LA-UR- *08-5139*

Title:	Making Almost Commuting Matrices Commute
Author(s):	Matthew Hastings/LANL/T-13/175388
Intended for:	Communications in Mathematical Physics

Los Alamos
NATIONAL LABORATORY
——— EST.1943 ———

Making Almost Commuting Matrices Commute

M. B. Hastings

Center for Nonlinear Studies and Theoretical Division,
Los Alamos National Laboratory, Los Alamos, NM, 87545

Suppose two Hermitian matrices A, B almost commute ($\|[A, B]\| \leq \delta$). Are they close to a commuting pair of Hermitian matrices, A', B', with $\|A - A'\|, \|B - B'\| \leq \epsilon$? A theorem of H. Lin[3] shows that this is uniformly true, in that for every $\epsilon > 0$ there exists a $\delta > 0$, independent of the size N of the matrices, for which almost commuting implies being close to a commuting pair. However, this theorem does not specify how δ depends on ϵ. We give uniform bounds relating δ and ϵ. The proof is constructive, giving an explicit algorithm to construct A' and B'. We provide tighter bounds in the case of block tridiagonal and tridiagonal matrices. Within the context of quantum measurement, this implies an algorithm to construct a basis in which we can make a *projective* measurement that approximately measures two approximately commuting operators simultaneously. Finally, we comment briefly on the case of approximately measuring three or more approximately commuting operators using POVMs (positive operator-valued measures) instead of projective measurements.

The problem of when two almost commuting matrices are close to matrices which exactly commute, or, equivalently, when a matrix which is close to normal is close to a normal matrix, has a long history. See, for example [1, 2], and other references in [3] where it is mentioned that the problem dates back to the 1950s or earlier. Finally in 1995, Lin[3] proved that for any $\epsilon > 0$, there is a $\delta > 0$ such that for all N, for any pair of Hermitian N-by-N matrices, A, B, with $\|A\|, \|B\| \leq 1$, and $\|[A, B]\| \leq \delta$, there exists a pair A', B' with $[A', B'] = 0$ and $\|A - A'\| \leq \epsilon$ and $\|B - B'\| \leq \epsilon$. This proof was later shortened and generalized by Friis and Rordam[4]. Interestingly, the same is not true for almost commuting unitary matrices[5] or for almost commuting triplets[6, 7].

The importance of the above results is that the bound is uniform in N. That is, δ depends only on ϵ. Unfortunately, the proofs do not give any bounds on how δ depends on ϵ. Further, the proofs of Lin and Friis and Rordam are nonconstructive, so there is no known way to find the matrices A' and B'. In this paper, we present a construction of matrices A' and B' which enables us to give lower bounds on how small δ must be to obtain a given error ϵ.

Specifically, we prove that

Theorem 1. *Let A and B be Hermitian, N-by-N matrices, with $\|A\|, \|B\| \leq 1$. Suppose $\|[A, B]\| \leq \delta$. Then, there exist Hermitian, N-by-N matrices A' and B' such that*

1: $[A', B'] = 0$.

2: $\|A' - A\| \leq \epsilon(\delta)$ *and* $\|B' - B\| \leq \epsilon(\delta)$, *with*

$$\epsilon(\delta) = E(1/\delta)\delta^{1/5}, \tag{1}$$

where the function $E(x)$ grows slower than any power of x. The function $E(x)$ does not depend on N.

The proof of theorem (1) involves first constructing a related problem involving a block tridiagonal matrix, H, and a block identity matrix X (we use the term "block identity matrix" to refer to a block diagonal matrix that is proportional to the identity matrix in each block). For such matrices we prove the theorem

Theorem 2. *Let X be a block identity Hermitian matrix and let H be a block tridiagonal matrix, with the j-th block of X equal to $c + j\Delta$ times the identity matrix, for some constants c and Δ. Let $\|H\|, \|X\| \leq 1$. Then, there exist Hermitian matrices A' and B' such that*

1: $[A', B'] = 0$.

2: $\|A' - H\| \leq \epsilon'(\Delta)$ *and* $\|B' - X\| \leq \epsilon'(\Delta)$, *with*

$$\epsilon'(\Delta) = E'(1/\Delta)\Delta^{1/4}, \tag{2}$$

where the function $E'(x)$ grows slower than any power of x. The function $E'(x)$ does not depend on N.

After proving these results, we prove a tighter bound in the case where H is a tridiagonal matrix, rather than a block tridiagonal matrix:

Theorem 3. *Let X be a diagonal Hermitian matrix and let H be a tridiagonal matrix, with the j-th diagonal entry of X equal to $c + j\Delta$, for some constants c and Δ. Let $\|H\|, \|X\| \leq 1$. Then, there exist Hermitian matrices A' and B' such that*

1: $[A', B'] = 0$.

2: $\|A' - H\| \leq \epsilon'(\Delta)$ *and* $\|B' - X\| \leq \epsilon'(\Delta)$, *with*

$$\epsilon'(\Delta) = E''(1/\Delta)\Delta^{1/2}, \tag{3}$$

where the function $E''(x)$ grows slower than any power of x. The function $E''(x)$ does not depend on N.

The proofs rely heavily on ideas relating to Lieb-Robinson bounds[8–11]. These bounds, combined with appropriately chosen filter functions, have been used in recent years in Hamiltonian complexity to study the dynamics and ground states of quantum systems, obtaining results such as a higher dimensional Lieb-Schultz-Mattis theorem[9], a proof of exponential decay of correlations[12], studies of dynamics out of equilibrium[13–15], new algorithms for simulation of quantum systems[16–20], an area law for entanglement entropy for general interacting systems[21], study of harmonic lattice systems[22], a Goldstone theorem with fewer assumption[23], and many others. The present paper represents a different application, to the study of almost commuting matrices.

Before beginning the proof, we give some discussion of physics intuition behind the result. The next few paragraphs are purely to motivate the problems from a physics viewpoint. In the last section on quantum measurement and in the discussion at the end we give additional applications to quantum measurement and construction of Wannier functions. As mentioned, we begin by relating this problem to the study of block tridiagonal matrices. We then interpret the matrix as H as a Hamiltonian for a single particle moving in one dimension, and apply the Lieb-Robinson bounds. The result (2) implies that we can construct a complete orthonormal basis of states which are simultaneously localized in both position (X) and energy (H). It is certainly easy to construct an overcomplete basis of states which is localized in both position and energy, by considering, for example, Gaussian wavepackets. The interesting result is the ability to construct an orthonormal basis which satisfies this.

