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# **Ensemble Phase Averaged Equations for Multiphase Flows in Porous Media, Part 1: the Bundle-of-Tubes Model**

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## **Abstract**

A bundle-of-tubes construct is used as a model system to study ensemble averaged equations for multiphase flow in a porous material. Momentum equations for the fluid phases obtained from the method are similar to Darcy's law, but with additional terms. We study properties of the additional terms, and the conditions under which the averaged equations can be approximated by the diffusion model or the extended Darcy's law as often used in models for multiphase flows in porous media. Although the bundle-of-tubes model is perhaps the simplest model for a porous material, the ensemble averaged equation technique developed in this paper assumes the very same form in more general treatments described in Part 2 of the present work (Zhang 2009). Any model equation system intended for the more general cases must be understood and tested first using simple models. The concept of ensemble phase averaging is dissected here in physical terms, without involved mathematics through its application to the idealized bundle-of-tubes model for multiphase flow in porous media.

## **1. Introduction**

Flow of immiscible fluids in various porous materials is of special importance to soil science, chemical, environmental, construction, and petroleum industries. In these

systems, capillary action plays a crucial role in driving the motion of fluids within the porous media.(Alava *et al.* 2004; Faybishenko 2004; Hassanizadeh and Gray 1993; Washburn 1921; Richards 1931; Scheidegger 1974; Ianson and Hoff 1986; Hall *et al.* 1984; Gray and Hassanizadeh 1991; Liu 1991) Extensive theoretical investigations and advanced experimental techniques, such as neutron radiography, have been applied to study the motion of the fluids.(El Abd *et al.* 2005; Lockington and Parlange 2003; Gummerson *et al.* 1979; Shiozawa and Fujimaki 2004; Guen and Kovscek 2006; Manthey *et al.* 2005; Tsakiroglou *et al.* 2003; Gray *et al.* 2002; Beliaev and Hassanizadeh 2001; Hassanizadeh and Gray 1993; Czachor 2007; Hilfer 2006; Hall 2007; Culligan *et al.* 2004; Moseley and Dhir 1996) Despite intensive interrogation, models for such flows are still largely empirical. Typically, the motion of fluids in a porous material are assumed either to be nonlinear diffusion processes,(Lockington and Parlange 2003; Pachepsky *et al.* 2003; El Abd *et al.* 2005) or to obey the same Darcy's law as in a single phase flow. (Richards 1931; Brooks and Corey 1964) The diffusion approach is based on the early work of Washburn (1921) This approach is mainly used to model liquid imbibition in porous solids. The main focus for this model is liquid motion, while the motion of the other fluid, typically a gas, is not emphasized. In this paper we explain the reason why the diffusion approach is more successful in liquid-gas systems than in liquid-liquid systems. In the Darcy's law approach, each fluid in the porous material is driven by the pressure gradient. The application of Darcy's law to multiphase flows in porous materials is a significant extension of Darcy's law originally developed for a single phase flow in porous materials. For the cases of two-phase flow, this extension requires a closure model to describe the pressure difference between phases. The pressure difference is usually taken to be the capillary pressure (Leverett 1941) with an implicit assumption that the pressure difference is caused solely by the interfacial surface tension between fluids. In liquid-gas two-phase flows, it was originally thought that the pressure difference was only a function of the liquid saturation. Later, experimental observations found that the pressure difference exhibits more complicated characters. The validity of both the diffusion approach and extensions of Darcy's law have been questioned. (Shiozawa and Fujimaki 2004; Tsakiroglou *et al.* 2003; Hillel 1980; Hassanizadeh and Gray 1993; Prat 1995; Le Guen and Kovscek 2006; Hilfer 2006; Hall

2007; Czachor 2007; Gummerson et al. 1979; Lockington and Parlange 2003; DiCarlo 2007)

To develop equations describing the motion of fluids in porous materials, the multiphase flows in a porous material are treated as special cases of multi-material interactions involving two fluid phases and one porous solid phase using ensemble phase averaging method (Zhang 2009; Zhang et al. 2007). The closure relations associated with the averaged equations are expressed as averages of the interactions on phase interfaces. The averaged equations derived this way are in a form similar to the extended Darcy's law, but with additional terms. The presentation of this work is divided into two parts appearing as separate papers in this issue of the Journal. In Part 1, we illustrate the ensemble phase averaging method using a simple but meaningful example. General theory for multiphase flows in porous material is described in Part 2.

This paper is Part 1, in which we apply the ensemble averaging method to a bundle-of-tubes model for two-phase flows in porous materials. The main objective of this paper is to understand the additional terms that arise in the averaged equations, and to study their properties and the possible closure of the equations using this simple example. Although a typical porous material has a much more complex morphology than that captured by the bundle-of-tubes model, the model does possess a unique advantage of being conceptually simple and amenable to analytical solution. This porous material model has been used by Dahle *et al.* (2005) to study the behavior of the capillary pressure. By using the bundle-of-tubes model, we hope to explain the physical meaning of the closure relationships.

We will show that, despite of its simplicity, dissecting the averaged equations associated with the bundle-of-tubes model provides interesting insight into characteristics common to more complex porous materials. From the point view of the averaged equations, any theoretical model intended for describing two-phase flows should first be validated in a simple system such as this. The study of this simple bundle-of-tubes provides a starting point to study more general cases described in Part 2 of the present work.

## **2. Flow in capillary tubes**

We consider a one-dimensional two-phase flow in a porous material consisting of parallel capillary tubes with various diameters as shown in Figure 1. Let  $L$  be the length of the capillary tubes,  $\phi$  be the diameters of the capillary tube, and  $\beta$  be the contact angle of fluid 1 on the solid wall, measured from the solid wall in contact with fluid 1 to the interface between fluids 1 and 2. For capillary tubes, the curvature radius  $a$  of the fluid interface can be calculated as  $a = \phi/(2 \cos \beta)$ . In this example, we suppose that there is a reservoir of fluid 1 on the left of the capillary tubes. Initially the capillary tubes are filled with fluid 2 and are connected to a reservoir of fluid 2 on the right ends. Let  $p_L$  and  $p_R$  denote the pressures in left and right reservoirs, respectively, as illustrated in Figure 1. Inside a given capillary tube the flow is a Poiseuille flow, except for in the region close to the interface between two phases. The momentum equations can be written as

$$\frac{32\mu_1}{\phi^2} \bar{u}_1 = -\nabla p_1 = \frac{p_L - p_1(x_I)}{x_I}, \quad x < x_I \quad (1)$$

and

$$\frac{32\mu_2}{\phi^2} \bar{u}_2 = -\nabla p_2 = \frac{p_2(x_I) - p_R}{L - x_I}, \quad x > x_I \quad (2)$$

where  $\mu_1$  and  $\mu_2$  are viscosities,  $p_1(x)$  and  $p_2(x)$  are pressures for fluids 1 and 2 within the tube,  $x_I$  is the interface location as measured from the left end of the capillary tube. Because of the curvature of the fluid interface in a capillary tube, the interface location can only be determined within an error of order of the diameter of the tube. In this example we assume that the length of the tube is much greater than the tube diameter,  $L \gg \phi$ . As a consequence of this assumption, an error of order  $\phi/L$  is expected in the quantities calculated in this paper. In formulating the equations above we have used the fact that the diameter of a tube is independent of location  $x$ , and the pressures varies linearly within a given tube. Across the fluid interface, the pressure difference caused by surface tension is

$$p_2(x_I) - p_1(x_I) = 2\Gamma_{21}/a = 4\Gamma_{21} \cos \beta / \phi. \quad (3)$$

Continuity of the fluid phases requires  $\bar{u}_1 = \bar{u}_2$  in a capillary tube. Using this relation, we can eliminate the pressures  $p_1$  and  $p_2$  at the interface from (1) and (2), and find

$$\bar{u}_1 = \bar{u}_2 = \frac{\phi^2 \Delta p + 4\Gamma_{21} \phi \cos \beta}{32\mu_1 x_I + 32\mu_2 (L - x_I)}, \quad (4)$$

where  $\Delta p = p_L - p_R$  is the pressure difference between the reservoirs at the ends of the capillary tubes.

