LA-UR- 08-4601

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Intended for: JOURNAL:
NUMERISCLE MATHEMATIK

Los Alamos
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EST. 1943

CONVERGENCE ANALYSIS OF THE HIGH-ORDER MIMETIC FINITE DIFFERENCE METHOD

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AMS subject classifications.

Abstract. We prove second-order convergence of the conservative variable and its flux in the high-order MFD method. The convergence results are proved for unstructured polyhedral meshes and full tensor diffusion coefficients. For the case of non-constant coefficients, we also develop a new family of high-order MFD methods. Theoretical result are confirmed through numerical experiments.

1. Introduction. High-order discretization methods are expected to be more efficient than low-order methods in modeling of miscaleneous physical processes. In this article we consider diffusion processes that are crucial for modeling heat transfer, migration of electrons in semiconductor chips, contaminant transport, etc. We consider mixed formulation of the diffusion problem and analyze both theoretically and numerically new high-order mimetic methods [4, 18] that provide second-order accurate numerical fluxes of the conservative variable (temperature, pressure, energy, etc.).

Modeling with polygonal and polyhedral meshes provides enormous geometric flexibility in describing complex geometries. These meshes are used in Earth study, computational fluid dynamics [16], electromagnetics [15], biological modeling, etc. Polyhedral meshes result in a more optimal partition of the computational domain than simplicial meshes. Locally refined meshes with hanging nodes are examples of polyhedral partitions that are admissible for mimetic methods. Development of new high-order methods on polyhedral meshes is a step towards more efficient numerical simulations.

The mimetic finite difference (MFD) method studied in this article belongs to a family of compatible discretization methods that includes finite element methods [22], spectral element methods, finite volume methods [17, 19, 26], etc. The MFD method mimics essential properties of PDEs and the fundamental identities of the vector and tensor calculus. It has been successfully employed for solving problems of continuum mechanics [24], electromagnetics [20], gas dynamics [11], two-phase flows in porous media [23], and linear diffusion on polygonal [21], polyhedral [8, 10, 25] and generalized polyhedral meshes [9]. The MFD method for diffusion problem in the aforementioned papers is a low-order approximation based on a piecewise constant representation of the scalar variable and its flux. It turns out that this method is second-order accurate for the approximation of cell-averages of the scalar variable due to a superconvergence effect, c.f. [8]. Nonetheless, it is only first-order accurate for the flux on families of general polyhedral meshes. In [4, 18], we developed a new high-order MFD method that is second order accurate for both primary variables. The method has been developed for piecewise constant diffusion coefficients only. It is based on different ideas than the high-order mimetic methods proposed in [13, 14]

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whose drawback is lose of accuracy on rough grids.

In this article, we develop the convergence analysis of the high-order MFD method [4, 18] under quite general assumptions on the mesh. For instance, the admissible mesh may include degenerate and non-convex polyhedral elements. We also extend the high-order MFD method to the case of *non-constant* tensor diffusion coefficients through the new local consistency condition (S2). Using this condition, we develop a new family of MFD scalar products resulting in a family of finite difference methods with similar properties. The analysis of this family will be the topic of future research. Finally, we prove error estimates for a post-processed solution. The accurate post-processed solution can be used in problems, such as the reactive transport in porous media, to evaluate solution inside mesh elements.

The paper outline is as follows. In Section 2, we introduce the high-order MFD method. In Section 3, we prove the second-order convergence estimate for the flux variable. In Section 4, we prove the second-order convergence estimates for the scalar variable and the post-processed solution. In Section 5, we develop methods for calculating the inner product matrices satisfying the theoretical assumptions. In Section 6, we present results of numerical experiments on polygonal meshes. Finally, in Section 7 we discuss some final remarks and summarize the conclusions.

2. High-order Formulation of the MFD Method.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded polyhedral domain for $d = 3$ or a polygonal domain for $d = 2$ with Lipschitz continuous boundary Γ. We consider the diffusion equation in mixed form for the scalar solution field p and the vector flux field $\boldsymbol{F}$ defined by

$$\boldsymbol{F} + \mathsf{K}\nabla p = 0 \qquad \text{in } \Omega, \tag{2.1a}$$

$$\operatorname{div}\boldsymbol{F} = f \qquad \text{in } \Omega, \tag{2.1b}$$

$$p = g \qquad \text{on } \Gamma. \tag{2.1c}$$

In (2.1a), f is the forcing term of the divergence equation, g is a boundary function accounting for non-homogeneous Dirichlet condition on Γ, and K is a constant full symmetric tensor describing material properties.

(K1) K is *strongly elliptic* [1], i.e. there exist two constants κ_* and κ^* such that

$$\kappa_* \|\boldsymbol{G}\| \leq \left\|\mathsf{K}^{1/2}\boldsymbol{G}\right\| \leq \kappa^* \|\boldsymbol{G}\| \qquad \forall \boldsymbol{G} \in \mathbb{R}^d. \tag{2.2}$$

(K2) All the components of K and K^{-1} are in $W^{2,\infty}(\Omega)$.

In view of condition (2.2) the $d \times d$ symmetric matrix K is strictly positive definite, and thus non-singular. Its inverse matrix K^{-1} is also symmetric and positive definite, i.e. strongly elliptic, and satisfies analogous bounds from above an below involving, respectively, κ_*^{-1} and κ^{*-1}.

Under suitable assumptions on the regularity of Ω and f, the well-posedness of problem (2.1) can be proved, thus implying existence and uniqueness of solution [1].

2.1. Notation and basic assumptions.

For exposition's sake, we find it convenient to adopt mesh notation and assumptions introduced in [7]. Let $\mathcal{T}_h$ be a conformal partition of Ω into non-overlapping polyhedral elements (polygons in 2-D). For every element E we denote its d-dimensional Lebesgue measure by $|E|$, its barycenter by E, its diameter by h_E, and the number of its faces by m_E. The notation ∂E may denote the boundary of the element E or the union of the element faces depending on the context. Similarly, for every face e we denote its $(d-1)$-dimensional Lebesgue measure by $|e|$, its barycenter by $\boldsymbol{x}_e$, its unit normal vector by $\mathbf{n}^e$, its diameter by h_e, the number of its edges by m_e, and the $(d-2)$ measure of each face edge $l \in \partial e$ by $|l|$. We assume that the orientation of each mesh face e is uniquely defined by the orientation of its unit normal vector $\mathbf{n}^e$. The mesh $\mathcal{T}_h$ is sub-indexed by the mesh size parameter defined, as usual, by $h = \sup_E h_E$. We indicate the set of mesh faces by $\mathcal{E}_h$, the subset of internal faces by $\mathcal{E}_h^I$ and the subset of boundary faces by $\mathcal{E}_h^D$. The basic assumptions that follow are formulated for $d = 3$; the restriction to $d = 2$ is straightforward.

(HG) [*Properties of the decomposition* $\mathcal{T}_h$] There exist two positive real numbers N_s and ρ_s such that every mesh $\mathcal{T}_h$ admits a sub-partition $\mathcal{S}_h$ into shape-regular tetrahedra such that

(HG1) every polyhedron $E \in \mathcal{T}_h$ admits a decomposition $\mathcal{S}_{h|E}$ made of less than N_s tetrahedra;

(HG2) the shape-regularity of the tetrahedra $\mathsf{K} \in \mathcal{S}_h$ is defined as follows: the ratio between the radius r_K of the inscribed sphere and the diameter h_K is bounded from below by the constant ρ_s:

$$\frac{r_\mathsf{K}}{h_\mathsf{K}} \geq \rho_s > 0;$$

(ME) [*Star-shaped elements*] there exists a positive number τ^* such that each element E is star-shaped with respect to all points of a ball of radius $\tau^* h_E$ and centered at an internal point of E.

From the above assumptions several properties of the mesh, which are useful in the mimetic formulation, can be derived. We list them below for the sake of the reader's convenience and for future reference in the paper.

(M1) There exist two positive integers N_E and N_e such that every element E has $m_E \leq N_E$ faces, and every face e has $m_e \leq N_e$ edges.

(M2) For any mesh element $E \in \mathcal{T}_h$, the quantities $|E|$, $|e|$ for $e \in \partial E$, and $|l|$ for $l \in \partial E$ properly scale with respect to h_E; in particular, there exists a positive constant a^* such that $a^* h_E^{d-1} \leq |e|$ and $a^* h_E \leq h_e \leq h_E$ for all faces e of E.

(M3) there exists a constant C^{Ag} independent of h_E and such that [8]:

$$\sum_{e \in \partial E} ||\phi||^2_{L^2(e)} \leq C^{\mathrm{Ag}} \left(h_E^{-1} ||\phi||^2_{L^2(E)} + h_E |\phi|^2_{H^1(E)} \right) \tag{2.3}$$

for any function $\phi \in H^1(E)$. We will refer to (2.3) as *the Agmon inequality*;

(M4) for any function $q \in H^3(E)$ there exists a *quadratic interpolant* $q_E^{(2)}$ and a constant C independent of h_E such that [6]:

$$||q - q_E^{(2)}||_{L^2(E)} + h_E |q - q_E^{(2)}|_{H^1(E)} + h_E^2 |q - q_E^{(2)}|_{H^2(E)} \leq C h_E^3 |q|_{H^3(E)}. \tag{2.4}$$

The formulation of the mimetic finite difference method is based on the notion of *discrete fields*, which are collections of real numbers representing the degrees of freedom of the numerical scheme. From the notation standpoint, we will indicate both continuous and discrete *vector fields* by bold letters and will use normal fonts for *scalar fields*. Hence, we will not explicitly distinguish between continuous and discrete fields, since the field's nature can always be contextually derived without ambiguity.

2.2. High-order formulation of the MFD method. We represent discrete scalar and vector fields by the finite dimensional linear spaces Q_h and X_h.

• Q_h provides the degrees-of-freedom of scalar fields associated to mesh cells; thus,

$$q \in Q_h \quad \text{means that} \quad q = \{q_E\}_{E \in \mathcal{T}_h} \quad \text{with} \quad q_E \in \mathbb{R}.$$

The dimension of Q_h equals the number of mesh cells. The interpolation operator for the scalar field $q \in L^1(\Omega)$ reads as:

$$\left(q^I\right)_E = \frac{1}{|E|}\int_E q\,dV \quad \text{for all } E \in \mathcal{T}_h.$$

• X_h provides the degrees-of-freedom of the vector fields associated to the elemental faces of the mesh; thus,

$$\boldsymbol{G} \in X_h \quad \text{means that} \quad \boldsymbol{G} = \left\{\left(G_0^e, \boldsymbol{G}_1^e\right)\right\}^{e \in \mathcal{E}_h} \quad \text{with } G_0^e \in \mathbb{R},\ \boldsymbol{G}_1^e \in \mathbb{R}^{d-1}.$$

The dimension of X_h equals d times the number of mesh faces. By using the vector degrees-of-freedom we obtain the linear representation of the discrete flux $\boldsymbol{G} \in X_h$ on each mesh face given by

$$G^{(e)}(\boldsymbol{\xi}) = G_0^e + \boldsymbol{G}_1^e \cdot \frac{\boldsymbol{\xi} - \boldsymbol{\xi}_e}{h_e}, \quad \text{for } \boldsymbol{\xi} \in e, \tag{2.5}$$

where $\boldsymbol{\xi} \in \mathbb{R}^{d-1}$ is the position vector of the face points with respect to a *local coordinate system* chosen on e, $\boldsymbol{\xi}_e \in \mathbb{R}^{d-1}$ is the barycenter of e with respect to such coordinate system, and we recall that h_e is a characteristic length size of the face e. We remark that the discrete flux continuity across inter-element faces is naturally embodied into the definition of X_h because the vector degrees-of-freedom are uniquely defined on each mesh face. The interpolation operator for the continuous vector field $\boldsymbol{G} \in (L^s(\Omega))^d$ with $s > 2$ and $\text{div}\boldsymbol{G} \in L^2(\Omega)$ reads as

$$\int_e (\boldsymbol{G}^I)^e(\boldsymbol{\xi})\left(\boldsymbol{\nu}\cdot(\boldsymbol{\xi} - \boldsymbol{\xi}_e)\right)^m dS = \int_e \mathbf{n}^e \cdot \boldsymbol{G}\left(\boldsymbol{\nu}\cdot(\boldsymbol{\xi} - \boldsymbol{\xi}_e)\right)^m dS \quad \text{for every } \boldsymbol{\nu} \in \mathbb{R}^{d-1} \text{ and } m = 0, 1. \tag{2.6}$$

For $m = 0$ we get the usual interpolation formula of the low-order MFD method:

$$\left(\boldsymbol{G}^I\right)_0^e = \frac{1}{|e|}\int_e \mathbf{n}^e \cdot \boldsymbol{G}\,dS,$$

while for $m = 1$ we get the condition defining the high-order discrete flux components:

$$\int_e (\boldsymbol{G}^I)_1^e \cdot \frac{\boldsymbol{\xi} - \boldsymbol{\xi}_e}{h_e}\left(\boldsymbol{\nu}\cdot(\boldsymbol{\xi} - \boldsymbol{\xi}_e)\right) dS = \int_e \mathbf{n}^e \cdot \boldsymbol{G}\left(\boldsymbol{\nu}\cdot(\boldsymbol{\xi} - \boldsymbol{\xi}_e)\right) dS \quad \text{for every } \boldsymbol{\nu} \in \mathbb{R}^{d-1}.$$

The mimetic *discrete divergence* operator $\text{div}_h : X_h \to Q_h$ is defined element-wise as $\text{div}_h\boldsymbol{G} = \left\{\text{div}_{h,E}\boldsymbol{G}\right\}_{E \in \mathcal{T}_h}$ where

$$\text{div}_{h,E}\boldsymbol{G} := \frac{1}{|E|}\sum_{e \in \partial E}\sigma_E^e \int_e G^{(e)}(\boldsymbol{\xi})\,dS = \frac{1}{|E|}\sum_{e \in \partial E}\sigma_E^e |e| G_0^e$$

for every $\boldsymbol{G} \in X_h$, $E \in \mathcal{T}_h$ and setting $\sigma_E^e = \mathbf{n}^e \cdot \mathbf{n}_E^e$. This definition is consistent with the Gauss divergence theorem, and a straightforward calculation shows that the commuting property of the interpolation operators still holds:

$$(\text{div}\boldsymbol{G})^I = \text{div}_h\boldsymbol{G}^I. \tag{2.7}$$

Let us now introduce the $L^2(E)$-orthogonal projection from $(H^2(E))^d$ onto the space of linear vectors defined on E, for each element $E \in \mathcal{T}_h$. The projection operator is denoted by $\mathcal{P}_E^{(1)}(\cdot)$ and formally defined for every $\boldsymbol{u} \in (L^2(E))^d$ by:

$$\int_E \left(\mathcal{P}_E^{(1)}(\boldsymbol{u}) - \boldsymbol{u}\right) \cdot \boldsymbol{v}\, dV = 0 \qquad \text{for every } \boldsymbol{v} \in \left(P_1(E)\right)^d. \tag{2.8}$$

This operator is clearly bounded, i.e. $\|\mathcal{P}_E^{(1)}(\boldsymbol{u})\|_{L^2(E)} \leq \|\boldsymbol{u}\|_{L^2(E)}$ and its approximation properties are characterized by the following lemma.

