\mathcal{W}_{\mathrm{r}}[t, \mathbf{k}] = \mathcal{W}_{\mathrm{r}}\left[0, \mathbf{\Phi}^{\mathrm{T}}(t)\mathbf{k}\right] e^{-\frac{1}{2}\mathbf{k}^{\mathrm{T}}\boldsymbol{\sigma}_T(t)\mathbf{k}}, \tag{III.40}$$

with

$$\boldsymbol{\sigma}_T(t) = 2\int_0^t d\tau\,\mathbf{\Phi}(t,\tau)\,\mathbf{D}(\tau)\,\mathbf{\Phi}^{\mathrm{T}}(t,\tau), \tag{III.41}$$

and note that $\mathbf{\Phi}(t,\tau)$ here does not have time-translational invariance for nonlocal dissipation where $\mathbf{\Phi}(t,\tau) = \mathbf{\Phi}(t)\,\mathbf{\Phi}^{-1}(\tau) \neq \mathbf{\Phi}(t-\tau)$.

Now we have shown that the form of solution from the master equation is equivalent to that derived from the Langevin equation in Sec. II B. What remains to be shown is that the

thermal covariances are indeed equivalent. To do this we can invert Eq. (III.41) to express the diffusion in terms of the thermal covariance

$$2\mathbf{D}(t) \;=\; \mathbf{H}(t)\,\boldsymbol{\sigma}_T(t) + \boldsymbol{\sigma}_T(t)\,\mathbf{H}^{\mathrm{T}}(t) + \dot{\boldsymbol{\sigma}}_T(t), \tag{III.42}$$

and from here it is straightforward to show that substitution of the Langevin thermal covariance (II.28) will yield the correct (symmetrized) diffusion coefficients we see in expression (III.24).

IV. EVOLUTION OF STATES

A. General Solutions

The evolution of states, whether derived via the Langevin equation II C or master equation III C, is most easily represented by the characteristic or moment generating function (Fourier transform) of the Wigner distribution.

$$\mathcal{W}_{\mathrm{r}}[t,\mathbf{k}] = \mathcal{W}_{\mathrm{r}}\big[0,\boldsymbol{\Phi}^{\mathrm{T}}(t)\mathbf{k}\big]\, e^{-\frac{1}{2}\mathbf{k}^{\mathrm{T}}\boldsymbol{\sigma}_T(t)\mathbf{k}}, \tag{IV.1}$$

with the thermal covariance defined

$$\boldsymbol{\sigma}_T(t) \;=\; \int_0^t d\tau \int_0^t d\tau'\, \boldsymbol{\Phi}(t-\tau)\,\boldsymbol{\nu}(\tau,\tau')\,\boldsymbol{\Phi}^{\mathrm{T}}(t-\tau'), \tag{IV.2}$$

and $\boldsymbol{\Phi}(t)$ is the phase-space propagator for the Langevin equation defined in Eq. (II.19).

The solution has two factors. The first one tends to unity in the long time limit and encodes the disappearance of the initial state (we will call it the *death factor*). The second factor describes the appearance of a Gaussian state that evolves in time and tends asymptotically to a state that corresponds to thermal equilibrium (we will refer to this as the *birth factor*). Assuming dissipation, all initial distributions evolve towards this final Gaussian state, with thermal covariance $\boldsymbol{\sigma}_T(t)$. This state does not look like the thermal state of a free harmonic oscillator because of the coupling to the environment. It results from considering the thermal equilibrium state for the whole system (system plus environment) including the system-environment interaction, which gives rise to a non-trivial correlation between them, and tracing out the environment.

The *death factor* contains the information of the initial conditions; it describes the gradual disappearance of the initial distribution and it is always temperature independent. The free

evolution of the Wigner function is rotation in phase space at constant angular velocity. Dissipation will modify the death factor to not only rotate but also inspiral from decay, or in the case of overdamping, strictly decay. Most generally, nonlocal dissipation will cause this factor to act as a parametrically damped oscillator, which could be quite complicated.

The *birth factor* describes the complicated birth and settlement of a state of thermal equilibrium. This factor is always Gaussian with a covariance matrix given by Eq. (IV.2), which involves a convolution of the noise kernel with propagators that reflect the natural oscillatory decay of the system. This covariance matrix vanishes at the initial time and tends at late times to an equilibrium covariance matrix which can more easily be determined by the Lyapunov equation (III.30). The thermal covariance matrix is always positive definite.

1. *Trajectories of the Cumulants*

As we have already mentioned, the Fourier transfom of the reduced Wigner function corresponds to its characteristic function, from which the correlation functions for the phase-space variables can be easily derived using Eq. (III.27). The general expressions for the cumulants can be obtained straightforwardly from the logarithm of the reduced Wigner function in Fourier space as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n!} \kappa^{(n)}_{i_1 \ldots i_n}(t) \prod_{l=1}^{n} \imath k^{i_l} = \log \mathcal{W}_{\mathrm{r}}(t, \boldsymbol{\kappa}), \tag{IV.3}$$

where k^{i_l} denotes the components of the vector $\mathbf{k}$ and we used the Einstein summation convention for pairs of repeated indices (*i.e.*, it is implicitly understood that a sum $\sum_{i_l=1}^{2}$ should be preformed over each pair of repeated indices i_l). $\boldsymbol{\kappa}^{(n)}$ is the n^{th} cumulant and acts as a tensor of order n contracted with n copies of $\mathbf{k}$. Using the result for $\mathcal{W}_{\mathrm{r}}(t, \boldsymbol{\kappa})$ from Eq. (IV.1) we have

$$\sum_{n=1}^{\infty} \frac{1}{n!} \kappa^{(n)}_{i_1 \ldots i_n}(t) \prod_{l=1}^{n} \imath k^{i_l} = \sum_{n=1}^{\infty} \frac{1}{n!} \kappa^{(n)}_{i_1 \ldots i_n}(0) \prod_{l=1}^{n} \imath \left(\boldsymbol{\Phi}^{\mathrm{T}}(t)\mathbf{k}\right)^{i_l} - \frac{1}{2}\mathbf{k}^{\mathrm{T}}\boldsymbol{\sigma}_T(t)\mathbf{k}, \tag{IV.4}$$

where $\kappa^{(n)}_{j_1 \ldots j_n}(0)$ are the cumulants associated with the initial distribution. Eq. (IV.4) implies

$$\kappa^{(n)}_{i_1 \ldots i_n}(t) = \kappa^{(n)}_{j_1 \ldots j_n}(0) \prod_{l=1}^{n} \imath \left(\boldsymbol{\Phi}^{\mathrm{T}}(t)\right)^{j_l i_l} + \delta_{n2}\, \sigma_T^{i_1 i_2}(t). \tag{IV.5}$$

We can see that the only cumulant with a non-vanishing asymptotic value, which is a consequence of the thermal fluctuations, is the covariance matrix (with $n = 2$). The closely

related second momenta of the distribution are given by

$$\langle \mathbf{q}\mathbf{q}^{\mathrm{T}} \rangle(t) = \mathbf{\Phi}(t) \, \langle \mathbf{q}\mathbf{q}^{\mathrm{T}} \rangle_0 \, \mathbf{\Phi}^{\mathrm{T}}(t) + \boldsymbol{\sigma}_T(t), \tag{IV.6}$$

where $\langle \cdots \rangle_0$ denotes expectation value with respect to the reduced Wigner function at the initial time.[1] All other cumulants experience whatever oscillator decay is inherent in the homogeneous solution of the Langevin equation. In particular, the expectation value

$$\langle \mathbf{q} \rangle(t) = \mathbf{\Phi}(t) \, \langle \mathbf{q} \rangle_0 \, , \tag{IV.7}$$

follows a trajectory plotted in Fig. 1 for local dissipation, where one can see that the trajectory of the expectation values $\langle x \rangle, \langle p \rangle$ for any initial distribution inspiral into the origin. This captures the behavior of Gaussians plotted by Unruh and Zurek [5].

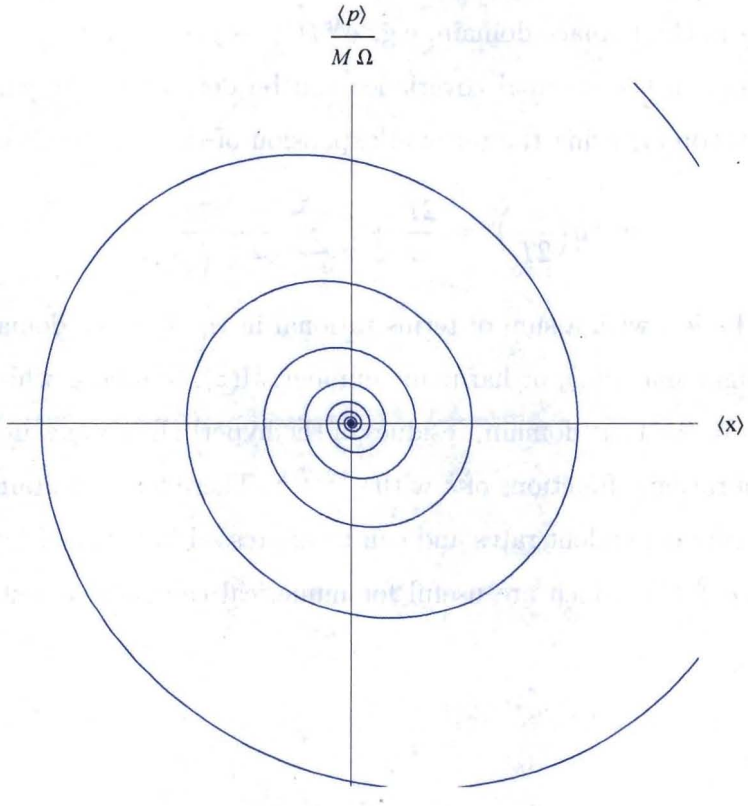

FIG. 1: The trajectory of the expectation values $\langle x \rangle, \langle p \rangle$.

[1] Note that the expectation value of any phase-space function with respect to the reduced Wigner function is equivalent to a quantum expectation value with respect to the corresponding reduced density matrix where the arguments x and p of the phase-space function are promoted to operators and the Weyl ordering prescription is employed. In particular, for the second-order cumulants this corresponds to considering symmetrized two-point quantum correlation functions.

2. *Thermal Covariance*

As we have now seen, aside from the propagator, the only other quantity that needs to be calculated is the thermal covariance. Here we discuss the full-time evolution of the thermal covariance. The much simpler late-time evolution will be discussed in Sec. IV B. The full-time thermal covariance is most easily expressed in the Laplace domain as

$$\hat{\boldsymbol{\sigma}}_T(s) \;=\; 2\int_0^\infty d\omega\, I(\omega)\coth\Big(\frac{\omega}{2T}\Big)\,\hat{\boldsymbol{\Phi}}(s+\imath\omega)\,\hat{\boldsymbol{\Phi}}^{\mathrm{T}}(s-\imath\omega), \tag{IV.8}$$

which can be derived by derived from Eq. (IV.2) by splitting the time integrals in half $\int_0^t d\tau \int_0^t d\tau' = \int_0^t d\tau \int_0^\tau d\tau' + \int_0^t d\tau' \int_0^\tau d\tau$, writing the cosine of the noise kernel in exponential form, manipulating the integrals until one has only Laplace convolutions, and then utilizing frequency shifting in the Laplace domain, e.g. $e^{\lambda t} f(t) \to \hat{f}(s-\lambda)$.

For analytic spectra the thermal covariance can be determined by contour integration (and the residue theorem) using the rational expansion of the hyperbolic cotangent

$$\coth\Big(\frac{\omega}{2T}\Big) \;=\; \frac{2T}{\omega} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\frac{\omega}{2\pi T}}{k^2+\big(\frac{\omega}{2\pi T}\big)^2}. \tag{IV.9}$$

One should then be left with a sum of terms rational in the Laplace domain, which can be contracted into digamma, $\psi(z)$, or harmonic number, $\mathrm{H}(z)$, functions which are asymptotically logarithmic. In the time domain, residues of the hyperbolic cotangent additionally give rise to products of rational functions of k with $e^{-2\pi T t k}$. These terms contain all effects which decay at temperature dependent rates and can be expressed in terms of Lerch transcendent functions, $\Phi\big(z,1,e^{-2\pi T t}\big)$, which are useful for numerical calculations but not particularly insightful.

3. *Initial Jolts*

In previous study of quantum Brownian motion, going all the way back to Unruh and Zurek[5], it was noticed that the diffusion coefficients exhibited cutoff sensitive transient behavior near the initial time or "jolts". These jolts were thought to be an incredible catalyst for decoherence. Unruh and Zurek worked in the limit of an ohmic spectra with local dissipation, but the local propagator does not apply near the initial time as $\mathbf{H}(t)$ is strictly parametric there. Later, the HPZ master equation[12] was thought to have the

correct master equation coefficients for arbitrary spectral density, but as we have shown in Sec. II B 2, the final time propagator does not take the form posited therein nor in other derivations.

To begin our discussion, first we note from Eq. (IV.2) that the thermal covariance is positive definite as the noise kernel is a positive definite function. We also note that the thermal covariance begins with $\boldsymbol{\sigma}_T(0) = 0$ and $\dot{\boldsymbol{\sigma}}_T(0) = 0$. Now given that this matrix is positive definite, the off-diagonal entries must be smaller than the average (arithmetic or geometric) diagonal entries. But the off-diagonal $\sigma_T^{xp}(t)$ is proportional to $\dot{\sigma}_T^{xx}(t)$ and therefore we have the constraint

$$|\dot{\sigma}_T^{xx}(t)| \leq \frac{2}{M}\sqrt{\sigma_T^{xx}(t)\sigma_T^{pp}(t)}, \tag{IV.10}$$

which is also generally less than the late-time uncertainty as both $\sigma_T^{xx}(t)$ and $\sigma_T^{pp}(t)$ begin increasing and then proceed to undergo damped oscillations, wherein each cycle there is a net increase in uncertainty. This constrains the growth of position uncertainty. If the uncertainty function takes reasonable values, then the position uncertainty must have reasonable growth.

There is a peculiar bootstrapping dilemma here. $\sigma_T^{xx}(t)$ begins at zero with zero derivative, but growth is constrained by the current uncertainty. If we consider some infinitesimal time Δt after the initial time, one would naively expect from the $\dot{\sigma}_T^{xx}(t)$ constraint that $\sigma_T^{xx}(t)$ remain zero and thus never increase from zero. The initial time evolution of $\sigma_T^{xx}(t)$ must be strictly non-analytic to overcome this. Also, due to this non-analytic bootstrapping evolution, one would expect $\sigma_T^{pp}(t)$ to initially accelerate relatively faster than $\sigma_T^{xx}(t)$.

An analogous constraint can be placed upon the growth in momentum uncertainty by considering the positive definite matrix $\dot{\boldsymbol{\Phi}} \cdot \boldsymbol{\nu} \cdot \dot{\boldsymbol{\Phi}}^{\mathrm{T}}$ which yields

$$|\dot{\sigma}_T^{pp}(t)| \leq 2M\sqrt{\sigma_T^{pp}(t)\left(\ddot{G}\cdot\nu\cdot\ddot{G}\right)(t)}, \tag{IV.11}$$

with psuedo-acceleration correlation

$$\left(\ddot{G}\cdot\nu\cdot\ddot{G}\right)(t) = \Omega^4\sigma_T^{xx}(t) + 4\Omega^2\left(G\cdot\nu\cdot\gamma\cdot\dot{G}\right)(t) + 4\left(\dot{G}\cdot\gamma\cdot\nu\cdot\gamma\cdot\dot{G}\right)(t). \tag{IV.12}$$

This constraint is only comparable to that of $\dot{\sigma}_T^{xx}(t)$ in the weak coupling limit. Otherwise the initial growth of momentum uncertainty appears less constrained.