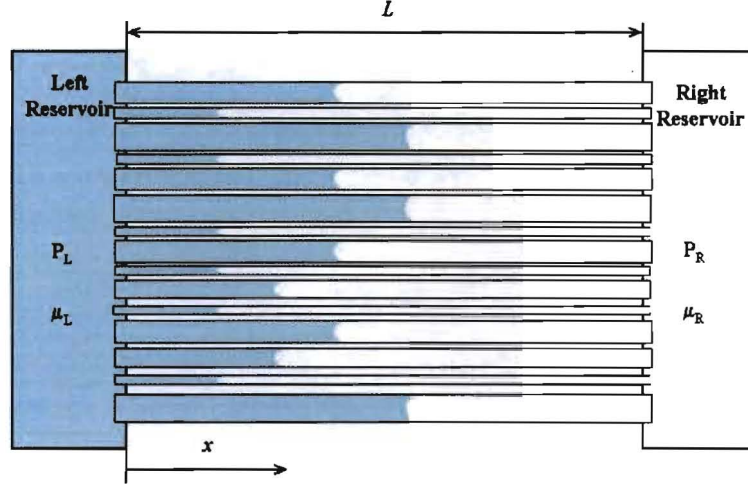


Figure 1. The illustration of ensemble of capillaries with different diameters.

For simplicity, we assume  $\Delta p \geq 0$  and  $\Gamma_{21} \geq 0$ , hence the velocity of the invading fluid  $I$  increases with the capillary diameter. The interface location can be calculated by solving  $dx_I/dt = \bar{u}_1$ . The solution is

$$\phi(\phi \Delta p + 4\Gamma_{21} \cos \beta)t = 16\mu_1 x_I^2 + 16\mu_2 (2Lx_I - x_I^2). \quad (5)$$

This solution relates the penetration of fluid  $I$  to the diameter of the capillary tube. Given a fluid interfacial position  $x$  and time  $t$ , we can use (5) to find a diameter  $\phi(x, t)$  of the tubes that the fluid interfaces are in. For a capillary tube with a specified diameter  $\phi$ , one can also use (5) to find the location of the fluid interface in the tube,  $x_I(\phi, t)$ . The interface location  $x_I(\phi, t)$  is an increasing function of  $\phi$ . The pressure within a capillary tube is found by substituting (4) into (1) and (2)

$$p_1(x, t, \phi) = p_L - \frac{\mu_1 (\Delta p + 4\Gamma_{21} \cos \beta / \phi) x}{\mu_1 x_I(\phi, t) + \mu_2 [L - x_I(\phi, t)]}, \quad x < x_I, \quad (6)$$

and

$$p_2(x, t, \phi) = p_R + \frac{\mu_2 (\Delta p + 4\Gamma_{21} \cos \beta / \phi) (L - x)}{\mu_1 x_I(\phi, t) + \mu_2 [L - x_I(\phi, t)]}, \quad x > x_I. \quad (7)$$

These results are obtained based on the Poiesuille flow in a circular capillary tube. Near a fluid interface the flow is not a true Poiesuille flow and the pressures are different. The size of the region where the pressures differ significantly is of the order of the diameter of the capillary tube and can be estimated from (6) and (7). When pressures derived from (6) and (7) are used to calculate the average pressures, or the phase interaction forces, an error of order  $\phi/L$  should be expected.

### 3. Ensemble Phase Averaged Equations

With the analytical solutions for pressures and velocities listed above, we can apply the ensemble phase averaging method to study properties of the resulting closure quantities in this simple geometry. The ensemble phase averaging method has been used previously to derive averaged equations for disperse multiphase flows. (Zhang and Prosperetti 1994, 1997) The method has also been extended for continuous multiphase flows with infinite number of degrees of freedom in the system. (Zhang *et al.* 2007) In the ensemble phase averaging method, at a given location  $x$  and time  $t$ , the average of a quantity pertaining to a specified phase is calculated by averaging over all the flows in which the specified location is occupied by that phase at the time. Although a general derivation of the averaged equations has been developed in Part 2 of this work (Zhang 2009), for the simple bundle-of-tubes geometry the derivation of the ensemble phase averaged equations can be significantly simplified. This simplified derivation illustrates the physical interactions on the phase interfaces with much less mathematical steps. In this section we present this derivation. For general derivations, readers are referred to the cited work (Zhang 2009; Zhang *et al.* 2007).

In the ensemble averaging method, the volume fraction  $\theta_i$  of fluid 1 at any point is calculated as the probability of the point occupied by the fluid. In the simple bundle-of-tubes model, this probability can be calculated geometrically as the ratio of the cross section area of the capillary tubes containing fluid 1 to the total area  $A_T$  of the cross section. Let  $N_T$  be the total number of tubes in the cross section and  $P(\phi)$  be the probability distribution of the tube diameters. For a specified location  $x$  and time  $t$ , the penetration depth of the invading fluid increases with the diameter of the tube. Thus fluid 1 will only occupy capillary tubes with a diameter larger than the  $\phi(x,t)$  calculated from

(5) by replacing  $x_i$  with  $x$ . The respective areas occupied by fluids 1 and 2 in the cross section are

$$A_1 = N_T \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} P(\phi) d\phi, \quad A_2 = N_T \int_0^{\phi(x,t)} \frac{\pi\phi^2}{4} P(\phi) d\phi, \quad (8)$$

and their volume fractions of fluids 1 and 2 can be calculated as  $\theta_i(x,t) = A_i / A_T$ , or

$$\theta_1(x,t) = n_A \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} P(\phi) d\phi, \quad \theta_2(x,t) = n_A \int_0^{\phi(x,t)} \frac{\pi\phi^2}{4} P(\phi) d\phi, \quad (9)$$

where  $n_A = N_T / A_T$  is the number of the tubes per unit cross section area. The gradient of the volume fractions can be calculated as

$$\nabla \theta_1(x,t) = -\nabla \theta_2(x,t) = -n_A \frac{\pi\phi^2}{4} P(\phi) \frac{\partial \phi(x,t)}{\partial x}. \quad (10)$$

The degree of saturation for fluid  $i$  can be defined as  $S_i = \theta_i / (\theta_1 + \theta_2)$  for  $i = 1, 2$ . Then,

$$S_1(x,t) = \int_{\phi(x,t)}^{\infty} \phi^2 P(\phi) d\phi / \int_0^{\infty} \phi^2 P(\phi) d\phi, \quad S_2(x,t) = \int_0^{\phi(x,t)} \phi^2 P(\phi) d\phi / \int_0^{\infty} \phi^2 P(\phi) d\phi. \quad (11)$$

For a given position and time, the average of a quantity pertaining to a phase is then calculated by averaging over all possible values of that quantity for which the spatial position of interest is occupied by the phase at the specified time. In this bundle-of-tubes model, the chance of a tube being selected as a sample is proportional to its cross section area. Hence in this example, the ensemble phase average of a quantity becomes a cross section area weighted average. Again, since the invading fluid only occupies the tubes with diameter greater than  $\phi(x,t)$ , the average pressure  $\langle p_1 \rangle$ , the average pressure gradient, and the average velocity  $\langle u_1 \rangle$  can be calculated as

$$\langle p_1 \rangle = \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} p_1(x,t,\phi) P(\phi) d\phi / \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} P(\phi) d\phi, \quad (12)$$

$$\langle \nabla p_1 \rangle = \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} \nabla p_1(x,t,\phi) P(\phi) d\phi / \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} P(\phi) d\phi, \quad (13)$$

$$\langle u_1 \rangle = \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} \bar{u}_1(x,t,\phi) P(\phi) d\phi / \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} P(\phi) d\phi. \quad (14)$$



Similarly the receding fluid occupies the tubes with the diameter less than  $\phi(x, t)$ , the cross section area weighted averages for the pressure, the pressure gradient and the velocity are given by

$$\langle p_2 \rangle = \int_0^{\phi(x,t)} \frac{\pi\phi^2}{4} p_2(x, t, \phi) P(\phi) d\phi \bigg/ \int_0^{\phi(x,t)} \frac{\pi\phi^2}{4} P(\phi) d\phi, \quad (15)$$

$$\langle \nabla p_2 \rangle = \int_0^{\phi(x,t)} \frac{\pi\phi^2}{4} \nabla p_2(x, t, \phi) P(\phi) d\phi \bigg/ \int_0^{\phi(x,t)} \frac{\pi\phi^2}{4} P(\phi) d\phi, \quad (16)$$

$$\langle u_2 \rangle = \int_0^{\phi(x,t)} \frac{\pi\phi^2}{4} \bar{u}_1(x, t, \phi) P(\phi) d\phi \bigg/ \int_0^{\phi(x,t)} \frac{\pi\phi^2}{4} P(\phi) d\phi. \quad (17)$$

The volume fractions and averages defined above, can be used to derive averaged equations. Using (9), we find

$$\theta_1(x, t) \langle p_1 \rangle(x, t) = n_A \int_{\phi(x,t)}^{\infty} \frac{\pi\phi^2}{4} p_1(x, t, \phi) P(\phi) d\phi. \quad (18)$$

where we employ the notation  $\langle p_1 \rangle(x, t) = \langle p_1(x, t, \phi) \rangle$ . Upon differentiation of (18) with respect to  $x$ , and using (9), (10), (12), and (13), we find

$$\theta_1 \langle \nabla p_1 \rangle = \theta_1 \nabla \langle p_1 \rangle - [\langle p_1 \rangle_I - \langle p_1 \rangle] \nabla \theta_1 \quad (19)$$

where  $\langle p_1 \rangle_I = p_1(x, t, \phi(x, t))$  is the pressure of fluid 1 on the phase interface with fluid 2.