Additional physics intuition can be obtained by considering the case where H is a tridiagonal matrix with 0 on the diagonal and elements just above and below the diagonal equal to 1, and where X is a diagonal matrix with entries $1/N, 2/N, \dots$. We refer to this as a uniform chain. In the uniform chain case, if we define a new matrix H' by randomly perturbing H, replacing each diagonal element of H with a small diagonal number chosen at random, the eigenvectors of H' are localized with high probability[24, 25]. Then, we can construct a matrix X' which exactly commutes with H' as follows: if v is an eigenvector of H', we choose it to have eigenvalue for X' equal to (v, Xv). Then, since the eigenvectors are localized, we find that $\|X - X'\|$ is small. The difference $\|X - X'\|$ depends on the localization length which depends inversely on the amount of disorder, while the difference $\|H - H'\|$ depends on the amount of disorder. Unfortunately, we do not have a good enough understanding of the effect of disorder for matrices H which are block tridiagonal, rather than just tridiagonal, to turn this approach into a proof for general H and X, and thus we rely on an alternative, constructive approach.

I. PROOF OF MAIN THEOREM

We now outline the proof of theorem (1). The proof is constructive, and is described by the following algorithm:

1: Construct H from A as described in section (II A) and lemma (1). H will be block diagonal in a basis of eigenvectors of A and $\|H - A\|$ will be bounded.

2: Construct X from B as described in section (II B). We will bound $\|X - A\|$. In a basis of eigenvalues of X, the matrix H will be block tridiagonal.

3: Construct a new basis as described in section (III) such that in this basis H is close to a block diagonal matrix. That is, we will bound the operator norm of the block off-diagonal part of H. The blocks will be different from the blocks considered in step **(2)** above and will be larger. Further, we will show that X is close to a block identity matrix in this basis.

4: Set A' to be the block diagonal part of H in the basis constructed in step **(3)** and set B' to the block identity matrix constructed in step **(3)**, so that $[A', B'] = 0$.

This algorithm involves several choices of constants. In a final section, (V), we indicate how to pick the constants to obtain the error bound (1). The key step will be step 3.

II. REDUCTION TO BLOCK TRIDIAGONAL PROBLEM

The first two steps of the proof above **(1,2)** reduce theorem (1) to theorem (2), while the last two steps **(3,4)** prove theorem (2). In this section we present the first two steps.

A. Construction of Finite-Range H

We begin by constructing matrix H as given in the following lemma, where the constant Δ will be chosen later.

Lemma 1. *Given Hermitian matrices A and B, with $\|[A,B]\| \leq \delta$, for any Δ there exists a matrix H with the following properties.*

   1*: $\|[H,B]\| \leq \delta$.*

   2*: For any two vectors v_1, v_2 which are eigenvectors of B with corresponding eigenvalues x_1, x_2, and with $|x_1 - x_2| \geq \Delta$, we have $(v_1, Hv_2) = 0$.*

   3*: $\|A - H\| \leq \epsilon_1$, with $\epsilon_1 = c_0 \delta / \Delta$, where c_0 is a numeric constant given below.*

Proof. We define

$$H = \Delta \int dt \exp(iBt) A \exp(-iBt) f(\Delta t), \tag{4}$$

where the function $f(t)$ is defined to have the Fourier transform

$$\tilde{f}(\omega) = (1 - \omega^2)^3, \quad |\omega| \leq 1 \tag{5}$$
$$\tilde{f}(\omega) = 0, \quad |\omega| \geq 1,$$

and hence the Fourier transform of $f(\Delta t)$ is supported on the interval $[-\Delta, \Delta]$. Properties (1) and (2) follow immediately from Eq. (4). Property (3) follows from

$$\begin{aligned}
\|A - H\| &\leq \int dt \| \exp(iBt) A \exp(-iBt) - A \| \Delta f(\Delta t) \tag{6} \\
&\leq \int dt\, t \|[A,B]\| \Delta f(\Delta t) \\
&= c_0 \delta / \Delta,
\end{aligned}$$

where

$$c_0 = \int dt \delta |t| f(t). \tag{7}$$

Note that since the first and second derivatives of $\tilde{f}(\omega)$ vanish at $\omega = \pm 1$, the function $f(t)$ decays as $1/t^3$ for large t and hence c_0 is finite. Note also that the precise form of the function $f(t)$ is unimportant: all we require is that $\tilde{f}(0) = 1$, that $\tilde{f}$ is supported on the interval $[-1, 1]$, and that $\tilde{f}$ is sufficiently smooth that $f(t)$ decays fast enough for the integral over t (7) to converge. $\qquad \square$

Remark: In a basis of eigenvectors of B, property (3) in the above lemma implies that H is "finite-range", in that the off-diagonal elements are vanishing for sufficiently large $|x_1 - x_2|$. The next theorem is a Lieb-Robinson bound for such finite range Hamiltonians, similar to those proven for many-body Hamiltonians[8–11]

We now introduce some terminology. Given two sets of real numbers, S_1, S_2, we define

$$\text{dist}(S_1, S_2) = \min_{x_1 \in S_1, x_2 \in S_2} |x_1 - x_2|. \tag{8}$$

Remark: The reason for introducing this "distance function" is that we think of H as defining the Hamiltonian for a one-dimensional, finite- range quantum system, with different "sites" of the system corresponding to different eigenvectors of B, and then the distance function is the distance between different sets of sites.

Further, we say that a vector w is "supported on set S for position operator B" if w is a linear combination of eigenvectors of B whose corresponding eigenvalues are in set S. Finally, for any set S we define a projection operator $P(S, B)$ to be the projector onto eigenvectors of B whose corresponding eigenvalues lie in set S. We now give the Lieb-Robinson theorem:

Theorem 4. *Let H have the properties*

> **1:** $\|H\| \leq 1$.

> **2:** *For any two vectors v_1, v_2 which are eigenvectors of B with corresponding eigenvalues x_1, x_2, and with $|x_1 - x_2| \geq \Delta$, we have $(v_1, Hv_2) = 0$.*

Define

$$v_{LR} = e^2 \Delta. \tag{9}$$

Then, for any vector v supported on a set S for position operator B, and for any projection operator $P(T, B)$, we have

$$|P(T, B) \exp(-iHt)v| \leq e^{-\mathrm{dist}S_1, S_2/\Delta} \tag{10}$$

for any

$$t \leq \mathrm{dist}(S_1, S_2)/v_{LR} \tag{11}$$

Proof. Expand $\exp(-iHt)v$ in a power series as $v - iHtv - (H^2/2)t^2 v + \dots$. Then, by assumption, $P(T, B)(-it)^n (H^n/n!)v$ vanishes for $n < \mathrm{dist}(S, T)/\Delta$. However,

$$
\begin{aligned}
\left| \sum_{n \geq m} (-it)^n (H^n/n!)v \right| &\leq \sum_{n \geq m} (t^n/n!)|v| \\
&\leq \frac{1}{e} \sum_{n \geq m} (et/n)^n |v| \\
&\leq \frac{1}{e}(et/m)^m \frac{1}{1 - et/m}.
\end{aligned} \tag{12}
$$

For the given v_{LR}, the result follows. $\qquad\square$

Remark: the proof of this Lieb-Robinson bound is significantly simpler than the proofs of the corresponding bounds for many-body systems considered elsewhere. The power series technique used here does not work for such systems.