LEMMA 2.1. *Under the regularity assumption (ME), the projection operator* $\mathcal{P}_E^{(1)}(\cdot)$ *provides a* second-order *accurate approximation of vector fields of* $(H^2(E))^d$*:*

$$\|\boldsymbol{u} - \mathcal{P}_E^{(1)}(\boldsymbol{u})\|_{L^2(E)} + h_E|\boldsymbol{u} - \mathcal{P}_E^{(1)}(\boldsymbol{u})|_{H^1(E)} \leq Ch_E^2|\boldsymbol{u}|_{H^2(E)}. \tag{2.9}$$

Proof. Let $\mathcal{I}_E^{(1)}(\cdot) : (H^2(E))^d \rightarrow (P_1(E))^d$ be the nodal linear interpolation operator acting on the components of the vectors of $(H^2(E))^d$. Adding and substracting $\mathcal{I}_E^{(1)}(\boldsymbol{u})$, using the triangular inequality, noting that $\mathcal{P}_E^{(1)} \circ \mathcal{I}_E^{(1)} = \mathcal{I}_E^{(1)}$, and using a standard estimate on star-shaped domains [6] for the interpolation error yield:

$$\begin{aligned}
\|\boldsymbol{u} - \mathcal{P}_E^{(1)}(\boldsymbol{u})\|_{L^2(E)} &\leq \left\|\boldsymbol{u} - \mathcal{I}_E^{(1)}(\boldsymbol{u})\right\|_{L^2(E)} + \left\|\mathcal{I}_E^{(1)}(\boldsymbol{u}) - \mathcal{P}_E^{(1)}(\boldsymbol{u})\right\|_{L^2(E)} \\
&\leq \left\|\boldsymbol{u} - \mathcal{I}_E^{(1)}(\boldsymbol{u})\right\|_{L^2(E)} + \left\|\mathcal{P}_E^{(1)}\left(\mathcal{I}_E^{(1)}(\boldsymbol{u}) - \boldsymbol{u}\right)\right\|_{L^2(E)} \\
&\leq 2\left\|\boldsymbol{u} - \mathcal{I}_E^{(1)}(\boldsymbol{u})\right\|_{L^2(E)} \\
&\leq 2h_E^2|\boldsymbol{u}|_{H^2(E)}.
\end{aligned} \tag{2.10}$$

Likewise, using the previous arguments and an inverse estimate in the third step yield:

$$\begin{aligned}
|\boldsymbol{u} - \mathcal{P}_E^{(1)}(\boldsymbol{u})|_{H^1(E)} &\leq \left|\boldsymbol{u} - \mathcal{I}_E^{(1)}(\boldsymbol{u})\right|_{H^1(E)} + \left|\mathcal{I}_E^{(1)}(\boldsymbol{u}) - \mathcal{P}_E^{(1)}(\boldsymbol{u})\right|_{H^1(E)} \\
&\leq \left|\boldsymbol{u} - \mathcal{I}_E^{(1)}(\boldsymbol{u})\right|_{H^1(E)} + \left|\mathcal{P}_E^{(1)}\left(\mathcal{I}_E^{(1)}(\boldsymbol{u}) - \boldsymbol{u}\right)\right|_{H^1(E)} \\
&\leq \left|\boldsymbol{u} - \mathcal{I}_E^{(1)}(\boldsymbol{u})\right|_{H^1(E)} + Ch_E^{-1}\left\|\mathcal{P}_E^{(1)}\left(\mathcal{I}_E^{(1)}(\boldsymbol{u}) - \boldsymbol{u}\right)\right\|_{L^2(E)} \\
&\leq Ch_E|\boldsymbol{u}|_{H^2(E)}.
\end{aligned} \tag{2.11}$$

Inequality (2.9) follows from (2.10)-(2.11). □

Now, we equip Q_h and X_h with the scalar products $\left[\cdot,\cdot\right]_{Q_h}$ and $\left[\cdot,\cdot\right]_{X_h}$, which assemble contributions of each mesh element. The elemental terms are always denoted, for simplicity, by $\left[\cdot,\cdot\right]_E$, even if their definition is different for scalars and vectors. The symbols $|||\cdot|||_{Q_h}$, $|||\cdot|||_{X_h}$, and $|||\cdot|||_E$ are the norms induced by these scalar products. The scalar product of Q_h is given by

$$\left[p,q\right]_{Q_h} = \sum_{E\in\mathcal{T}_h} \left[p,q\right]_E \quad \text{where} \quad \left[p,q\right]_E = |E|p_E q_E \quad \text{for all } p,q \in Q_h, \tag{2.12}$$

and corresponds to the L^2-scalar product for piecewise constant functions. The scalar product of X_h is given by

$$\left[\boldsymbol{F},\boldsymbol{G}\right]_{X_h} = \sum_{E\in\mathcal{T}_h} \left[\boldsymbol{F},\boldsymbol{G}\right]_E \quad \text{for all } \boldsymbol{F},\boldsymbol{G} \in X_h,$$

where the *local* scalar product $\left[\cdot,\cdot\right]_E$ in $X_h|_E$ is required to satisfy the two following conditions:

(S1) *spectral stability*: there exist two constants σ_*, $\sigma^* > 0$ independent of h such that for all $\boldsymbol{G} \in X_h$ and for every element E there holds

$$\sigma_* h_E \sum_{e\in\partial E} \int_e |G^{(e)}(\boldsymbol{\xi})|^2\,dS \le \left[\boldsymbol{G},\boldsymbol{G}\right]_E \le \sigma^* h_E \sum_{e\in\partial E} \int_e |G^{(e)}(\boldsymbol{\xi})|^2\,dS,$$

where $G^{(e)}(\boldsymbol{\xi})$ is the local affine function defined on e by (2.5);

(S2) *local consistency*: for every element E, every discrete vector field $\boldsymbol{G} \in X_h$, and every polynomial function $q^{(2)} \in P_2(E)$ there holds:

$$\left[\left(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q^{(2)})\right)^I,\boldsymbol{G}\right]_E + \int_E q^{(2)}\,\mathrm{div}_{h,E}\boldsymbol{G}\,dV = \sum_{e\in\partial E} \sigma_E^e \int_e G^{(e)}(\boldsymbol{\xi})q^{(2)}(\boldsymbol{\xi})\,dS,$$

where $\mathrm{div}_{h,E}\boldsymbol{G}$ is the discrete divergence operator previously defined.

To ensure the symmetry of the resulting scalar product, condition (S2) considers the projected field $\left(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q^{(2)})\right)^I$ instead of $(\mathsf{K}\nabla q^{(2)})^I$ as is done in the low-order formulation. This fact is thoroughly discussed in sub-section 5.1.

REMARK 2.1. *In the case of piece-wise constant diffusion tensor condition (S2) becomes*

$$\left[\left(\mathsf{K}\nabla q^{(2)}\right)^I,\boldsymbol{G}\right]_E + \int_E q^{(2)}(\boldsymbol{x})\mathrm{div}_{h,E}\boldsymbol{G}\,dV = \sum_{e\in\partial E} \sigma_E^e \int_e G^{(e)}(\boldsymbol{\xi})q^{(2)}(\boldsymbol{\xi})\,dS,$$

which is the condition considered in [4]. Therefore, the analysis presented in this paper includes the method of [4].

We terminate the presentation of the mimetic setting by introducing the following bilinear form whose arguments are a function of $L^1(\Gamma)$ and a vector of X_h:

$$\begin{aligned}\langle g,\boldsymbol{G}\rangle_\Gamma &= \sum_{e\in\mathcal{E}_h^D} \int_e G^{(e)}(\boldsymbol{\xi})g(\boldsymbol{\xi})\,dS \\ &= \sum_{e\in\mathcal{E}_h^D} \left(|e|G_0^e \int_e g\,dS + \boldsymbol{G}_1^e \cdot \int_e g(\boldsymbol{\xi})\frac{\boldsymbol{\xi}-\boldsymbol{\xi}_e}{h_e}\,dS\right).\end{aligned} \tag{2.13}$$

The dual mimetic formulation of problem (2.1) reads as:

Find $(\boldsymbol{F}_h, p_h) \in X_h \times Q_h$ *such that:*

$$\left[\boldsymbol{F}_h,\boldsymbol{G}\right]_{X_h} - \left[p_h,\mathrm{div}_h\boldsymbol{G}\right]_{Q_h} = -\langle g,\boldsymbol{G}\rangle_\Gamma \quad \text{for every } \boldsymbol{G} \in X_h, \tag{2.14a}$$

$$\left[\mathrm{div}_h\boldsymbol{F}_h, q\right]_{Q_h} = \left[f^I, q\right]_{Q_h} \quad \text{for every } q \in Q_h. \tag{2.14b}$$

The dual mimetic formulation (2.14) does not require the introduction of a *discrete flux* (or gradient) operator. However, such operator can be defined, as usual in mimetic discretizations, by duality with respect to the discrete divergence operator and the scalar products of Q_h and X_h.

3. Convergence of flux variable. The main result of this section is proved in Theorem 3.4, which states the convergence of the flux approximation $\boldsymbol{F}_h$ to the interpolant $\boldsymbol{F}^I$ of the exact flux $\boldsymbol{F}$ and provides an error estimate in the norm $|||\cdot|||_{X_h}$. The theorem's proof is based on some technical lemmas that are proved below.

LEMMA 3.1 (Agmon inequalities). *Let E be a mesh element, e be a mesh face that belongs to ∂E, and q a function in $H^3(E)$. Under Assumptions (HG)-(ME) it follows that:*

$$\sum_{e\in\partial E} ||q - q_E^{(2)}||^2_{L^2(e)} \leq \widetilde{C} h_E^5 ||q||^2_{H^3(E)}, \tag{3.0a}$$

$$\sum_{e\in\partial E} \left|\left|\mathsf{K}\nabla q - \mathcal{P}_E^{(1)}\left(\mathsf{K}\nabla q_E^{(2)}\right)\right|\right|^2_{L^2(e)} \leq 2\widetilde{C} C_{\mathsf{K}} h_E^3 ||q||^2_{H^3(E)}. \tag{3.0b}$$

where $q_E^{(2)}$ is the quadratic interpolant of q in E satisfying (2.4), *$\widetilde{C}$ depends only on the constants appearing in (M1)-(M4), and $C_{\mathsf{K}} = ||\mathsf{K}||^2_{W^{2,\infty}(E)}$.*

Proof. We prove inequality (3.0a) by applying the Agmon inequality with $\phi = q - q_E^{(2)}$ and using a standard estimate for the interpolation error (with error constant C^{Ip}). We have

$$\begin{aligned}\sum_{e\in\partial E} ||q - q_E^{(2)}||^2_{L^2(e)} &\leq C^{\mathrm{Ag}} \left(h_E^{-1}||q - q_E^{(2)}||^2_{L^2(E)} + h_E|q - q_E^{(2)}|^2_{H^1(E)}\right) \\ &\leq C^{\mathrm{Ag}} \left(h_E^{-1}\left(C^{\mathrm{Ip}} h_E^3 |q|_{H^3(E)}\right)^2 + h_E\left(C^{\mathrm{Ip}} h_E^2 |q|_{H^3(E)}\right)^2\right) \\ &\leq \widetilde{C} h_E^5 ||q||^2_{H^3(E)},\end{aligned} \tag{3.1}$$

where the constant $\widetilde{C}$ includes the Agmon and the interpolation constant.