Using this relation and averaging over the momentum equation (1), we obtain

$$n_A \int_{\phi(x,t)}^{\infty} 8\mu_1 \pi \bar{u}_1 P(\phi) d\phi = -\theta_1 \nabla \langle p_1 \rangle + F_{12}, \quad (20)$$

where

$$F_{12} = [\langle p_1 \rangle_I - \langle p_1 \rangle] \nabla \theta_1. \quad (21)$$

The left hand side of equation (20) represents the viscous drag acting on fluid 1. If we write the viscous drag on the left hand side of (20) as  $\theta_1 C_{1s} \mu_1 \langle u_1 \rangle$  then the drag coefficient can be calculated as

$$C_{1s} = \int_{\phi(x,t)}^{\infty} 32 \bar{u}_1 P(\phi) d\phi \bigg/ \int_{\phi(x,t)}^{\infty} \phi^2 \bar{u}_1(x, t, \phi) P(\phi) d\phi \quad (22)$$

With this definition, we have the averaged momentum equation for fluid 1

$$\theta_1 C_{1s} \mu_1 \langle u_1 \rangle = -\theta_1 \nabla \langle p_1 \rangle + F_{12}. \quad (23)$$

Similarly, the averaged momentum equation and the drag coefficient for fluid 2 can be written in the similar forms,

$$\theta_2 C_{2s} \mu_2 \langle u_2 \rangle = -\theta_2 \nabla \langle p_2 \rangle + F_{21} \quad (24)$$

$$C_{2s} = \int_0^{\phi(x,t)} 32 \bar{u}_2 P(\phi) d\phi \bigg/ \int_0^{\phi(x,t)} \phi^2 \bar{u}_2(x,t,\phi) P(\phi) d\phi, \quad (25)$$

with  $F_{21}$  defined as

$$F_{21} = [p_2(x,t,\phi(x,t)) - \langle p_2 \rangle] \nabla \theta_2 = -[\langle p_2 \rangle_I - \langle p_2 \rangle] \nabla \theta_1, \quad (26)$$

where  $\langle p_2 \rangle_I = p_2(x,t,\phi(x,t))$  is the pressure of fluid 2 on the interface. Using (21) and (26), and noting that  $p_1(x,t,\phi(x,t))$  and  $p_2(x,t,\phi(x,t))$  are evaluated at the interface, we find

$$F_{12}(x,t) + F_{21}(x,t) = [\langle p_2 \rangle(x,t) - \langle p_1 \rangle(x,t) - 4\Gamma_{21} \cos \beta / \phi(x,t)] \nabla \theta_1 \quad (27)$$

after using (3).

Although in this paper we derived momentum equations (23) and (24) in the bundle-of-tubes model for a porous material, the functional forms of these momentum equations are quite general since similar equations are obtained in a more general treatment (see Part 2) after neglecting inertial terms.

If Darcy's law were assumed to be valid for each fluid phase, the force densities  $F_{12}$  and  $F_{21}$  have to vanish simultaneously implying  $\langle p_2 \rangle - \langle p_1 \rangle = 4\Gamma_{21} \cos \beta / \phi(x,t)$ . This is in agreement with the original concept of capillary pressure. According to Dahle *et al.* (2005), the quantity  $4\Gamma_{21} \cos \beta / \phi(x,t)$  is a static part of the capillary pressure; and  $\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma_{21} \cos \beta / \phi(x,t)$  is a dynamic part of the capillary pressure. Equations (23) and (24) show that the dynamic part of the capillary pressure not only affects the pressure difference but also presents itself as a term in the momentum equations.

These momentum equations together with the continuity equations

$$\frac{\partial \theta_i}{\partial t} + \nabla \cdot (\theta_i \langle u_i \rangle) = 0, \quad i = 1, 2 \quad (28)$$

and the condition  $\theta_1 + \theta_2 = \theta_p$ , where  $\theta_p$  is the porosity, form a closed system of

equations provided a closure relation for  $\langle p_2 \rangle - \langle p_1 \rangle$  and  $F_{12}$  or  $F_{21}$  can be found. Although this averaging method does not give the functional forms for these closure quantities, it does provide an explicit way to calculate the closure relationship using related quantities evaluated at the phase interfaces. This is a significant advantage of the averaging method. For this simple bundle-of-tubes model, we have the analytical solutions for the quantities needed to specify the closure relationship. For more complex pore morphologies, numerical results can be used for this purpose. Of course, calculation of closure quantities is not required if the flow details in the pores are already known. It is hoped that by explicitly calculating and studying the closure quantities in the selected (simple) cases, one can obtain better understanding of the transport process and then formulate physically-based closure models for the more complicated cases.

#### 4. Properties of the closure relationships

In the bundle-of-tubes model, the key closure quantities, (i.e. the pressure difference, the drag coefficients, and the force densities  $F_{12}$  and  $F_{21}$ ), can be calculated explicitly for a specified tube diameters distribution  $P(\phi)$ . In this section, we take this advantage and calculate the closure quantities. To facilitate the study of the relative magnitudes of the terms in the averaged equations, we non-dimensionalize the key terms. Length is non-dimensionalized by the characteristic length  $L$  of the capillary tube; force is non-dimensionalized by  $\Gamma_{21}L$ ; and time is non-dimensionalized by  $\mu_1 L / \Gamma_{21}$ . The length of the capillary tube, the viscosity of fluid 1 and the surface tension between the fluids are thereby set to unity. Table 1 shows the value of other quantities under this non-dimensionalization scheme. In the following calculations, we assume that the probability distribution for the tube diameters is uniform between the smallest pore size  $\phi_s$  and the largest pore size  $\phi_L$  as

$$P(\phi) = 1 / (\phi_L - \phi_s) \quad (29)$$

Table 1. Parameters for bundle of tube models. The values of the dimensionless  $L$ ,  $\mu$  and  $\Gamma$  are equal to 1.

Parameter	Description	Value
$L$	Length of tube	1
$\mu_1$	Viscosity of fluid 1	1
$\Gamma$	Surface tension	1
$\phi_s$	Lower cut-off pore diameter	$10^{-8}L$
$\phi_L$	Upper cut-off pore diameter	$10^{-5}L$
$\beta$	Contact angle	0 (radians)

#### 4.1. Receding fluid with negligible viscosity

We first study a case in which the viscosity of fluid 2 is negligible and the pressures in both reservoirs are set to zero. This case resembles the scenario in which water replaces air in the capillary tubes at ambient conditions. Since the fluid 2 is inviscid, we have  $p_2 = \langle p_2 \rangle = 0$  and  $F_{21} = 0$ . Using (27) we have

$$F_{12} = -[\langle p_1 \rangle(x, t) + 4\Gamma_{21} \cos \beta / \phi(x, t)] \nabla \theta_1 \quad (30)$$

In this case, the interface location  $x$  in a specified tube can be solved from (5) and is proportional to the square root of time  $\sqrt{t}$ . As a consequence, at a given  $x$  and  $t$ , the pressure  $p_1(x, t)$  for fluid 1 as calculated in (6) can be expressed in terms of the grouping  $x/\sqrt{t}$ . The same is true for the tube diameter  $\phi$  solved from (5), the volume fraction  $\theta_1$  calculated using (9), the averaged pressure of fluid 1  $\langle p_1 \rangle$  calculated using (12), and the drag coefficient  $C_{1r}$  calculated using (22). The volume fraction, the saturation (defined as  $S_1 = \theta_1 / (\theta_1 + \theta_2)$ ), the drag coefficient, the average pressure, and the tube diameter can all be expressed as a single valued functions of  $x/\sqrt{t}$  for different time  $t$ , as shown in Figure 2. Similar profiles of  $S_1$  versus  $x/\sqrt{t}$  are commonly reported for water imbibition in building materials (Hall *et al.* 1984; Hall 2007; El Abd *et al.* 2005; Ridgway *et al.* 2006; Lockington and Parlange 2003).