In conclusion, growth in position uncertainty is well constrained while growth in momentum is less constrained. Corresponding to this, we will show in Sec. IV B 4 that the

late-time momentum uncertainty has much more sensitivity to the high frequency modes of the bath. In terms of ohmic coupling, the initial cutoff jolts and late-time logarithmic cutoff sensitivity only occurs in the momentum uncertainty. The position uncertainty is relatively well behaved in both respects, having only initial logarithmic jolts and no late-time cutoff sensitivity at all.

4. Linear Entropy

In this subsection we investigate the linear entropy [20], which can be easily obtained from the Wigner distribution as follows:

$$S_{\rm L} = 1 - {\rm Tr}(\hat{\rho}_{\rm r}^2) = 1 - 2\pi \int d^2\mathbf{q}\, W_{\rm r}^2(\mathbf{q}). \tag{IV.13}$$

In Fourier space it becomes

$$S_{\rm L} = 1 - \frac{1}{2\pi} \int d^2\mathbf{k}\, |\mathcal{W}_{\rm r}(\mathbf{k})|^2, \tag{IV.14}$$

and using the result in Eq. (IV.1) we finally get

$$S_{\rm L} = 1 - \frac{1}{2\pi} \int d^2\mathbf{k}\, \left|\mathcal{W}_{\rm r}\big(0, \mathbf{\Phi}^{\rm T}(t)\mathbf{k}\big)\right|^2 e^{-\mathbf{k}^{\rm T}\boldsymbol{\sigma}_T(t)\mathbf{k}}. \tag{IV.15}$$

At the initial time the linear entropy is that of the initial state, and at late times it tends to $S_L = 1 - (1/2)(\det \boldsymbol{\sigma}_T^{\infty})^{-1/2}$.

Alternatively, one can express the linear entropy in terms of an integral of the Fourier-transformed reduced Wigner function at the initial time by introducing the change of variables $\mathbf{k}_0 = \mathbf{\Phi}^{\rm T}(t)\mathbf{k}$. Eq. (IV.15) can then be written as

$$\begin{aligned} S_{\rm L} &= 1 - \frac{1}{2\pi} \int d^2\mathbf{k}_0\, \det\big[\mathbf{\Phi}^{\rm T}(t)\big]\, |\mathcal{W}_{\rm r}(0, \mathbf{k}_0)|^2\, e^{-\mathbf{k}_0^{\rm T}\mathbf{\Phi}(t)\,\boldsymbol{\sigma}_T(t)\,\mathbf{\Phi}^{\rm T}(t)\mathbf{k}_0} \\ &= 1 - \frac{1}{2\sqrt{\det[\boldsymbol{\sigma}_T(t)]}} \int d^2\mathbf{k}_0\, |\mathcal{W}_{\rm r}(0, \mathbf{k}_0)|^2\, N\Big(\mathbf{0}, \big(2\,\mathbf{\Phi}(t)\,\boldsymbol{\sigma}_T(t)\,\mathbf{\Phi}(t)^{\rm T}\big)^{-1}; \mathbf{k}_0\Big), \end{aligned} \tag{IV.16}$$

where $N(\boldsymbol{\mu}, \boldsymbol{\sigma}; \mathbf{k}_0)$ is a normalized Gaussian distribution for the variable $\mathbf{k}_0$ with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\sigma}$. For small times this integral is similar to that for the initial state, whereas for long times the normalized Gaussian distribution becomes increasingly close to a delta function.

For a Gaussian initial state $\mathcal{W}_{\rm r}(0,\mathbf{k}_0) = \exp\left(-\mathbf{k}_0^{\rm T}\boldsymbol{\sigma}_0\mathbf{k}_0 - i\mathbf{k}_0^{\rm T}\langle\mathbf{q}\rangle_0\right)$ the integral in Eq. (IV.15) can be explicitly computed:

$$\begin{aligned} S_{\rm L} &= 1 - \frac{1}{2\pi}\int d^2\mathbf{k}\, e^{-\mathbf{k}^{\rm T}\left(\mathbf{\Phi}(t)\,\boldsymbol{\sigma}_0\,\mathbf{\Phi}(t)^{\rm T}+\boldsymbol{\sigma}_T(t)\right)\mathbf{k}} \\ &= 1 - \frac{1}{2\sqrt{\det[\mathbf{\Phi}(t)\,\boldsymbol{\sigma}_0\,\mathbf{\Phi}(t)^{\rm T}+\boldsymbol{\sigma}_T(t)]}}. \end{aligned} \tag{IV.17}$$

For these Gaussian states, reasonable linear entropy is synonymous with reasonable uncertainty functions (*i.e.*, the linear entropy will be positive if and only if the Heisenberg uncertainty principle is satisfied). We will find that the late time uncertainty is well behaved. The uncertainty at the initial and intermediate times should not violate the Heisenberg uncertainty principle either.

5. *Decoherence of a Quantum Superposition*

In this section we will illustrate how one can get a useful qualitative picture of the phenomenon of environment-induced decoherence from the the solutions of the master equation given by Eqs. (IV.1)-(IV.2). In order to do that we will consider a quantum superposition, $|\psi\rangle = \left(|\psi_+\rangle + |\psi_-\rangle\right)/\sqrt{K}$, of a pair of states $|\psi_\pm\rangle$ which correspond to a pair of Gaussian wavefunctions in position space separated by a distance $2\delta_x$ and K is an appropriate normalization constant. Specifically, we have

$$\psi_\pm(x) = \psi_0(x \mp \delta_x), \tag{IV.18}$$

$$\psi_0(x) = \sqrt{N(0,\sigma_0^{xx};x)}. \tag{IV.19}$$

where $N(\mu,\sigma^2;x)$ is a normalized Gaussian distribution for the variable x with mean μ and variance σ^2, and $\psi_0(x)$ is a reference Gaussian state centered at the origin.

Taking into account the definition of the Wigner function,

$$W(x,p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dy\, e^{ipy}\rho(x-y/2, x+y/2), \tag{IV.20}$$

and applying it to the density matrix $\rho(x,x') = \langle x|\psi\rangle\langle\psi|x'\rangle$ we get

$$W(\mathbf{q}) = \frac{1}{K}\Big(W_+(\mathbf{q}) + W_-(\mathbf{q}) + 2\cos(2\delta_x p)\, W_0(\mathbf{q})\Big). \tag{IV.21}$$

This Wigner function, plotted in Fig. 2, exhibits oscillations of size $1/\delta_x$ along the p direction. These oscillations are closely connected to the coherence of the quantum superposition (and

the existence of non-diagonal terms in the density matrix) and are absent in the Wigner function for the incoherent mixture $W(\mathbf{q}) = (1/2)\Big(W_+(\mathbf{q}) + W_-(\mathbf{q})\Big)$.

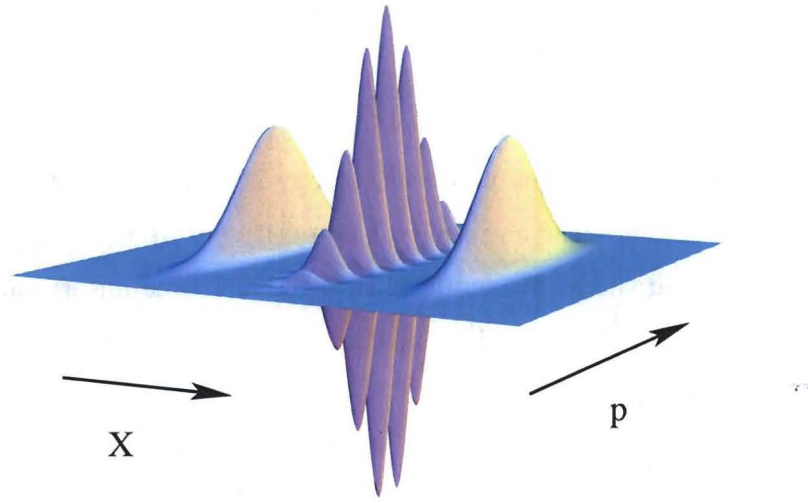

FIG. 2: Wigner function associated with a state $|\psi\rangle = \big(|\psi_1\rangle + |\psi_2\rangle\big)/\sqrt{K}$ which corresponds to the coherent quantum superposition of two Gaussian wavefunctions in position space shifted by a distance δ_x.

In this context the decoherence effect due to the interaction with the environment corresponds to the washing-out of the oscillations in the reduced Wigner function as it evolves according to the master equation. This can be seen rather simply from the result for the solutions of the master equation obtained in this section and given by Eqs. (IV.1)-(IV.2). Taking into account that the inverse Fourier transform of Eq. (IV.1) corresponds to a convolution of the homogeneously evolving initial state and a thermal Gaussian, the Wigner function can then be expressed

$$W_{\rm r}(t,\mathbf{q}) \;=\; \int d^2\mathbf{q}'\, S(\mathbf{q}-\mathbf{q}')\, W_{\rm r}(0, \mathbf{\Phi}(t)\,\mathbf{q}')\,, \tag{IV.22}$$

$$S(t,\mathbf{q}) \;\equiv\; N(\mathbf{0}, \boldsymbol{\sigma}_T(t); \mathbf{q}), \tag{IV.23}$$

where the thermal Gaussian now acts as a Gaussian smearing function which starts as a delta function at the initial time and broadens as time passes. It is well known that a Gaussian smearing with variance σ^2 will average out details of size σ. Therefore as the characteristic size of $\boldsymbol{\sigma}_T(t)$ along the direction of the oscillations becomes comparable to $1/\delta_x$, the oscillations of the reduced Wigner function get washed out and the Wigner function

becomes equivalent to that of the completely incoherent mixture. The time it takes for this to happen is known as the decoherence time, $t_{\rm D}$.

Obviously, ordinary dissipation will destroy these interference terms, but ordinary dissipation does not distinguish between coherence and incoherence. If we want to distinguish ordinary dissipation from decoherence we consider a co-rotating, co-decaying view of the Wigner function, $W_{\rm r}(t, \mathbf{\Phi}^{-1}(t)\,\mathbf{q})$, which would be equivalent to the initial state smeared by a co-rotating Gaussian with covariance $\boldsymbol{\sigma}_{\rm D}(t) = \mathbf{\Phi}^{-1}(t)\,\boldsymbol{\sigma}_T(t)\,\mathbf{\Phi}^{-{\rm T}}(t)$. This is not terribly difficult to work out given that we can express the coherent state as a linear combination of (complex) Gaussian functions.

$$W(\mathbf{q}) \;=\; \frac{1}{K}\left(W_+(\mathbf{q}) + W_-(\mathbf{q}) + 2\,{\rm Re}\left[W_0\left(x, p + \imath\frac{\delta_x}{2\sigma_0^{xx}}\right)\right] e^{-\frac{\delta_x^2}{2\sigma_0^{xx}}}\right). \qquad \text{(IV.24)}$$

The time evolution of the state can then be easily determined from the time evolution of the covariance. The interference terms in the co-rotating perspective work out to be

$$2\cos\left(\frac{\delta_x\left(\sigma_{\rm rot}^{xx}(t)p - \sigma_{\rm rot}^{xp}(t)x\right)}{2\det\left[\boldsymbol{\sigma}_{\rm rot}(t)\right]\sigma_0^{xx}}\right) W_0(\mathbf{q})\, e^{-\frac{\delta_x^2}{2\sigma_0^{xx}}\left(1-\frac{\sigma_{\rm rot}^{xx}(t)}{4\det[\boldsymbol{\sigma}_{\rm rot}(t)]\sigma_0^{xx}}\right)}, \qquad \text{(IV.25)}$$

where $\boldsymbol{\sigma}_{\rm rot}(t)$ is the co-rotating covariance $\boldsymbol{\sigma}_{\rm rot}(t) = \mathbf{\Phi}^{-1}(t)\,\boldsymbol{\sigma}(t)\,\mathbf{\Phi}^{-{\rm T}}(t) = \boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_{\rm D}(t)$. Time evolution decreases the frequency of interference and introduces exponential suppression in addition to the dissipation that we have ignored in the co-rotating, co-decaying perspective. This decoherence factor is only present in the interference terms; it begins at unity and will asymptote to $e^{-\frac{\delta_x^2}{2\sigma_0^{xx}}}$ in accord with physical expectations. Given that we have this factor explicitly, for this problem, we may define the decoherence time $t_{\rm D}$ as the time at which the exponent surpasses order unity.

$$4\det[\boldsymbol{\sigma}_{\rm rot}(t_{\rm D})] \;\gg\; \left(1 - \frac{2\sigma_0^{xx}}{\delta_x^2}\right)^{-1} \frac{\sigma_{\rm rot}^{xx}(t_{\rm D})}{\sigma_0^{xx}}. \qquad \text{(IV.26)}$$

The analogous problem of momentum superposition would have yielded the asymptotic decoherence factor $e^{-\frac{\delta_p^2}{2\sigma_0^{pp}}}$ with the decoherence time

$$4\det[\boldsymbol{\sigma}_{\rm rot}(t_{\rm D})] \;\gg\; \left(1 - \frac{2\sigma_0^{pp}}{\delta_p^2}\right)^{-1} \frac{\sigma_{\rm rot}^{pp}(t_{\rm D})}{\sigma_0^{pp}}. \qquad \text{(IV.27)}$$

Obviously in both cases there is a general tendency towards decoherence as the determinant receives four copies of the inverse propagator (which grows in time) whereas the right hand side of the equations only receive two copies of the inverse propagator. The question is if there

exists a very short decoherence timescale. Let us consider the limit in which decoherence outpaces dissipation and system frequency. We can then expand our inequalities perturbative in $t_{\rm D}$. It is then a straight forward calculation to find the decoherence time to be

$$t_{\rm D} \sim \frac{1}{M D_{pp} \delta_x^2} \left(1 + \mathcal{O}\left(\frac{\sigma_0^{xx}}{\delta_x^2}\right)\right), \tag{IV.28}$$

for a superposition of positions. But in the case of a superposition of momenta, a fast decoherence timescale does not readily emerge.

ALBERT: we need some references here right?

Also, this is counterintuitive because we know momentum gets the jolts and position doesn't. But the effect here is reversed.

B. Late-Time Evolution and Local Dissipation

We now focus our attention on the simplest kind of system dynamics, when the psuedo-Hamiltonian $\mathbf{H}$ is effectively constant and the homogeneous evolution of the system becomes exponential and thus time-translation invariant, $\mathbf{\Phi}(t) = e^{-t\,\mathbf{H}}$.

This is a feature an ohmic spectral density, $I(\omega) \propto \omega$, in the high cutoff limit, which directly produces local dissipation, $\gamma(t-\tau) = \gamma_0\,\delta(t-\tau)$. But this is also a more general feature of the late time evolution for fairly arbitrary spectra wherein $\mathbf{H}(t)$ asymptotes to some constant value. Moreover, we will show that the local dissipation of ohmic coupling is more correctly viewed as a late time limit calculation, specifically with respect to the cutoff timescale.