B. Construction of X

In this subsection, we construct the operator X from B. We define a function $Q(x)$ by

$$Q(x) = \Delta \lfloor x/\Delta + 1/2 \rfloor. \tag{13}$$

Then, we set

$$X = Q(B). \tag{14}$$

Note that $|Q(x) - x| \leq \Delta/2$ for all x, and $Q(x)/\Delta$ is always an integer. Then,

$$\|X - B\| \leq \epsilon_2, \tag{15}$$

with

$$\epsilon_2 = \Delta/2. \tag{16}$$

By (2) in lemma (1), the matrix H is a block tridiagonal matrix when written in a basis of eigenstates of X, with eigenvalues of X ordered in increasing order.

III. CONSTRUCTION OF NEW BASIS

In this section we construct the basis to make H close to a block diagonal matrix and X close to a block identity matrix. This completes step (3) of the construction of A' and B'. We refer to the basis constructed in this step as the "new basis" and we refer to the basis in which X is diagonal as the "old basis".

There will be a total of n_{cut} different blocks in the new basis, where n_{cut} is chosen later. Consider the interval I_i, where $I_i = [-1 + 2(i-1)/n_{cut}, -1 + 2i/n_{cut})$ for $i < n_{cut}$ and $I_i = [-1 + 2(i-1)/n_{cut}, -1 + 2i/n_{cut}]$ for $i = n_{cut}$. Let J_i be the matrix given by projecting H onto the subspace of eigenvalues of X lying in this interval, and call this subspace $\mathcal{B}_i$. Then, in the old basis of eigenvalues of X, H is block tridiagonal with at least L different blocks, where $L = \lfloor (2/n_{cut})/\Delta) - 1 \rfloor$ blocks.

We claim that:

Lemma 2. *Let J be a Hermitian block tridiagonal matrix, with $\|J\| \leq 1$ acting on a space $\mathcal{B}$. Let there be L blocks. Let $\mathcal{V}_j$ denote the subspace in the j-th block. Then, there exists a space $\mathcal{W}$ which is a subspace of $\mathcal{B}$ with the following properties:*

- **1:** *The projection of any normalized vector $v \in \mathcal{V}_1$ onto the orthogonal complement of $\mathcal{W}$ has norm bounded by ϵ_3 where ϵ_3 is equal to $1/L^{1/3}$ times a function growing slower than any power of L.*

- **2:** *For any normalized vector $w \in W$, the projection of Jw onto the orthogonal complement of $\mathcal{W}$ has norm bounded by ϵ_4, where ϵ_4 is equal to $1/L^{1/3}$ times a function growing slower than any power of L.*

- **3:** *The projection of any normalized vector $v \in \mathcal{V}_L$ onto $\mathcal{W}$ has norm bounded by ϵ_5, where ϵ_5 is equal to a function decaying faster than any power of L.*

Proof. This lemma is the key step in the proof of the main theorem, and the proof of this lemma is given in the next section. $\square$

For a given choice of i, we reference to the space $\mathcal{W}$ as constructed in lemma (2) as $\mathcal{W}_i$. We refer to the subspaces $\mathcal{V}_j$ defined in lemma (2) as $\mathcal{V}_j(i)$. Let $\mathcal{B}_i$ have dimension $D_B(i)$ and let $\mathcal{W}_i$ as constructed dimension $D_W(i)$. Let $\mathcal{W}_i^\perp$ denote the $D_B(i) - D_W(i)$-dimensional space which is the orthogonal complement of $\mathcal{W}_i$. By properties **(1,2)** in lemma (2), $D_B(i) \geq d_1(i)$ and $D_B(i) \leq D_W(i) - d_L(i)$.

The new basis has n_{cut} blocks. For $1 < i \leq n_{cut}$, we define the i-th block to be the space spanned by $\mathcal{W}_i$ and $\mathcal{W}_{i-1}^\perp$. For $i = 1$, the i-th block is the space spanned by $\mathcal{W}_i$.

Then, the matrix H is block tridiagonal in this new basis. The block-off-diagonal terms arise from three sources. First, the matrix J_i contains non-vanishing matrix elements between the spaces $\mathcal{W}_i$ and $\mathcal{W}_i^\perp$, and those spaces are now in different blocks. However, by property **(2)** in lemma (2), these matrix elements are bounded by ϵ_4. Second, there are non-vanishing matrix elements between the subspace $\mathcal{W}_{i-1}^\perp$ and $\mathcal{V}_1^i$, and $\mathcal{V}_1^i$ may not be completely contained in subspace $\mathcal{W}_i$. However, by property **(1)** in lemma (2), these contribute only ϵ_3 to the norm of the block-off-diagonal terms of H in the new basis. of J in the old basis. Third, there are non-vanishing matrix elements between $\mathcal{W}_i$ and $\mathcal{V}_L^i$, and $\mathcal{V}_L^i$ may not be completely contained in subspace $\mathcal{W}_i^\perp$. However, by property **(1)** in lemma (2), these contribute only ϵ_5 to the norm of the block-off-diagonal terms of H in the new basis. Therefore, the block-off-diagonal terms in H are bounded in operator norm by

$$2(\epsilon_3 + \epsilon_4 + \epsilon_5). \tag{17}$$

Define B' to be the block identity matrix (in the new basis) which is equal to $-1 + 2i/n_{cut}$ times the identity matrix in the i-th block. Since each block i in the new basis lies within the space spanned by $\mathcal{B}_i$ and $\mathcal{B}_{i-1}$ we have

$$\|B' - B\| \leq 2/n_{cut}. \tag{18}$$

IV. PROOF OF LEMMA 2

Let the space $\mathcal{V}_1$ be M dimensional, with orthonormal basis vectors $v_1, ..., v_M$. Let S denote the matrix whose columns are these basis vectors, so that S is a similarity.

Define a function $\sigma(\omega)$ as follows. Let $\sigma(\omega) = 1$ for $\omega \leq 0$. Let $\sigma(\omega) = 0$ for $\omega \geq 1$. Finally, for $0 \leq \omega \leq 1$, choose $\sigma(\omega)$ to be infinitely differentiable so that the Fourier transform of $\sigma(\omega)$, which we write $\tilde{\sigma}(t)$, is bounded by a function which decays faster than any polynomial. We also impose $\sigma(\omega) + \sigma(1 - \omega) = 1$.
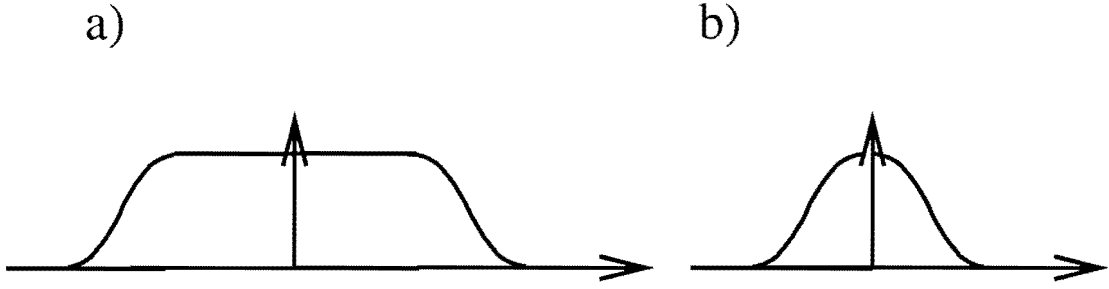
FIG. 1: Sketch of (a) $\mathcal{F}(0,0,1,\omega)$ and (b) $\mathcal{F}(0,1,1,\omega)$.