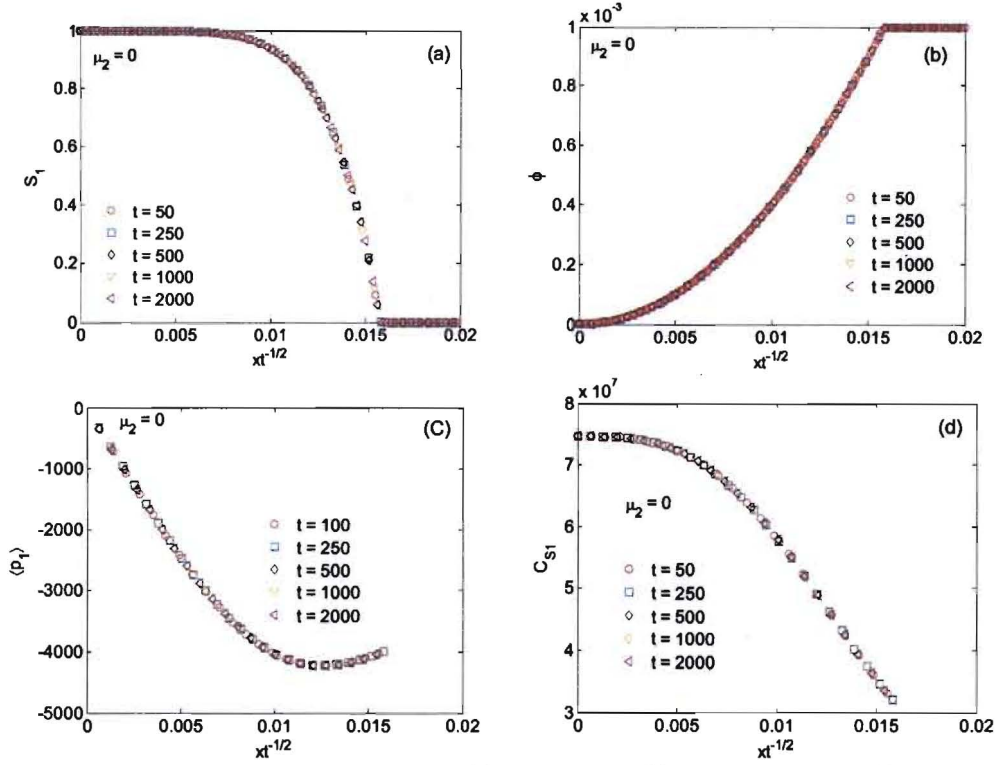


Figure 2. The correlations of  $S_I$  vs.  $x/\sqrt{t}$  (a),  $\phi$  vs.  $x/\sqrt{t}$  (b),  $\langle p_1 \rangle$  vs.  $x/\sqrt{t}$  (c), and  $C_{S1}$  vs.  $x/\sqrt{t}$  (d) at different imbibition times.

Figure 2(c), the plot of  $\langle p_1 \rangle$  vs.  $x/\sqrt{t}$ , suggests that the ensemble averaged pressure  $\langle p_1 \rangle$  is not a monotonic function of  $x/\sqrt{t}$ . The response seen in Figure 2(c) can be rationalized by referring to Figure 1 and recalling that in this case both  $p_L$  and  $p_R$  are zero. For a specified time, ensemble averaging over the tubes at small values of  $x$  involves tubes (of various diameters) that all contain the invading fluid 1. Within in each of these individual tubes the pressure decreases with increasing  $x$ . Thus the ensemble averaged pressure decreases with increasing  $x$  in this regime. However, when the ensemble average is performed at larger values of  $x$ , the tubes containing the invading fluid 1 are those tubes of larger diameter. This is a consequence of the fact that the velocity of invading fluid is slower in the smaller diameter tubes and thus at the specified time, and large enough values of  $x$ , the small diameter tubes are not yet filled by the invading fluid. At even larger values of  $x$ , only the very largest diameter tubes are filled with fluid 1. In such tubes the pressure drop due to friction with walls is relatively small

compared to that in smaller tubes; and the pressure in forefront of the invading fluid is close to the pressure on the fluid interface, which is  $-4\Gamma_{21} \cos \beta / \phi(x,t)$ , an increasing function of  $x$ , because  $\phi(x,t)$  increases with  $x$ . Since only the filled tubes are counted in the ensemble phase averaging procedure in the calculation of the average pressure of fluid 1. The resulting ensemble average then increases with  $x$  as shown in Figure 2(c). In this case, the extended Darcy's law is invalid in the forefront of the invading fluid since it would predict negative velocities for fluid 1, if used. Difficulties associated with direct application of Darcy's law to two-phase flows in a porous medium have been recognized previously (Hall *et al.* 1996; Nordbotten *et al.* 2008). In an attempt to amend Darcy's law, a "macroscopic pressure" is sometimes defined as a linear combination of the spatial derivatives of the volume averaged pressure of various orders (Nordbotten *et al.* 2008). In the case of this bundle-of-tubes model, the ensemble phase average can be regarded as a volume average with the representative volume being a slab perpendicular to the tube direction with infinitesimal thickness in the direction of flow. For the case of inviscid fluid 2, the "macroscopic pressure" of the fluid vanishes, and flux of fluid 1 is then proportional the gradient of the "macroscopic pressure" of fluid 1 according to equation (33) of Nordbotten *et al.* (2008). Since the flux is positive in this example, this requires the gradient of the "macroscopic pressure" of fluid 1 to be negative. However, as shown in Figure 2(c), the gradient of ensemble phase averaged pressure, which equals to the intrinsic volume averaged pressure, is positive. In other words, the gradient of the so-called "macroscopic pressure" and the gradient of the intrinsic volume averaged pressure have different signs; while the lowest order approximation to the "macroscopic pressure" is thought to be the intrinsic average pressure (Nordbotten *et al.* 2008).

From (11) we find that saturation  $S_1$  can be written as a single-valued function of diameter  $\phi$ . As mentioned above the diameter  $\phi(x,t)$  is a single-valued function of  $x/\sqrt{t}$ . Therefore the variable  $x/\sqrt{t}$  can also be regarded as a single-valued function of the fluid 1 saturation  $S_1$  in the region where the saturation  $0 < S_1 < 1$ , as shown in Figure 2(a). In this region, the diameter  $\phi(x,t)$ , the averaged pressure  $\langle p_1 \rangle$  and the drag

coefficient  $C_{1s}$  can be expressed as single-valued functions of the volume fraction  $\theta_1$ , or saturation  $S_1$ . With these relations we can rewrite the momentum equation for fluid 1 as

$$\theta_1 \langle u_1 \rangle = -D_1(S_1) \nabla S_1, \quad 0 < S_1 < 1, \quad (31)$$

where

$$D_1(S_1) = \frac{\theta_p}{C_{1s}(S_1)\mu_1} \left[ S_1 \frac{d\langle p_1 \rangle}{dS_1} + \langle p_1 \rangle(S_1) + 4\Gamma_{21} \cos \beta / \phi(S_1) \right]. \quad (32)$$

In this way the volume flux  $\theta_1 \langle u_1 \rangle$  per unit cross section area can be expressed in a form similar to Fick's law of diffusion with a saturation-dependent diffusion coefficient  $D_1(S_1)$ . This imbibition flow can be described as a diffusion process because the averaged pressure  $\langle p_1 \rangle$ , the surface tension term  $4\Gamma_{21} \cos \beta / \phi$ , and the drag coefficient  $C_{1s}$  depend only on the saturation. This explains the success of the diffusion approach in modeling imbibition of liquids when gas viscosity is negligible (Washburn 1921). However, this condition is only satisfied in the cases where the viscosity of the receding fluid is negligible and in the region where  $S_1$  is less than one. In cases where  $S_1 = 1$ , as shown in Figure 2(a), we have  $\nabla S_1 = 0$ , while the fluid flux  $\theta_1 \langle u_1 \rangle > 0$ ; therefore (31) is incorrect. In this region,  $F_{12} = 0$  according to (21), and thus the fluid is purely driven by the pressure gradient term in the momentum equation (23). Indeed, in this region, the average pressure  $\langle p_1 \rangle$  decreases linearly with  $x$  as shown in Figure 2(c) while the saturation remains constant. The derivative  $d\langle p_1 \rangle / dS_1$  then becomes undefined and the diffusivity defined by (32) becomes infinity as shown in Figure 3(a). This explains that many reported experimental values for  $D_1$  increase significantly (as much as  $10^4$  times), as the saturation approaches unity. (El Abd *et al.* 2005; Meyer and Warrick 1990) The fact that the diffusivity becomes undefined in the region of full saturation, highlights a limitation of the diffusion approach of describing fluid imbibition processes in porous materials.

This example demonstrates the importance of the additional force density term,  $F_{12}$ , in the ensemble averaged momentum equation (23). The force density  $F_{12}$  can be calculated from the pressure difference  $\langle p_1 \rangle_l - \langle p_1 \rangle$  by using (21). In Figures 3(b), (c),

and (d), the relation between the pressure difference  $\langle p_1 \rangle_I - \langle p_1 \rangle$ , the surface tension, and the drag coefficient are plotted as functions of the saturation  $S_1$  of fluid 1. In this case of negligible viscosity of fluid 2, all of these closure quantities for fluid 1 are single-valued functions of  $S_1$ .