1. *Late-Time Propagator*

The late-time psuedo-Hamiltonian operator takes the form

$$\mathbf{H} = \begin{bmatrix} 0 & -\frac{1}{M} \\ M\Omega_{\rm R}^2 & 2\Gamma \end{bmatrix}, \tag{IV.29}$$

and can be effectively represented as arising from the propagator

$$\hat{G}_{\rm R}(s) = \frac{\frac{1}{M}}{s^2 + 2\Gamma s + \Omega_R^2}, \tag{IV.30}$$

$$G_{\rm R}(t) = \frac{1}{M\tilde{\Omega}_{\rm R}} \sin\left(\tilde{\Omega}_{\rm R} t\right) e^{-\Gamma t}, \tag{IV.31}$$

with $\tilde{\Omega}_{\rm R} = \sqrt{\Omega_{\rm R}^2 - \Gamma^2}$. This effective propagator $G_{\rm R}(t)$ is not equivalent to the late time limit of the true propagator $G(t)$, but they should share the same asymptotic dynamics. Specifically if one is to take the asymptotic expansion

$$G(t) \;=\; G_\infty(t) + \Delta G(t), \tag{IV.32}$$

where $G_\infty(t)$ contains the asymptotic limiting behavior and $\Delta G(t)$ contains the early time corrections, then $G_\infty(t)$ should directly yield $\tilde{\Omega}_{\rm R}$ and Γ in its arguments. For analytic spectra, one only need to solve the characteristic equation

$$f^2 + 2\hat{\gamma}(f)\, f + \Omega^2 = 0, \tag{IV.33}$$

for the rates f with smallest negative real part.

2. Initial State Distortion

In any limit wherein one has local dissipation, there will be a distorted (but equivalent) view of the initial state. Contrast the exact (non-local, full time) solutions to the Langevin equation

$$\hat{x}(s) = \frac{s x_0 + \dot{x}_0}{s^2 + 2\hat{\gamma}(s) s + \Omega^2}\,, \tag{IV.34}$$

to solutions one would obtain from the corresponding local differential equation at late time

$$\hat{x}(s) = \frac{s x_0 + \dot{x}_0 + 2\Gamma x_0}{s^2 + 2\Gamma s + \Omega_{\rm R}^2}\,. \tag{IV.35}$$

In addition to other effects, there is always a finite, position-dependent shift in the initial momentum between these two states: $\dot{x}_0 \to \dot{x}_0 + 2\Gamma x_0$. Following the approach in Ref. [19] one can easily see that this kick translates into a distortion of the Wigner distribution from the bare initial state to a shifted one

$$W_{\rm bare}(x,p) \to W_{\rm ren}(x,p) = W_{\rm bare}(x, p + 2M\Gamma x). \tag{IV.36}$$

This phase-space transformation has a Jacobian with determinant equal to one:

$$\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 2M\Gamma & 1 \end{bmatrix} \qquad \det \mathbf{K} = 1. \tag{IV.37}$$

Therefore, it is simple to calculate renormalized expectation values in terms of bare expectation values and vice versa:

$$\langle A(x,p)\rangle_{\text{ren}} = \iint dxdp\, A(x,p) W_{\text{ren}}(x,p), \tag{IV.38}$$

$$\langle A(x,p)\rangle_{\text{ren}} = \langle A(x,p-2M\Gamma x)\rangle_{\text{bare}}\,. \tag{IV.39}$$

We can immediately see that the normalization, linear entropy (see Sec. IV A 4) and state overlap are all unchanged by the kick. We can also check that the Heisenberg uncertainty relation is also preserved as follows. First, we start with the covariance matrix for x and p corresponding to the Wigner distribution

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xp} \\ \sigma_{px} & \sigma_{pp} \end{bmatrix}, \tag{IV.40}$$

with $\sigma_{xx} = \langle xx\rangle_{\text{ren}}$, $\sigma_{xp} = \sigma_{px} = \langle xp\rangle_{\text{ren}}$ and $\sigma_{pp} = \sigma_{pp} = \langle pp\rangle_{\text{ren}}$, and which transforms in the following way under linear phase space transformations:

$$\boldsymbol{\sigma} \to \mathbf{K}\,\boldsymbol{\sigma}\,\mathbf{K}^{\mathrm{T}}\,. \tag{IV.41}$$

Hence, from Eq. (IV.37) we have

$$\det \boldsymbol{\sigma}_{\text{bare}} = \det \boldsymbol{\sigma}_{\text{ren}}\,. \tag{IV.42}$$

Finally, one takes into account that

$$(\det \boldsymbol{\sigma}) \geq \frac{\hbar^2}{4}, \tag{IV.43}$$

corresponds to the formulation in terms of the Wigner function of the generalized Heisenberg uncertainty relation due to Schrödinger [8, 9]:

$$\langle \Delta x^2\rangle \langle \Delta p^2\rangle - \left\langle \frac{1}{2}\{\Delta x, \Delta p\}\right\rangle^2 \geq \frac{\hbar^2}{4}, \tag{IV.44}$$

where $\{\hat{A},\hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$.

Furthermore, by switching to the density matrix formalism, we can see that pure states are mapped to pure states and positivity is preserved. It is a straight forward calculation to show that

$$\boldsymbol{\rho}_{\text{bare}}(x,y) \to \boldsymbol{\rho}_{\text{ren}}(x,y) = e^{+\imath M\Gamma x^2}\boldsymbol{\rho}_{\text{bare}}(x,y)e^{-\imath M\Gamma y^2}. \tag{IV.45}$$

Therefore if we start in a pure state, which acts as a projection operator

$$\boldsymbol{\rho}_{\text{bare}}^2 = \boldsymbol{\rho}_{\text{bare}} \tag{IV.46}$$

then it is fairly easy to see that this will hold for the distorted state.

Additionally, given the positivity condition

$$\mathbf{A}^\dagger \boldsymbol{\rho}_{\text{bare}} \mathbf{A} \geq 0 \tag{IV.47}$$

for all vectors $\mathbf{A}$, then it is easy to see that the distorted state will also fulfill this condition by simply considering the vectors $e^{+\imath M\Gamma x^2}\mathbf{A}$.

Therefore we can confidently say that these are two equivalent perspectives. If one is interested in studying the evolution of a certain state of the system properly correlated with the environment, one can always consider the reduced Wigner function associated with that state and invert Eq. (IV.36) to obtain the corresponding initial Wigner function before the interaction was switched on.[2]

3. *Late-Time Diffusion*

Given the effectively local dissipation of the late-time limit, one can then compute the master equation coefficients using Eq. (III.24) sans the nonlocal contribution. This will be prone to initial jolts, but here we are only interested in the late-time coefficients. The full-time diffusion coefficients are also more easily calculated from the dynamics of the thermal covariance, Eq. (III.42).

The relations for the diffusion coefficients can be simplified by writing the cosine of the noise kernel in exponential form, manipulating the time integral until one has a Laplace convolution, and then utilizing frequency shifting in the Laplace domain, e.g. $e^{\lambda t} f(t) \rightarrow$

[2] Using this approach the system-environment correlations at high frequencies will be the same as those of other properly correlated states (such as the global equilibrium states considered in Ref. [3] or states prepared from those in a finite time). However, in general the correlations for low frequencies will differ and the states of the whole system plus environment will not be equivalent even if their reduced Wigner functions are the same. In particular this implies that even if the reduced Wigner functions of the two states coincide at some given time, they will in general evolve differently (until thermal equilibrium is reached).

$\hat{f}(s-\lambda)$. After some algebraic manipulation they can be written as

$$\hat{D}_{xp}(s) = -\frac{1}{s}\int_0^\infty d\omega\, I(\omega)\coth\Big(\frac{\omega}{2T}\Big)\mathrm{Re}\Big[\hat{G}_{\mathrm{R}}(s+\imath\omega)\Big], \tag{IV.48}$$

$$\hat{D}_{pp}(s) = +\frac{1}{s}\int_0^\infty d\omega\, I(\omega)\coth\Big(\frac{\omega}{2T}\Big)\mathrm{Re}\Big[\hat{\dot{G}}_{\mathrm{R}}(s+\imath\omega)\Big]. \tag{IV.49}$$

Using the final value theorem, Eq. (B.8), for Laplace transforms one gets

$$D_{xp}(\infty) = -\int_0^\infty d\omega\, I(\omega)\coth\Big(\frac{\omega}{2T}\Big)\mathrm{Re}\Big[\hat{G}_{\mathrm{R}}(\imath\omega)\Big], \tag{IV.50}$$

$$D_{pp}(\infty) = +\int_0^\infty d\omega\, I(\omega)\coth\Big(\frac{\omega}{2T}\Big)\mathrm{Re}\Big[\hat{\dot{G}}_{\mathrm{R}}(\imath\omega)\Big]. \tag{IV.51}$$

These expressions, and similar formulations, become very easy to work with when one has analytic spectra. For analytic spectra, the spectral density is odd and therefore these integrals can be represented as contour integrals. Furthermore, $\hat{G}_{\mathrm{R}}(s)$ is always rational, an analytic $I(\omega)$ can be constructed to be rational, and the hyperbolic cotangent has the rational expansion given by Eq. (IV.9). Therefore one is left with a single sum of contour integrals of purely rational functions which can trivially be resolved via the method of residues.

4. Late-Time Thermal Covariance and Uncertainty Function

Given late time master equation coefficients which have all taken their asymptotic values, from Eq. (III.41) it is a straight forward calculation to show that the thermal covariance can be expressed

$$\boldsymbol{\sigma}_T(t) = \boldsymbol{\sigma}_T^\infty - \boldsymbol{\Phi}(t)\,\boldsymbol{\sigma}_T^\infty\,\boldsymbol{\Phi}^{\mathrm{T}}(t), \tag{IV.52}$$

$$\boldsymbol{\sigma}_T^\infty = \begin{bmatrix} \frac{1}{M\Omega_{\mathrm{R}}^2}\left(\frac{1}{2\Gamma}D_{pp}^\infty - D_{xp}^\infty\right) & 0 \\ 0 & \frac{M}{2\Gamma}D_{pp}^\infty \end{bmatrix}, \tag{IV.53}$$

where $\boldsymbol{\sigma}_T^\infty$ indeed coincides with the stationary covariance of our Lyapunov equation (III.30).

As we have now well proven, any specific features of the initial distribution decay away and at late times the state tends generically to a Gaussian with a covariance matrix given by Eq. (IV.53). Therefore, from Eq. (IV.6) it follows that at late times $(\Delta x)^2 = (\boldsymbol{\sigma}_T^\infty)_{xx}$ and $(\Delta p)^2 = (\boldsymbol{\sigma}_T^\infty)_{pp}$. We can therefore express the position and momentum uncertainties at late times as

$$(\Delta x)^2 = \frac{1}{M\Omega_{\mathrm{R}}^2}\left(\frac{1}{2\Gamma}D_{pp}^\infty - D_{xp}^\infty\right), \tag{IV.54}$$

$$(\Delta p)^2 = \frac{M}{2\Gamma}D_{pp}^\infty. \tag{IV.55}$$

Let us inspect the high frequency contributions to our uncertainties. From Eqs. (IV.50)-(IV.51), the high frequency contribution of any spectra takes the form

$$\Delta D_{xp}^{\infty} = \frac{2}{\pi}\int^{\infty} d\omega \frac{I(\omega)}{\omega^2} + \cdots, \tag{IV.56}$$

$$\Delta D_{pp}^{\infty} = 2\Gamma\, \Delta D_{xp}^{\infty} + \cdots, \tag{IV.57}$$

and we can immediately see that these high frequency contributions will never affect the late-time uncertainty in position, but only momentum. This will include the logarithmic cutoff sensitivity corresponding to ohmic coupling.

C. Example Spectra: Ohmic with Finite Cutoff

1. *Nonlocal Propagator*

The simplest ohmic dissipation (with finite cut-off) one can construct corresponds to the following damping kernel:

$$\hat{\gamma}(s) = \frac{\gamma_0}{1+\frac{s}{\Lambda}}. \tag{IV.58}$$

This damping kernel is ohmic at frequencies much less than the cutoff, but at very high frequencies it vanishes as it should. The corresponding spectral density also happens to have a rational cut-off.

$$I(\omega) = \frac{2}{\pi} M\gamma_0\, \omega \left[1+\left(\frac{\omega}{\Lambda}\right)^2\right]^{-1}. \tag{IV.59}$$

Calculating the Green function is no more difficult than factoring a cubic polynomial. Specifically one must factor $(s^2+\Omega^2)(s+\Lambda)+2\gamma_0\Lambda s$ in the denominator of the Green function. The effect of the finite cutoff is to shift the system relaxation and oscillation timescales slightly,

$$\gamma_\star = \gamma_0\left[1+2\frac{\gamma_0}{\Lambda}+\mathcal{O}\left(\frac{1}{\Lambda^2}\right)\right], \tag{IV.60}$$

$$\tilde{\Omega}_\star = \tilde{\Omega}\left[1+\left(1-\left(\frac{\gamma_0}{\tilde{\Omega}}\right)^2\right)\frac{\gamma_0}{\Lambda}+\mathcal{O}\left(\frac{1}{\Lambda^2}\right)\right], \tag{IV.61}$$

with $\tilde{\Omega}_\star = \sqrt{\Omega_\star^2-\gamma_\star^2}$, and to add an additional relaxation timescale that corresponds to the cut-off,

$$\Lambda_\star = \Lambda - 2\gamma_0 + \mathcal{O}\left(\frac{1}{\Lambda}\right). \tag{IV.62}$$

The fully nonlocal Green function can be expressed

$$\hat{G}(s) = \frac{1}{M}\frac{s+\Lambda}{(s+\Lambda_\star)(s^2+2\gamma_\star s+\Omega_\star^2)}. \tag{IV.63}$$

Partial fraction decomposition then affords us the asymptotic expansion described in Sec. IV B 1. One is left with two terms: one term describes pure decay at the rate $\Lambda_\star$, while the other describes oscillatory decay at the rates $\tilde{\Omega}_\star$ and $\gamma_\star$. The late time master equation coefficients are therefore

$$\Gamma = \gamma_\star, \quad \Omega_{\rm R} = \Omega_\star, \tag{IV.64}$$

for times much greater than $\Lambda_\star^{-1}$. The full-time, nonlocal propagator can be expressed

$$G(t) = \frac{\Lambda_\star^2+\Omega_\star^2}{(\Lambda_\star-\gamma_\star)^2+\tilde{\Omega}_\star^2}\left[G_{\rm R}(t) - \frac{2\gamma_\star}{\Lambda_\star^2+\Omega_\star^2}\left(\dot{G}_{\rm R}(t) - \frac{e^{-\gamma_\star}}{M}\right)\right], \tag{IV.65}$$

where $G_{\rm R}(t)$ is the late-time, local propagator.

In the high cutoff limit one recovers the usual coefficients γ_0 and Ω, but it should be noted that effective propagator $G_{\rm R}(t)$ still does not apply to the initial state at $t = 0$. $G_{\rm R}(t)$ only applies to early times approaching the initial time. For this particular propagator, the local approximation is good for $G(t)$ but not $\ddot{G}(t)$ which happens to appear in $\dot{\mathbf{\Phi}}(t)$ and thus the nonlocal corrections to the diffusion coefficients (III.24).