Define a function $\mathcal{F}(\omega_0, r, w, \omega)$ by

$$\mathcal{F}(\omega_0, r, w, \omega) = \sigma((|\omega - \omega_0| - r)/w). \tag{19}$$

Then $\mathcal{F}(\omega_0, r, w, \omega) = 0$ for $|\omega - \omega_0| \geq r + w$. and $\mathcal{F}(\omega_0, r, w, \omega) = 1$ for $|\omega - \omega_0| \leq r$, and for $r \geq 0$ and $w > 0$, the function $\mathcal{F}(\omega_0, r, w, \omega)$ is infinitely differentiable with respect to ω everywhere. The functions $\mathcal{F}(0,0,1,\omega)$ and $\mathcal{F}(0,1,1,\omega)$ are sketched in Fig. 1a,b; the variable r denotes the width of the flat part at the center of the function, while w denotes the width of the changing part of the function. Since $\mathcal{F}(\omega)$ is infinitely differentiable, there is a function $T(x)$ which decays faster than any polynomial such that:

$$\int_{|t| \geq t_0} \mathrm{d}t \tilde{\mathcal{F}}(\omega_0, w, w, t) \leq T(wt), \tag{20}$$

$$\int_{|t| \geq t_0} \mathrm{d}t \tilde{\mathcal{F}}(\omega_0, 0, w, t) \leq T(wt).$$

The operator norm of J is bounded by 1. The idea of the proof is to divide the interval of eigenvalues of J, which is $[-1, 1]$, into various small overlapping windows. Then, for each interval centered on a frequency ω, we will construct vectors given by approximately projecting vectors in $\mathcal{V}_1$ onto the space spanned by eigenvectors of J with eigenvalues lying in that interval; we call the spaces of these vectors $\mathcal{X}_i$, where i labels the particular window. Then, each of these projected vectors x will have the property that Jx is close to ωx. This will be the key step in ensuring property (2) in the claims of the lemma. The idea of *approximate* projection is important here. In fact, we will use the smooth filter functions $\mathcal{F}(\omega_0, r, w, \omega)$ above. The smoothness will be essential to ensure that the vectors x have most of their amplitude in the first blocks rather than the last blocks. Because the windows overlap, the vectors may not be orthogonal to each other; the overlap between vectors is something we will need to bound (see Eq. (27) below). To control the overlap, we choose $\mathcal{W}$ to be a subpace of the space spanned by the $\mathcal{X}_i$ as explained below.

Let n_{win} be some even integer chosen later. We will choose

$$n_{win} = L/F(L), \tag{21}$$

where the function $F(L)$ is a function that grows slower than any power of L and is defined further below. The choice of function $F(L)$ will depend only on the function $T(x)$ defined above.

For each $i = 0, ..., n_{win} - 1$, define

$$\omega(i) = -1 + 2i/(n_{win} - 1). \tag{22}$$

A. Construction of Spaces $\mathcal{X}_i$

To construct $\mathcal{X}_i$, we define the matrix τ_i by

$$\tau_i = \mathcal{F}(\omega(i), 0, 2/n_{win}, J)S. \tag{23}$$

Define

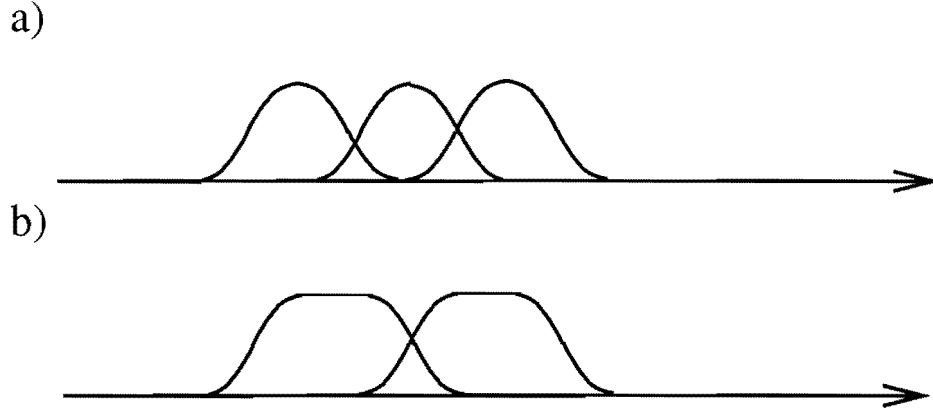$$\lambda_{min} = 1/(n_{win} L^2). \tag{24}$$

FIG. 2: a) Sketch of overlapping windows. b) Re-arrangement of windows as discussed in section on tridiagonal matrices.

Compute the eigenvectors of the matrix $\tau_i^\dagger \tau_i$. For each eigenvector x_a with eigenvalue greater than or equal to λ_{min} compute $y_a = \tau_i x_a$. Let $\mathcal{X}_i$ be the space spanned by all such vectors y_a. Let Z_i project onto the eigenvectors x_a with eigenvalue less than λ_{min}; the projector Z_i will be used later in computing the error estimates.

Remark: To understand this construction, in Fig. 2a we sketch the functions $\mathcal{F}(\omega(i-1), 0, 2/n_{win}, \omega)$, $\mathcal{F}(\omega(i), 0, 2/n_{win}, \omega)$, and $\mathcal{F}(\omega(i+1), 0, 2/n_{win}, \omega)$, which form partially overlapping windows. Note that the vectors $\mathcal{F}(\omega(i), 0, 2/n_{win}, J)S$ and $\mathcal{F}(\omega(i \pm 1), 0, 2/n_{win}, J)S$ need not be orthogonal.

B. Construction of $\mathcal{W}$

We now construct the space $\mathcal{W}$. Let each space $\mathcal{X}_i$ have dimension D_i. In each space $\mathcal{X}_i$ we can find an orthonormal basis of vectors, $v_{i,b}$, for $b = 1, ..., D_i$. Note that for even $|i - j| > 1$, the spaces $\mathcal{X}_i$ and $\mathcal{X}_j$ are orthogonal. We define a block tridiagonal matrix ρ of inner products of vectors $v_{i,b}$ as follows: the i-th block has dimension D_i, and on the diagonal the matrix is equal to the identity matrix. Above the diagonal, the block in the i-th row and $i+1$-st column is equal to the matrix of inner products $(v_{i,b}, v_{i+1,c}$ for $b = 1, ..., D_i$ and $c = 1, ..., D_{i+1}$. We define a new vector space $\mathcal{R}$ to be a space of dimension $\sum_{i=0}^{n_{win}-1} D_i$, and then A is a linear operator from $\mathcal{R}$ to $\mathcal{B}$.