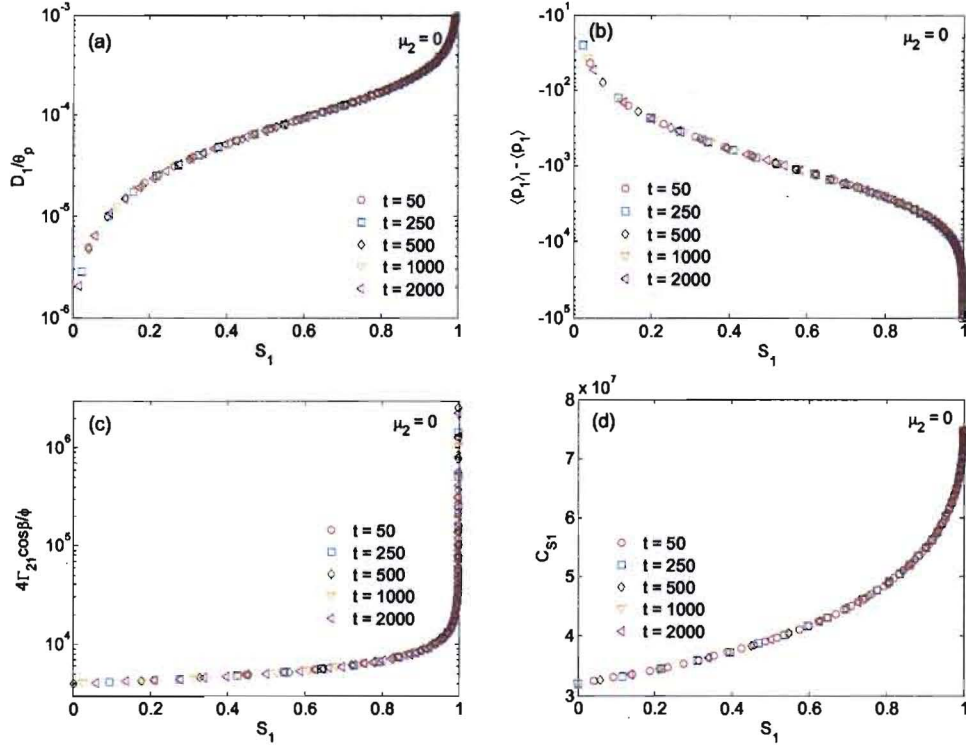


Figure 3. The correlations of  $D_1/\theta_p$  vs.  $S_1$  (a),  $\langle p_1 \rangle_I - \langle p_1 \rangle$  vs.  $S_1$  (b),  $4\Gamma_{21} \cos \beta / \phi$  vs.  $S_1$  (c), and  $C_{S1}$  vs.  $S_1$  (d).

#### 4.2. Receding fluid with finite viscosity

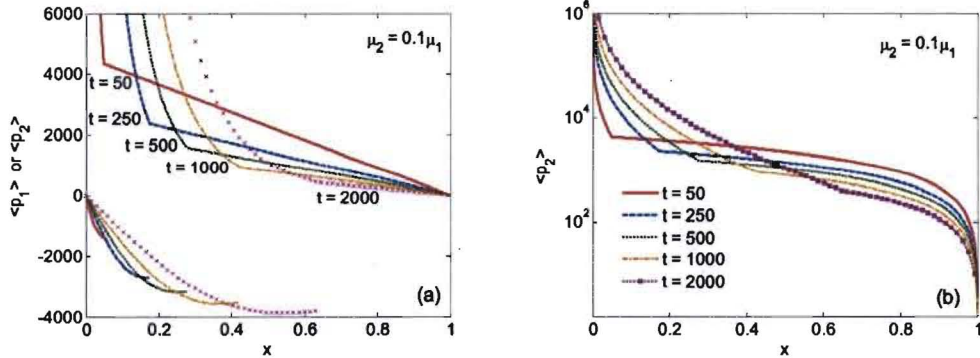
In this subsection we study a case in which the pressure difference between reservoirs is still zero, but viscosity  $\mu_2$  of fluid 2 is not negligible. The viscosity ratio,  $\mu_2/\mu_1$  between fluid 2 and fluid 1 is set to be in a range from 0.01 to 1. In this case, surface tension generates the pressure gradient needed to drive fluid 2, hence  $p_2 > 0$  and  $\langle p_2 \rangle > 0$ . Figure 4(a) plots the average pressures as a function of  $x$  at different time  $t$ . We note that there are kinks in the curves for the average pressure  $\langle p_2 \rangle$  of fluid 2. The curve for the



average pressure  $\langle p_1 \rangle$  stops at the  $x$ -coordinate of the kink point for the corresponding average pressure  $\langle p_2 \rangle$ . This kink point is the deepest penetration for the invading fluid at the specified time. After that point, the fluid 1 is not present and the average pressure of fluid 1 is undefined. After the kink, the average pressure  $\langle p_2 \rangle$  varies linearly with  $x$ . To explain this phenomenon, we note that similar to (19), we have

$$\theta_2 \nabla \langle p_2 \rangle = \theta_2 \langle \nabla p_2 \rangle + [\langle p_2 \rangle_I - \langle p_2 \rangle] \nabla \theta_2 \quad (33)$$

where  $\langle p_2 \rangle_I$  is the pressure of fluid 2 at the phase interface. When  $x$  is larger than the  $x$ -coordinate of the kink point, all the tubes are filled with fluid 2, and  $\nabla \theta_2 = 0$ . In each tube, according to (2), the pressure gradient  $\nabla p_2$  is independent of  $x$ , and so is its average  $\langle \nabla p_2 \rangle$ . The gradient of the averaged pressure calculated from (33) is then independent of  $x$ ; and  $\langle p_2 \rangle$  varies linearly with  $x$  as shown in Figure 4(a). In the coexisting region for both phases, the average pressures are highly nonlinear. Figure 4(b) shows that the magnitude of  $\langle p_2 \rangle$  is much larger than that of  $\langle p_1 \rangle$  because fluid 2 preferentially occupies small tubes. Figure 4(c) and (d) show the effect of viscosity ratio the average pressure profiles for the two phases at a specified time. As the viscosity ratio increases, the non-monotonic behavior of  $\langle p_1 \rangle$  disappears (e.g.  $\mu_2 = \mu_1$  as shown in Figure 4(c)).



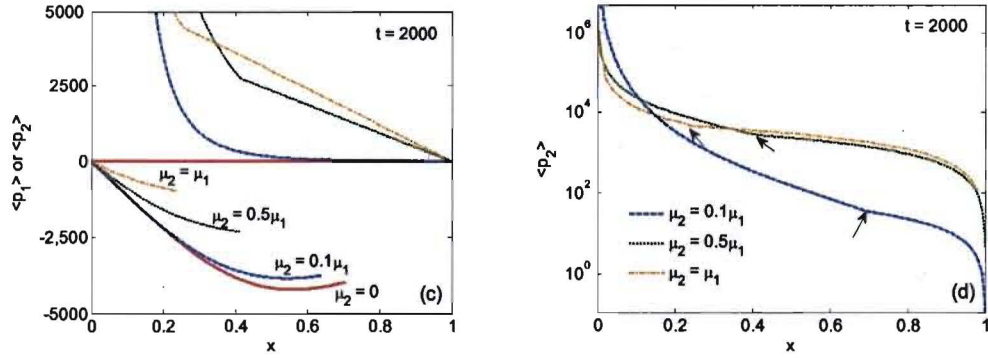


Figure 4. The profiles of the average pressures of two phases ( $\langle p_1 \rangle$  or  $\langle p_2 \rangle$ ) vs.  $x$  without the external pressure difference, (a) - (b) imbibition time effect, and (c) - (d) viscosity ratio effect (arrows in the plot indicate the kinks)

Figure 5(a) shows that the saturation is not a single valued function  $x/\sqrt{t}$  when the viscosity ratio of two phases is non-zero. Figure 5(b) and (c) show that the pressure difference  $\langle p_1 \rangle_i - \langle p_1 \rangle$  of fluid 1 at the interface and the pressure difference  $\langle p_2 \rangle - \langle p_1 \rangle$  between two phases depend not only on the saturation but also on time  $t$ . Figure 5(d) shows the drag coefficient  $C_{1s}$  as a weak function of time  $t$ .

