LLNL-PROC-412721

LAWRENCE
LIVERMORE
NATIONAL
LABORATORY

Using the Sherman Morrison Woodbury formula for coupling external circuits with FEM for Simulation of eddy current problems

D. White

May 1, 2009

IEEE International Magnetics Conference 2009
Sacramento, CA, United States
May 4, 2009 through May 8, 2009

Disclaimer

Using the Sherman-Morrison-Woodbury Formula for Coupling External Circuits with FEM for Simulation of Eddy Current Problems

Daniel A. White
Lawrence Livermore National Laboratory, Livermore, CA 94550

***Abstract*—Simulation of three-dimensional transient eddy current problems is important to numerous applications. The Finite Element Method (FEM) has proven be to be powerful numerical technique for solving the Partial Differential Equations (PDE) describing eddy currents. In order to solve the PDE, boundary conditions must be provided, and in many applications the boundary conditions are not known explicitly but can be provided by a Resistor-Inductor-Capacitor (RLC) circuit model. The emphasis of this paper is on an efficient and exact coupling of the RLC network equations with the FEM equations. The coupling is based on an exact linear algebra identity known as the Sherman-Morrison-Woodbury (SMW) formula. One advantage of this approach is that the FEM matrices are not modified. This is important if a fast "black-box" solver is available for the FEM matrices, these solvers typically require that the matrices have certain mathematical properties and these properties are not modified by the SMW approach. A second advantage is that the SMW approach is valid for an arbitrary number of independent external circuits.**

***Index Terms*—Eddy currents, finite element, circuits, pulse power.**

I. Introduction

SIMULATION of three-dimensional transient eddy current problems is important to numerous applications. The particular class of eddy current problems considered here are problems that are driven by a pulsed power supply such as a capacitor bank. Example applications of interest include electromagnetic metal forming, in which eddy currents are used to drive sheet metal into a mold at high velocity; equation of state research, in which pulse magnetic fields are used to compress materials to high pressure; and railguns, in which pulsed magnetic fields are used to accelerate a projectile. In each of these applications the complete problem can be decomposed into a complex dynamic load and a pulsed power supply. The power supply consists of, for example, a collection of capacitors, cables, snubbers, and switches. The power supply is modeled by a Resistor-Inductor-Capacitor (RLC) network, i.e. a lumped parameter model. On the other hand the load is modeled by a three-dimensional Partial Differential Equation (PDE), and this PDE is solved using the Finite Element Method (FEM). For a trivial time-independent load, it may be possible to compute the resistance and inductance of the load using FEM and then use these values in an RLC model of the entire system. But the focus here is on non-trivial loads, meaning that the load is modified as power is being delivered, either through deformation or melting or both. The emphasis of this paper is on an exact and efficient computational algorithm for coupling the external RLC circuit equations with the FEM equations for this latter case.

An early approach for coupling RLC circuits with FEM equations was an iterative scheme in which the circuit equations and the FEM equations were updated in an operator split (or leapfrog) manner [1]. This approach is appealing, however it is not robust, and in fact it can be shown to be unstable. This is discussed more in Section II below. A more robust approach is to simultaneously solve the circuit equations with the FEM equations, this results in a large asymmetric indefinite system of equations to be solved at every time step [2] [3] [4] [5]. This may be satisfactory for modestly sized problems, but it is intractable for problems involving millions of degrees of freedom. Modern FEM programs for transient eddy current simulation use very efficient so-called "fast solvers", such as multigrid, to compute the fields. However in order to apply these fast methods the FEM matrix must have certain properties, and these properties are destroyed by the simplistic direct addition of the circuit equations.

II. Fully Coupled Formulation

A. FEM Disrectization of the Eddy Current Equations

As mentioned in the Introduction the overarching approach is to decompose the problem into the power supply, modeled as one or more RLC circuits, and the load, modeled as a PDE. This is illustrated in 1. The key coupling variables are the port current and the port voltage. The RLC circuit equations will be solved using Modified Nodal Analysis. In MNA, the nodal voltages are computed given the port currents, i.e. current is the input to MNA and voltage is the output. The load PDE is solved using the $H(curl)$-conforming electric field FEM, and in this formulation the port current is computed given the port voltage, i.e. voltage is the input to the model and current is the output. Clearly, for a self consistent coupling , the two models must agree on the port current and the port voltage.