On a final note, if we parameterize everything in terms of the phenomenological frequencies

$$\Lambda = \Lambda_\star + 2\gamma_\star, \tag{IV.66}$$

$$\gamma_0 = \frac{(\Lambda_\star+\gamma_\star)^2+\tilde{\Omega}_\star^2}{(\Lambda_\star+2\gamma_\star)^2}\gamma_\star, \tag{IV.67}$$

then we never have to factor the cubic.

2. *Moderate-Time Diffusion for Finite Cutoff*

Given our late time Green function (IV.31) with late time coefficients (IV.64), which are valid at times later than the cutoff, and the rational expansion of the hyperbolic cotangent (IV.9), the frequency integrals over ω in the diffusion coefficients in the Laplace domain (IV.48)-(IV.49) become sums over k of trivial contour integrals. Abiding in the Laplace

domain, these sums can be identified as harmonic number or equivalently digamma functions. [3]

$$\hat{D}_{xp}(s) = \frac{2\gamma_0}{\pi\tilde{\Omega}_R s}\,\mathrm{Im}\left[\frac{\Gamma_s + \imath\tilde{\Omega}_R}{1-\left(\frac{\Gamma_s+\imath\tilde{\Omega}_R}{\Lambda}\right)^2}\left\{\mathrm{H}\left(\frac{\Lambda}{2\pi T}\right) - \mathrm{H}\left(\frac{\Gamma_s+\imath\tilde{\Omega}_R}{2\pi T}\right)\right\}\right] - \frac{2\gamma_0 T}{\Lambda s}\frac{1}{\left(1+\frac{\Gamma_s}{\Lambda}\right)^2+\left(\frac{\tilde{\Omega}_R}{\Lambda}\right)^2}, \tag{IV.68}$$

$$\hat{D}_{pp}(s) = \frac{2\gamma_0}{\pi\tilde{\Omega}_R s}\,\mathrm{Im}\left[\frac{\left(\Gamma+\imath\tilde{\Omega}_R\right)\left(\Gamma_s+\imath\tilde{\Omega}_R\right)}{1-\left(\frac{\Gamma_s+\imath\tilde{\Omega}_R}{\Lambda}\right)^2}\left\{\mathrm{H}\left(\frac{\Lambda}{2\pi T}\right) - \mathrm{H}\left(\frac{\Gamma_s+\imath\tilde{\Omega}_R}{2\pi T}\right)\right\}\right] + \frac{2\gamma_0 T}{s}\frac{1+\frac{s}{\Lambda}}{\left(1+\frac{\Gamma_s}{\Lambda}\right)^2+\left(\frac{\tilde{\Omega}_R}{\Lambda}\right)^2}, \tag{IV.69}$$

where $\Gamma_s = \Gamma + s$, making the late time limit easy to take here: we simply strip the over all factor of s^{-1} and take $\Gamma_s \to \Gamma$. The initial time limit is also easy to take; all terms vanish at initial time, although some very weakly like $\lim_{s\to\infty}\frac{\log(s)}{s}$. This weak convergence corresponds to apparent initial jolts in the diffusion coefficients, though we should be very careful to note that the local Green function used to derive these coefficients is not accurate at the cutoff timescale where said jolts would occur. To truly probe to the cutoff timescale, one must use the fully nonlocal Green function and our newly derived diffusion coefficient formulas as discussed in Sec. IV A 3.

The H functions are harmonic number functions discussed in Appendix A 1. These terms make up, among other things, the well known $\log\frac{\Lambda}{\Omega}$ divergence. They behave asymptotically like logarithms but with $\mathrm{H}(0) = 0$, making both their high and zero temperature limits trivial. At high temperature, all of the harmonic number functions vanish, leaving only the second terms which are proportional to temperature. At zero temperature, all of the harmonic number functions can be equivalently evaluated as logarithms. These and other limits will be detailed in the next section.

Alternatively we can work in the time domain, though the sums over k cannot in general

[3] Many of the expressions derived throughout this paper assume underdamping, *i.e.*, $\Gamma < \Omega_R$ with $\tilde{\Omega}_R = \sqrt{\Omega_R^2 - \Gamma^2}$. They can be used for the overdamping regime by making the following analytical continuation: $\tilde{\Omega}_R \to \imath\tilde{\Gamma}$ with $\tilde{\Gamma} = \sqrt{\Gamma^2 - \Omega_R^2}$ real.
Therefore, Eqs. (IV.68)-(IV.69) can be applied to the overdamping case if the Im and Re terms are first expanded assuming $\tilde{\Omega}_R$ is real, and then the analytical continuation $\tilde{\Omega}_R \to \imath\tilde{\Gamma}$ is made.

be identified with any simply behaved special functions.

$$D_{xp}(t) = D_{xp}(\infty) - M\gamma_0 \left\{ \dot{G}_{\rm R}(t) + G_{\rm R}(t) \left(2\Gamma - \frac{d}{dt} \right) \right\} {\rm DF}(t), \tag{IV.70}$$

$$D_{pp}(t) = D_{pp}(\infty) - M\gamma_0 \left\{ \dot{G}_{\rm R}(t) \left(\Gamma + \frac{d}{dt} \right) + G_{\rm R}(t)\Omega^2 \right\} {\rm DF}(t), \tag{IV.71}$$

with the thermal decay function

$${\rm DF}(t) = -\frac{\cot\left(\frac{\Lambda}{2T}\right) e^{-\Lambda t}}{\left(1+\frac{\Gamma}{\Lambda}\right)^2 + \left(\frac{\tilde{\Omega}_{\rm R}}{\Lambda}\right)^2} + \frac{2}{\pi}{\rm TS}(t), \tag{IV.72}$$

$${\rm TS}(t) = \sum_{k=1}^{\infty} \frac{\left(\frac{\Lambda}{2\pi T}\right)^2}{\left(\frac{\Lambda}{2\pi T}\right)^2 - k^2} \frac{k\, e^{-2\pi T k t}}{\left(k+\frac{\Gamma}{2\pi T}\right)^2 + \left(\frac{\tilde{\Omega}_{\rm R}}{2\pi T}\right)^2}. \tag{IV.73}$$

For numerical purposes, we can express this thermal sum in terms of Lerch transcendent functions

$${\rm TS}(t) = {\rm Re}\left[\frac{1 - \imath\frac{\Gamma}{\tilde{\Omega}_{\rm R}}}{1 - \left(\frac{\Gamma+\imath\tilde{\Omega}_{\rm R}}{\Lambda}\right)^2} \Phi_1\left(\frac{\Gamma + \imath\tilde{\Omega}_{\rm R}}{2\pi T}; 2\pi T t \right) \right] - {\rm Sy}_\Lambda \left[\frac{\Phi_1\left(\frac{\Lambda}{2\pi T}; 2\pi T t\right)}{\left(1-\frac{\Gamma}{\Lambda}\right)^2 + \left(\frac{\tilde{\Omega}_{\rm R}}{\Lambda}\right)^2} \right] \tag{IV.74}$$

which with the definitions of the Lerch Φ_1 and symmetric part being

$$\Phi_1(z;\lambda) = \sum_{k=1}^{\infty} \frac{e^{-\lambda}}{k+z}, \tag{IV.75}$$

$${\rm Sy}_z\left[f(z)\right] = \frac{f(+z)+f(-z)}{2}. \tag{IV.76}$$

We will foster a more qualitative understanding of the decay function shortly.

Both diffusion coefficients can be expressed as their asymptotic values plus damped, oscillating differential operators acting on the same decay function ${\rm DF}(t)$. The decay function is such that at the initial time it causes cancelation with the asymptotic values and the diffusion coefficients vanish. In this (asymptotic) high temperature perspective, the decay function contains two terms. The first decays at a cut-off dependent and can be expressed in closed form. The second decays with primarily temperature dependent rates and cannot be expressed in closed form with intuitive functions. It contains the initial time cancelation to the $\log\frac{\Lambda}{\Omega}$ divergence. Although well convergent at moderate times, the sum contribution to the regular diffusion coefficient is very slow to converge at the initial time, even for moderate temperatures. Though to be fair, the initial time behavior of these coefficients is not meaningful down to the cutoff timescale.

It is important to note that the coefficients $D_{xp}(t)$ and $D_{pp}(t)$ both exhibit logarithmic divergences in the limit $\Lambda \to \infty$. This has been pointed out for $D_{xp}(t)$ in Ref. [11], where the coefficients of the master equation were calculated perturbatively to second order in the system-environment coupling constants (linear order in γ_0). The fact that there is also a logarithmic divergence in $D_{pp}(t)$ was not seen in that reference because it is quartic in the system-environment coupling constants (quadratic in γ_0). Moreover, such kinds of perturbative calculations cannot be employed to study the long time behavior since they are only valid for $t \ll \Gamma^{-1}$ and they miss for instance the exponential decay of the second and third terms on the right-hand side of Eqs. (IV.70)-(IV.71).

While our expressions (IV.68)-(IV.71) can easily give us the zero temperature diffusion coefficients at asymptotically late time, they can not easily give us the corresponding moderate time behavior in closed form. And unfortunately, the zero temperature limit of $\coth\left(\frac{\omega}{2T}\right) \to 1$ combined with an odd spectral density means that our diffusion coefficient integrals cannot be cast as closed contour integrals. Even so, the integrals can be performed and the results expressed in terms of exponential integrals with predictable time scales. At zero temperature (and in the high cutoff limit) we find the decay function to take the form

$$\lim_{T\to 0} \mathrm{DF}(t) \;=\; \frac{2}{\pi}\frac{d}{dt}\left\{ \mathrm{Re}\left[\frac{\mathrm{E}_1\Big(\big[\gamma_0 + \imath\tilde{\Omega}\big]\, t\Big)}{\imath\tilde{\Omega}\, e^{-(\gamma_0+\imath\tilde{\Omega})t}}\right] - \mathrm{Sy}_\Lambda\left[\frac{\mathrm{E}_1(\Lambda t)}{\Lambda\, e^{-\Lambda t}}\right]\right\}, \tag{IV.77}$$

where E_1 is the exponential integral, App. A 2, whose asymptotic behavior is like $\frac{e^{-z}}{z}$. It should be noted that, unlike the asymptotic limits of the diffusion coefficients, the full time behavior is highly sensitive to the form of the cut-off regulator at low temperature. For our smooth regulator, we find relatively smoothly evolving diffusion coefficients (such as in HPZ at $T = 10\Omega$) all the way down to zero temperature. Whereas a hard cut-off will produce the same average behavior, but with a slowly decaying envelope of considerable cut-off frequency oscillations.

Analogous functions appear when we approximate the thermal sum in (IV.73) as an integral with a comparably soft cutoff:

$$\sum_{k=1}^{\infty} \frac{\left(\frac{\Lambda}{2\pi T}\right)^2}{\left(\frac{\Lambda}{2\pi T}\right)^2 - k^2} \;\approx\; \int_{k_i}^{\infty} dk \frac{\left(\frac{\Lambda}{2\pi T}\right)^2}{\left(\frac{\Lambda}{2\pi T}\right)^2 + k^2}, \tag{IV.78}$$

where $k_i \approx 1$. Still in the high cutoff limit, we find this approximation of the decay function

to be

$$\mathrm{DF}(t) \approx \frac{2}{\pi}\frac{d}{dt}\left\{\mathrm{Re}\left[\frac{\mathrm{E}_1\left(\left[2\pi T k_i + \gamma_0 + \imath\tilde{\Omega}\right] t\right)}{\imath\tilde{\Omega}\, e^{-(\gamma_0+\imath\tilde{\Omega})t}}\right] - \mathrm{Sy}_\Lambda\left[\frac{\mathrm{E}_1([2\pi T k_i + \imath\Lambda]\, t)}{\imath\Lambda\, e^{-\imath\Lambda t}}\right]\right\} \quad \text{(IV.79)}$$

where we have discarded all finite terms at the initial time which decay at cut-off rates, as our approximation ultimately ruins the behavior of $\mathrm{DF}(t)$ there (not that it was meaningful prior). Thus when using this approximate decay function, the time-dependent, decaying part of the diffusion coefficients must be clamped at the initial time.

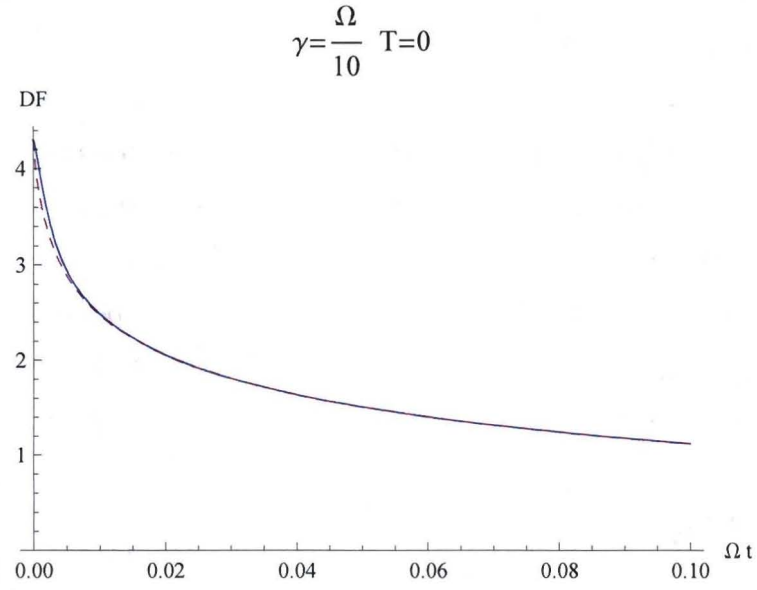

FIG. 3: Zero temperature decay functions for · exact, ··· approximate at $\Lambda = 10^3\Omega$. The slopes differ near the initial time (within the cut-off time scale).

At moderate times, our approximation reveals the exact same form of exponential integral behavior as in the zero temperature limit. But the temperature enters in such a way that the exponential decay inherent in E_1 is not balanced out with a $e^{-2\pi T k_i t}$ factor. Therefore temperature is an inherently stronger relaxation time-scale here, though do not forget that there are additional $e^{-\Gamma t}$ factors from $G_{\mathrm{R}}(t)$ functions in the full diffusion coefficients.