To prove inequality (3.0b), we add and substract $\mathsf{K}\nabla q_E^{(2)}$ to its left-hand side and use the triangle inequality to obtain:

$$\begin{aligned}&\left|\left|\mathsf{K}\nabla q - \mathcal{P}_E^{(1)}\left(\mathsf{K}\nabla q_E^{(2)}\right)\right|\right|^2_{L^2(e)} \\ &\qquad \leq 2\left(||\mathsf{K}\nabla(q - q_E^{(2)})||^2_{L^2(e)} + ||\mathcal{P}_E^{(1)}\left(\mathsf{K}\nabla q_E^{(2)}\right) - \mathsf{K}\nabla q_E^{(2)}||^2_{L^2(e)}\right).\end{aligned} \tag{3.2}$$

The first term of the right-hand side of (3.2) is bounded through the Agmon inequality by taking $\phi = \mathsf{K}\nabla(q - q_E^{(2)})$ and then using the estimate for the interpolation error (the error constant is still denoted by C^{Ip}):

$$\begin{aligned}&\sum_{e\in\partial E} ||\mathsf{K}\nabla(q - q_E^{(2)})||^2_{L^2(e)} \\ &\qquad \leq C^{\mathrm{Ag}}\kappa^{*2}\left(h_E^{-1}||\nabla(q - q_E^{(2)})||^2_{L^2(E)} + h_E|\nabla(q - q_E^{(2)})|^2_{H^1(E)}\right) \\ &\qquad \leq C^{\mathrm{Ag}}\kappa^{*2}\left(h_E^{-1}\left(C^{\mathrm{Ip}} h_E^2 |q|_{H^3(E)}\right)^2 + h_E\left(C^{\mathrm{Ip}} h_E |q|_{H^3(E)}\right)^2\right) \\ &\qquad \leq \widetilde{C}\kappa^{*2} h_E^3 ||q||^2_{H^3(E)},\end{aligned}$$

where the constant $\widetilde{C}$ still includes the Agmon and the interpolation constant. Similarly, the second term of the right-hand side of (3.2) is bounded by taking $\phi =$

$\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_E^{(2)}) - \mathsf{K}\nabla q_E^{(2)}$ in (2.3), applying (2.9) with $\boldsymbol{u} = \mathsf{K}\nabla q_E^{(2)}$, and using (2.4):

$$\begin{aligned}
&\sum_{e\in\partial E} \left\| \mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_E^{(2)}) - \mathsf{K}\nabla q_E^{(2)} \right\|^2_{L^2(e)} \\
&\quad \leq C^{\mathrm{Ag}}\Big(h_E^{-1} \left\| \mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_E^{(2)}) - \mathsf{K}\nabla q_E^{(2)} \right\|^2_{L^2(E)} + h_E \left| \mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_E^{(2)}) - \mathsf{K}\nabla q_E^{(2)} \right|^2_{H^1(E)} \Big) \\
&\quad \leq \widetilde{C} h_E^3 |\mathsf{K}\nabla q_E^{(2)}|^2_{H^2(E)} \\
&\quad \leq \widetilde{C} h_E^3 \|\mathsf{K}\|^2_{W^{2,\infty}(E)} \|q_E^{(2)}\|^2_{H^3(E)} \\
&\quad \leq \widetilde{C} h_E^3 \|\mathsf{K}\|^2_{W^{2,\infty}(E)} \|q\|^2_{H^3(E)}.
\end{aligned}$$

The lemma follows by taking $C_{\mathsf{K}} = \max(\kappa^{*2}, \|\mathsf{K}\|^2_{W^{2,\infty}(E)}) = \|\mathsf{K}\|^2_{W^{2,\infty}(E)}$. ◻

LEMMA 3.2. *Let $q \in H^3(\Omega)$ and $q_E^{(2)}$ the piecewise quadratic polynomial that interpolates $q|_E$ in the mesh element E. There exists a positive constant C_1 independent of q and h such that for every $\boldsymbol{G} \in X_h$ there holds:*

$$\sum_{E\in\mathcal{T}_h} \sum_{e\in\partial E} \sigma_E^e \int_e q_E^{(2)}(\boldsymbol{\xi}) G^{(e)}(\boldsymbol{\xi})\, dS - \left\langle q|_\Gamma, \boldsymbol{G} \right\rangle_\Gamma \leq C_1 h^2 \|q\|_{H^3(\Omega)} |||\boldsymbol{G}|||_{X_h}. \quad (3.3)$$

Proof. Since the trace of q in $H^3(\Omega)$ is continuous at every internal face and $\sigma_E^e G^{(e)}$ takes opposite values at the two sides of every internal face, we have that:

$$\sum_{E\in\mathcal{T}_h} \sum_{e\in\partial E} \sigma_E^e \int_e q(\boldsymbol{\xi}) G^{(e)}(\boldsymbol{\xi})\, dS = \left\langle q|_\Gamma, \boldsymbol{G} \right\rangle_\Gamma. \quad (3.4)$$

By using (3.4) and the Cauchy-Schwarz inequality twice we get:

$$\begin{aligned}
&\sum_{E\in\mathcal{T}_h} \sum_{e\in\partial E} \sigma_E^e \int_e q_E^{(2)}(\boldsymbol{\xi}) G^{(e)}(\boldsymbol{\xi})\, dS - \left\langle q|_\Gamma, \boldsymbol{G} \right\rangle_\Gamma \\
&\quad = \sum_{E\in\mathcal{T}_h} \sum_{e\in\partial E} \sigma_E^e \int_e \left(q_E^{(2)}(\boldsymbol{\xi}) - q(\boldsymbol{\xi}) \right) G^{(e)}(\boldsymbol{\xi})\, dS \\
&\quad \leq \sum_{E\in\mathcal{T}_h} \sum_{e\in\partial E} \|q_E^{(2)} - q\|_{L^2(e)} \|G^{(e)}\|_{L^2(e)} \\
&\quad \leq \sum_{E\in\mathcal{T}_h} \left(\sum_{e\in\partial E} \|q_E^{(2)} - q\|^2_{L^2(e)} \right)^{1/2} \left(\sum_{e\in\partial E} \|G^{(e)}\|^2_{L^2(e)} \right)^{1/2}
\end{aligned}$$

Inequality (3.0a), Assumption (S1), and setting $C_1 = \widetilde{C}/{\sigma_*}^{1/2}$ allow us to get the final developments:

$$\sum_{E\in\mathcal{T}_h}\sum_{e\in\partial E}\sigma_E^e\int_e q_E^{(2)}(\boldsymbol{\xi})G^{(e)}(\boldsymbol{\xi})\,dS - \langle q|_\Gamma, \boldsymbol{G}\rangle_\Gamma$$

$$\leq \sum_{E\in\mathcal{T}_h}\widetilde{C}h_E^{2+1/2}\|q\|_{H^3(E)}\left(\sigma_* h_E\right)^{-1/2}|||\boldsymbol{G}|||_E$$

$$\leq C_1 h^2\left(\sum_{E\in\mathcal{T}_h}\|q\|^2_{H^3(E)}\right)^{1/2}\left(\sum_{E\in\mathcal{T}_h}|||\boldsymbol{G}|||^2_E\right)^{1/2}$$

$$= C_1 h^2\|q\|_{H^3(\Omega)}|||\boldsymbol{G}|||_{X_h}.$$

□

LEMMA 3.3. *Let $q\in H^3(\Omega)$ and $q^{(2)}$ the $\mathcal{T}_h$-piecewise polynomial such that $q_E^{(2)}$ is the quadratic interpolant of $q|_E$ for every mesh element E. There exists a positive constant C_2 independent of q and h such that for every $\boldsymbol{G}\in X_h$ there holds:*

$$\left[(\mathsf{K}\nabla q)^I - \left(\mathcal{P}_E^{(1)}\left(\mathsf{K}\nabla q^{(2)}\right)\right)^I, \boldsymbol{G}\right]_{X_h} \leq C_2 h^2\|q\|_{H^3(\Omega)}|||\boldsymbol{G}|||_{X_h}. \tag{3.5}$$

Proof. To ease notation, let us introduce the symbol

$$\boldsymbol{\mathcal{G}}_q = \mathsf{K}\nabla q - \mathcal{P}_E^{(1)}\left(\mathsf{K}\nabla q_E^{(2)}\right). \tag{3.6}$$

Since $((\boldsymbol{\mathcal{G}}_q)^I)^e$ is the $L^2(e)$-orthogonal projection of $\mathbf{n}^e\cdot\boldsymbol{\mathcal{G}}_q$ onto $P_1(e)$, we have that $\|(\boldsymbol{\mathcal{G}}_q^I)^e\|_{L^2(e)} \leq \|\boldsymbol{\mathcal{G}}_q\|_{L^2(e)}$. Therefore, using spectral stability (S1), Agmon inequality (3.0b), and setting $(C_2)^2 = \sigma^*\widetilde{C}C_{\mathsf{K}}$ yield:

$$|||\boldsymbol{\mathcal{G}}_q^I|||^2_E \leq \sigma^* h_E\sum_{e\in\partial E}\|(\boldsymbol{\mathcal{G}}_q^I)^e\|^2_{L^2(e)} \leq \sigma^* h_E\sum_{e\in\partial E}\|\boldsymbol{\mathcal{G}}_q\|^2_{L^2(e)} \leq (C_2)^2 h_E^4\|q\|^2_{H^3(E)}.$$

We estimate the global bound for $|||\boldsymbol{\mathcal{G}}_q^I|||_{X_h}$ as follows:

$$|||\boldsymbol{\mathcal{G}}_q^I|||^2_{X_h} = \sum_{E\in\mathcal{T}_h}|||\boldsymbol{\mathcal{G}}_q^I|||^2_E \leq (C_2)^2\sum_{E\in\mathcal{T}_h}h_E^4\|q\|^2_{H^3(E)} = (C_2)^2 h^4\|q\|^2_{H^3(\Omega)}. \tag{3.7}$$

Finally, we apply the Cauchy-Schwarz inequality and use (3.7) to obtain

$$\left[\boldsymbol{\mathcal{G}}_q^I, \boldsymbol{G}\right]_{X_h} \leq |||\boldsymbol{\mathcal{G}}_q^I|||_{X_h}|||\boldsymbol{G}|||_{X_h} \leq C_2 h^2\|q\|_{H^3(\Omega)}|||\boldsymbol{G}|||_{X_h},$$

which is inequality (3.5) because of definition (3.6). □

THEOREM 3.4. *Let $(\boldsymbol{F}, p)$ be the exact solution of (2.1) with $p\in H^3(\Omega)$, and $(\boldsymbol{F}_h, p_h)$ the mimetic solution in $X_h\times Q_h$. Then, it holds*

$$|||\boldsymbol{F}^I - \boldsymbol{F}_h|||_{X_h} \leq Ch^2\|p\|_{H^3(\Omega)} \tag{3.8}$$

where $C = \max(C_1, C_2)$ is independent of h and only depends on the constants appearing in (M1)-(M4) and (S1)-(S2).

Proof. Let $p^{(2)}$ be the $\mathcal{T}_h$-piecewise polynomial such that $p_E^{(2)}\in P_2(E)$ is the quadratic interpolant of $p|_E$. Using equation (2.14a) with $\boldsymbol{G} = (\boldsymbol{F}^I - \boldsymbol{F}_h)$ and $g = p|_\Gamma$, equation (2.14b) with $q = p_h$, and equation (2.7) with $\boldsymbol{G} = \boldsymbol{F}$ yield

$$\begin{aligned} \left[\boldsymbol{F}_h, \boldsymbol{F}^I - \boldsymbol{F}_h\right]_{X_h} &= \left[p_h, \mathrm{div}_h(\boldsymbol{F}^I - \boldsymbol{F}_h)\right]_{Q_h} - \langle p|_\Gamma, \boldsymbol{F}^I - \boldsymbol{F}_h\rangle_\Gamma \\ &= \left[p_h, f^I\right]_{Q_h} - \left[p_h, f^I\right]_{Q_h} - \langle p|_\Gamma, \boldsymbol{F}^I - \boldsymbol{F}_h\rangle_\Gamma \\ &= -\langle p|_\Gamma, \boldsymbol{F}^I - \boldsymbol{F}_h\rangle_\Gamma. \end{aligned} \tag{3.9}$$

Note that the numerical error $(\boldsymbol{F}^I - \boldsymbol{F}_h)$ is orthogonal to $\boldsymbol{F}_h$ with respect to the scalar product of X_h if $g = 0$.
We transform the expression of the flux error by using identity (3.9), substituting $\boldsymbol{F}^I = (-\mathsf{K}\nabla p)^I$ and adding and substracting the term $-\big(\mathcal{P}^{(1)}_E(\mathsf{K}\nabla p^{(2)}_E)\big)^I$ to obtain

$$\begin{aligned} |||\boldsymbol{F}^I - \boldsymbol{F}_h|||^2_{X_h} &= \left[\boldsymbol{F}^I - \boldsymbol{F}_h, \boldsymbol{F}^I - \boldsymbol{F}_h\right]_{X_h} \\ &= \left[\boldsymbol{F}^I, \boldsymbol{F}^I - \boldsymbol{F}_h\right]_{X_h} + \langle p|_\Gamma, \boldsymbol{F}^I - \boldsymbol{F}_h\rangle_\Gamma \\ &= \left[(-\mathsf{K}\nabla p)^I, \boldsymbol{F}^I - \boldsymbol{F}_h\right]_{X_h} + \langle p|_\Gamma, \boldsymbol{F}^I - \boldsymbol{F}_h\rangle_\Gamma \\ &= A_1 + A_2 \end{aligned}$$

where

$$\begin{aligned} A_1 &= \left[-\big(\mathcal{P}^{(1)}_E(\mathsf{K}\nabla p^{(2)})\big)^I, \boldsymbol{F}^I - \boldsymbol{F}_h\right]_{X_h} + \langle p|_\Gamma, \boldsymbol{F}^I - \boldsymbol{F}_h\rangle_\Gamma\ , \\ A_2 &= \left[-(\mathsf{K}\nabla p)^I + \big(\mathcal{P}^{(1)}_E(\mathsf{K}\nabla p^{(2)}_E)\big)^I, \boldsymbol{F}^I - \boldsymbol{F}_h\right]_{X_h}. \end{aligned}$$

The term $|A_1|$ is bounded by noting that

$$\left[\big(\mathcal{P}^{(1)}_E(\mathsf{K}\nabla p^{(2)}_E)\big)^I, \boldsymbol{F}^I - \boldsymbol{F}_h\right]_E = \sum_{e\in\partial E} \sigma^e_E \int_e p^{(2)}_E(\boldsymbol{\xi})(\boldsymbol{F}^I - \boldsymbol{F}_h)^e(\boldsymbol{\xi})\, dS,$$

which follows from local consistency assumption (S2) because $\mathrm{div}_h(\boldsymbol{F}^I) = f^I = \mathrm{div}_h\boldsymbol{F}_h$, and then using inequality (3.3) with $\boldsymbol{G} = (\boldsymbol{F}^I - \boldsymbol{F}_h)$. The term $|A_2|$ is bounded by using inequality (3.5) with $\boldsymbol{G} = (\boldsymbol{F}^I - \boldsymbol{F}_h)$. Finally, inequality (3.8) follows by combining the bounds for $|A_1| + |A_2|$ and taking $C = \max(C_1, C_2)$. □

4. Convergence of scalar variable and post-processing. In this section, we derive an a priori estimate for the approximation error of the scalar solution and for the $\mathcal{T}_h$-piecewise quadratic field built by the post-processing technique proposed in [4]. Our post processing technique is based on an element-wise reconstruction of a linear gradient field from the discrete flux solution, and generalizes the analogous technique for the low-order mimetic scheme [3, 12] to the high-order case. The post processed scalar field p^*_h is defined as the unique $\mathcal{T}_h$-piecewise quadratic polynomial satisfying

$$\int_E p^*_h\, dV = \int_E p_h\, dV \tag{4.1a}$$

$$\int_E \nabla p^*_h \cdot \nabla q\, dV = -\left[\boldsymbol{F}_h, (\nabla q)^I\right]_E \quad \text{for each } q \in \mathcal{P}^2(E)\backslash\mathbb{R} \tag{4.1b}$$

for all $E \in \mathcal{T}_h$.