This matrix ρ is Hermitian and positive semidefinite. It is equal to $\rho = A^\dagger A$, for some matrix A which has entries only on the block diagonal and on the diagonal above the block diagonal. We define spaces Y_i, for $i = 0, ..., n_{blk}$, as follows, where n_{blk} is equal to $\lceil n_{win}/l_b \rceil$ with the "block length" l_b being an integer equal to

$$l_b = \lfloor n_{win}^{1/3} \rfloor. \tag{25}$$

We pick Y_i to be the subspace of $\mathcal{R}$ containing the blocks from the $i * l_b$-th block to the $(i+2) * l_b - 1$-th block. We claim that

Lemma 3. *There exist spaces N_i, for $i = 0, ..., n_{blk}$ with the properties that:*

1: N_i is a subspace of $\mathcal{Y}_i$.

2: For any vector $v \in N_i$, the quantity $(v, \rho v)$ is bounded by $1/l_b^2$ times a function growing slower than any power of l_b.

3: For any vector v which is an eigenvector of ρ with eigenvalue less than $1/l_b^2$, the projection of v onto the space orthogonal to the space spanned by N_i is bounded by a function decaying faster than any negative power of l_b.

Proof. □

For each even i, consider the projector P_i which projects onto the subspace of $\mathcal{R}$ containing the blocks from the $i * l_b$-th block to the $(i+1) * l_b - 1$-th block. By Jordan's lemma for pairs of projectors[26], we can find an orthonormal basis for N_i, with basis vector $n_{i,b}$ such that $P_i n_{i,b}$ is orthogonal to $n_{i,c}$ for $b \neq c$. Let N_i^L be the space spanned by vectors $n_{i,b}$ such that $|P_i n_{i,b}| \geq 1/2$, and let N_i^R be the space spanned by vectors $n_{i,b}$ such that $|P_i n_{i,b}| < 1/2$.

For each odd i, define N_i' to be the subspace of N_i which is orthogonal to the space spanned by N_{i-1}^R and N_{i+1}^L. Consider the projector P_i which projects onto the subspace of $\mathcal{R}$ containing the blocks from the $i * l_b$-th block to the $(i+1) * l_b - 1$-th block. By Jordan's lemma for pairs of projectors[26], we can find an orthonormal basis for N_i', with basis vector $n_{i,b}$ such that $P_i n_{i,b}$ is orthogonal to $n_{i,c}$ for $b \neq c$. Let N_i^L be the space spanned by vectors $n_{i,b}$ such that $|P_i n_{i,b}| \geq 1/2$, and let N_i^R be the space spanned by vectors $n_{i,b}$ such that $|P_i n_{i,b}| < 1/2$.

We now define $\mathcal{W}_i$, for $i = 0, ..., n_{blk}$ as follows. Take the space projected onto by P_i. Consider the subspace of that space which is orthogonal to the space spanned by N_i^L and N_{i+1}^R. Act on this space with A. The result is the space $\mathcal{W}_i$. The space $\mathcal{W}$ is the space spanned by the $\mathcal{W}_i$. Let P be the projector onto $\mathcal{W}$.

Note that for any i, for any $v \in \mathcal{W}_i$, we have

$$|(1 - P_i)Jv| \leq l_b(2/n_{win}) \tag{26}$$

Further, by construction, for any vectors $v \in \mathcal{W}$, with $v = \sum_i v_i$ with $v_i \in \mathcal{W}_\rangle$,we have we have

$$|v|^2 \geq (1 - \epsilon(l_b))(1/l_b)^2 \sum_i |v_i|^2, \tag{27}$$

where $\epsilon(l_b)$ is a function decaying faster than any negative power of l_b.

We also claim that for any vector v in the space spanned by $\mathcal{X}_i$, that

$$|Pv - v| \leq (2/l_b))|v|. \tag{28}$$

C. Verification of Claims

We now verify the claims regarding the subspace $\mathcal{W}$.

Proof of First Claim— To prove (**1**), note that for any vector $v \in \mathcal{B}$ we have

$$v = \sum_{i=0}^{n_{win}-1} \mathcal{F}(\omega(i), 0, 2n_{win}, J)v. \tag{29}$$

For any $v \in \mathcal{V}_1$, with $|v| = 1$, we can write $v = Sx$ with $|x| = 1$, and then

$$\begin{aligned}|v - \sum_{i=0}^{n_{win}-1} \tau_i(1 - Z_i)x|^2 &= |\sum_{i=0}^{n_{win}-1} \tau_i Z_i x|^2 \\ &\leq n_{win}\lambda_{min} \\ &\leq 1/L^2.\end{aligned} \tag{30}$$

Since $\sum_{i=0}^{n_{win}-1} \tau_i(1 - Z_i)x$ is in the space spanned by the $\mathcal{X}_i$, by Eq. (28), this verifies the first claim, given that $F(L)$ is chosen to grow slower than any power of L.

Proof of Second Claim— To prove the second claim (**2**), consider any vector $v \in \mathcal{W}$, with $v = \sum_i v_i$ with $v_i \in \mathcal{W}_i$. By Eqs. (26,27), and by the fact that $\mathcal{W}_i$ is orthogonal to $\mathcal{W}_j$ for $|i - j| > 1$, we have

$$\begin{aligned}|(1 - P)v|^2 &\leq 2\sum_i |(1 - P)v_i|^2 \\ &\leq 2\sum_i \left(l_b(2/n_{win})\right)^2 \sum_i |v_i|^2 \\ &\leq 2\frac{1}{(1 - \epsilon(l_b)(1/l_b)^2} \left(l_b(2/n_{win})\right)^2 |v|^2 \\ &= 2\frac{1}{(1 - \epsilon(l_b)} \left(l_b^2(2/n_{win})\right)^2 |v|^2,\end{aligned} \tag{31}$$

verifying the second claim.