3. Late-Time Diffusion Coefficients

For ease of use, we offer the late time diffusion coefficients in a few simple regimes. We start with the asymptotically high temperature regime as it is the most common approxi-

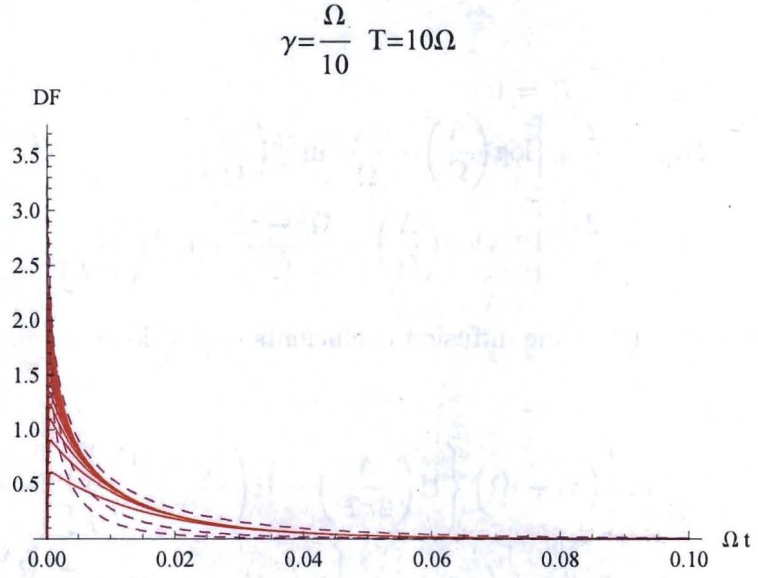

FIG. 4: Moderate termperature decay functions for · a sequence of the first 50 high temperature sums, ··· approximate solutions for $k_i = \frac{1}{2}, 1, 1\frac{1}{2}$ at $\Lambda = 10^5\Omega$. The high temperature sums are very slow to converge at the initial time.

mation.

$$\Omega \ll \Lambda \ll T$$

$$D_{xp} = 0, \tag{IV.80}$$

$$D_{pp} = 2\gamma_0 T. \tag{IV.81}$$

Next we look at the weak coupling coefficients, which were thought to have been correctly derived by less rigorous means.

$$\gamma_0 \ll \Omega \ll \Lambda$$

$$D_{xp} = \frac{2}{\pi}\gamma_0 \,\mathrm{Re}\left[\mathrm{H}\left(\frac{\Lambda}{2\pi T}\right) - \mathrm{H}\left(\frac{i\Omega}{2\pi T}\right)\right], \tag{IV.82}$$

$$D_{pp} = \gamma_0 \Omega \coth\left(\frac{\Omega}{2T}\right). \tag{IV.83}$$

In comparison to the weak coupling master equation of Caldeira *et al.*[4], the normal diffusion coefficient is the same to lowest order in the coupling, but the anomalous diffusion coefficient is *completely absent* in CCR.

Now we look at the zero temperature regime, previously explored by Unruh and Zurek

[5].

$$\Omega \ll \Lambda \quad T = 0$$

$$D_{xp} = \frac{2}{\pi}\gamma_0 \left[\log\left(\frac{\Lambda}{\Omega}\right) - \frac{\gamma_0}{\tilde{\Omega}} \tan^{-1}\left(\frac{\gamma_0}{\tilde{\Omega}}\right)\right], \tag{IV.84}$$

$$D_{pp} = \frac{2}{\pi}\gamma_0 \left[2\gamma_0 \log\left(\frac{\Lambda}{\Omega}\right) + \frac{\tilde{\Omega}^2 - \gamma_0^2}{\tilde{\Omega}} \tan^{-1}\left(\frac{\gamma_0}{\tilde{\Omega}}\right)\right]. \tag{IV.85}$$

Finally we look at the late time diffusion coefficients only taking the high cut-off limit.

$$\Omega \ll \Lambda$$

$$D_{xp} = \frac{2}{\pi}\frac{\gamma_0}{\tilde{\Omega}} \mathrm{Im}\left[\left(\gamma_0 + i\tilde{\Omega}\right)\left\{\mathrm{H}\left(\frac{\Lambda}{2\pi T}\right) - \mathrm{H}\left(\frac{\gamma_0 + i\tilde{\Omega}}{2\pi T}\right)\right\}\right], \tag{IV.86}$$

$$D_{pp} = 2\gamma_0 T + \frac{2}{\pi}\frac{\gamma_0}{\tilde{\Omega}} \mathrm{Im}\left[\left(\gamma_0 + i\tilde{\Omega}\right)^2\left\{\mathrm{H}\left(\frac{\Lambda}{2\pi T}\right) - \mathrm{H}\left(\frac{\gamma_0 + i\tilde{\Omega}}{2\pi T}\right)\right\}\right]. \tag{IV.87}$$

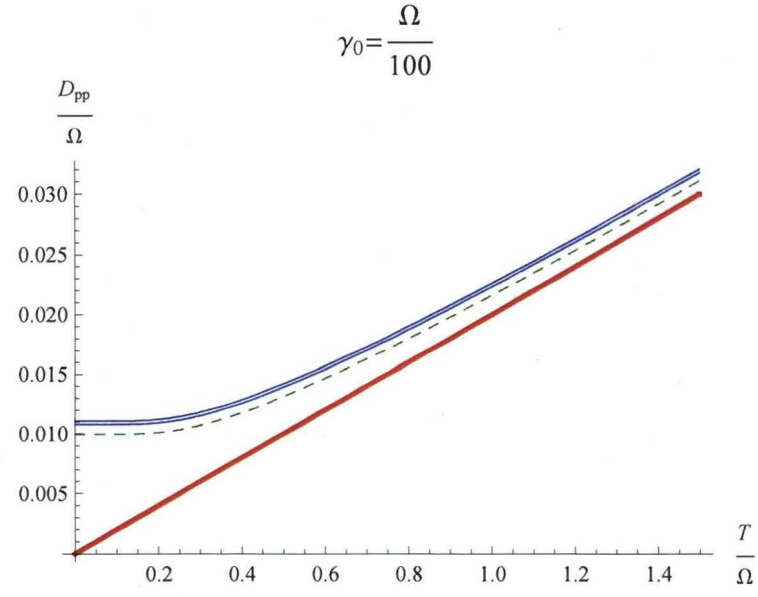

FIG. 5: Late time D_{pp} for • high temperature, ··· Caldeira, · HPZ at $\Lambda = 10^3\Omega$ and $\Lambda = 10^4\Omega$.

One can see from Fig. 5 that, ignoring the contribution from the cut-off, the CCR approximation matches extremely well with our exact results for the *normal diffusion coefficient* at weak coupling. The CCR error from neglecting the cut-off dependence is determined by the order of magnitude of the cut-off scale since it is logarithmic in Λ. In any case, the CCR approximation underestimates the magnitude of the diffusion coefficient.

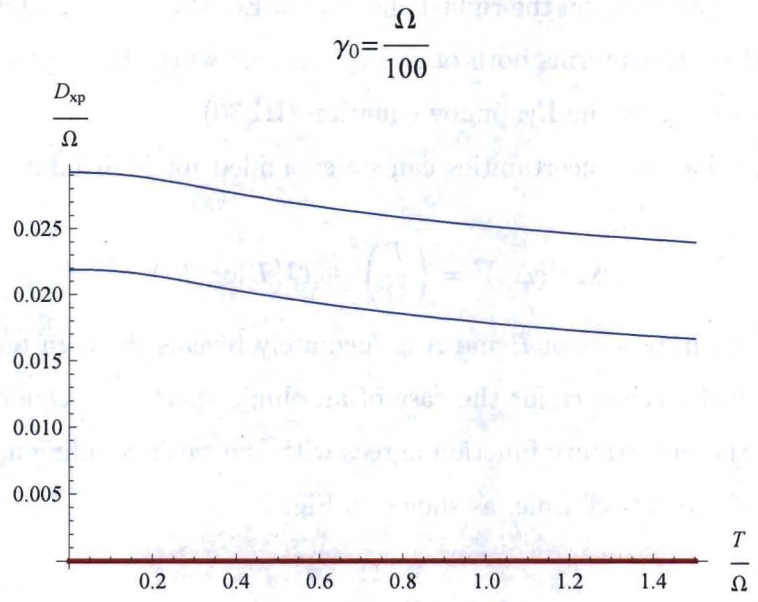

FIG. 6: Late time D_{xp} for • high temperature or equivalently Caldeira, · HPZ at $\Lambda = 10^3\Omega$ and $\Lambda = 10^4\Omega$.

The difference is more severe for the *anomalous diffusion coefficient* which is absent in CCR. The largest contribution (in the weak coupling regime) to the anomalous diffusion coefficient comes from the cut-off and it does not vanish at finite temperature (see Fig. 6). This logarithmic sensitivity does not enter into the normal diffusion coefficient until second order, but in the anomalous diffusion coefficient it is only proportional to one power of the coupling constant, which is the order to which CCR's master equation should be valid.

4. Late-Time Uncertainty Function

Given the late-time limit of the thermal covariance (IV.53) determined by the asymptotic limits ($t \gg \Gamma^{-1}$) of the diffusion coefficients in Sec. IV C 3, we now have sufficient information to study the late-time uncertainty function. Note that in a weak-coupling perturbative expansion in powers of γ_0 both D^∞_{xp} and D^∞_{pp} are of order γ_0 plus higher-order corrections, but they give contributions of different orders to $\boldsymbol{\sigma}^\infty_T$. Whereas D^∞_{pp} gives contributions of order 1 because it appears multiplied by a factor $1/\gamma_0$, D^∞_{xp} gives contributions of order γ_0. That is why the correct thermalization in the weak-coupling limit was obtained in Ref. [4] despite having completely neglected the anomalous diffusion coefficient. The origin of the

mixed orders in γ_0 appearing on the right-hand side of Eq. (IV.53) can be ultimately traced to the fact that $\mathbf{H}$ contains terms both of order 1 and γ_0, whose implication for $\boldsymbol{\sigma}_T^\infty$ can be straightforwardly seen from the Lyapunov equation (III.30).

The product of the two uncertainties can be expanded for high temperature and large cut-off as

$$(\Delta x)^2(\Delta p)^2 = \left(\frac{T}{\Omega}\right)^2 + \mathcal{O}\big(T\log(\Lambda)\big). \tag{IV.88}$$

Inspecting the terms in powers of T and Λ immediately reveals the high-temperature result of classical statistical mechanics for the case of an ohmic spectrum. One can also see that at weak coupling the uncertainty function agrees with the weak coupling approximation for moderate values of the cut-off scale, as shown in Fig. 7.

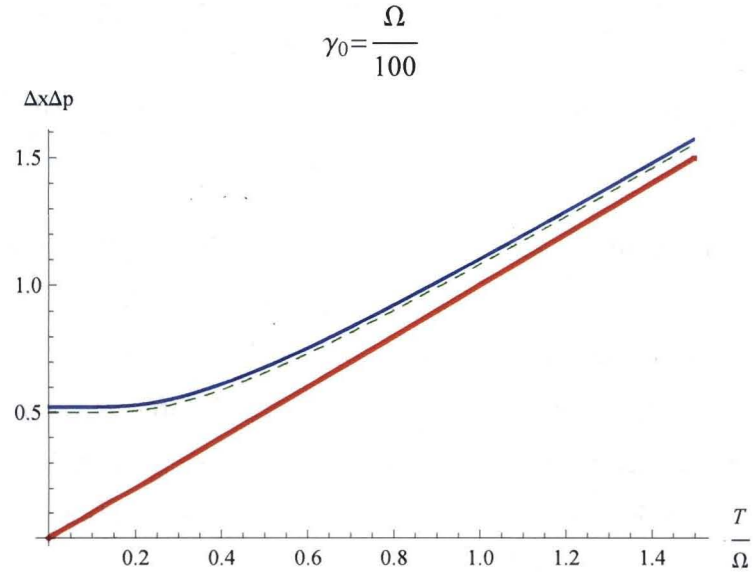

FIG. 7: Late time $\Delta x\Delta p$ for • high temperature, classical statistical mechanics, $\cdots$ weak coupling approximation $\frac{1}{2}\coth\frac{\Omega}{2T}$ · HPZ at $\Lambda = 10^3\Omega$ and $\Lambda = 10^4\Omega$.

Had one naively tried to have finite diffusion coefficients in the limit $\Lambda \to \infty$ subtracting by hand the $\log(\Lambda/\Omega)$ term, one would find a violation of the Heisenberg uncertainty principle at low temperature and strong coupling (see Fig. 8), which renders the theory unphysical. Of course this does not happen with the unsubtracted theory, as seen in Fig. 9. It is thus clear that the logarithmic dependence on the ultraviolet cut-off that appears in the diffusion coefficients is a physically important parameter and not something that can be subtracted away.

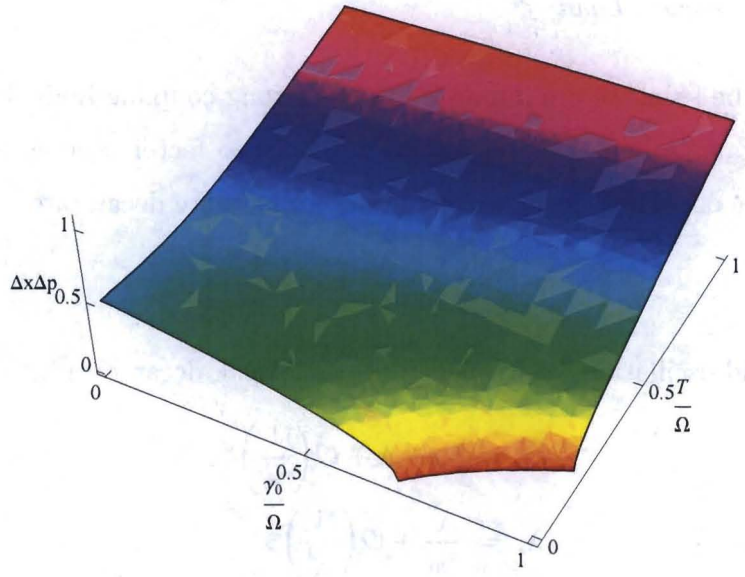

FIG. 8: Late time $\Delta x \Delta p$ for the unphysical, subtracted theory.

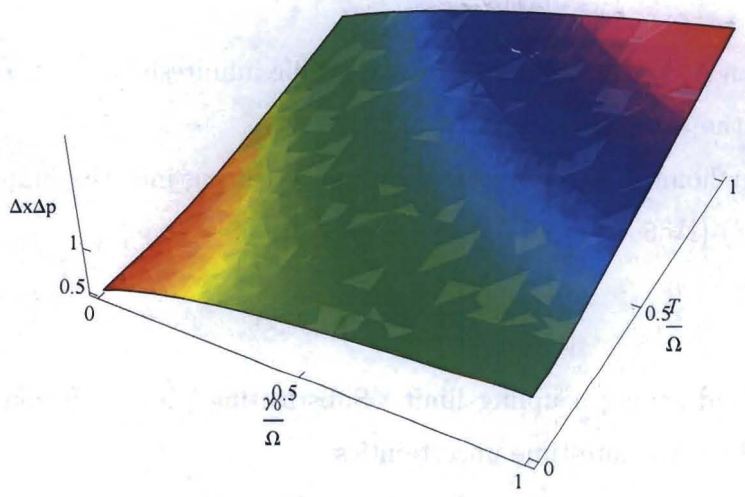

FIG. 9: Late time $\Delta x \Delta p$ for the $\Lambda = 10^3 \Omega$ theory.

Substituting Eqs. (IV.86)-(IV.87) into Eqs. (IV.54)-(IV.55), one can see that only the momentum uncertainty contains a logarithmically divergent cut-off dependence. In contrast, the position uncertainty is much smaller and finite in the limit $\Lambda \to \infty$ (this had already been noticed for Gaussian wave-packets in Ref. [5]). The anomalous diffusion coefficient actually acts as an "anti-diffusion" term in doing this.