Since the present mimetic formulation approximates the scalar solution field by a piecewise constant function, the convergence rate of p^*_h to the exact solution field p

is expected to equal the convergence rate of the low-order scheme [12]. Therefore, we have

$$\|p_h^* - p\|_{L^2(\Omega)} \leq Ch^2,$$

in accordance with the behavior that has been experimentally observed in the examples of [4]. Nevertheless, we get a better approximation of the gradient of the solution field within each mesh element by exploiting the more accurate representation of the solution flux. Note that the computational cost of the post-processing procedure is negligible since it is calculated element by element and, in addition, the related local matrix to be inverted turns out to be diagonal. Details concerning implementation can be found in [4].

4.1. Assumptions and preliminaries.

Let $E \in \mathcal{T}_h$. There exist an *elemental lifting operator* $R_E : X_h|_E \to H(\mathsf{div}, E)$ such that:

(L1), for all $\boldsymbol{G} \in X_h$, it holds

$$\text{div} R_E(\boldsymbol{G}_E) = (\text{div}_h \boldsymbol{G})_E \qquad \text{in } E \tag{4.2b}$$

$$R_E(\boldsymbol{G}_E)|_e \cdot \mathbf{n}_e = G_E^e \qquad \forall e \in \partial E \tag{4.2a}$$

(L2), for all vector fields whose components have a linear restriction on E, i.e. $\boldsymbol{G}_E = \boldsymbol{G}|_E \in (P_1(E))^d$, it holds

$$R_E(\boldsymbol{G}_E^I) = \boldsymbol{G}_E \ .$$

(L3), for all $\boldsymbol{G} \in X_h$, it holds

$$\rho_* h_E \sum_{e \in \partial E} \int_e |G^{(e)}(\boldsymbol{\xi})|^2 \, dS \leq \|R_E(\boldsymbol{G})\|^2_{L^2(E)} \leq \rho^* h_E \sum_{e \in \partial E} \int_e |G^{(e)}(\boldsymbol{\xi})|^2 \, dS \ ,$$

with the constant factors ρ_* and ρ^* independent of E.

Throughout the paper, we will refer to (L2) as the *P_1-compatibility condition* and to the operators that verify this condition as *P_1-compatible operators.* We also define the *global lifting operator* $R : X_h \to \left(L^2(\Omega)\right)^3$ that combines all element-wise contributions from the local lifting operators R_E.

We emphasize that *we do not need to give the explicit form of the lifting operator* R_E since the convergence analysis just requires its existence.

Note that the requirements of the lifting operator above are weaker than the ones assumed in [8]. Indeed, our operator R_E only needs to be stable with respect to the elemental norm, while it is not required to reproduce the scalar product exactly as given by condition (5.7) of Reference [8]. More precisely, it is unlikely that an operator R_E exists that satisfies Assumptions (L1) and (L2) and is also such that

$$\left[\boldsymbol{F}, \boldsymbol{G}\right]_E := \int_E \widetilde{\mathsf{K}}_E^{-1} R_E(\boldsymbol{F}) \cdot R_E(\boldsymbol{G}) \, dV \quad \forall \boldsymbol{F}, \boldsymbol{G} \in X_h \tag{4.3}$$

for all elements E, where $\widetilde{\mathsf{K}}_E$ is a constant approximation of K over the element E. We can see that by simply taking $\boldsymbol{F} = \left(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q)\right)^I$, where q is a second-order polynomial with zero average on E. Indeed, using definition (4.3) and (L2) we find:

$$\begin{aligned}[\boldsymbol{F},\boldsymbol{G}]_E &= \int_E \widetilde{\mathsf{K}}_E^{-1} R_E(\boldsymbol{F}) \cdot R_E(\boldsymbol{G})\, dV \\ &= \int_E \widetilde{\mathsf{K}}_E^{-1} R_E(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q)^{\boldsymbol{I}}) \cdot R_E(\boldsymbol{G})\, dV \\ &= \int_E \widetilde{\mathsf{K}}_E^{-1} \mathcal{P}_E^{(1)}(\mathsf{K}\nabla q) \cdot R_E(\boldsymbol{G})\, dV.\end{aligned}$$

On the other hand, the local consistency condition (L1) gives

$$\begin{aligned}[\boldsymbol{F},\boldsymbol{G}]_E &= \left[\left(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q)\right)^{\boldsymbol{I}}, \boldsymbol{G}\right]_E \\ &= -\int_E q \,\mathrm{div}_{h,E} \boldsymbol{G}\, dV + \sum_{e\in\partial E} \int_e \boldsymbol{G}^e(\boldsymbol{\xi}) q(\boldsymbol{\xi})\, dS \\ &= 0 + \int_{\partial E} \mathbf{n}_E \cdot R_E(\boldsymbol{G}) q\, dS \\ &= \int_E \mathrm{div}\big(q R_E(\boldsymbol{G})\big)\, dV \\ &= \int_E \nabla q \cdot R_E(\boldsymbol{G})\, dV + \int_E q \,\mathrm{div} R_E(\boldsymbol{G})\, dV \\ &= \int_E \nabla q \cdot R_E(\boldsymbol{G})\, dV + \mathrm{div}_{h,E}(\boldsymbol{G}) \int_E q\, dV \\ &= \int_E \nabla q \cdot R_E(\boldsymbol{G})\, dV + 0,\end{aligned}$$

where in the last step we also use that $q \in P_2(E)\backslash\mathbb{R}$. The comparison of the two above identities reveals that for every $q \in P_2(E)\backslash\mathbb{R}$, there must hold

$$\int_E \mathsf{K}^{-1}\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q) \cdot R_E(G)\, dV = \int_E \nabla q \cdot R_E(G)\, dV, \qquad \forall G \in X_h.$$

This condition is quite strong and may lead to a contradiction unless K is constant on the element E.

Although, for simplicity, we prefer to keep (L1)-(L3) as assumptions, from (HG)-(ME) it can be easily proved that such a lifting operator always exists. For example, one could directly build R_E by solving an *ad hoc* discrete $BDM_1 - P_0$ problem in E in accordance with the framework proposed in [2, 18, 22].

For each elemental lifting operator satisfying (L1)-(L3) we have the following approximation result.

LEMMA 4.1. *There exists a constant C independent of h_E such that for any $\boldsymbol{G} \in (H^2(E))^d$ there holds:*

$$\|\boldsymbol{G} - R_E(\boldsymbol{G}^{\boldsymbol{I}})\|_{L^2(E)} \leq C h_E^2 \|\boldsymbol{G}\|_{H^2(E)} \quad \forall E \in \mathcal{T}_h\ . \tag{4.4}$$

Proof. Let $E \in \mathcal{T}_h$. We add and substract $\boldsymbol{G}^{(1)}$, a linear interpolant of $\boldsymbol{G}$ in E, and we apply the triangle inequality and the estimate of the interpolation error to obtain

$$\|\boldsymbol{G} - R_E(\boldsymbol{G}^I)\|_{L^2(E)} \leq \|\boldsymbol{G} - \boldsymbol{G}^{(1)}\|_{L^2(E)} + \|\boldsymbol{G}^{(1)} - R_E(\boldsymbol{G})^I\|_{L^2(E)}$$
$$\leq C' h_E^2 \|\boldsymbol{G}\|_{H^2(E)} + \|\boldsymbol{G}^{(1)} - R_E(\boldsymbol{G}^I)\|_{L^2(E)} \,. \qquad (4.5)$$

We transform the second term of inequality (4.5) through properties (L2)-(L3) and noting that $\|(\boldsymbol{G}^{(1)} - \boldsymbol{G})^I_e\|_{L^2(e)} \leq \|\boldsymbol{G}^{(1)} - \boldsymbol{G}\|_{L^2(e)}$ since the component of the interpolation vector (2.6) for any given face e is the orthogonal projection of $\mathbf{n}_E^e \cdot (\boldsymbol{G}^{(1)} - \boldsymbol{G})$ onto $P_1(e)$, the space of linear functions defined on that face. We have:

$$\|\boldsymbol{G}^{(1)} - R_E(\boldsymbol{G}^I)\|^2_{L^2(E)} = \|R_E(\boldsymbol{G}^{(1)} - \boldsymbol{G}^I)\|^2_{L^2(E)}$$
$$\leq \rho^* h_E \sum_{e\in\partial E} \int_e |(\boldsymbol{G}^{(1)} - \boldsymbol{G})^I_e|^2 \, dS$$
$$\leq \rho^* h_E \sum_{e\in\partial E} \|\boldsymbol{G}^{(1)} - \boldsymbol{G}\|^2_{L^2(e)}.$$

Now, we apply Agmon inequality (3.1) with $\phi = (\boldsymbol{G}^{(1)} - \boldsymbol{G})_i$, i.e. for each spatial component labelled by $i = 1, \ldots, d$, thus leading to

$$\|\boldsymbol{G}^{(1)} - R_E(\boldsymbol{G}^I)\|^2_{L^2(E)} \leq h_E^4 C'' C^{\mathrm{AI}} \|\boldsymbol{G}\|^2_{H^2(E)} \,, \qquad (4.6)$$

where the constant C^{AI} is independent of h_E. Inequality (4.4) eventually follows by combining(4.5) and (4.6) and setting $C = 2\max\left(C', (C''C^{\mathrm{AI}})^{1/2}\right)$. □

The existence of a lifting operator satisfying conditions (L1)-(L3) makes it possible to reformulate the convergence result of the flux approximation stated in Theorem (3.4) as follows.

PROPOSITION 4.2. *Let $(\boldsymbol{F}, p)$ be the exact solution of* (2.1) *with $p \in H^3(\Omega)$, and $(\boldsymbol{F}_h, p_h)$ the mimetic solution in $X_h \times Q_h$. Under Assumptions (HG)-(ME) and (S1)-(S2), we have the error bound*

$$\|\boldsymbol{F} - R(\boldsymbol{F}_h)\|_{L^2(\Omega)} \leq Ch^2 \|p\|_{H^3(\Omega)}, \qquad (4.7)$$

where the constant C is independent of h.

Proof. Using (L3) and (S1) we have that:

$$\|R(\boldsymbol{F}^I - \boldsymbol{F}_h)\|^2_{L^2(\Omega)} \leq \rho^* \sum_{E\in\mathcal{T}_h} h_E \sum_{e\in\partial E} \|\boldsymbol{F}^I - \boldsymbol{F}_h\|^2_{L^2(e)}$$
$$\leq \frac{\rho^*}{\sigma_*} \sum_{E\in\mathcal{T}_h} |||\boldsymbol{F}^I - \boldsymbol{F}_h|||^2_E = \frac{\rho^*}{\sigma_*} |||\boldsymbol{F}^I - \boldsymbol{F}_h|||^2_{X_h}. \qquad (4.8)$$

We add and substract $R(\boldsymbol{F}^I)$ in the left-hand side of (4.7), apply the triangle inequality and inequality (4.8), and obtain:

$$\|\boldsymbol{F} - R(\boldsymbol{F}_h)\|_{L^2(\Omega)} \leq \|\boldsymbol{F} - R(\boldsymbol{F}^I)\|_{L^2(\Omega)} + \|R(\boldsymbol{F}^I - \boldsymbol{F}_h)\|_{L^2(\Omega)}$$
$$\leq \|\boldsymbol{F} - R(\boldsymbol{F}^I)\|_{L^2(\Omega)} + \sqrt{\frac{\rho^*}{\sigma_*}} |||\boldsymbol{F}^I - \boldsymbol{F}_h|||_{X_h}. \qquad (4.9)$$

The first term in the right-hand side of (4.9) is an interpolation error that is controlled by decomposing the L^2 norm on the partition $\mathcal{T}_h$ and applying the estimate provided by Lemma 4.1 with $\boldsymbol{G} = \boldsymbol{F}|_E$ for each element $E \in \mathcal{T}_h$. The second term in the right-hand side of (4.9) is the flux error estimated in Theorem 3.4. Proposition inequality (4.7) follows by combining these two results and properly setting the constant C. □

Let $q \in \cup_{E\in\mathcal{T}_h} H^1(E)$. We denote the jump of q across the internal face e by $[\![q]\!]_e$, and extend this definition to the boundary faces by taking $[\![q]\!]_e = q|_e$ for $e \subset \Gamma$. A priori estimates will be given using the norm:

$$||q||^2_{1,h} = \sum_{E\in\mathcal{T}_h} \Big(||\nabla q||^2_{L^2(E)} + \sum_{e\in\partial E} h_e^{-1} ||\, [\![q]\!]_e \,||^2_{L^2(e)} \Big). \tag{4.10}$$

When the trace of q is a continuous field at the internal faces of the mesh and $q|_\Gamma = 0$, the norm $||q||_{1,h}$ coincides with the H^1-seminorm of q, which is also the norm of $H^1_0(\Omega)$. Thus, norm (4.10) can be interpreted as a discrete extension of the H^1 Sobolev norm to the "broken" Sobolev space $\cup_{E\in\mathcal{T}_h} H^1(E)$. Note, indeed, that both numerical solutions and their post-processed counterparts are $\mathcal{T}_h$-piecewise discontinuous functions and for this reason they do not belong to $H^1(\Omega)$.

The *inf-sup* condition proved in [8], that we state for the sake of reference in the following lemma, still holds.