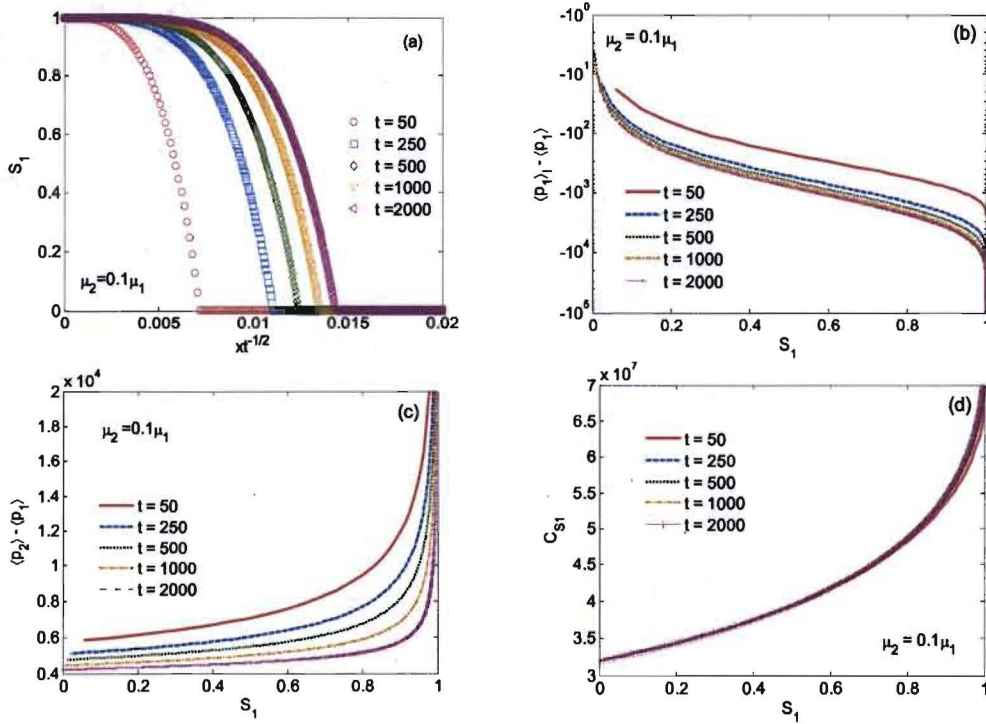


Figure 5. The correlations of  $S_l$  vs.  $x/\sqrt{t}$  (a),  $\langle p_1 \rangle_l - \langle p_1 \rangle$  vs.  $S_l$  (b),  $\langle p_2 \rangle - \langle p_1 \rangle$  vs.  $S_l$ , and  $C_{S_l}$  vs.  $S_l$  (d).

These results imply that a parameter in addition to the saturation  $S_l$  is needed to uniquely determine the closure relationships. In introducing the concept of dynamic capillary pressure, Hassanizadeh and Gray (1993), assumed that the dynamic part of the capillary pressure was proportional to the local time derivative  $\partial S_l / \partial t$ . This idea can be generalized to model the closure quantities as nonlinear functions of the saturation  $S_l$  and its local time derivative  $\partial S_l / \partial t$ . In the bundle-of-tubes model, for a given pair of  $S_l$  and  $\partial S_l / \partial t$ , there is a unique corresponding pair of  $x$  and  $t$ . With a uniform distribution of diameters (given by the probability distribution) between  $\phi_s$  and  $\phi_L$ , the required relationship can be found as follows. By differentiating (5) and (11) with respect to  $t$ , and then eliminating  $\partial \phi / \partial t$ , from the resulting relations we find

$$\frac{1}{t} = \frac{(\phi_L^3 - \phi_s^3)(2\phi\Delta p + 4\Gamma_{21} \cos \beta)}{3\phi^3(\phi\Delta p + 4\Gamma_{21} \cos \beta)} \frac{\partial S_l}{\partial t}. \quad (34)$$

Using (11) the diameter  $\phi$  can be expressed in terms of saturation  $S_l$  as

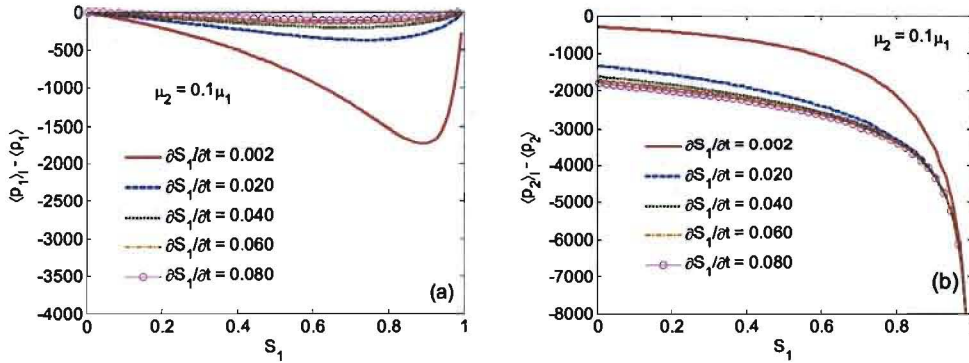
$$\phi(S_l) = \sqrt[3]{\phi_L^3 - S_l(\phi_L^3 - \phi_s^3)}. \quad (35)$$

With (34) and (35), the time  $t$  can then be expressed as a function of saturation  $S_l$  and its local time derivative  $\partial S_l / \partial t$ . Using (5) and (35), position  $x$  can also be expressed as a function of the saturation  $S_l$  and  $\partial S_l / \partial t$ . The closure quantities calculated at  $(x, t)$  can then be expressed as functions of these two primary variables,  $S_l$  and  $\partial S_l / \partial t$ . These functions are shown in Figure 6. For a fixed  $\partial S_l / \partial t$ , the relation between  $\langle p_1 \rangle_l - \langle p_1 \rangle$  and  $S_l$  is non-monotonic as shown in Figure 6(a). At a small  $\partial S_l / \partial t$ , this non-monotonic behavior is more pronounced. For a given  $\partial S_l / \partial t$ , initially the pressure difference  $\langle p_1 \rangle_l - \langle p_1 \rangle$  decreases with the saturation  $S_l$ , but as the saturation approaches to unity, the pressure difference starts to increase. This behavior may seem to be in contradiction to the pressure difference plotted in Figure 5(b). To explain this apparent contradiction, one needs to recall that these two figures are obtained under different conditions. Figure 5(b) is obtained at a constant time whereas Figure 6(a) is obtained at a fixed  $\partial S_l / \partial t (=$

$-\partial S_2/\partial t$ ). According to (4), flow velocity increases with the tube diameter and the invading fluid occupies the large tubes first. As the saturation  $S_1$  approaches unity and saturation  $S_2$  approaches zero, the diameters of the tubes containing fluid 2 (and thus the velocity of fluid 2) becomes small. Thus, the second term in the continuity equation for fluid 2 becomes small and negligible

$$\frac{\partial S_2}{\partial t} + S_2 \nabla \cdot \langle \mathbf{u}_2 \rangle + \langle \mathbf{u}_2 \rangle \cdot \nabla S_2 = 0. \quad (36)$$

The pressure difference  $\langle p_1 \rangle_I - \langle p_1 \rangle$  in Figure 6(a) is obtained with fixed  $\partial S_2/\partial t$ , thus as velocity  $\langle \mathbf{u}_2 \rangle$  decreases, the magnitude of  $|\nabla S_2|$  increases. Since the change in the saturation is caused by moving across the fluid interfaces, the large saturation gradient indicates a point close to fluid interfaces. Therefore the pressure difference  $\langle p_1 \rangle_I - \langle p_1 \rangle$  in Figure 6(a) is calculated with the average pressure  $\langle p_1 \rangle$  evaluated at a point close to the interface; and thus the pressure difference is of lower magnitude as seen in Figure 6(a). The same is true for the pressure difference  $\langle p_2 \rangle_I - \langle p_2 \rangle$  of fluid 2, however, it is not shown in Figure 6(b), because the minimum for this pressure difference occurs at saturation  $S_2 \approx 10^{-8}$  according to the assumed probability distribution (29) of the tube diameters. Figure 6(c) shows the average pressure difference between two phases as a function of  $S_1$  and  $\partial S_1/\partial t$ .





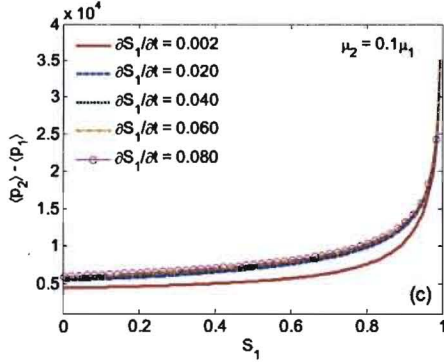


Figure 6. The effect of  $\partial S_2/\partial t$  on the profile of  $\langle p_1 \rangle_I - \langle p_1 \rangle$  vs.  $S_I$  (a),  $\langle p_2 \rangle_I - \langle p_2 \rangle$  vs.  $S_I$  (b), and  $\langle p_2 \rangle - \langle p_1 \rangle$  vs.  $S_I$  (c).