Proof of Third Claim— To verify the last claim, we use the Lieb-Robinson bounds. Let $\tilde{\mathcal{F}}(\omega_0, r, w, t)$ denote the Fourier transform of $\mathcal{F}(\omega_0, r, w, \omega)$ with respect to the last variable ω. Then, for any x with $(1 - Z_i)x = x$, we find that $y = \tau_i x = \mathcal{F}(\omega(i), 0, 2n_{win}, J)Sx$ is equal to

$$y = \int \mathrm{d}t \tilde{\mathcal{F}}(\omega(i), 2/3n_{win}, 2/3n_{win}, t) \exp(iJt)Sx. \tag{32}$$

We use the Lieb-Robinson bounds for matrix J, by defining a position matrix which is equal to i in the i-th block. Using the Lieb-Robinson bounds, for time $t \leq L/2v_{LR}$, with $v_{LR} = e^2$, we find that the norm of the projection of $\exp(iJt)Sx$ onto the space $\mathcal{V}_L$ is bounded by $\exp(-L/2)$. At the same time, the integral $\int_{|t|\geq L/2v_{LR}} dt \tilde{\mathcal{F}}(\omega(i), 2/3n_{win}, 2/3n_{win}, t)$ is bounded by $T(L/6v_{LR}n_{win}) = T(F(L)/6e^2)$. Since $T(x)$ decays faster than any negative power of x, we can choose an $F(x)$ which grows slower than any power of x such that $T(F(L)/6e^2)$ still decays faster than any negative power of L. Thus, since $|y| \geq \lambda_{min}|x|$ by construction, for this choice of $F(x)$ the projection of any vector $y \in \mathcal{W}_i$ onto $\mathcal{V}_L$ is bounded by $|y|$ times a function decaying faster than any negative power of L. Using Eq. (27), we find that the projection of any vector $v \in \mathcal{W}$ onto $\mathcal{V}_L$ is bounded by $|v|$ times a function decaying faster than any negative power of L, verifying the third claim.

This completes the proof of Lemma (2). After giving the error bounds in the next section, we explain some of the motivation behind the above construction, and comment on the easier case in which J is a tridiagonal matrix, rather than a block tridiagonal matrix.

V. ERROR BOUNDS

We finally give the error bounds to obtains theorems (1,2). To obtain (2), we pick

$$n_{cut} = \Delta^{-1/4}, \tag{33}$$

so that $L = \lfloor (2/n_{cut})/\Delta) - 1 \rfloor$ is of order $2/\Delta^{3/4}$. Then, from lemma (2) and Eq. (17), in the new basis the block-off-diagonal terms in H are bounded in operator norm by a constant times $\Delta^{1/4}$ times a function growing slower than any power of $1/\Delta$. By Eq. (18), the difference between B and B' is bounded in operator norm by a constant times $\Delta^{1/4}$. Therefore, theorem (2) follows. To obtain theorem (1), we pick

$$\Delta = \delta^{4/5} \tag{34}$$

in lemma (1).

We omit the detailed analysis, but it is possible to choose $E(x)$ to be a polylog as follows. We can pick $T(x)$ to decay like $\exp(-x^\eta)$, for any $\eta < 1$[27, 28]. Then we can pick $F(L)$ to equal $\log(L)^\theta$, for $\theta > 1/\eta$, so that $T(F(L)) \sim \exp(-(\log(L))^{\theta/\eta})$ decays faster than any power.

VI. TRIDIAGONAL MATRICES

In this section, we present tighter bounds for the case in which H is a tridiagonal matrix, rather than a block tridiagonal matrix.

Remark: The difficulty we face is that the even and odd space are not orthogonal to each other. If they were orthogonal, then many of the estimates would be easier. Consider the case in which J is a block diagonal matrix, so that $\mathcal{V}_1$ is one dimensional. Let $\rho(E)$ be a smoothed density of states at energy E: $\rho(E) = \text{tr}(S^\dagger \mathcal{F}(E, 1/L, 1/L, J)S)$. Suppose $\rho(E)$ is such that it has a peak in the crossing points of Fig. 2a (the points where one function $\mathcal{F}$ is decreasing and the other is increasing and they cross). Then, with the overlapping windows as shown, we find that most of the smoothed density of states lies in the overlap between the windows, rather than in the windows themselves. The overlap between the vectors in different windows becomes much larger now. In the case of a tridiagonal matrix, we can rearrange the windows as shown in Fig. 2b to reduce the overlap; this general idea will motivate the construction in this section.

We prove that

Lemma 4. *Let J be an L-by-L Hermitian tridiagonal matrix, with $\|J\| \leq 1$ acting on a space $\mathcal{B}$. Let V_j denote the vector with a 1 in the j-th entry and zeroes elsewhere. Then, there exists a space $\mathcal{W}$ which is a subspace of $\mathcal{B}$ with the following properties:*

(1): The projection of V_1 onto the orthogonal complement of $\mathcal{W}$ has norm bounded by ϵ_3 where ϵ_3 is equal to a constant times $1/L$.

(2): For any normalized vector $w \in W$, the projection of Jw onto the orthogonal complement of $\mathcal{W}$ has norm bounded by ϵ_4, where ϵ_4 is equal to $1/L$ times a function growing slower than any power of L.

(3): The projection of V_L onto $\mathcal{W}$ has norm bounded by ϵ_5, where ϵ_5 is a function decaying faster than any power of L.

This lemma implies theorem (3): we construct A', B' as before, following steps (3) to construct the new basis, but because of the tighter bounds in lemma (4) we can choose $n_{cut} = \Delta^{-1/2}$ when constructing the new basis. Now, in step (4), we find that A', B' are diagonal matrices, rather than just block diagonal matrices.

For each $i = 0, 1, ..., n_{win} - 1$, define

$$\omega(i) = -1 + 2i/(n_{win} - 1), \tag{35}$$

as before. Define

$$\rho_i = S^\dagger \mathcal{F}(\omega(i), 0, 2/n_{win}, J)S. \tag{36}$$

Set

$$\lambda_{min} = 1/(n_{win}L^2), \tag{37}$$

as before with

$$n_{win} = L/F(L) \tag{38}$$

as before. To prove Lemma (4), we use the following algorithm. There are $n_{win} + 1$ windows, labelled $0, ..., n_{win}$. We label various windows as "marked" or "unmarked" as follows.

1: Set $i = 0$. Initialize a real variable x to 0. Initialize all windows to unmarked.

2: If $\rho_i < \lambda_{min}$, increment i by one and go to step **6**.

3: Mark window i.

4: Set x to $x + \rho_i$. If $x \geq 9\rho_{i+1}$ and $\rho_{i+2} \geq \lambda_{min}$, set i to $i + 2$, set x to 0, and go to step **6**. If $x \geq 2\rho_{i+1}$ and $\rho_{i+2} < \lambda_{min}$, mark window $i + 1$, set i to $i + 3$, set x to 0, and go to step **6**.

5: Increment i by one.

6: If $i > n_{win}$, terminate. Otherwise go to step **2**.

After running this algorithm, the there will be a sequences of marked windows, separated by sequences of unmarked windows. Note that the length of a marked of windows is at most $\log_{10/9}(1/\lambda_{min})$, since at the start of such a sequence x is at least λ_{min}, x grows exponentially along the sequence (otherwise in step **4** the state will eventually change unless $\rho_{i+1} > 2x$) and x can be at most 1.

Let the total number of sequences be n_{seq}. Note that $n_{seq} \leq n_{win}$.