LEMMA 4.3. *For any $q \in Q_h$ there exists a vector $\boldsymbol{G} \in X_h$ such that:*

$$\big[q, \mathrm{div}_h \boldsymbol{G}\big]_{Q_h} = |||q|||^2_{Q_h} \quad \textit{and} \quad |||\boldsymbol{G}|||_{X_h} \leq C |||p_h - p^I|||_{Q_h}. \tag{4.11}$$

Furthermore, an *inf-sup* condition holds when using the norm $||\cdot||_{1,h}$.

LEMMA 4.4. *There exists a positive constant C' independent of h such that for any $q \in Q_h$ there exists a discrete vector $\boldsymbol{G} \in X_h$ satisfying*

$$\big[\mathrm{div}_h \boldsymbol{G}, q\big]_{Q_h} = ||q||^2_{1,h} \quad \textit{and} \quad ||R(\boldsymbol{G})||_{L^2(\Omega)} \leq C' ||q||_{1,h}\ . \tag{4.12}$$

Proof. Let $q \in Q_h$. We define the discrete vector $\boldsymbol{G} \in X_h$ through

$$G^{(e)} = \frac{1}{h_e} [\![q]\!]_e \quad \text{for every } e \in \mathcal{E}_h. \tag{4.13}$$

From to (2.12) and (4.2b), integrating by parts and summing over all elements and edges, we get

$$\begin{aligned} \big[\mathrm{div}_h \boldsymbol{G}, q\big]_{Q_h} &= \sum_{E\in\mathcal{T}_h} \int_E q_E \, \mathrm{div} R_E(\boldsymbol{G})\, dV = \sum_{E\in\mathcal{T}_h} \sum_{e\in\partial E} \int_E q_E \, R_E(\boldsymbol{G}) \cdot \mathbf{n}^e_E \, dS \\ &= \sum_{e\in\mathcal{E}_h} \int_e R(\boldsymbol{G}) \cdot \mathbf{n}^e [\![q]\!]_e \, dS\ . \end{aligned} \tag{4.14}$$

Using (4.2a) and (4.13), identity (4.14) becomes

$$\big[\mathrm{div}_h \boldsymbol{G}, q\big]_{Q_h} = \sum_{e\in\mathcal{E}_h} \frac{1}{h_e} \int_e [\![q]\!]^2_e \, dS = ||q||^2_{1,h},$$

thus proving the identity on the left of (4.12).

Using Assumption (L3), definition (4.13) and noting that $a^* h_E \leq h_e$ from (M2) allow us to obtain:

$$\begin{aligned} ||R_E(\boldsymbol{G})||^2_{L^2(E)} &\leq \rho^* h_E \sum_{e\in\partial E} \int_e \big|G^{(e)}(\boldsymbol{\xi})\big|^2 dS = \rho^* h_E \sum_{e\in\partial E} \frac{1}{h_e^2} \int_e \big|[\![q]\!]_e\big|^2 dS \\ &\leq \frac{\rho^*}{a^*} \sum_{e\in\partial E} \frac{1}{h_e} \int_e \big|[\![q]\!]_e\big|^2 dS\ . \end{aligned} \tag{4.15}$$

The right inequality in (4.12) is finally derived by summing inequality (4.15) over all the elements $E \in \mathcal{T}_h$, setting $C' = (\rho^*/a^*)^{1/2}$ and recalling the definition of $||q||_{1,h}$. □

4.2. Convergence of the scalar solution field.

The main result of the present subsection is given in Theorem 4.7, where we prove the error estimates for the approximation of p through the norms $||\cdot||_{1,h}$ and $|||\cdot|||_{Q_h}$. The theorem's proof relies on the error formula for the constitutive equation discussed in Lemma 4.5 and the inequality of Proposition 4.6. This latter, in particular, is stated in a more general form than that required by Theorem 4.7. This generality will be useful in the analysis of the post-processed numerical solution of the next sub-section.

LEMMA 4.5 (Constitutive error equation). *Let $(\boldsymbol{F}, p)$ be the exact solution of* (2.1), *and $(\boldsymbol{F}_h, p_h)$ the mimetic solution in $X_h \times Q_h$. Under Assumptions (L1)-(L3), for every $\boldsymbol{G} \in X_h$ there holds:*

$$\left[p_h - p^I, \mathrm{div}_h \boldsymbol{G}\right]_{Q_h} = \left[\boldsymbol{F}_h, \boldsymbol{G}\right]_{X_h} + \int_\Omega \nabla p \cdot R(\boldsymbol{G})\, dV.$$

Proof. Using (??) we have:

$$\left[p_h - p^I, \mathrm{div}_h \boldsymbol{G}\right]_{Q_h} = \left[\boldsymbol{F}_h, \boldsymbol{G}\right]_{X_h} + \langle g, \boldsymbol{G}\rangle_\Gamma - \left[p^I, \mathrm{div}_h \boldsymbol{G}\right]_{Q_h}. \tag{4.16}$$

From (2.7), noting that $\mathrm{div} R(\boldsymbol{G}|_E)$ is constant on each element $E \in \mathcal{T}_h$, integrating by parts with $p|_\Gamma = g$ and using the flux equation (2.1a) yield the developments:

$$\begin{aligned} \left[p^I, \mathrm{div}_h \boldsymbol{G}\right]_{Q_h} &= \int_\Omega p^I\, \mathrm{div} R(\boldsymbol{G})\, dV = \int_\Omega p\, \mathrm{div} R(\boldsymbol{G})\, dV \\ &= -\int_\Omega \nabla p \cdot R(\boldsymbol{G})\, dV + \int_\Gamma g\, \mathbf{n} \cdot R(\boldsymbol{G})\, dS\,. \end{aligned} \tag{4.17}$$

Equation (4.2b) and (2.13) allow us to transform the last term in (4.17) as follows:

$$\int_\Gamma g\, \mathbf{n} \cdot R(\boldsymbol{G})\, dS = \sum_{e \subset \Gamma} \int_e g \mathbf{n}^e \cdot R(\boldsymbol{G})\, dS = \sum_{e \subset \Gamma} \int_e g(\boldsymbol{\xi}) G^{(e)}(\boldsymbol{\xi})\, dS = \langle g, \boldsymbol{G}\rangle_\Gamma. \tag{4.18}$$

Substituting (4.17) into (4.16), using (4.18) to cancel the boundary term, and the Cauchy-Schwarz inequality yield:

$$\left[p_h - p^I, \mathrm{div}_h \boldsymbol{G}\right]_{Q_h} = \left[\boldsymbol{F}_h, \boldsymbol{G}\right]_{X_h} + \int_\Omega \nabla p \cdot R(\boldsymbol{G})\, dV.$$

□

PROPOSITION 4.6. *Let $(\boldsymbol{F}, p)$ be the exact solution of* (2.1) *with $p \in H^3(\Omega)$, and $(\boldsymbol{F}_h, p_h)$ the mimetic solution in $X_h \times Q_h$. Then, there exists a constant C independent of h such that for every collection of vectors $\{\boldsymbol{G}_E\}_E$ with every $\boldsymbol{G}_E$ being in the local restriction $X_{h,E}$ there holds:*

$$\begin{aligned} &\sum_{E \in \mathcal{T}_h} \left(\left[\boldsymbol{F}_h, \boldsymbol{G}_E\right]_E + \int_E \nabla p \cdot R_E(\boldsymbol{G}_E)\, dV \right) \\ &\qquad \le C h^2 ||p||_{H^3(\Omega)} \left(\sum_{E \in \mathcal{T}_h} ||R_E(\boldsymbol{G}_E)||^2_{L^2(E)} \right)^{1/2}. \end{aligned} \tag{4.19}$$

Proof. Let us consider for every element $E \in \mathcal{T}_h$ the quadratic interpolant of p defined on E, which we denote by $p_E^{(2)} \in P_2(E)$. We add and substract $\nabla p_E^{(2)}$ to the argument of the elemental integral of (4.19) to obtain:

$$\int_E \nabla p \cdot R_E(\boldsymbol{G}_E)\,dV = \int_E \nabla p_E^{(2)} \cdot R_E(\boldsymbol{G}_E)\,dV + \int_E \nabla(p - p_E^{(2)}) \cdot R_E(\boldsymbol{G}_E)\,dV \quad (4.20)$$

Through the integration by parts, Assumptions(L1)-(L2) and local consistency (S2), the first term of the right-hand side of (4.20) becomes:

$$\begin{aligned}\int_E \nabla p_E^{(2)} \cdot R_E(\boldsymbol{G}_E)\,dV &= -\int_E p_E^{(2)} \mathrm{div}(R_E(\boldsymbol{G}_E))\,dV + \int_{\partial E} p_E^{(2)} \mathbf{n}_E^e \cdot R_E(\boldsymbol{G}_E)\,dS \\ &= -\int_E p_E^{(2)} \mathrm{div}_{h,E} \boldsymbol{G}_E\,dV + \sum_{e \in \partial E} \int_e p_E^{(2)}(\boldsymbol{\xi})(\boldsymbol{G}_E)^e(\boldsymbol{\xi})\,dS \\ &= \left[\left(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla p_E^{(2)})\right)^{\boldsymbol{I}}, \boldsymbol{G}_E\right]_E . \end{aligned} \quad (4.21)$$

Adding and substracting $\boldsymbol{F}^{\boldsymbol{I}}$ and using (4.20)-(4.21) makes it possible to reformulate the left-hand side of (4.19) as follows:

$$\begin{aligned}&\sum_{E \in \mathcal{T}_h} \left(\left[\boldsymbol{F}_h, \boldsymbol{G}_E\right]_E + \int_E \nabla p \cdot R_E(\boldsymbol{G}_E)\,dV \right) = \sum_{E \in \mathcal{T}_h} \Bigg(\left[\boldsymbol{F}_h - \boldsymbol{F}^{\boldsymbol{I}}, \boldsymbol{G}_E\right]_E \\ &\quad + \left[\left(\boldsymbol{F} + \mathcal{P}_E^{(1)}(\mathsf{K}\nabla p_E^{(2)})\right)^{\boldsymbol{I}}, \boldsymbol{G}_E\right]_E + \int_E \nabla(p - p_E^{(2)}) \cdot R_E(\boldsymbol{G}_E)\,dV \Bigg) \\ &\quad = T_1 + T_2 + T_3 .\end{aligned}$$

After using Cauchy-Schwarz inequality twice and by combining (S1) and (L3), we control T_1 through the error estimate of Theorem 3.4:

$$\begin{aligned} T_1 &\le \sum_{E \in \mathcal{T}_h} |||\boldsymbol{F}_h - \boldsymbol{F}^{\boldsymbol{I}}|||_E |||\boldsymbol{G}_E|||_E \\ &\le \left(\sum_{E \in \mathcal{T}_h} |||\boldsymbol{F}_h - \boldsymbol{F}^{\boldsymbol{I}}|||_E^2 \right)^{1/2} \left(\sum_{E \in \mathcal{T}_h} |||\boldsymbol{G}_E|||_E^2 \right)^{1/2} \\ &\le |||\boldsymbol{F}_h - \boldsymbol{F}^{\boldsymbol{I}}|||_{\mathcal{T}_h} \left(\frac{\sigma^*}{\rho_*} \sum_{E \in \mathcal{T}_h} ||R_E(\boldsymbol{G}_E)||_{L^2(E)}^2 \right)^{1/2} \\ &\le C' h^2 ||p||_{H^3(\Omega)} \left(\sum_{E \in \mathcal{T}_h} ||R_E(\boldsymbol{G}_E)||_{L^2(E)}^2 \right)^{1/2} , \end{aligned}$$

where C' absorbs the constant factor $\left(\sigma^*/\rho_*\right)^{1/2}$ and the constant of inequality (3.8), and is thus independent of h.

By applying the Cauchy-Schwarz inequality twice to T_2 we obtain:

$$\begin{aligned} T_2 &\le \sum_{E \in \mathcal{T}_h} |||\left(\boldsymbol{F} + \mathcal{P}_E^{(1)}(\mathsf{K}\nabla p_E^{(2)})\right)^{\boldsymbol{I}}|||_E\, |||\boldsymbol{G}_E|||_E \\ &\le \left(\sum_{E \in \mathcal{T}_h} |||\left(\boldsymbol{F} + \mathcal{P}_E^{(1)}(\mathsf{K}\nabla p_E^{(2)})\right)^{\boldsymbol{I}}|||_E^2 \right)^{1/2} \left(\sum_{E \in \mathcal{T}_h} |||\boldsymbol{G}_E|||_E^2 \right)^{1/2} . \end{aligned} \quad (4.22)$$

By using Assumption (S1), substituting $\boldsymbol{F} = -\mathsf{K}\nabla p$, and using Agmon inequality (3.0b) we control the argument of the first summation in (4.22) as follows:

$$\begin{aligned}
\left|\left|\left| \left(\boldsymbol{F} + \mathcal{P}^{(1)}_E(\mathsf{K}\nabla p^{(2)}_E)\right)^{\mathcal{I}} \right|\right|\right|^2_E &\leq \sigma^\star h_E \sum_{e\in\partial E} \|\left(\boldsymbol{F} + \mathcal{P}^{(1)}_E(\mathsf{K}\nabla p^{(2)}_E)\right)^{\mathcal{I}}\|^2_{L^2(e)} \\
&\leq \sigma^\star h_E \sum_{e\in\partial E} \|\mathsf{K}\nabla p - \mathcal{P}^{(1)}_E(\mathsf{K}\nabla p^{(2)}_E)\|^2_{L^2(e)} \\
&\leq \sigma^\star \widetilde{C} C_{\mathsf{K}} h^4_E \|p\|^2_{H^3(E)} \,. \qquad (4.23)
\end{aligned}$$

Combining (4.22) and (4.23), using Assumptions (S1) and (L3), and noting that $|||\boldsymbol{G}_E|||_E \leq (\sigma^\star/\rho_\star)^{1/2}\|R_E(\boldsymbol{G}_E)\|_{L^2(E)}$ give the inequality:

$$\begin{aligned}
T_2 &\leq \left(\sum_{E\in\mathcal{T}_h} \sigma^\star \widetilde{C} C_{\mathsf{K}} h^4_E \|p\|^2_{H^3(E)}\right)^{1/2} \left(\frac{\sigma^\star}{\rho_\star} \sum_{E\in\mathcal{T}_h} \|R_E(\boldsymbol{G}_E)\|^2_{L^2(E)}\right)^{1/2} \\
&\leq C'' h^2 \|p\|_{H^3(\Omega)} \left(\sum_{E\in\mathcal{T}_h} \|R_E(\boldsymbol{G}_E)\|^2_{L^2(E)}\right)^{1/2}
\end{aligned}$$

where $C'' = \sigma^\star \left(\widetilde{C} C_{\mathsf{K}}/\rho_\star\right)^{1/2}$ is independent of h.