Similar to the observation in Figure 5(d), Figure 7(a) and 7(b) show that  $C_{IS}$  and  $C_{2S}$  are functions of  $S_I$  but are almost independent of  $\partial S_2/\partial t$ . This explains why the permeability (the inverse of these drag coefficients) is often reported not to be rate dependent, although it is often reported to be dependent on the degree of saturation (Brooks and Corey 1965; Braun *et al.* 2005).

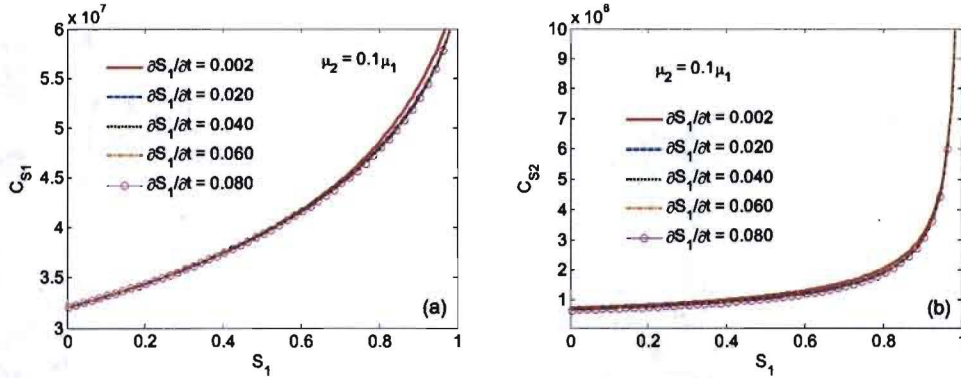


Figure 7. The effect of  $\partial S_2/\partial t$  on the profile of  $C_{S1}$  vs.  $S_I$  (a) and  $C_{S2}$  vs.  $S_I$  (b).

In this simple bundle-of-tubes model, the static part of the capillary pressure (Dahle *et al.* 2005),  $4\Gamma_{21} \cos \beta / \phi$ , is only a function of the saturation. This is because the tube diameter  $\phi$  is only a function of the saturation as shown in (11) and (35). In Figure 8, we display the dynamic part of the capillary pressure  $\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma \cos \beta / \phi$ . Initially it was thought (Hassanizadeh and Gray 1993) that the dynamic part of the capillary pressure was proportional to  $\partial S_1/\partial t$ . Later Dahle *et al.* (2005) found that when

$\partial S_1 / \partial t = 0$ , the dynamic part of the capillary pressure was not zero. They then modified the dynamic capillary pressure to contain two terms. The first term is a function of the saturation  $S_1$  only. The second term is proportional to  $\partial S_1 / \partial t$  with a coefficient depending on  $S_1$ . For cases with  $\mu_2 \ll \mu_1$ ,  $\langle p_2 \rangle = 0$  and  $\langle p_1 \rangle$  is a function of the saturation, as explained above. The dynamic part of the capillary pressure is then a function of saturation only and is independent of  $\partial S_1 / \partial t$  as shown in Figure 8. However, for the cases where  $\mu_2$  is not negligible, our results show that the capillary pressure depends on  $\partial S_1 / \partial t$  in a non-linear manner. The strongest response occurs at small values of  $\partial S_1 / \partial t$ . As  $\partial S_1 / \partial t$  increases, the value of  $\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma \cos \beta / \phi$  approaches a constant value.

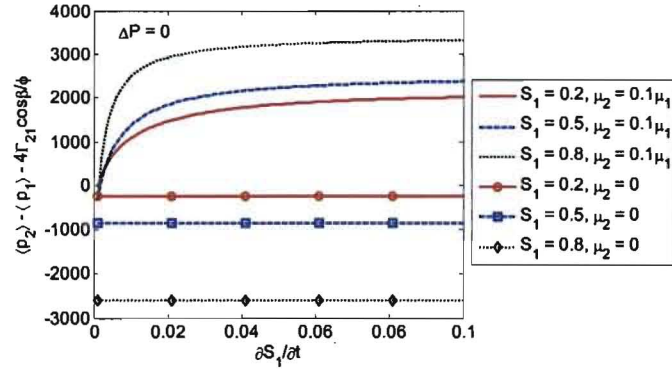


Figure 8. The correlation between  $\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma \cos \beta / \phi$  and  $\partial S_1 / \partial t$  at targeted  $S_1$  and  $\Delta P = 0$ .

To study the effects of the viscosity ratio, we fix the value of  $S_1$  at 0.5 and then plot  $\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma \cos \beta / \phi$  as a function of  $\partial S_1 / \partial t$  in Figure 9. The dynamic pressure is sensitive to  $\partial S_1 / \partial t$  for a small  $\partial S_1 / \partial t$  and approaches a constant for a large  $\partial S_1 / \partial t$ . When the viscosity ratio between the receding and the invading fluids decreases, the saturation rate dependent region of  $\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma \cos \beta / \phi$  expands.

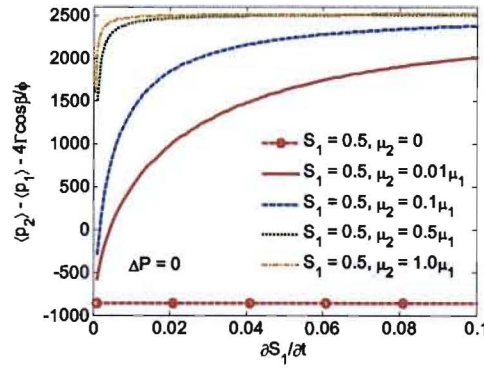
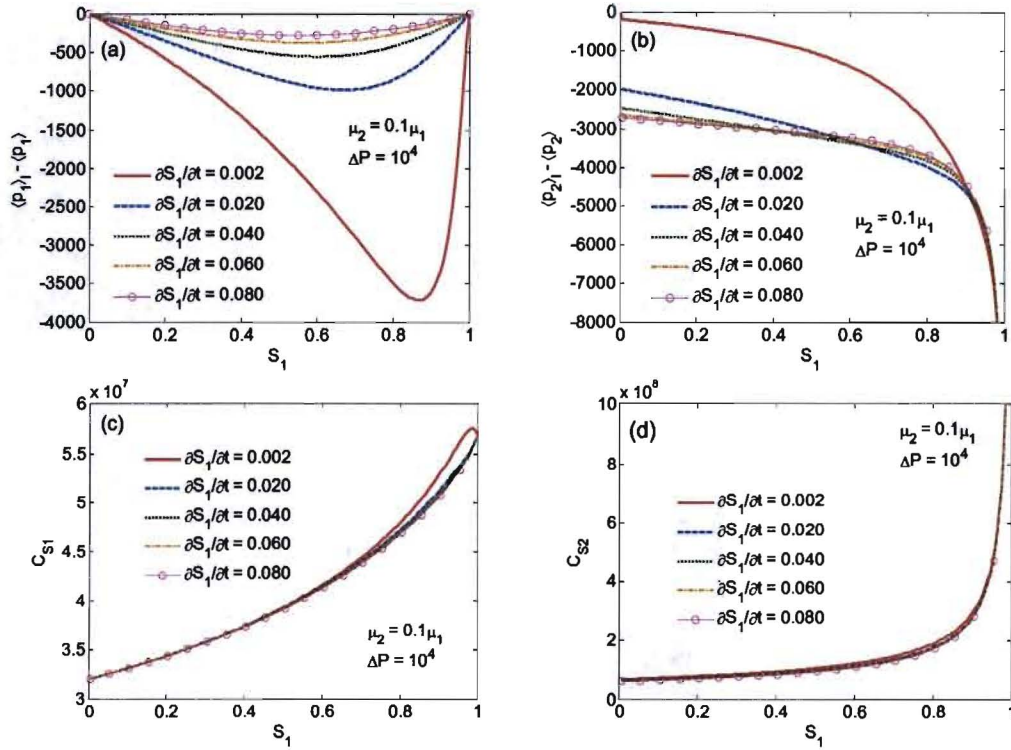


Figure 9. The correlation between  $\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma \cos \beta / \phi$  and  $\partial S_1 / \partial t$  at different viscosity at  $S_1 = 0.5$ .

#### 4.3. Effects of reservoir pressure difference

To examine the effects of the pressure difference in the reservoir, we recalculate all the quantities in Figure 6, 7 and 8 with  $\Delta P = 10^4$  and plot the results in Figure 10.