5. The Strong Coupling Limit

For reasons to be seen, we are interested in the strong coupling limit, but with the cut-off still finite. Again, solving this means we just have to factor a cubic polynomial. The timescales here are of a different nature. There is a moderate decay rate

$$\Lambda_\star = \Lambda - 2\frac{\Lambda^2}{\gamma_0} + \mathcal{O}\left(\frac{1}{\gamma_0^2}\right), \tag{IV.89}$$

and there are rapid oscillations of frequency $\gamma_\star$, with weak decay rate $\lambda_\star$

$$\gamma_\star = \gamma_0 + \Lambda + \mathcal{O}\left(\frac{1}{\gamma_0}\right), \tag{IV.90}$$

$$\lambda_\star = \frac{\Lambda^2}{\gamma_0} + \mathcal{O}\left(\frac{1}{\gamma_0^2}\right). \tag{IV.91}$$

The propagator, in the strong coupling limit, takes the familiar form

$$G(t) = \frac{1}{M\gamma_\star}\sin(\gamma_\star t)\, e^{-\lambda_\star t}, \tag{IV.92}$$

$$\Gamma = \lambda_\star, \quad \Omega_{\mathrm{R}} = \gamma_\star, \tag{IV.93}$$

for times later than $\Lambda_\star^{-1}$. The the decay rate $\lambda_\star$, while infinitesimal, is necessary for convergence in deriving the thermal state.

Now taking our homogeneous evolution and substituting into the finite cutoff diffusion coefficients, (IV.68)-(IV.69) we obtain:

$$D_{xp}^{\infty} = 0 \quad D_{pp}^{\infty} = \Lambda^2 \tag{IV.94}$$

in the late time and strong coupling limit. Substituting these diffusion coefficients into (IV.53), we then have the late-time uncertainties

$$(\Delta x)^2 = \frac{1}{2}\frac{1}{M\gamma_0}\left(1 + \mathcal{O}\left(\frac{\Lambda}{\gamma_0}\right)\right), \tag{IV.95}$$

$$(\Delta p)^2 = \frac{1}{2}M\gamma_0\left(1 + \mathcal{O}\left(\frac{\Lambda}{\gamma_0}\right)\right), \tag{IV.96}$$

Therefore, for this model of strong coupling to the environment, the Brownian particle will become arbitrarily localized in position at late time. But note that although the particle is localized in position, the uncertainty principle is not violated; in fact, it is minimized. Also interesting here is that the localization could happen slowly.

D. Example Spectra: Analytic with High Cufoff

In this section we consider the late-time and high cutoff limiting evolution of analytic supra-ohmic spectra, analytic monotonically increasing spectra, and the high frequency contribution of more general analytic spectra. By *analytic spectra* we refer to the special class of environmental couplings whose spectral density $I(\omega)$ and damping kernel $\hat{\gamma}(s)$ are both analytic, possibly up to a finite number of simple poles in the complex plane. Not any analytic function can be used for the spectral density. The spectral density must also be odd to correspond to an analytic damping kernel, as can be seen from Eq. (II.16). This class of spectral functions generates a restricted class of functions for the damping kernel: not any analytic function can be used for the damping kernel. For instance, $\hat{\gamma}(s) = s$ does not correspond to any analytic spectral density, nor does any odd function, although the damping kernel is not restricted to even functions.

Next we consider what limiting behavior our spectral density must have. At low frequency, the spectral density must be at least proportional to ω in order to counteract the reciprocal $\sinh(\omega/2T)$ of the noise kernel so that there is no infrared (IR) divergence in the frequency integrals for the diffusion coefficients. In fact, one could consider spectral densities down to $I(\omega) \propto \omega^0$ at low frequency with some IR cut-off and this would only result in a logarithmic dependence on the cut-off. However, the corresponding damping kernel would not be analytic and we will not consider this case. At high frequency, since the spectral density is analytic, it must at least fall off to a constant for the diffusion coefficient integrals to converge like $\int^\infty d\omega/\omega^2$. Moreover, since a constant cannot be an odd function, within the kind of spectral distributions that we are considering, it must at least vanish. And physically one would want the coupling to vanish at arbitrarily high frequency regardless.

Let us inspect the high frequency contribution to the diffusion coefficients associated with a spectral density of this kind which includes ohmic plus supra-ohmic contributions. The midrange behavior, corresponding to frequencies above the temperature T and Ω, and up to the cut-off scale Λ, is where the dominant contribution from individual supra-ohmic terms, and more generally any monotonically increasing spectral density, will come from [assuming that there is some regulator factor, such as $1/(1+\omega^2/\Lambda^2)$ or $\exp(-\omega^2/\Lambda^2)$, which guarantees the convergence of the integrals in the infinite frequency limit]. Hence, to leading order in Λ

the high frequency contribution to the late-time diffusion coefficients (Sec. IV B 4) becomes

$$\Delta D_{xp}^{\infty} = \frac{2}{\pi}\int^{\Lambda} d\omega \frac{\mathrm{Re}[\hat{\gamma}(\imath\omega)]}{\omega} + \cdots, \tag{IV.97}$$

$$\Delta D_{pp}^{\infty} = 2\Gamma\,\Delta D_{xp}^{\infty} + \cdots, \tag{IV.98}$$

where we have discarded small terms that will not contribute to leading order in Λ, and we did not explicitly include the regulator since it only becomes important for frequencies larger than Λ, where it suppresses the integrand with increasing strength (completely doing so in the limit $\omega \to \infty$).

If we do not want the diffusion coefficients to be highly sensitive to the cut-off, we will need to suppress supra-ohmic contributions by some factors of $1/\Lambda$. Recasting the damping kernel as a potentially infinitesimal function of frequency but finite function of the ratio of frequency to the cut-off scale, $\hat{\gamma}(s/\Lambda)$, will make these integrals fairly insensitive to the cut-off.

$$\Delta D_{xp}^{\infty} = \frac{2}{\pi}\int^{\Lambda} d\omega \frac{\mathrm{Re}\left[\hat{\gamma}\left(\imath\frac{\omega}{\Lambda}\right)\right]}{\omega} + \cdots, \tag{IV.99}$$

and now one has in the high cutoff limit $\Omega_{\mathrm{R}} = \Omega$ and $\Gamma = \hat{\gamma}(0)$ as the supra-ohmic terms can no longer significantly affect the homogeneous dynamics. Any less suppression would cause considerable cut-off sensitivity and any more suppression would give no significant contribution. The ohmic terms must be integrated in their entirety since their leading-order contributions are $\mathcal{O}(\log\Lambda)$ and their next-order terms are still $\mathcal{O}(1)$. However, for the supra-ohmic terms only their leading-order terms need to be computed in the large cut-off limit.

Given some dominant midrange behavior

$$\mathrm{Re}\left[\hat{\gamma}\left(\imath\frac{\omega}{\Lambda}\right)\right] \approx \gamma_0 + \sum_{n=1}^{\infty}\gamma_{2n}\left(\frac{\omega}{\Lambda}\right)^{2n}, \tag{IV.100}$$

with the assumption that there is also a regulator strong enough to make the integrals converge , the high frequency, supra-ohmic contributions to the late time diffusion coefficients in the large cut-off limit become

$$\Delta D_{xp}^{\text{s.o.}}(\infty) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\gamma_{2n}}{2n} + \mathcal{O}\left(\frac{1}{\Lambda}\right), \tag{IV.101}$$

$$\Delta D_{pp}^{\text{s.o.}}(\infty) = 2\gamma_0\Delta D_{xp}^{\text{s.o.}}(\infty) + \mathcal{O}\left(\frac{1}{\Lambda}\right), \tag{IV.102}$$

where we did not include the γ_0 term as it corresponds to the $\log \Lambda$ divergence already included in our expressions for the ohmic case, but we did include all other terms. And we again note that for individual supra-ohmic contributions, or more generally monotonically increasing spectral densities, these are the dominant contributions in the high cutoff limit. If the spectral density is not monotonically increasing, then the lower frequency contributions can also be significant.

E. Example Spectra: Sub-Ohmic with no Cutoff

As a final example, we will consider the most common and well behaved sub-ohmic spectral density, $I(\omega) \propto \sqrt{\omega}$, which requires no cutoff, high or low. We will express our spectral density suggestively with two timescales as

$$I(\omega) = \frac{2}{\pi} M \Gamma \sqrt{\omega_0 \, \omega}, \tag{IV.103}$$

where ω_0 will be chosen such that Γ is the appropriate late-time master equation coefficient, but for now it is merely a parameter. It is then a straightforward calculation to find the propagator to be

$$\hat{G}(s) = \frac{\frac{1}{M}}{s^2 + 2\Gamma\sqrt{2\omega_0 \, s} + \Omega^2}, \tag{IV.104}$$

which is amenable to partial fraction decomposition as s is strictly positive. If we let $\omega_0^2 = \Omega^2 + \Gamma^2$, the roots of the quartic denominator $\sqrt{s_k}$ can then be shown to be the conjugate pairs $\frac{1}{\sqrt{2}}\left(+\sqrt{\omega_0} \pm \imath\sqrt{\omega_0 + 2\Gamma}\right)$ and $\frac{1}{\sqrt{2}}\left(-\sqrt{\omega_0} \pm \imath\sqrt{\omega_0 - 2\Gamma}\right)$. After partial fraction decomposition, we then have the propagator

$$\hat{G}(s) = \sum_{k=1}^{4} \frac{A_k}{M} \frac{1}{\sqrt{s} - \sqrt{s_k}}, \tag{IV.105}$$

with inverse Laplace transform

$$G(t) = \sum_{k=1}^{4} \frac{A_k}{M} \sqrt{s_k} \, e^{s_k t} \operatorname{erfc}\left(-\sqrt{s_k t}\right), \tag{IV.106}$$

where $\operatorname{erfc}(z)$ is the cumulative error function of the normal distribution, i.e. $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty d\zeta \, e^{-\zeta^2}$. There are additional terms from the individual Laplace transforms like $t^{-1/2}$, but they vanish in the sum.

The error function has a well known asymptotic expansion, e.g.

$$e^z \mathrm{erfc}(-\sqrt{z}) = 2e^z - \frac{1}{\sqrt{\pi z}} + \frac{1}{2\sqrt{\pi z^3}} - \cdots \quad \text{for} \quad z \gg 1, \tag{IV.107}$$

which we can use to expand the Green function asymptotically as discussed in Sec. IV B 1. From $G_\infty(t)$ we can extract the late time dynamics: the dissipation rate is Γ by construction

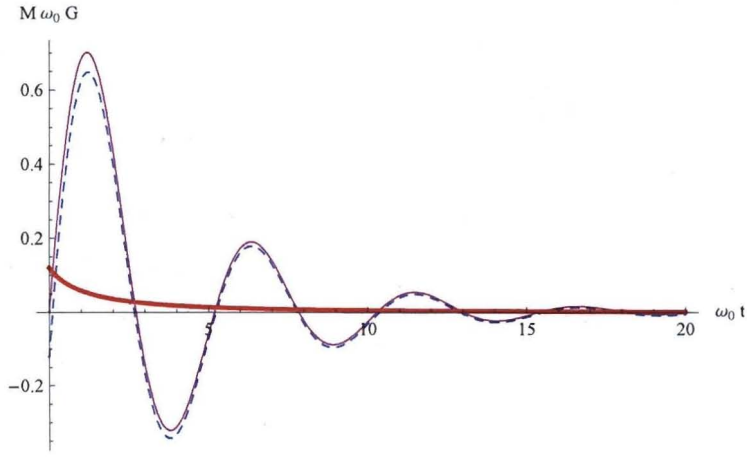

FIG. 10: Late time expansion of sub-ohmic · propagator $G(t)$ into ··· asymptotic $G_\infty(t)$ and • early time contribution $\Delta G(t)$ for $\Gamma = \frac{\omega_0}{4}$. The early time contribution $\Delta G(t)$ is decisively anharmonic.

and $\Omega_R = \omega_0 + \Gamma$. Therefore we have in the weak coupling limit $\Omega_R = \Omega + \mathcal{O}(\Gamma)$ as expected. Also note that $f = -\Gamma \pm \imath\sqrt{\omega_0(\omega_0 + 2\Gamma)}$ is the solution to the characteristic rate equation (IV.33) with smallest *negative* real part, even though this equation was only proven for analytic spectra.

Now that we have the late-time $\mathbf{H}$, we can more easily compute the late-time diffusion coefficients from Eqs. (IV.50)-(IV.51). The resulting expressions are no longer as simple as in the case with analytic spectra; we only report the late-time zero temperature coefficients, but a high temperature expansion is also possible.

$$D_{xp} = +\frac{\Gamma}{\sqrt{\Omega_R + \Gamma}} \mathrm{Re}\left[\sqrt{+\tilde{\Omega}_R + \imath\Gamma} - \sqrt{-\tilde{\Omega}_R + \imath\Gamma}\right], \tag{IV.108}$$

$$D_{pp} = -\frac{\Gamma\,\tilde{\Omega}_R}{\sqrt{\Omega_R + \Gamma}} \mathrm{Im}\left[\sqrt{+\tilde{\Omega}_R + \imath\Gamma} + \sqrt{-\tilde{\Omega}_R + \imath\Gamma}\right] + \frac{\Gamma^2}{\sqrt{\Omega_R + \Gamma}} \mathrm{Re}\left[\sqrt{+\tilde{\Omega}_R + \imath\Gamma} - \sqrt{-\tilde{\Omega}_R + \imath\Gamma}\right]. \tag{IV.109}$$

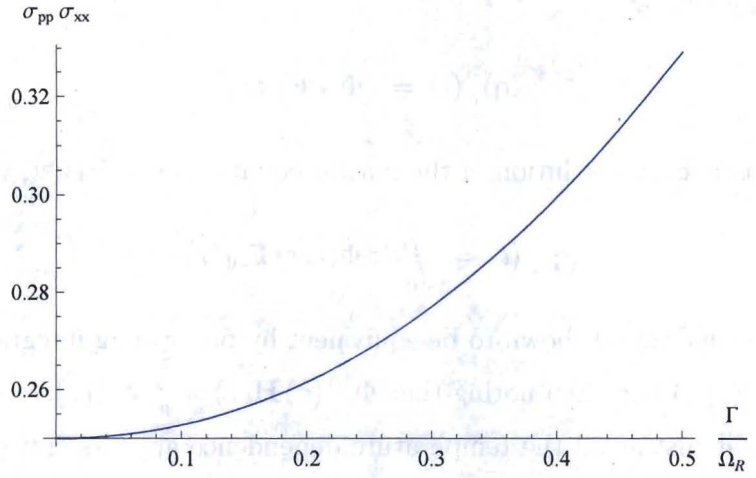

FIG. 11: Late-time sub-ohmic uncertainty function at zero temperature. In the weak coupling limit, one has the minimal uncertainty ground state (zero temperature thermal state) as expected.

V. GENERALIZATIONS OF THE THEORY

A. Influence of a Classical Force

In this section we consider the case of a classical force $F(t)$ acting on the quantum oscillator. This is done by adding a time-dependent potential $-F(t)x$ to the system Lagrangian:

$$L_{\rm s} = \frac{1}{2}M\left(\dot{x}^2 - \Omega^2 x^2\right) + F(t)x. \tag{V.1}$$

Following our master equation derivation in Sec. III B, it is easy to show that a classical force in the Langevin equation adds a forcing term $\mathbf{F}_{\rm eff}(t)$ to the master equation

$$\frac{\partial}{\partial t}W_{\rm r}(\mathbf{q};t) \;=\; \left\{\boldsymbol{\nabla}_{\mathbf{q}}^{\rm T}\,\mathbf{H}(t)\,\mathbf{q} - \boldsymbol{\nabla}_{\mathbf{q}}^{\rm T}\,\mathbf{F}_{\rm eff}(t) + \boldsymbol{\nabla}_{\mathbf{q}}^{\rm T}\,\mathbf{D}(t)\,\boldsymbol{\nabla}_{\mathbf{q}}\right\}W_{\rm r}(\mathbf{q};t)\,, \tag{V.2}$$

where the effective force is given by

$$\mathbf{F}_{\rm eff}(t) \;\equiv\; \mathbf{F}(t) + \int_0^t d\tau\left\{\left[\frac{d}{dt} + \mathbf{H}(t)\right]\boldsymbol{\Phi}(t-\tau)\right\}\mathbf{F}(\tau), \tag{V.3}$$

and this second term vanishes for local dissipation.