We control T_3 as follows by applying the Cauchy-Schwarz inequality twice and using an estimate for the interpolation error (with constant C^I):

$$\begin{aligned}
T_3 &\leq \sum_{E\in\mathcal{T}_h} \|\nabla(p - p^{(2)}_E)\|_{L^2(E)} \|R_E(\boldsymbol{G}_E)\|_{L^2(E)} \\
&\leq \left(\sum_{E\in\mathcal{T}_h} \|\nabla(p - p^{(2)}_E)\|^2_{L^2(E)}\right)^{1/2} \left(\sum_{E\in\mathcal{T}_h} \|R_E(\boldsymbol{G}_E)\|^2_{L^2(E)}\right)^{1/2} \\
&\leq \left(\sum_{E\in\mathcal{T}_h} C^I h^4_E |p|^2_{H^3(E)}\right)^{1/2} \left(\sum_{E\in\mathcal{T}_h} \|R_E(\boldsymbol{G}_E)\|^2_{L^2(E)}\right)^{1/2} \\
&\leq C''' \|p\|_{H^3(\Omega)} \left(\sum_{E\in\mathcal{T}_h} \|R_E(\boldsymbol{G}_E)\|^2_{L^2(E)}\right)^{1/2}
\end{aligned}$$

and $C''' = C^I$.

The proposition eventually follows by taking $C = \max\left(C', C'', C'''\right)$. □

Note that the convergence proof in the norm $|||\cdot|||_{Q_h}$, which is given in the following theorem, is somewhat more general than the similar approximation result considered in [8]. More precisely, the proof discussed below does not require additional regularity assumptions on the shape of Ω, e.g. Ω is convex, neither that the flux inner product is defined through a lifting operator nor that the source term of the divergence equation belongs to $H^1(\Omega)$. Nonetheless, the higher-order approximation of the fluxes imposes the $H^3(\Omega)$ regularity to the exact scalar solution field in order to achieve optimal convergence rate. This requirement holds for the evaluation of the error in both $\|\cdot\|_{1,h}$ and $|||\cdot|||_{Q_h}$ norm.

THEOREM 4.7. *Let $p \in H^3(\Omega)$ be the exact solution of* (2.1) *and $p_h \in Q_h$ its mimetic approximation. Under Assumptions (K1)-(K2), (HG)-(ME), (L1)-(L3), (S1)-(S2), there exists a constant C independent of h such that:*

$$(i) \quad ||p_h - p^I||_{1,h} \leq Ch^2 ||p||_{H^3(\Omega)}$$

$$(ii) \quad |||p_h - p^I|||_{Q_h} \leq Ch^2 ||p||_{H^3(\Omega)}$$

The constant C only depends on the various constant factors introduced in (K1)-(K2), (M1)-(M4), (L1)-(L3), (S1)-(S2).

Proof. Both relations (i) and (ii) are proved through the same argument. The proof starts from the error equation provided by Lemma 4.5, then we apply the corresponding *inf-sup* condition, c.f. Lemmas 4.3-4.4, and finally we bound the flux error through the estimate given by the inequality of Proposition 4.6.

(i). Let $\boldsymbol{G} \in X_h$ be the vector associated to $q = p_h - p^I$ by the discrete *inf-sup* condition of Lemma 4.4. Lemma 4.5 and Proposition 4.6 imply:

$$||p_h - p^I||^2_{1,h} = \left[p_h - p^I, \mathrm{div}_h \boldsymbol{G}\right]_{Q_h} \leq Ch^2 ||p||_{H^3(\Omega)} \, ||R(\boldsymbol{G})||_{L^2(\Omega)}$$

and item (i) follows by using the inequality given in (4.12).

(ii). Let $\boldsymbol{G} \in X_h$ be the vector associated to $q = p_h - p^I$ by the discrete *inf-sup* condition of Lemma 4.3. Lemma 4.5 and Proposition 4.6 imply:

$$|||p_h - p^I|||^2_{Q_h} = \left[p_h - p^I, \mathrm{div}_h \boldsymbol{G}\right]_{Q_h} \leq Ch^2 ||p||_{H^3(\Omega)} \, ||R(\boldsymbol{G})||_{L^2(\Omega)}$$

and item (ii) follows by using the inequality given in (4.11) and Assumption (L3). ☐

4.3. Convergence of the post-processed solution.

Let us first define the scalar field p^* as the unique $\mathcal{T}_h$-piecewise quadratic polynomial satisfying

$$\int_E p^* \, dV = \int_E p \, dV \tag{4.24a}$$

$$\int_E \nabla p^* \cdot \nabla q \, dV = \int_E \nabla p \cdot \nabla q \, dV \qquad \text{for each } q \in \mathcal{P}^2(E) \backslash \mathbb{R} \tag{4.24b}$$

for all $E \in \mathcal{T}_h$. From the interpolation theory on star-shaped domains [6] the following estimate holds:

$$h_E^{-1} ||p - p^*||_{L^2(E)} + ||\nabla(p - p^*)||_{L^2(E)} \leq Ch_E^2 |p|_{H^3(E)}. \tag{4.25}$$

LEMMA 4.8. *Let $\mathcal{T}_h^e = \{E \in \mathcal{T}_h$ such that $e \in \partial E\}$. Then, there exists a constant C independent of h such that for every $e \in \mathcal{E}_h$ there holds*

$$h_e^{-1} || \, [\![p - p^*]\!]_e \, ||^2_{L^2(e)} \leq C \sum_{E \in \mathcal{T}_h^e} h_E^4 |p|^2_{H^3(E)}. \tag{4.26}$$

Proof. Let us derive (4.26) for the case an internal mesh face e, i.e. $\mathcal{T}_h^e = \{E_1, E_2\}$, and note that the case of a boundary mesh face, i.e. $\mathcal{T}_h^e = \{E\}$, can be treated by simply adapting the same argument. Separating the contributions from E_1 and E_2 into the jump term and using the triangular inequality give:

$$h_e^{-1} || \, [\![p - p^*]\!]_e \, ||^2_{L^2(e)} \leq 2 \sum_{E \in \mathcal{T}_h^e} h_e^{-1} ||(p - p^*)|_E||^2_{L^2(e)}.$$

By using Agmon inequality (2.3) from (M4) with $\phi = (p - p^\star)|_E$ and recalling that $h_e^{-1} \leq h_E^{-1} \big(a^\star\big)^{-1/2}$ from (M2), we have

$$h_e^{-1} \|(p - p^\star)|_E\|^2_{L^2(e)} \leq \big(a^\star\big)^{-1/2} \left(h_E^{-2} \|(p - p^\star)|_E\|^2_{L^2(E)} + |(p - p^\star)|_E|^2_{H^1(E)} \right),$$

and (4.26) follows from interpolation estimate (4.25). □

THEOREM 4.9.

$$\|p - p_h^\star\|_{1,h} \leq Ch^2 \|p\|_{H^3(\Omega)}$$

Proof. The approximation error for the post-processed solution field is split into two terms as follows:

$$\begin{aligned} \|p - p_h^\star\|^2_{1,h} &= \sum_{E \in \mathcal{T}_h} \left(\|\nabla(p - p_h^\star)\|^2_{L^2(E)} + \sum_{e \in \partial E} h_e^{-1} \| \,[\![\, p - p_h^\star \,]\!]_e \|^2_{L^2(e)} \right) \\ &= T_1 + T_2 \ . \end{aligned} \tag{4.27}$$

Let us add and substract $p^\star$ to the argument of T_1 and use the triangle inequality to obtain

$$\begin{aligned} T_1 &\leq \sum_{E \in \mathcal{T}_h} \|\nabla(p - p^\star)\|^2_{L^2(E)} + \sum_{E \in \mathcal{T}_h} \|\nabla(p^\star - p_h^\star)\|^2_{L^2(E)} \\ &= T_{1,1} + T_{1,2} \ . \end{aligned}$$

We bound $T_{1,1}$ through (4.25). To bound $T_{1,2}$, we first observe that from (4.1b) with $\nabla q = \nabla(p^\star - p_h^\star)$ and (4.24b) it follows that

$$\begin{aligned} T_{1,2} &= \sum_{E \in \mathcal{T}_h} \left(\int_E \nabla p^\star \cdot \nabla(p^\star - p_h^\star)\, dV - \int_E \nabla p_h^\star \cdot \nabla(p^\star - p_h^\star)\, dV \right) \\ &= \sum_{E \in \mathcal{T}_h} \left(\int_E \nabla p^\star \cdot \nabla(p^\star - p_h^\star)\, dV + \big[\boldsymbol{F}_h, \big(\nabla(p^\star - p_h^\star)\big)^{I}\big]_E \right) \\ &= \sum_{E \in \mathcal{T}_h} \left(\int_E \nabla p \cdot \nabla(p^\star - p_h^\star)\, dV + \big[\boldsymbol{F}_h, \big(\nabla(p^\star - p_h^\star)\big)^{I}\big]_E \right). \end{aligned} \tag{4.28}$$

We control the right-hand side of (4.28) through Proposition 4.6, i.e. inequality (4.19) with $\boldsymbol{G}|_E = \nabla(p^\star - p_h^\star)^I_E$. Assumption (L2) implies that

$$R_E(\nabla(p^\star - p_h^\star)^I_E) = \nabla(p^\star - p_h^\star)|_E$$

because the restriction of $\nabla(p^\star - p_h^\star)$ to E is a linear vector field. Therefore, we obtain that

$$\begin{aligned} T_{1,2} &\leq Ch^2 \|p\|_{H^3(\Omega)} \, \|R(\nabla(p^\star - p_h^\star)^I)\|_{L^2(\Omega)} \\ &= Ch^2 \|p\|_{H^3(\Omega)} \, \|\nabla(p^\star - p_h^\star)\|_{L^2(\Omega)} \\ &\leq Ch^2 \|p\|_{H^3(\Omega)} \, \|\nabla(p - p_h^\star)\|_{L^2(\Omega)}, \end{aligned} \tag{4.29}$$

where the last step follows from (4.24b) since $p_h^\star|_E \in \mathcal{P}^2(E) \backslash \mathbb{R}$. Combining (4.29) and the bound of $T_{1,1}$, it follows:

$$T_1 \leq Ch^2 \|p\|_{H^3(\Omega)} \, \|\nabla(p - p_h^\star)\|_{L^2(\Omega)} \leq Ch^2 \|p\|_{H^3(\Omega)} \, \|p - p_h^\star\|_{1,h} \ .$$

To bound the term T_2 in (4.27) we introduce the splitting

$$p_h^* = p_h + \widehat{p}_h \quad \text{and} \quad p^* = p^I + \widehat{p} \quad \text{with} \quad \widehat{p}_h, \widehat{p} \in \mathcal{P}^2(E)\backslash\mathbb{R}, \tag{4.30}$$

from which it readily follows that

$$[\![p^* - p_h^*]\!]_e = [\![p^I - p_h]\!]_e + [\![\widehat{p} - \widehat{p}_h]\!]_e.$$

Then, we control T_2 by the chain of inequalities:

$$\begin{aligned}
T_2 &\leq 2 \sum_{e\in\mathcal{E}_h} h_e^{-1} || [\![p - p_h^*]\!]_e ||^2_{L^2(e)} \\
&\leq \sum_{e\in\mathcal{E}_h} h_e^{-1} \Big(|| [\![p - p^*]\!]_e ||^2_{L^2(e)} + || [\![p^* - p_h^*]\!]_e ||^2_{L^2(e)} \Big) \\
&\leq \sum_{e\in\mathcal{E}_h} h_e^{-1} \Big(|| [\![p - p^*]\!]_e ||^2_{L^2(e)} + 2|| [\![p^I - p_h]\!]_e ||^2_{L^2(e)} + 2|| [\![\widehat{p}_h - \widehat{p}]\!]_e ||^2_{L^2(e)} \Big) \\
&= T_{2,1} + T_{2,2} + T_{2,3}.
\end{aligned}$$

To bound $T_{2,1}$, we use (4.26) and obtain:

$$T_{2,1} = \sum_{e\in\mathcal{E}_h} h_e^{-1} || [\![p - p^*]\!]_e ||^2_{L^2(e)} \leq C \sum_{E\in\mathcal{T}_h} h_E^4 |p|^2_{H^3(E)} \leq Ch^4 ||p||^2_{H^3(\Omega)}.$$

To bound $T_{2,2}$, we first note that

$$T_{2,2} = \sum_{e\in\mathcal{E}_h} h_e^{-1} || [\![p^I - p_h]\!]_e ||^2_{L^2(e)} \leq ||p^I - p_h||^2_{1,h},$$

and then we use the result of Theorem 4.7.

To bound $T_{2,3}$, we start separating the side contributions to the jump argument as in the proof of Lemma 4.8 and applying the Agmon inequality to each side term. Hence, by summing over all mesh faces we get:

$$\sum_{e\in\mathcal{E}_h} h_e^{-1} ||[\![\widehat{p}_h - \widehat{p}]\!]||^2_{L^2(e)} \leq C^{\mathrm{Ag}} N_E \sum_{E\in\mathcal{T}_h} \Big(h_E^{-2} ||\widehat{p}_h - \widehat{p}||^2_{L^2(E)} + |\widehat{p}_h - \widehat{p}|^2_{H^1(E)} \Big). \tag{4.31}$$

By definition $\widehat{p}_h^I = \widehat{p}^I = 0$; thus, using a standard interpolation estimate makes it possible to develop the first summation argument as follows:

$$h_E^{-2} ||\widehat{p}_h - \widehat{p}||^2_{L^2(E)} = h_E^{-2} ||(\widehat{p}_h - \widehat{p}) - (\widehat{p}_h - \widehat{p})^I ||^2_{L^2(E)} \leq C|\widehat{p}_h - \widehat{p}|^2_{H^1(E)}. \tag{4.32}$$

Using bound (4.32) into (4.31) and noting that $\nabla\big((\widehat{p}_h - \widehat{p})|_E\big) = \nabla\big((p_h^* - p^*)|_E\big)$, c.f. (4.30), yield:

$$T_{2,3} \leq C \sum_{E\in\mathcal{T}_h} ||\nabla(\widehat{p}_h - \widehat{p})||^2_{L^2(E)} = C \sum_{E\in\mathcal{T}_h} ||\nabla(p_h^* - p^*)||^2_{L^2(E)} = CT_{1,2}.$$

The estimate of $T_{2,3}$ terminates by using (4.29) to control $T_{1,2}$. □

The convergence result for the numerical approximation to the scalar solution p is completed by the following result that is an obvious corollary of the above theorem.