For each sequence of marked windows, from window i to j, construct the vector $y = \mathcal{F}((\omega(i) + \omega(j))/2, (\omega(j) - \omega(i))/2, 2/n_{win}, J)Sx$. We define y_a, for $a = 1, ..., n_{seq}$, to be the vector y constructed from the a-th sequence. By construction, the norm square of y_a when projected onto eigenvectors of J with eigenvalues greater than $\omega(j)$ is bounded by $1/9$ of the norm square of y_a. The vector y_{a+1} has vanishing projection onto eigenvectors of J with eigenvalues less than or equal to $\omega(j)$. Therefore,

$$(y_a, y_{a+1}) \leq (1/3)|y_a||y_{a+1}|. \tag{39}$$

We define $\mathcal{W}$ to be the space spanned by all such vectors y_a, and we define P to project onto $\mathcal{W}$. Consider any vector $v \in \mathcal{W}$, with

$$v = \sum_{a=1}^{n_{seq}} v_a, \tag{40}$$

with v_a parallel to y_a. By Eq. (39)

$$|v|^2 \geq \frac{1}{3} \sum_{a=1}^{n_{seq}} |v_a|^2. \tag{41}$$

Remark: The function $\mathcal{F}((\omega(i) + \omega(j))/2, (\omega(j) - \omega(i))/2, 2/n_{win}, \omega)$ is equal to unity for $\omega(i) \leq \omega \leq \omega(j)$.

We now prove the Lemma (4) as follows: to prove the first claim, note that by construction,

$$|Pv_1 - v_1| \leq n_{win}\lambda_{min} \tag{42}$$
$$\leq 1/L^2.$$

To prove the second claim, consider the a-th sequence of marked windows, from window i to window j. Let $\omega(i) = (\omega^-(i) + \omega^+(j))/2$. Then,

$$|(J - \omega(i))y_a| \leq \frac{2 + \log_{10/9}(1/\lambda_{min})}{n_{win}}|y_a| \tag{43}$$

which is bounded by $1/L$ times a function growing slower than any power of L. Therefore,

$$|(1 - P)Jy_a| \leq \frac{2 + \log_{10/9}(1/\lambda_{min})}{n_{win}}|y_a| \tag{44}$$

Using the bound Eq. (41), for any vector $v \in \mathcal{W}$,

$$|(1 - P)Jv| \leq \sqrt{3}\frac{2 + \log_{10/9}(1/\lambda_{min})}{n_{win}}|v|, \tag{45}$$

which is bounded by $1/L$ times a function growing slower than any power of L, verifying the second claim.

The proof of the third claim is identical to the previous case.

VII. QUANTUM MEASUREMENT

The constructions above can be applied to operators which arise in various physical quantum systems. For example, consider a quantum spin for a large spin S. Then, the operators S_x/S and S_y/S have operator norm 1 and have a commutator that is of order $1/S$. Thus, we can find a basis in which both operators are almost diagonal. While it is well known that one can use a POVM (positive operator-valued measure) to approximately measure S_x and S_y at the same time, the existence of the given basis implies that one can approximately measure S_x and S_y simultaneously with a single *projective* measurement. Interestingly, while the operator S_z^2 is also almost diagonal in this basis (since it equals $S(S + 1) - S_x^2 - S_y^2$), it is not possible to find a basis in which S_x, S_y, and S_z are all almost diagonal (this obstruction is similar to that in [6]). Therefore, to approximately measure S_x, S_y, and S_z simultaneously will require a POVM, rather than a projective measurement.

For completeness, we now briefly show how to construct a POVM to approximately measure several almost commuting operators simultaneously. Consider any number N of Hermitian matrices, labelled $A_1, ..., A_N$, with $\|[A_i, A_j]\| \leq \delta$ for all i, j and with $\|A_i\| \leq 1$ for all i. We now construct a POVM to approximately measure all N operators simultaneously.

For $i = 1, ..., N$ and $n = 0, ..., n_{max}$, define

$$M(i, n) = \sqrt{\mathcal{F}(-1 + 2i/n_{max}, 2/3n_{max}, 2/3n_{max}, A_i)}. \tag{46}$$

Define

$$O_{(n_1, n_2, ..., n_N)} = \Big(M(1, n_1)M(2, n_2)...M(N, n_N)\Big)\Big(M(N, n_N)...M(2, n_2)M(1, n_1)\Big). \tag{47}$$

Then,

$$\sum_{n_1, n_2, ...=1}^{n_{max}} O(n_1, n_2, ..., n_N) = \mathbb{1}, \tag{48}$$

and all of the operators O are positive semidefinite by construction. Therefore, the operators $O(n_1, n_2, ..., n_N)$ form a POVM.

We now show that this approximately measures all operators simultaneously. That is, we show that for any density matrix ρ, if the outcome of the measurement is $n_1, n_2, ..., n_N$, then if we perform a subsequent measurement of any operator A_i, the outcome will be close to $-1 + 2n_i/n_{max}$ with high probability. We show this by computing the

expectation value $(A_i - (-1 + 2n_i/n_{max}))^2$ averaged over all measurement outcomes. For any density matrix ρ, for any i, the average over all outcomes of $(A_i - (-1 + 2n_i/n_{max})^2$ is equal to

$$\sum_{n_1, n_2, \ldots = 1}^{N} \text{tr}((A_i - (-1 + 2n_i/n_{max}))^2 M(1, n_1) M(2, n_2) \ldots \rho \ldots M(2, n_2) M(1, n_1)) \tag{49}$$

Note that for $i > j$

$$\sum_{n_j = 1}^{N} \|M(j, n_j)(A_i - (-1 + 2n_i/n_{max})^2 M(j, n_j) - (A_i - (-1 + 2n_i/n_{max})^2\| \leq \text{const.} \times \delta^2 n_{max}^2. \tag{50}$$

To show Eq. (50), write

$$M(j, n_j) = \int dt \exp(iA_j t) \sqrt{\widetilde{\mathcal{F}(-1 + 2j/n_{max}, 2/3n_{max}, 2/3n_{max}, t)}}, \tag{51}$$

where $\sqrt{\widetilde{\mathcal{F}(., ., ., t)}}$ denotes the Fourier transform of the square-root of $\mathcal{F}$. Then, since $\mathcal{F}(., ., ., \omega)$ is infinitely differentiable, the Fourier transform decays faster than any power of t. Then since $\| \exp(iA_j t)A_i - A_i \exp(iA_j t)t - it[A_j, A_i]\| \leq \text{const.} \times (t^2 \delta^2)$, we find that

$$\|M(j, n_j)A_i - A_i M(j, n_j)\| \leq \| \int dt\, it[A_j, A_i] \exp(iA_j t) \sqrt{\widetilde{\mathcal{F}(-1 + 2j/n_{max}, 2/3n_{max}, 2/3n_{max}, t)}} \| \tag{52}$$
$$+ \text{const.} \times n_{win}^2 \delta^2$$
$$\leq \text{const.} \times n_{win}^2 \delta^2,$$

where the last line follows from the symmetry of $\sqrt{\widetilde{\mathcal{F}(., ., ., t)}}$ in t. Also,

$$\sum_{n_i = 1}^{N} \|M(i, n_i)(A_i - (-1 + 2n_i/n_{max})^2 M(j, n_j) - (A_i - (-1 + 2n_i/n_{max})^2\| \leq \text{const.} \times (1/n_{max}^2). \tag{53}$$