Following our solution via the Langevin equation in Sec. II C, it is even easier to show that the solutions take the form

$$\mathcal{W}_{\rm r}[t,\mathbf{k}] = \mathcal{W}_{\rm r}\left[0,\boldsymbol{\Phi}^{\rm T}(t)\mathbf{k}\right]e^{-\frac{1}{2}\mathbf{k}^{\rm T}\boldsymbol{\sigma}_T(t)\mathbf{k}}\,e^{-i\mathbf{k}^{\rm T}\langle\mathbf{q}\rangle_F(t)}, \tag{V.4}$$

with forced mean given by

$$\langle \mathbf{q} \rangle_F(t) \;=\; (\mathbf{\Phi} * \mathbf{F})(t), \tag{V.5}$$

while a characteristic curves solution of the master equation, Sec. III C 1, would suggest.

$$\langle \mathbf{q} \rangle_F(t) \;=\; \int_0^t d\tau\, \mathbf{\Phi}(t,\tau)\, \mathbf{F}_{\text{eff}}(\tau). \tag{V.6}$$

These two expressions can be shown to be equivalent by performing integration by parts on the derivative in $\mathbf{F}_{\text{eff}}(t)$ and then noting that $\mathbf{\Phi}^{-1}(\tau)\,\mathbf{H}(\tau) = \frac{d}{dt}\mathbf{\Phi}^{-1}(\tau)$.

One can see that just as all the temperature dependence appears entirely in the second cumulant, or covariance, the external force only affects the first cumulant, or mean. Eq. (V.2) shows that the mean, $\langle \mathbf{q} \rangle(t)$, is shifted by $\langle \mathbf{q} \rangle_F(t)$, which characterizes the response to the driving force. In fact, using Eq. (II.19) one can immediately see that it corresponds to shifting $\langle x \rangle$ and $\langle p \rangle$ respectively by $(G * F)(t)$ and $(M\dot{G} * F)(t)$, as one would expect.

B. N-Oscillator Master Equation

Here we wish to show that, as we have used a very robust matrix notation, our master equation and solutions trivially generalize to multiple system oscillators $\mathbf{x}$ with arbitrarily bilinear coupling to themselves and the bath oscillators $\mathbf{x}_\text{b}$. We first generalize the system-bath interaction Hamiltonian to the general bilinear form

$$H_{\text{SB}} \;=\; \mathbf{x}^{\text{T}}\, \mathbf{c}\, \mathbf{x}_\text{b}, \tag{V.7}$$

where $\mathbf{c}$ connects system positions in the first index to bath positions in the second index.

The effect is a simple generalization of the spectral density and related kernels

$$I_{ij}(\omega) \;=\; \sum_k \delta(\omega - \omega_k)\, \frac{c_{ik} c_{jk}}{2 m_k \omega_k}, \tag{V.8}$$

$$\boldsymbol{\nu}(t,\tau) \;=\; \int_0^\infty d\omega\, \mathbf{I}(\omega) \coth\!\Big(\frac{\omega}{2T}\Big) \cos[\omega(t-\tau)]\,, \tag{V.9}$$

$$\mathbf{M}\boldsymbol{\gamma}(t,\tau) \;=\; \int_0^\infty d\omega \frac{\mathbf{I}(\omega)}{\omega} \cos[\omega(t-\tau)]\,, \tag{V.10}$$

Our master equation and solution naturally extend to this more general scenario, although solving for $\mathbf{\Phi}(t)$ can be exceedingly difficult analytically.

The spectral density must be a positive definite matrix as its diagonal must be positive in all coordinate systems. Technically, the indices of the spectral density matrix only apply to system momenta and skip system position as is the case for one system oscillator. The noise and damping kernels also operate strictly between system momenta. We leave more thorough discussion to future work.

VI. SUMMARY OF RESULTS

Quantum Brownian motion of an oscillator coupled to a thermal reservoir of quantum oscillators has been the canonical model for studying the environmental effects on a quantum system, even of macroscopc scale, such as quantum dissipation, diffusion, decoherence and entanglement. It also provides important information on quantum measurement, such as noise, fluctuations, correlations, uncertainty relation and standard quantum limit in mesoscopic systems. Many experiments have been carried out for testing these processes. Fifteen years ago an exact master equation [12] was claimed to have been found for the reduced density matrix of the system with a general environment of arbitrary spectral density and temperature. Subsequently there have been claims of exact solutions [16]. We have found these previous derivations to be correct for local dissipation, but containing errors or omissions for nonlocal dissipation; in their place we have presented the most complete and correct derivation of the QBM master equation to date. In this paper we report on solutions to this equation for a fairly general set of physical conditions and a generalization of the QBM master equation to higher dimensional systems. Previous derivations required one to solve integro-differential equations whereas we have reduced everything to quadrature. We expect these results to be useful in realistic settings for the analysis of many problems which can be described by this model.

We have calculated the master equation coefficients and addressed concerns over their potentially divergent behaviors. An initial kick to the system occurs from cutoff sensitive frequency renormalization. This can be avoided by a more well behaved choice of system-environment coupling, for which the phenomenological system frequency never changes. Another initial kick to the system occurs with misapplication of local propagators to the initial state, to which they should never apply. Both of these kicks can be viewed as a distorted view of the initial state which maps pure states to pure states and is thus equivalent.

The final initial time divergences are the initial jolts in the diffusion coefficients and corresponding thermal covariance which were thought to be responsible for rapid decoherence. But said jolts occur at the short timescale, which is exactly where the dynamics are strictly nonlocal. In our correct nonlocal treatment, we find that significant jolting only occurs in the growth of the momentum uncertainty (Sec IV A 3). Lastly, for ohmic spectra, there is the logarithmic cutoff sensitivity in the diffusion coefficients, which we also find only to affect the momentum uncertainty. This divergence cannot be consistently subtracted and conditions the possibility of obtaining meaningful results upon the existence of a physically well-motivated cut-off.

For ohmic spectra, we have the master equation coefficients for all temperature ranges, beyond weak coupling, and for finite cutoff at all times beyond the cutoff timescale. This has enabled us to clarify the validity and shortcomings of previously obtained approximations and provide more general results. Perhaps the most useful are our simple expressions for the master equation coefficients at late time [Eq. (IV.68)-(IV.69)] and Sec. IV C 3. We additionally consider supra-ohmic spectra in the high cutoff limit and the sub-ohmic spectra $I(\omega) \propto \sqrt{\omega}$. More generally we have reduced everything to quadrature or more simply contour integration for analytic spectra. All initial states evolve into a given Gaussian state corresponding to thermal equilibrium. We have obtained the covariance matrix for this state and the corresponding uncertainty function for local dissipation (Sec. IV C 4) and also nonlocal dissipation. As one might expect, arbitrarily strong coupling to the reservoir will tend to localize the position with arbitrary precision.

We have generalized the QBM master equation to include the influence of classical forces. This modifies the dynamics by driving the mean position and momentum just as with a classical driven system. In this model we found that the force has no effect upon the width of the wave-packet or any cumulant other than the mean. These results may be useful for the study of low-temperature measurements of forced oscillators, which are relevant for experiments with nanomechanical resonators [23, 24]. They also play a crucial role in future schemes for the detection of gravitational waves with high-intensity laser interferometers, where the radiation pressure effects on the cavity mirrors are important [25, 26].

Finally, we have extended the model of one quantum oscillator bilinearly coupled to a thermal reservoir of oscillators to a model of multiple oscillators bilinearly coupled to themselves and the bath in an arbitrary fashion. With this generalization, the potential for

application becomes almost endless and we leave further study to the future research.

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APPENDIX A: SPECIAL FUNCTIONS

1. Harmonic Number

The Harmonic Number $\mathrm{H}(n)$ is a function similar to a logarithm, whose definition and main properties are

$$\mathrm{H}(n) = \sum_{k=1}^{n} \frac{1}{k}, \qquad n \in \mathbb{Z}^{+} \tag{A.1}$$

$$\mathrm{H}(0) = 0, \tag{A.2}$$

$$\gamma_{\mathrm{E}} = \lim_{n\to\infty} (\mathrm{H}(n) - \log(n)), \tag{A.3}$$

where γ_{E} is known as the Euler-Mascheroni constant. Its generalization to the complex plane exhibits similar properties and is given by

$$\mathrm{H}(z) = \gamma_{\mathrm{E}} + \psi(z+1), \qquad z \in \mathbb{C}, \tag{A.4}$$

where $\psi(z)$ is the digamma function, defined as

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{A.5}$$

It satisfies the recurrence relation

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \tag{A.6}$$

and its Taylor expansion near 1 as well as its asymptotic expansion for $|z| \to \infty$ are given respectively by

$$\psi(z+1) = -\gamma_{\mathrm{E}} + \sum_{k=1}^{\infty} \zeta(k+1)(-z)^k, \tag{A.7}$$

$$\psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \cdots \quad \text{if} \quad |\arg(z)| < \pi, \tag{A.8}$$

where $\zeta(n)$ is the Riemann zeta function.

2. Exponential Integral

The exponential integral is a special function asymptotically like $\frac{e^{-z}}{z}$. The following formula are true for $|\arg(z)| < \pi$

$$\mathrm{E}_1(z) = \int_z^\infty \frac{e^{-z'}}{z'} dz', \tag{A.9}$$

$$\mathrm{E}_1(z) = \frac{e^{-z}}{z + \frac{1}{1+\frac{1}{z+\cdots}}}, \tag{A.10}$$

and along $|\arg(z)| = \pi$ the exponential integral has a branch cut.

APPENDIX B: SOME PROPERTIES OF LAPLACE TRANSFORMS

Given a real function $f(t)$, defined for all real numbers $t \geq 0$, its Laplace transform is defined as

$$\hat{f}(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt. \tag{B.1}$$

The main properties used in the paper are the following. First, the Laplace transform of a derivative is given by

$$\mathcal{L}\left\{\dot{f}(t)\right\}(s) = s\hat{f}(s) - f(0). \tag{B.2}$$

Second, multiplying $f(t)$ by an exponential corresponds to a translation of the Laplace transform:

$$\mathcal{L}\left\{e^{at} f(t)\right\}(s) = \hat{f}(s-a). \tag{B.3}$$

Third, if the inverse Laplace transform of $\hat{f}(s)$ is $f(t)\,\theta(t)$, multiplying $f(s)$ by an exponential corresponds to a translation of the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{e^{as}\hat{f}(s)\right\}(s) = f(t+a)\,\theta(t+a). \tag{B.4}$$

Fourth, the Laplace transform of a *Laplace convolution* is given by the product of the Laplace transforms:

$$\mathcal{L}\{(f*g)(t)\}(s) = \hat{f}(s)\,\hat{g}(s), \tag{B.5}$$

where

$$(f*g)(t) = \int_0^t dt' f(t-t')g(t'). \tag{B.6}$$

Fifth, the *initial value theorem* relates the initial value of a function $f(t)$ and the infinite limit of its Laplace transform as follows:

$$f(0^+) = \lim_{s\to\infty} s\hat{f}(s). \tag{B.7}$$

Sixth, the *final value theorem* relates the infinite limit of a function $f(t)$ and the initial value of its Laplace transform as follows:

$$f(\infty) = \lim_{s\to 0} s\hat{f}(s), \tag{B.8}$$

provided that all the poles of $\hat{f}(s)$ are on the $\mathrm{Re}(s) < 0$ half of the s complex plane. Seventh, the inverse Laplace transform of $\hat{f}(s)$ can be calculated using *Bromwich's integral*, which involves an analytical continuation of $\hat{f}(s)$ in the complex plane:

$$f(t) = \mathcal{L}^{-1}\{\hat{f}(s)\}(s) = \int_{\alpha-\imath\infty}^{\alpha+\imath\infty} e^{st}\hat{f}(s)dt, \tag{B.9}$$

where α is a real number chosen so that the integration path lies within the region of convergence of $\hat{f}(s)$, i.e., $\alpha > s_j$ for every singularity s_j of $\hat{f}(s)$.

Bromwich's integral illustrates the close relationship between the Laplace transform and the Fourier transform through analytical continuation. However, even if all the singularities of $\hat{f}(s)$ lie on the $\mathrm{Re}(s) < 0$ half of the complex plane, the relation is not direct because the Laplace transform involves an integral with domain $[0, \infty)$ rather than $(-\infty, \infty)$. The precise relationship can be understood as follows. Consider a real function $f(t)$ defined for all real values of t and whose Fourier transform is $\tilde{f}(\omega)$. It is useful to define the following additional Fourier transforms:

$$\tilde{f}_\pm(\omega) = \int_{-\infty}^{\infty} dt\, e^{-\imath\omega t} f(t)\, \theta(\pm t)\,, \tag{B.10}$$

such that $\tilde{f}(\omega) = \tilde{f}_+(\omega) + \tilde{f}_-(\omega)$ and which satisfy the property $\tilde{f}_\pm(-\omega) = \big(\tilde{f}_\pm(\omega)\big)^*$ since $f(t)$ is real. Assuming that the Laplace transform $\hat{f}(s)$ has no singularities for $\mathrm{Re}(s) > 0$, it can be related by analytical continuation to $\tilde{f}_+(\omega)$:

$$\tilde{f}_+(\omega) = \lim_{\epsilon\to 0} \hat{f}(\epsilon + \imath\omega). \tag{B.11}$$

If $f(t)$ is an even function, one has $\tilde{f}_-(\omega) = \tilde{f}_+(-\omega)$, and using Eq. (B.11) one can then write

$$\tilde{f}(\omega) = \tilde{f}_+(\omega) + \tilde{f}_+(-\omega) = \lim_{\epsilon\to 0}\Big(\hat{f}(\epsilon + \imath\omega) + \hat{f}(\epsilon - \imath\omega)\Big). \tag{B.12}$$

Similarly, if $f(t)$ is an even function, one has $\tilde{f}_-(\omega) = -\tilde{f}_+(-\omega)$, which implies

$$\tilde{f}(\omega) = \tilde{f}_+(\omega) - \tilde{f}_+(-\omega) = \lim_{\epsilon \to 0} \Big(\hat{f}(\epsilon + \imath\omega) - \hat{f}(\epsilon - \imath\omega) \Big). \tag{B.13}$$

APPENDIX C: SYSTEM-ENVIRONMENT INTERACTION AND RENORMALIZATION

1. Divergent Contributions from the Dissipation Kernel

For any reasonable spectral function $I(\omega)$ (which does not give raise to IR divergences and grows at most like a power-law at high frequencies) the dissipation kernel

$$\mu(\tau) = -\int_0^\infty d\omega \sin(\omega\tau) I(\omega), \tag{C.1}$$

is well defined even in the infinite cut-off limit (at least as a distribution). However, its contribution to the equation of motion (II.5) involves a multiplication by a step function and, even if $\mu(\tau)$ is a well-defined distribution, in general $\mu(\tau)\theta(\tau)$ is not necessarily a well-defined one [27]. This will also imply that the Laplace transform $\hat{\mu}(s)$ and the Fourier transforms $\tilde{\mu}_\pm(\omega)$ [as defined in Eq. (B.10)] are divergent even though $\tilde{\mu}(\omega)$ is finite. Indeed, from Eq. (C.1)] one can easily calculate the Laplace transform

$$\hat{\mu}(s) = -\int_0^\infty d\omega\, I(\omega) \frac{\omega}{\omega^2 + s^2}. \tag{C.2}$$

which is in general divergent in the limit $\Lambda \to \infty$. Nevertheless, those divergences can be cancelled out by an appropriate choice of the bare frequency and one can define a finite renormalized Laplace transform of the dissipation kernel as follows:

$$\hat{\mu}_{\rm ren}(s) = \hat{\mu}(s) + M\delta\Omega^2, \tag{C.3}$$

$$\delta\Omega^2 = \frac{1}{M}\hat{\mu}(0) = \frac{1}{M}\int_0^\infty d\omega\, \frac{I(\omega)}{\omega}, \tag{C.4}$$

$$\hat{\mu}_{\rm ren}(s) = \int_0^\infty d\omega \frac{I(\omega)}{\omega} \frac{s^2}{\omega^2 + s^2}. \tag{C.5}$$

Since $\hat{\mu}(s) + M\Omega^2_{\rm bare} = \hat{\mu}_{\rm ren}(s) + M\Omega^2$ (with $\Omega^2_{\rm bare} = \Omega^2 + \delta\Omega^2$), the integro-differential equation (II.5) can be equivalently rewritten in terms of the renormalized quantities.