COROLLARY 4.10. *Under the same assumptions of Theorems 4.7 and 4.9 there holds:*

$$||p - p_h^*||_{L^2(\Omega)} + ||p - p_h^*||_{1,h} \leq Ch^2 ||p||_{H^3(\Omega)}.$$

The constant C only depends on the various constant factors introduced in (K1)-(K2), (M1)-(M4), (L1)-(L3), (S1)-(S2).

Proof. This corollary is an immediate consequence of the superconvergence result in Q_h-norm given by the second item of Theorem 4.7 and the convergence of the gradient approximation given in Theorem 4.9. □

5. Scalar product implementation. Without loss of generality, we reformulate Assumption (S2) on $P_2(E)\backslash\mathbb{R}$, which is the linear space of polynomials up to degree 2 having zero average on E. The dimension of $P_2(E)\backslash\mathbb{R}$ equals $\widetilde{m}_{P_2} = m_{P_2}-1$, where $m_{P_2} = (d+1)(d+2)/2$ is the dimension of $P_2(E)$. Since $\mathrm{div}_{h,E}\boldsymbol{G}$ is constant on E and the elemental integral of $q^{(2)}$ is zero, we obtain

$$\left[\left(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q^{(2)})\right)^I, \boldsymbol{G}\right]_E = \sum_{e\in\partial E} \sigma_E^e \int_e G^{(e)}(\boldsymbol{\xi}) q^{(2)}(\boldsymbol{\xi})\, dS,$$

which holds for every $\boldsymbol{G} \in X_h$ and $q^{(2)} \in P_2(E)\backslash\mathbb{R}$, and is equivalent to (S2).

To ease notation, we introduce the vector $\boldsymbol{G}_E$ representing the degrees-of-freedom of $\boldsymbol{G} \in X_h$ of the faces $e \in \partial E$. Low-order components are taken in $\boldsymbol{G}_E$ before the high-order components and all entries follow the local numbering of the element faces, e.g. $e_i \in \partial E$ for $i = 1,\ldots,m_E$. The structure of this vector is given by

$$\boldsymbol{G}_E^T = \left(G_0^{e_1},\ldots,G_0^{e_{m_E}},\quad \boldsymbol{G}_1^{e_1,T},\ldots,\boldsymbol{G}_1^{e_{m_E},T}\right), \tag{5.1}$$

and its size is equal to $d \times m_E$. Consistently with this notation, the restriction of $\boldsymbol{G}^I \in X_h$ to the faces of ∂E is given by the vector $\boldsymbol{G}_E^I$ having the structure of (5.1).

Now, the scalar product between the vectors $\boldsymbol{F}$ and $\boldsymbol{G}$ of X_h is locally implemented by means of the symmetric positive definite matrix M_E acting on their elemental restrictions $\boldsymbol{F}_E$ and $\boldsymbol{G}_E$:

$$\left[\boldsymbol{F},\boldsymbol{G}\right]_{X_h} = \sum_{E\in\mathcal{T}_h} \left[\boldsymbol{F},\boldsymbol{G}\right]_E \quad\text{with}\quad \left[\boldsymbol{F},\boldsymbol{G}\right]_E = \boldsymbol{G}_E^T\mathsf{M}_E\boldsymbol{F}_E. \tag{5.2}$$

The elemental matrix M_E is built as follows by Assumption (S2).

Let $\{q_i\}$ be some set of polynomials that form a basis for $P_2(E)\backslash\mathbb{R}$. The set of linearly independent vectors $\{\nabla q_i\}$ generates *a subspace* of $(P_1(E))^d$ formed by constant and linear vector fields on E. Following [4], we require that $\{\nabla q_i\}$ is an orthogonal set:

$$\int_E \nabla q_i \cdot \nabla q_j\, dV = |E|\delta_{ij}, \quad \text{for } i,j, = 1\ldots,\widetilde{m}_{P_2}. \tag{5.3}$$

Exploiting (5.3), it is possible to show that there exists a positive constant C_q independent of h_E such that

$$||q_i||_{L^\infty(E)} \leq C_q h_E \quad\text{and}\quad ||\nabla q_i||_{L^\infty(E)} \leq C_q \tag{5.4}$$

for every $i = 1,\ldots,\widetilde{m}_{P_2}$. The construction of a set of polynomials $\{q_i\}$ satisfying (5.3), and, consequently, having properties (5.4) is detailed in [4].

From Assumption (S2) on $q^{(2)} \in P_2(E)\backslash\mathbb{R}$ yields:

$$\begin{aligned}\left[\left(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q^{(2)})\right)^I, \boldsymbol{G}\right]_E &= \sum_{e\in\partial E}\int_e G^{(e)} q^{(2)}\, dS \\ &= \sum_{e\in\partial E} G_0^e \int_e q^{(2)}\, dS + \sum_{e\in\partial E} \boldsymbol{G}_1^e \cdot \int_e \frac{\boldsymbol{\xi}-\boldsymbol{\xi}_e}{h_e} q^{(2)}\, dS.\end{aligned} \tag{5.5}$$

We rewrite the right-hand side of (5.5) after introducing the j-th basis function q_j of $P_2(E)\backslash\mathbb{R}$ as

$$\sum_{e\in\partial E} G_0^e \int_e q_j\,dS + \sum_{e\in\partial E} \boldsymbol{G}_1^e \cdot \int_e \frac{\boldsymbol{\xi}-\boldsymbol{\xi}_e}{h_e} q_j\,dS = \boldsymbol{G}_E^T \mathsf{R}|_j, \tag{5.6}$$

where $\mathsf{R}|_j$ is the j-th column of the matrix

$$\mathsf{R} = \begin{pmatrix} \mathsf{R}_0 \\ \mathsf{R}_1 \end{pmatrix}. \tag{5.7}$$

The decomposition of R in (5.7) is induced by the structure of the elemental vectors in (5.1): the two submatrices R_0 and R_1 act, respectively, on the low- and high-order components of such vectors. Their j-th columns are written as:

$$\mathsf{R}_0|_j = \begin{pmatrix} \displaystyle\int_{e_1} q_j\,dS \\ \vdots \\ \displaystyle\int_{e_{m_E}} q_j\,dS \end{pmatrix} \quad\text{and}\quad \mathsf{R}_1|_j = \begin{pmatrix} \displaystyle\int_{e_1} \frac{\boldsymbol{\xi}-\boldsymbol{\xi}_{e_1}}{h_{e_1}} q_j\,dS \\ \vdots \\ \displaystyle\int_{e_{m_E}} \frac{\boldsymbol{\xi}-\boldsymbol{\xi}_{e_{m_E}}}{h_{e_{m_E}}} q_j\,dS \end{pmatrix}.$$

We now introduce the matrix N, whose j-th column is given by the degrees-of-freedom of $\big(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)\big)^I$ on the faces of ∂E:

$$\mathsf{N}|_j = \begin{pmatrix} \big(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)\big)_0^I \\ \big(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)\big)_1^I \end{pmatrix}, \tag{5.8}$$

and reformulate (5.2) as follows:

$$\big[\big(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)\big)^I, \boldsymbol{G}\big]_E = \boldsymbol{G}_E^T \mathsf{M}_E \mathsf{N}|_j.$$

Comparing (5.8) and (5.6) and using equality (5.5) yields the matrix relation:

$$\mathsf{M}_E \mathsf{N} = \mathsf{R}. \tag{5.9}$$

Let us now consider the vector $\boldsymbol{G}|_E = (\mathsf{K}\nabla q_i)^I = \mathsf{N}|_i$ in (5.5). Note that $\big((\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j))^I\big)^e$ is the L^2-projection of $\mathbf{n}^e \cdot \mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)$ onto the linear polynomials defined on e. As any linear combination of the components of ∇q_i is a linear function, these two terms coincide. Let

$$\overline{\mathsf{K}}_{ij} = \frac{1}{|E|}\int_E \mathsf{K}\nabla q_j \cdot \nabla q_i\,dV \tag{5.10}$$

for $i,j = 1,\ldots,\widetilde{m}_{P_2}$. The matrix $\overline{\mathsf{K}} = (\overline{\mathsf{K}}_{ij})$ is clearly symmetric. From the assumptions of strong ellipticity and orthogonality, i.e. (2.2) and (5.3), it follows that $\overline{\mathsf{K}}$ is positive definite and that its spectrum is included in the range $[\kappa_*, \kappa^*]$, c.f. [4]. As $\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)$ is a linear vector field, its divergence is constant on E. Using (S2), noting that the elemental average of the polynomial field q_i is zero, substituting (2.8) with

$\boldsymbol{v} = \nabla q_i$, and, using definition (5.10) in the last step we obtain:

$$
\begin{aligned}
\left[(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j))^{\mathcal{I}}, (\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_i))^{\mathcal{I}}\right]_E &= \sum_{e\in\partial E}\int_e \left((\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j))^{\mathcal{I}}\right)_E^e q_i\, dS \\
&= \sum_{e\in\partial E}\int_e \sigma_E^e \mathbf{n}^e \cdot \mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j) q_i\, dS \\
&= \int_E \mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)\cdot \nabla q_i\, dV + \int_E \operatorname{div}\big(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)\big)\, q_i\, dV \\
&= \int_E \mathsf{K}\nabla q_j \cdot \nabla q_i\, dV = |E|\overline{\mathsf{K}}_{ij}.
\end{aligned} \tag{5.11}
$$

Noting that

$$
\left[\big(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_j)\big)^{\mathcal{I}}, \big(\mathcal{P}_E^{(1)}(\mathsf{K}\nabla q_i)\big)^{\mathcal{I}}\right]_E = \mathsf{N}|_i^T \mathsf{M}_E \mathsf{N}|_j = \mathsf{N}|_i^T \mathsf{R}|_j, \tag{5.12}
$$

and comparing (5.11) and (5.12) for all columns of N and R yields the matrix relation:

$$
\mathsf{N}^T\mathsf{R} = |E|\overline{\mathsf{K}}. \tag{5.13}
$$

A simple calculation shows that

$$
\mathsf{N}^T\left(\mathsf{R}\frac{\overline{\mathsf{K}}^{-1}}{|E|}\mathsf{R}^T\right) = \mathsf{R}^T = \mathsf{N}^T\mathsf{M}_E \tag{5.14}
$$

because $\mathsf{M}_E = \mathsf{M}_E^T$. By comparison with (5.9), we find that a possible choice of M_E satisfying the local consistency (S2) is given by:

$$
\mathsf{M}_E = \mathsf{R}\frac{\overline{\mathsf{K}}^{-1}}{|E|}\mathsf{R}^T + \widetilde{\mathsf{M}}_E, \tag{5.15}
$$

where $\widetilde{\mathsf{M}}_E$ can be any real symmetric matrix of size $\widetilde{m}_{P_2}\times\widetilde{m}_{P_2}$ whose columns belong to the null space of N, i.e. $\mathsf{N}^T\widetilde{\mathsf{M}}_E = 0$. We take

$$
\widetilde{\mathsf{M}}_E = \mathsf{CUC}^T, \tag{5.16}
$$

where the columns of C form a basis set for $\mathsf{ker}(\mathsf{N})$ and U is any symmetric positive definite matrix (with product compatible size). The matrix U plays the role of a free coefficient matrix, and its optimal design is still an open issue even for the low order scheme. The size of the columns of N is given by $m_E \times d$ because it equals the total number of flux degrees-of-freedom of the elemental faces forming ∂E and the high-order flux approximation requires to specify d unknowns per elemental face. Recalling that N has $\widetilde{m}_{P_2}$ columns, we have

$$
\mathsf{dim}(\mathsf{ker}(\mathsf{N})) = m_E \times d - \widetilde{m}_{P_2} = \mathsf{rank}(\widetilde{\mathsf{M}}_E).
$$

The major properties of this construction are stated in the following theorem. The proof follows by repeating the stability analysis presented in [4] for the case of piecewise constant K, and, particularly, of Theorems 3.3 and 4.3 given therein. For this reason, it is omitted.

THEOREM 5.1.

(*i*) *Let* $\widetilde{\mathsf{M}}_E$ *be the matrix given by* (5.16). *Then,* M_E *defined by* (5.15) *is a symmetric positive definite matrix.*

(ii) Furthermore, assume that there exists two positive constants s_{U} and S_{U} independent of E such that

$$s_{\mathsf{U}}|E|\,\|\boldsymbol{v}\|^2 \le \left\|\mathsf{U}^{1/2}\mathsf{C}^T\boldsymbol{v}\right\|^2 \qquad \textit{for every } \boldsymbol{v} \in \textit{img}(\mathsf{C}),$$

and

$$\left\|\mathsf{U}^{1/2}\mathsf{C}^T\boldsymbol{v}\right\|^2 \le S_{\mathsf{U}}|E|\,\|\boldsymbol{v}\|^2 \qquad \textit{for every } \boldsymbol{v} \in \mathbb{R}^{d\,m_E - \tilde{m}_{P_2}}.$$

Then, there exist two positive constants σ_* and σ^* independent of h and U such that the matrix M_E satisfies Assumption (S1), and there holds

$$|E|\min\left\{\frac{1}{2}s_{\mathsf{U}},\sigma_*\right\}\|\boldsymbol{v}\|^2 \le \left\|\mathsf{M}_E^{1/2}\boldsymbol{v}\right\|^2 \le |E|\max\left\{\frac{1}{2}S_{\mathsf{U}},\sigma^*\right\}\|\boldsymbol{v}\|^2$$

for every $\boldsymbol{v} \in \mathbb{R}^{d\,m_E}$.