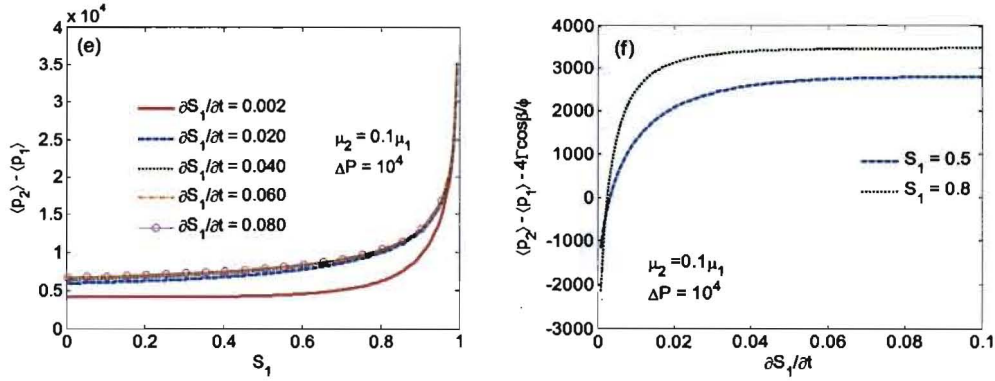


Figure 10. Plots  $\langle p_1 \rangle_I - \langle p_1 \rangle$  vs.  $S_I$  (a),  $\langle p_2 \rangle_I - \langle p_2 \rangle$  vs.  $S_I$  (b),  $C_{S1}$  vs.  $S_I$  (c),  $C_{S2}$  vs.  $S_I$  (d), and  $\langle p_2 \rangle - \langle p_1 \rangle$  vs.  $S_I$  (e) at different  $\partial S_I / \partial t$  with  $\Delta P = 10^4$ . The correlation between  $\langle p_2 \rangle - \langle p_1 \rangle - 4\Gamma \cos \beta / \phi$  and  $\partial S_I / \partial t$  at different  $S_I$  is plotted in (f).

With the positive pressure difference  $\Delta p$ , there is a larger velocity increase in large tubes than that in small tubes as one would expect. The increase in the penetration of fluid 1 is also more significant in larger tubes than that in small tubes, resulting in a larger spread in the co-existence regions of fluids. This leads to an increase in the average distance from a point  $x$  (where the average pressure is evaluated) to the interface where  $\langle p_1 \rangle_I$  and  $\langle p_2 \rangle_I$  are evaluated. This results in more significant pressure differences  $\langle p_1 \rangle_I - \langle p_1 \rangle$  and  $\langle p_2 \rangle_I - \langle p_2 \rangle$  in Figures 10(a) and (b) than the pressure differences in Figures 6 (a) and (b). The difference in the pressure difference  $\langle p_2 \rangle - \langle p_1 \rangle$  between phases shown in Figure 10(e), however, is almost the same as in Figure 6(c), because they are not directly related to external pressures. The drag coefficients of  $C_{S1}$  and  $C_{S2}$  in Figures 10(c) and (d) are almost the same as those in Figures 7(a) and (b). Again, the behavior of dynamic part of the capillary pressure shows a nonlinear dependence on  $\partial S_I / \partial t$  in Figure 10(f).

The average pressure difference  $\langle p_2 \rangle - \langle p_1 \rangle$  between phases is commonly called the capillary pressure implying an assumption that it is a result of surface tension. We now calculate a case without the surface tension. We keep all other parameters the same as in the last case except we set  $\Gamma = 0$ . The results of  $\langle p_1 \rangle_I - \langle p_1 \rangle$  are plotted in Figure 11(a). The curves behave similarly to the case plotted in Figure 6(a), but with smaller



magnitude due to the zero surface tension. The results of  $\langle p_2 \rangle_I - \langle p_2 \rangle$  are plotted in Figure 11(b). The pressure difference  $\langle p_2 \rangle_I - \langle p_2 \rangle$  approaches zero as the saturation  $S_I$  approaches unity. This property is also true for Figure 6(b) and Figure 11(b), but was not shown in those figures since the minimum value of the pressure difference occurs at the saturation  $S_I$  too close to unity. As shown in Figure 11(c), the average pressure difference  $\langle p_2 \rangle - \langle p_1 \rangle$  is not zero. In this case, this difference in average pressures is caused by the viscosity difference between the fluids, not by surface tension. If the viscosity is the same as the cases studied by Dahle *et al.* (2005), then the average pressures of the two phases are in fact the same. In other words, not only the surface tension, but also viscosity difference contributes to the pressure difference, or the “capillary pressure”.

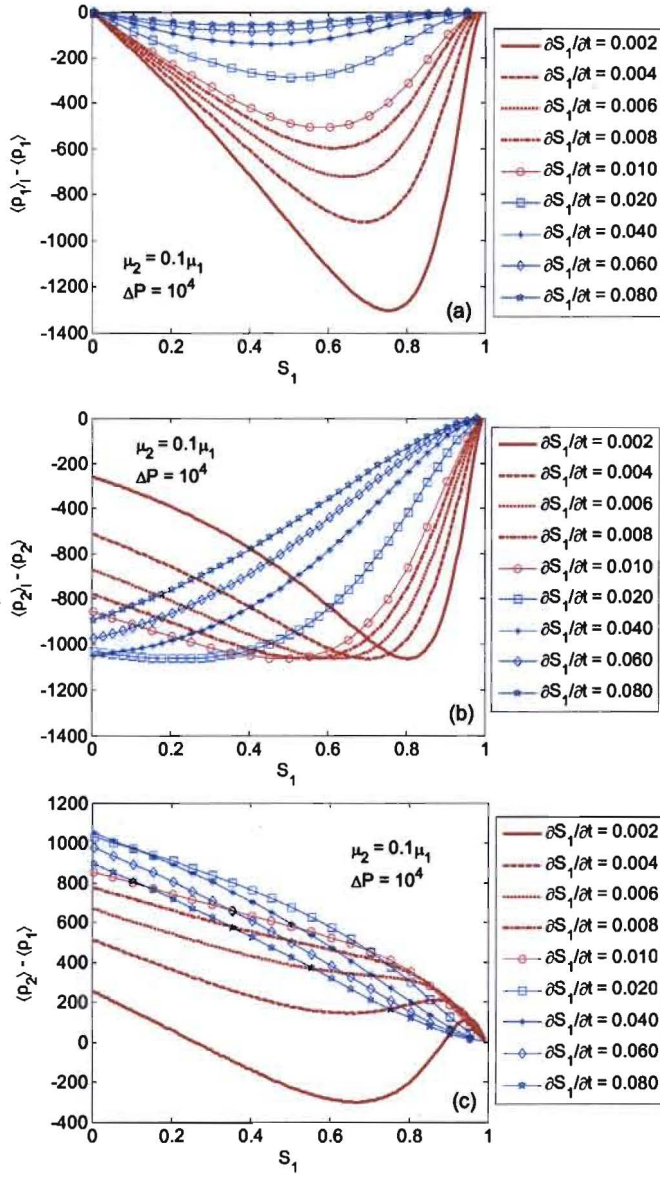


Figure 11. Plots  $\langle p_1 \rangle_I - \langle p_1 \rangle$ ,  $\langle p_2 \rangle_I - \langle p_2 \rangle$ , and  $\langle p_2 \rangle - \langle p_1 \rangle$  vs.  $S_1$  at different  $\partial S_1 / \partial t$  in the case of  $\Gamma_{2I} = 0$ .

## 5. Conclusions

In this paper, an ensemble phase averaging technique for continuous multi-material interactions is applied to derive averaged equations for multiphase flows in porous media. The ensemble averaged equations are found to have terms in addition to those commonly used in Darcy's law. Based on the bundle-of-tubes model, we studied properties of these

additional terms. We find these new terms are in principle capable of correcting many deficiencies in models based on the straightforward extensions of Darcy's law. We also studied the relations between these new terms and several recent models for two-phase flows in porous media.

Closure relationships were derived for the simple bundle-of-tubes model. The closure can be written as functions of saturation and the local time derivative of the saturation. The drag coefficients were found to be almost independent of the local time derivative of the saturation. Despite the name "capillary pressure", the difference in the average pressures of two fluids is not necessarily related to surface tension effects. Without surface tension, the average pressures of the two phases are not necessarily the same. If the pressure difference can be decomposed into a static part, representing surface tension effects, and a dynamic part, as suggested by Dahle *et al.* (2005), then the dynamic part of the capillary pressure not only affects the pressure difference, but also appears as terms in the averaged momentum equations.

Although it is commonly assumed that a fluid imbibition process can be modeled as a diffusion process, in the example we show that this is not generally true. It is not necessary that the velocity of the invading fluid decreases as  $1/\sqrt{t}$ . If the fluid being displaced is more viscous than the invading fluid, the velocity can even increase with time as fluid in the pores is replaced by the less viscous invading fluid.

The results and conclusions obtained in this paper are based on the simple geometry of the bundle-of-tubes model. Although we have reason to believe that the closure relations obtained here share many common features and trends in more complicated systems, more work is needed before these conclusions can be generalized.

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