Note that in principle supra-ohmic spectral densities with sufficiently high powers of ω would require the renormalization of the coefficients of higher-order derivative terms in

addition to the frequency (corresponding to terms with even positive powers of s in Laplace space). However, as seen in Sec. IV D, requiring that the diffusion coefficients do not contain contributions proportional to positive powers of the cut-off restricts the coefficients of the higher frequency terms of the spectral density (which must be suppressed by sufficiently high powers of the cut-off) in such a way that those renormalizations are no longer necessary.

The damping kernel $\gamma(\tau)$ is defined by

$$\mu(\tau) = M\frac{\partial}{\partial t}\gamma(\tau), \tag{C.6}$$

and using Eq. (C.1) it can be written as

$$\gamma(\tau) = \frac{1}{M}\int_0^\infty d\omega \frac{I(\omega)}{\omega}\cos(\omega\tau). \tag{C.7}$$

In Laplace space Eq. (C.6) becomes

$$\hat{\mu}(s) = Ms\hat{\gamma}(s) - M\gamma(0). \tag{C.8}$$

Taking into account that $\gamma(0) = \delta\Omega^2$, as follows from Eq. (C.7), we finally have

$$\hat{\mu}_{\rm ren}(s) = Ms\,\hat{\gamma}(s). \tag{C.9}$$

2. Frequency Renormalization

When studying linear QBM models, the following Lagrangian is usually considered:

$$\mathcal{L} = \frac{1}{2}M\left(\dot{x}^2 - \Omega_{\rm bare}^2 x^2\right) + \sum_n \frac{1}{2}m_n\left(\dot{x}_n^2 - \omega_n^2 x_n^2\right) - \sum_n c_n x x_n. \tag{C.10}$$

The corresponding equation of motion for the system oscillator is

$$\ddot{x} + 2\gamma * \dot{x} + \left(\Omega_{\rm bare}^2 - \delta\Omega^2\right)x = 0 \quad \left(t \gg \Lambda^{-1}\right), \tag{C.11}$$

where $\delta\Omega^2$ is the cut-off sensitive frequency renormalization defined in Eq. (C.4). For an ohmic distribution it is roughly proportional to the coupling strength and high-frequency cut-off Λ. Therefore, in order for the coupled system oscillator to have a frequency Ω fairly insensitive to the value of the cut-off in the large cut-off limit, one needs to start with a bare frequency for the isolated oscillator given by $\Omega_{\rm bare}^2 = \Omega^2 + \delta\Omega^2$, which diverges linearly in Λ when $\Lambda \to \infty$. Note that Eq. (C.11) is valid for times much larger than Λ^{-1} (a detailed analysis for early times is provided in the next subsection).

Alternatively one can consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}M\left(\dot{x}^2 - \Omega^2 x^2\right) + \sum_n \frac{1}{2}m_n\left(\dot{x}_n^2 - \omega_n^2\left(x_n - \frac{c_n}{m_n\omega_n^2}x\right)^2\right), \tag{C.12}$$

which can be written in the same form as Eq. (C.11), but where the square in the last term gives a contribution corresponding to $-(1/2)M\delta\Omega^2 x^2$. Thus, if one views the whole sum on the right-hand side of Eq. (C.12) as part of the environment Lagrangian including the system-environment interaction, one should no longer consider a bare frequency for the isolated oscillator, whose frequency now coincides with that of the coupled oscillator. Indeed, the equation of motion for the system oscillator derived from Eq. (C.12) (for times much larger than Λ^{-1}) is given by

$$\ddot{x} + 2\gamma * \dot{x} + \Omega^2 x \;=\; 0 \quad \left(t \gg \Lambda^{-1}\right). \tag{C.13}$$

3. Initial Time Divergences, Coupling Switch-on and Initial State Renormalization

a. Initial Time Divergences and Coupling Switch-on

The derivation of the HPZ master equations relies upon the key assumption that the system and environment are initially uncorrelated. For an ohmic environment, this gives rise to initial "jolts" in the normal diffusion coefficient of the master equation with a characteristic time-scale of order Λ^{-1} and an amplitude proportional to Λ. Similarly, without considering an interaction such as in Eq. (C.12), the frequency in the master equation $\Omega_{\rm R}^2(t)$ starts with a large value of the order of Λ and decreases to moderate values in a time of order Λ^{-1}.

The physical origin of the jolts in the coefficients of the master equation as well as other initial time divergences, such as the divergent contributions to correlation functions of system observables that are due to divergent boundary terms at the initial time (see Appendix D in Ref. [18]), can be understood as follows. In general when a system couples to an environment with an infinite number of modes, well-behaved states exhibit correlations with arbitrarily high-frequency modes. In contrast, states that are uncorrelated for sufficiently high frequencies (and hence completely factorizable states in particular) are pathological. For instance, in the limit of infinite cut-off they have infinite energy (even with an origin of energies such that the ground state of the whole system has vanishing energy) and their Hilbert space is unitarily inequivalent to the space of physical states, spanned by the basis

of energy eigenvectors of the whole system Hamiltonian including the system-environment interaction. (Of course for a finite UV cut-off there are no divergences, but the potentially divergent terms are very sensitive to changes in the value of the cut-off.) Physically acceptable initial states that correspond to the thermal equilibrium state for the whole system can be obtained using Euclidean path integrals [3]. However, the instantaneous preparation functions employed in Ref. [3] to produce other states in addition to the thermal equilibrium state still give rise to initial divergences, as explained in Ref. [6]. In order to obtain finite results, one needs to prepare the new initial state within a non-vanishing time [7], which corresponds to a physically more realistic situation. The alternative approach that we follow here is to switch on the interaction smoothly within a time t_s much longer than Λ^{-1} but shorter than any other relevant time-scale of the problem. In this way the factorized initial state, which is perfectly acceptable in the uncoupled case, becomes adequately correlated with the arbitrarily high-frequency modes in a regular fashion.

When adding the short time switch-on function to the spectral density to turn on the interaction gradually, as in Eq. (II.2), the initial jolt is no longer present in the results for the master equation coefficients, which behave smoothly during the switch-on time. Furthermore, for times much longer than t_s the contribution to Eqs. (??)-(??) from the switch-on period is negligible and one can simply use Eqs. (??)-(??) without including any switch-on function in the calculations. This point will be used throughout the paper: our explicit calculations of the diffusion coefficients will not take into account the switch-on functions and our results for those coefficients should only be regarded as valid for times sufficiently larger than t_s, while their values during that period should be smoothly interpolated to zero at the initial time.

b. Initial Kick

We start by considering the case in which there is **no switch-on** function and analyze the effect of the transition from the bare frequency to the renormalized one. The equation of motion is

$$\ddot{x}(\tau) + \frac{2}{M}\int_0^\tau \mu(\tau-\tau')x(\tau')d\tau' + \Omega^2_{\rm bare}x(\tau) = 0, \qquad \text{(C.14)}$$

with $\mu(\tau-\tau')$ given by Eq. (??). Taking Eq. (??) into account and integrating by parts, we get

$$\ddot{x}(\tau)+2\int_0^\tau \gamma(\tau-\tau')\dot{x}(\tau')d\tau'+\Big(\Omega^2_{\rm bare}-2\gamma(0)\Big)x(\tau)=-2\gamma(\tau)x(0). \tag{C.15}$$

For an ohmic environment we have $\gamma(\tau)=2\gamma_0\delta_\Lambda(\tau)$, where $\delta_\Lambda(\tau)$ reduces to a Dirac delta function in the limit $\Lambda\to\infty$. In that limit, Eq. (C.15) becomes

$$\ddot{x}(\tau)+2\gamma_0\dot{x}(\tau)+\Omega^2x(\tau)=-4\gamma_0\delta(\tau)x(0), \tag{C.16}$$

where we used the fact that $\Omega^2_{\rm bare}-2\gamma(0)=\Omega^2$. The term on the right-hand side can be interpreted as a frequency shift of the form $\delta\Omega^2(\tau)=4\gamma_0\delta(\tau)$ whose effect is an initial kick to the homogeneous solutions of the equation of motion, which can be seen as follows:

$$\dot{x}=v \qquad\qquad \Delta x(0)=\lim_{\epsilon\to0}\int_0^\epsilon d\tau\dot{x}=0 \tag{C.17}$$

$$\dot{v}=-2\gamma_0 v-\Omega^2_{\rm R}(\tau)x \qquad\qquad \Delta v(0)=\lim_{\epsilon\to0}\int_0^\epsilon d\tau\dot{v}=-2\gamma_0x(0) \tag{C.18}$$

with $\Omega^2_{\rm R}(\tau)=\Omega^2+4\gamma_0\delta(\tau)$.

Thus, the classical paths experience a finite velocity change within an infinitesimal time. One expects that this result, which corresponds to replacing $\delta_\Lambda(\tau)$ with a Dirac delta in Eq. (C.15), is a good approximation whenever all the relevant time-scales (Ω^{-1}, γ_0^{-1} and τ) are much larger than Λ^{-1}.

Next, we consider the Lagrangian in Eq. (II.2), which includes a **switch-on** function $\theta_{\rm s}(t)$ that turns on the system-environment interaction gradually. This corresponds to replacing $\mu(\tau-\tau')$ with $\mu(\tau-\tau')\theta_{\rm s}(\tau)\theta_{\rm s}(\tau')$ and $\Omega^2_{\rm bare}$ with $\Omega^2+\delta\Omega^2\theta^2_{\rm s}(\tau)$ in Eq. (C.14). Proceeding analogously and integrating by parts, we get the following result in terms of the damping kernel:

$$\ddot{x}(\tau)+2\int_0^\tau \gamma(\tau-\tau')\theta_{\rm s}(\tau)\theta_{\rm s}(\tau')\dot{x}(\tau')d\tau'+\Omega^2x(\tau)=-2\int_0^\tau \gamma(\tau-\tau')\theta_{\rm s}(\tau)\dot{\theta}_{\rm s}(\tau')x(\tau')d\tau', \tag{C.19}$$

where we took into account that $\delta\Omega^2=2\gamma(0)$ and $\theta_{\rm s}(0)=0$. Since we will consider a switch-on time $t_{\rm s}$ is much larger than Λ^{-1}, for an ohmic environment we can approximate $\gamma(\tau-\tau')$ by $2\gamma_0\delta(\tau-\tau')$ so that Eq. (C.19) becomes

$$\ddot{x}(\tau)+2\gamma_0\dot{x}(\tau)+\Omega^2x(\tau)=-2\gamma_0\delta_+(\tau)x(0), \tag{C.20}$$

where we introduced the function $\delta_+(\tau) = d\big(\theta_s^2(\tau)\big)/d\tau$ which becomes ... but its entire support ... so that $\int$

On the other hand, when taking into account the short time switch-on function which turns on the system-environment interaction gradually, the equation of motion exhibits a smooth transition between the decoupled and the coupled regimes without the term proportional to $\delta_\Lambda(\tau)$ on the right-hand side:

$$\ddot{x}(\tau) + \Omega^2 x(\tau) = 0, \quad \tau \ll t_s \tag{C.21}$$

$$\ddot{x}(\tau) + 2\gamma_0 \dot{x}(\tau) + \Omega_{\rm ren}^2 x(\tau) = 0, \quad t_s \ll \tau. \tag{C.22}$$

However, since the constant bare frequency Ω^2 is of order Λ so as to cancel out the divergent contribution that arises when integrating out the environment and give a finite value of the renormalized frequency at late times (larger than the switch-on time), the frequency will change significantly in a short period of time of order t_s. As long as t_s is much shorter than all the other relevant time-scales in the problem (except for Λ^{-1}), the dynamics for $\tau \gg t_s$ can be obtained by approximating the time-dependent frequency by a delta function. More specifically, since the renormalized frequency changes within a characteristic time-scale of order t_s and with an amplitude of order Λ, it can be written as

$$\Omega_{\rm R}^2(\tau) = \Omega_{\rm ren}^2 - 2c\Lambda t_s\, \delta_{t_s}(\tau), \tag{C.23}$$

where $\Omega_{\rm ren}^2$ is the asymptotic constant value for times much larger than t_s, c is a constant of order one, and $\delta_{t_s}(\tau)$ is a function peaked at $t = 0$ with amplitude of order t_s^{-1} and width of order t_s that becomes a Dirac delta in the limit $t_s \to 0$. Therefore, for $\tau \gg t_s$ the dynamics is governed by

$$\ddot{x}(\tau) + 2\gamma_0 \dot{x}(\tau) + \Omega_{\rm ren}^2 x(\tau) = 2c\,\Lambda t_s\, \delta(\tau) x(0). \tag{C.24}$$

Note that $\Lambda t_s \gg 1$ if the switch-on time is sufficiently large to cure the initial divergences discussed above.

The term on the right-hand side of Eq. (C.24) has the same form as in Eq. (C.15). They give the same kind of initial kick to the solutions of the equation and generate the same kind of transformation of the reduced Wigner function as considered in Sec. ??:

$$W_{\rm bare}(x,p) \to W_{\rm ren}(x,p) = W_{\rm bare}(x, p - c\,\Lambda t_s M x), \tag{C.25}$$

and as stated previously, this kind of transformation conserves normalization, positivity, generalized uncertainty, and pure states. For a finite switch-on time, the frequency change

simply corresponds to unitary evolution associated with a time-dependent Hamiltonian, which leaves the eigenvalues of the density matrix non-negative. By continuity, the limit of a very short switch-on time cannot make these eigenvalues negative.

c. Initial State Renormalization

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