A precise expression for σ_* and σ^* in terms of the constants appearing in (M1)-(M4) can be derived by repeating the proof given in [4].

5.1. Further remarks on the scalar product formulation. From (5.13)-(5.14) it follows immediately that

$$\mathsf{N}^T\mathsf{M}_E\mathsf{N} = \mathsf{N}^T\mathsf{R},$$

which is also in accordance with [4, 10]. Consequently, if M_E is a symmetric matrix, $\mathsf{N}^T\mathsf{R}$ must be a symmetric matrix. Note that $\mathsf{N}^T\mathsf{R}$ is independent of the particular choice of the inner product $[\cdot,\cdot]_E$. Therefore, the condition that $\mathsf{N}^T\mathsf{R}$ be a symmetric matrix is a necessary condition for the symmetry of M_E that should come from the consistency condition. Let us consider the consistency condition of the original formulation [8, 10]

$$\begin{aligned}\left[(\mathsf{K}\nabla q^{(2)})^I, \boldsymbol{G}\right]_E + \int_E q^{(2)}\,\mathrm{div}_{h,E}\boldsymbol{G}\,dV \\ = \sum_{e\in\partial E}\sigma_E^e\int_e G^{(e)}(\boldsymbol{\xi})q^{(2)}(\boldsymbol{\xi})\,dS,\end{aligned} \tag{5.17}$$

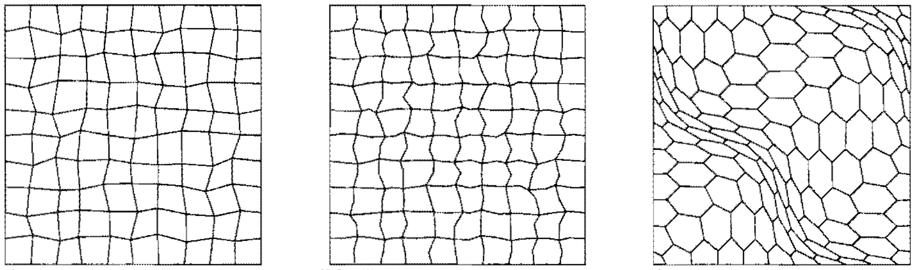
which is stated without the projection operator $\mathcal{P}_E^{(1)}(\,\cdot\,)$. We emphasize that taking (5.17) instead of (S2) is equivalent to defining the columns of the matrix N as

$$\mathsf{N}|_j = \begin{pmatrix} (\mathsf{K}\nabla q_j)_0^I \\ (\mathsf{K}\nabla q_j)_1^I \end{pmatrix}$$

instead of (5.8). The variation of K inside E can be taken into account through a high-order quadrature rule of Gaussian type. After this choice, the symmetry of $\mathsf{N}^T\mathsf{R}$ can be simply checked by hand calculation for some given element E and linear non-constant tensors K and ... it is generally false! So the point here is that whenever K is non constant the usual consistency condition (5.17) of the low order formulation is incompatible with the symmetry requirement of the scalar product matrix. Moreover, condition (5.17) is not satisfied by the classical BDM_1 finite element on triangles-tetrahedra when K is non constant.

It is worth noting that an alternative choice for the projection operator is given by:

$$\int_E \left(\mathcal{P}_E^{(1)}(\boldsymbol{u}) - \boldsymbol{u}\right)\cdot\mathsf{K}^{-1}\boldsymbol{v}\,dV = 0 \qquad \text{for every } \boldsymbol{v} \in P_1(E).$$



(a) $\mathcal{M}_1$: **randomly perturbed mesh with *convex* elements**

(b) $\mathcal{M}_2$: **randomly perturbed mesh with possibly *non-convex* elements**

(c) $\mathcal{M}_3$: **smooth mesh with mostly *hexagonal* elements**

FIG. 5.1. *The base mesh of the mesh families $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$.*

$$||\mathcal{P}_E^{(1)}(\boldsymbol{u})||_{L^2(E)} \leq C_{\mathsf{K}} ||\boldsymbol{u}||_{L^2(E)}$$

with $C_{\mathsf{K}} = \kappa^*/\kappa_*$. Indeed,

$$\begin{aligned} ||\mathcal{P}_E^{(1)}(\boldsymbol{u})||^2_{L^2(E)} &\leq \kappa^* \int_E \mathcal{P}_E^{(1)}(\boldsymbol{u}) \mathsf{K}^{-1} \mathcal{P}_E^{(1)}(\boldsymbol{u})\, dV \\ &= \kappa^* \int_E \mathcal{P}_E^{(1)}(\boldsymbol{u}) \mathsf{K}^{-1} \boldsymbol{u}\, dV \\ &\leq \kappa^* ||\mathsf{K}^{-1/2} \mathcal{P}_E^{(1)}(\boldsymbol{u})||_{L^2(E)}\, ||\mathsf{K}^{-1/2}\boldsymbol{u}||_{L^2(E)} \\ &\leq \left(\kappa^*/\kappa_*\right) ||\mathcal{P}_E^{(1)}(\boldsymbol{u})||_{L^2(E)}\, ||\boldsymbol{u}||_{L^2(E)}. \end{aligned}$$

The approximation properties of Lemma 2.1 still hold but in this case the constant C in (2.9) depends on C_{K}.

This second option offers the advantage that the scalar product $\left[\cdot,\cdot\right]_{X_h}$ provides an exact formula for the K^{-1}-weighted scalar product of linear vectors. Therefore, it exactly holds:

$$\left[\boldsymbol{v}^I, \boldsymbol{w}^I\right]_E = \int_E \boldsymbol{v} \cdot \mathsf{K}^{-1} \boldsymbol{w}\, dV \qquad \text{for every } \boldsymbol{v}, \boldsymbol{w} \in \left(P_1(E)\right)^d.$$

On the other hand, it requires the numerical integration of rational functions due to the expression of K^{-1}. From a rather extensive suite of experiments, we can reasonably claim that this latter formulation seems to perform as well as the former based on (2.8).

6. Numerical Experiments. The numerical experiments presented in this section are aimed to confirm the optimal behavior of the flux approximation provided by the current high-order mimetic formulation. We also characterize the convergence behavior of the post-processed solution p_h^* to the exact solution p. To this purpose, we solve (2.1) on the domain $\Omega =]0,1[\times]0,1[$ by applying the present high-order formulation and the low-order MFD method [10] to the benchmark problem having exact solution

$$p(x,y) = \sin(2\pi\, x)\sin(2\pi\, y) + x^3 + x^2\, y + x\, y^2 + y^3,$$

	i	$\#E$	$\#e$	$\#\mathsf{v}$	h
	1	100	220	121	$1.350\,10^{-1}$
	2	400	840	441	$6.969\,10^{-2}$
$\mathcal{M}_1$	3	1600	3280	1681	$3.572\,10^{-2}$
	4	6400	12960	6561	$1.772\,10^{-2}$
	5	25600	51520	25921	$8.901\,10^{-3}$
	1	100	440	341	$7.969\,10^{-2}$
	2	400	1680	1281	$3.968\,10^{-2}$
$\mathcal{M}_2$	3	1600	6560	4961	$2.069\,10^{-2}$
	4	6400	25920	19521	$1.040\,10^{-2}$
	5	25600	103040	77441	$5.253\,10^{-3}$
	1	121	400	280	$9.655\,10^{-2}$
	2	441	1400	960	$4.941\,10^{-2}$
$\mathcal{M}_3$	3	1681	5200	3520	$2.496\,10^{-2}$
	4	6561	20000	13440	$1.251\,10^{-2}$
	5	25921	78400	52480	$6.260\,10^{-3}$

TABLE 6.1
Run parameters for the mesh suites $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$

and smoothly variable diffusion tensor

$$\mathsf{K}(x,y) = \begin{pmatrix} (x+1)^2 + y^2 & -x\,y \\ -x\,y & (x+1)^2 \end{pmatrix}.$$

These examples are solved by a C^{++} program based on a variant of P2MESH [5], a public domain library designed to manage data structures of two-dimensional unstructured meshes.

Table 6.2 report the approximation errors and convergence rates obtained by solving the model equation on three different sets of successively refined meshes. The mesh construction is detailed in [4, 8, 18]. Mesh details about number of elements, edges, vertices and mesh size parameters are reported for each mesh considered in these experiments in Table 6.1. The first mesh of each mesh family is also shown in Figure 5.1. Approximation errors are measured by the following quantities:

$$\mathcal{E}_{Q_h}(p_h) = \frac{|||p^I - p_h|||_{Q_h}}{|||p^I|||_{Q_h}}, \qquad \text{and} \qquad \mathcal{E}_{X_h}(\boldsymbol{F}_h) = \frac{|||\boldsymbol{F}^I - \boldsymbol{F}_h|||_{X_h}}{|||\boldsymbol{F}^I|||_{X_h}},$$

where $|||\cdot|||_{Q_h}$ and $|||\cdot|||_{X_h}$ are the norms induced by the scalar products in Q_h and X_h, c.f. Section (2). For the post-processed solution, we consider the relative errors given by

$$\mathcal{E}_{1,h}(p_h^*) = \frac{||p - p_h^*||_{1,h}}{||p||_{1,h_0}},$$

where $||\cdot||_{1,h}$ is the norm defined in (4.10), and the denominator in the second formula is calculated by using the coarsest mesh. The convergence rate is evaluated from the relative errors with respect to the mesh size parameter h. Quadratic convergence rate is clearly seen in the flux approximation and for the post-processed solution gradient.

7. Conclusion. We considered a stationary diffusion problem with a full tensor coefficient discretized through the MFD method from [4, 18]. Under quite general assumptions on polygonal and polyhedral meshes, we proved second-order convergence

i	h	$\mathcal{E}_{Q_h}(p_h)$	Rate	$\mathcal{E}_{X_h}(\boldsymbol{F}_h)$	Rate	$\mathcal{E}_{1,h}(p_h^*)$	Rate
1	$1.350\,10^{-1}$	$2.161\,10^{-2}$	--	$6.358\,10^{-2}$	--	$7.858\,10^{-2}$	--
2	$6.969\,10^{-2}$	$6.437\,10^{-3}$	1.832	$1.115\,10^{-2}$	2.634	$1.850\,10^{-2}$	2.188
3	$3.572\,10^{-2}$	$1.671\,10^{-3}$	2.017	$2.293\,10^{-3}$	2.365	$4.409\,10^{-3}$	2.145
4	$1.772\,10^{-2}$	$4.241\,10^{-4}$	1.956	$5.463\,10^{-4}$	2.046	$1.088\,10^{-3}$	1.996
5	$8.901\,10^{-3}$	$1.064\,10^{-4}$	2.008	$1.340\,10^{-4}$	2.041	$2.708\,10^{-4}$	2.020

(*a*) *Approximation errors obtained using* $\mathcal{M}_1$

i	h	$\mathcal{E}_{Q_h}(p_h)$	Rate	$\mathcal{E}_{X_h}(\boldsymbol{F}_h)$	Rate	$\mathcal{E}_{1,h}(p_h^*)$	Rate
1	$7.969\,10^{-2}$	$2.274\,10^{-2}$	--	$5.910\,10^{-2}$	--	$6.692\,10^{-2}$	--
2	$3.968\,10^{-2}$	$6.462\,10^{-3}$	1.804	$1.167\,10^{-2}$	2.327	$1.670\,10^{-2}$	1.990
3	$2.069\,10^{-2}$	$1.659\,10^{-3}$	2.087	$2.702\,10^{-3}$	2.245	$4.104\,10^{-3}$	2.154
4	$1.040\,10^{-2}$	$4.187\,10^{-4}$	2.001	$6.594\,10^{-4}$	2.051	$1.023\,10^{-3}$	2.020
5	$5.253\,10^{-3}$	$1.049\,10^{-4}$	2.026	$1.635\,10^{-4}$	2.041	$2.555\,10^{-4}$	2.031

(*b*) *Approximation errors obtained using* $\mathcal{M}_2$

i	h	$\mathcal{E}_{Q_h}(p_h)$	Rate	$\mathcal{E}_{X_h}(\boldsymbol{F}_h)$	Rate	$\mathcal{E}_{1,h}(p_h^*)$	Rate
1	$9.655\,10^{-2}$	$2.327\,10^{-2}$	--	$5.247\,10^{-2}$	--	$6.532\,10^{-2}$	--
2	$4.941\,10^{-2}$	$7.284\,10^{-3}$	1.734	$1.214\,10^{-2}$	2.185	$1.833\,10^{-2}$	1.896
3	$2.496\,10^{-2}$	$1.990\,10^{-3}$	1.899	$2.942\,10^{-3}$	2.075	$4.785\,10^{-3}$	1.966
4	$1.251\,10^{-2}$	$5.155\,10^{-4}$	1.955	$7.327\,10^{-4}$	2.012	$1.217\,10^{-3}$	1.983
5	$6.260\,10^{-3}$	$1.309\,10^{-4}$	1.979	$1.837\,10^{-4}$	1.998	$3.063\,10^{-4}$	1.991

(*c*) *Approximation errors obtained using* $\mathcal{M}_3$

TABLE 6.2

Relative approximation errors for the conservative variable p_h*, the numerical flux* $\boldsymbol{F}_h$*, and the post-processed numerical solution* p_h^* *obtained by applying the high-order MFD method to the meshes of (a)* $\mathcal{M}_1$*, (b)* $\mathcal{M}_2$*, and (c)* $\mathcal{M}_3$*.*

of the conservative variable and its flux. The admissible polyhedral meshes can have degenerate and non-convex elements. We also developed a new family of high-order MFD methods for the case of non-constant coefficients. The theoretical results were confirmed by numerical experiments.

Acknowledgements. The work of the second author was supported by the DOE Office of Science Advanced Scientific Computing Research (ASCR) Program in Applied Mathematics Research.

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