Therefore,

$$\sum_{n_1, n_2, \ldots = 1}^{N} \text{tr}((A_i - (-1 + 2n_i/n_{max}))^2 M(1, n_1) M(2, n_2) \ldots \rho \ldots M(2, n_2) M(1, n_1)) \leq \text{const.} \times (N\delta^2 n_{max}^2 + 1/n_{max}^2). \tag{54}$$

Choosing

$$n_{max} = \delta^{-1/2} N^{-1/4}, \tag{55}$$

we find that we measure all operators to within a mean-square error of order $\delta \sqrt{N}$.

VIII. DISCUSSION

The main result is an explicit construction of a pair of exactly commuting matrices which are close to a pair of almost commuting matrices. The construction of the matrix is explicit and can be handled easily on a computer for modest N. We have in fact implemented the construction in Lemma (2) for the uniform chain.

We gave above applications to quantum measurement. Another application of this result is to construct Wannier functions for any two dimensional quantum system for a spectral gap. In [29], it was pointed out that given a two dimensional quantum system with a gap between bands, one could define an operator G which projected onto the bands below the gap. Then, define the operator X and Y to measure X and Y position of particles, and define GXG and GYG as projections of X and Y into the lowest band. Let $\|X\|, \|Y\| = L$, where L is the linear size of the system. Since the operator G was constructed in [29] as a short-range operator, the commutator $\|[GXG, GYG]\|$ is small compared to L^2, and thus we can use the results here to construct a basis of Wannier functions which is localized in both the x- and y-directions.

---

[1] P. R. Rosenthal: Are almost commuting matrices near commuting pairs?. AMS Monthly **76**, 925 (1969).

[2] P. R. Halmos: Some unsolved problems of unknown depth about operators on Hilbert space. Proc. Roy. Soc. Edinburgh A **76**, 67 (1976).

[3] H. Lin: Almost commuting self-adjoint matrices and applications. Fields. Inst. Commun. **13**, 193 (1995).

[4] P. Friis and M. Rordam: Almost commuting self-adjoint matrices—a short proof of Huaxin Lin's theorem. J. Reine Angew. Math. **479**, 121 (1996).

[5] D. Voiculescu: Asymptotically commuting finite rank unitaries without commuting approximants. Acta Sci. Math. **451**, 429 (1983).

[6] D. Voiculescu: Remarks on the singular extension in the C^*-algebra of the Heisenberg group. J. Op. Thy. **5**, 147 (1981).

[7] K. R. Davidson: Almost commuting Hermitian matrices. Math. Scand. **56**, 222 (1985).

[8] E. H. Lieb and D. W. Robinson: The finite group velocity of quantum spin systems. Commun. Math. Phys. **28**, 251 (1972).

[9] M. B. Hastings: Lieb-Schultz-Mattis in higher dimensions. Phys. Rev. B **69** 104431 (2004).

[10] M. B. Hastings and T. Koma: Spectral gap and exponential decay of correlations. Commun. Math. Phys. **265**, 781 (2006).

[11] B. Nachtergaele and R. Sims: Lieb-Robinson bounds and the exponential clustering theorem. Commun. Math. Phys. **265**, 119 (2006).

[12] M. B. Hastings: Locality in Quantum and Markov Dynamics on Lattices and Networks. Phys. Rev. Lett. **93**, 140402 (2004).

[13] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne: Exact relaxation in a class of nonequilibrium lattice systems. Phys. Rev. Lett. **100**, 030602 (2008).

[14] J. Eisert and T. J. Osborne: General entanglement scaling laws from time evolution. Phys. Rev. Lett. **97**, 150404 (2006).

[15] S. Bravyi, M. B. Hastings, and F. Verstraete: Lieb-Robinson Bounds and the Generation of Correlations and Topological Quantum Order. Phys. Rev. Lett. **97**, 050401 (2006).

[16] T. J. Osborne: A Renormialization-Group Algorithm for Eigenvalue Density Functions of Interacting Quantum Systems. arXiv:cond-mat/0605194.

[17] T. J. Osborne: Efficient Approximation of the Dynamics of One-Dimensional Quantum Spin Systems. Phys. Rev. Lett. **97**, 157202 (2006).

[18] M. B. Hastings: Quantum Belief Propagation. Phys. Rev. B Rapids **76**, 201102 (2007).

[19] T. J. Osborne: Efficient approximation of the dynamics of one-dimensional quantum spin systems. Phys. Rev. A **75**, 042306 (2007).

[20] M. B. Hastings: Observations Outside the Light-Cone: Algorithms for Non-Equilibrium and Thermal States. Phys. Rev. B **77**, 144302 (2008).

[21] M. B. Hastings: An Area Law for One Dimensional Quantum Systems. J. Stat. Mech., P08024 (2007).

[22] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims: Lieb-Robinson Bounds for Harmonic and Anharmonic Lattice Systems. arXiv:0712.3820.

[23] M. B. Hastings: Quasi-Adiabatic Continuation in Gapped Spin and Fermion Systems: Goldstone's Theorem and Flux Periodicity. J. Stat. Mech., P05010 (2007).

[24] R. Carmona, A. Klein, and F. Martinelli: Anderson localization for Bernoulli and other singular potentials. Commun. Math. Phys. **108**, 41 (1987).

[25] H. von Dreifus and A. Klein: A new proof of localization in the Anderson tight binding model. Commun. Math. Phys. **124**, 285 (1989). C. Shubin, R. Vakilian, and T. Wolff, Some harmonic analysis questions suggested by Anderson-Bernoulli models. Geom. Funct. Anal. **8**, 932 (1998).

[26] C. Jordan: Essai sur la géométrie à n dimensions. Bulletin de la S. M. F. **3**, 103 (1875).

[27] R. Gervais, Q. I. Rahman, and G. Schmeisser: A bandlimited function simulating a duration-limited one, *Anniversary volume on approximation theory and functional analysis*, Schiftenreihe Numer. Math., 355 (1984).

[28] T. Strohmer and J. Tanner: Implementations of Shannon's sampling theorem, a time-frequency approach. Sampling Thy. in Signal and Image Processing, **4**, 1 (2005); Fritz John, *Partial Differential Equations*, Springer-Verlag, New York, 1991.

[29] M. B. Hastings: Topology and Phases in Fermionic Systems. J. Stat. Mech., L01001 (2008).