The Ampere-Faraday equations are a pair of coupled first-order PDE's for electric and magnetic fields. The well-known eddy current approximation (i.e. low-frequency, good-conductor) is used. These equations can be expressed in terms

This work was performed under the auspices of the U.S. Department of Energy by the University of California, Lawrence Livermore National Laboratory under contract DE-AC52-07NA27344

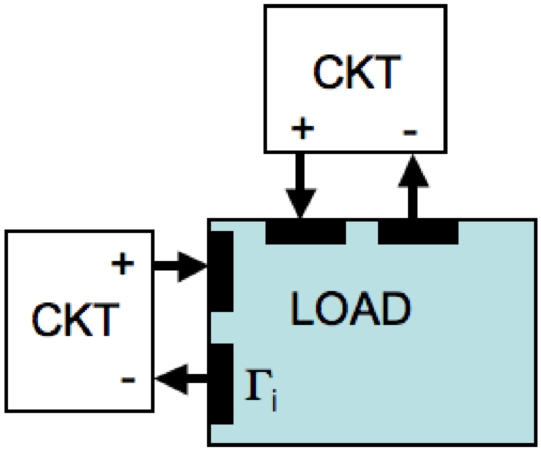

Fig. 1. Decomposition of the problem into RLC circuits and a load. Each circuit is coupled to the load via a port, with a port voltage and a port current. The load is modeled using FEM. A RLC circuit node is associated with a set of faces of the finite element mesh, denoted in the illustration as Γ_i.

of the electric field $\vec{E}$ and the magnetic flux density $\vec{B}$ as

$$\vec{\nabla} \times \frac{1}{\mu}\vec{B} = \sigma\vec{E} + \vec{J_s} \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

where μ is the magnetic permeability, σ is the electric conductivity, and $\vec{J_s}$ is a source term. There are also divergence equations for both $\vec{B}$ and $\vec{E}$, but since these area automatically satisfied by $H(curl)$-conforming FEM they need not be discussed further [6]. The source term $\vec{J_s}$ represents the current generated by an applied voltage,

$$\vec{J_s} = -\sigma\nabla\Phi \tag{3}$$

$$\nabla \cdot \nabla\Phi = 0, ;\ \Phi = g_i \ on\, \Gamma_i, \tag{4}$$

where g_i is the voltage applied to surface Γ_i, as shown in Figure 1. The fields $\vec{B}$ and $\vec{E}$ are approximated in basis function expansions

$$\vec{B} \approx \sum_{i=1}^{n} b_i \vec{W}_i^2$$

$$\vec{E} \approx \sum_{i=1}^{m} e_i \vec{W}_i^1$$

$$\Phi \approx \sum_{i=1}^{k} \phi_i \vec{W}_i^0$$

where W^2 denotes a $H(div)$-conforming basis function, W^1 denotes a $H(curl)$-conforming basis function, and W^0 denotes the standard scalar H^1-conforming basis function. For the case of lowest order basis functions, k, m, and n denote the number of mesh nodes, edges, and faces, respectively. The Galerkin finite element procedure, combined with backward Euler time integration, results in the fully discrete system of equations,

$$\mathbf{S}\mathbf{v}_{n+1} = \mathbf{g}_n \tag{5}$$

$$(\mathbf{M} + \Delta t\mathbf{Y})\,\mathbf{e}_{n+1} = (\mathbf{K})^T\mathbf{b}_n - \Delta t\mathbf{D}\mathbf{v}_{n+1} \tag{6}$$

$$\mathbf{b}_{n+1} = \mathbf{b}_n - \Delta t\mathbf{K}\mathbf{e}_{n+1} \tag{7}$$

$$\mathbf{i}_{n+1} = \mathbf{R}\mathbf{e}_{n+1} + \mathbf{P}\phi_{n+1} \tag{8}$$

The derivation of this fully discrete system of equations is given in [7]. The vectors ϕ, $\mathbf{e}$, and $\mathbf{b}$ are the degrees-of-freedom for the scalar potential, electric field, and magnetic flux density, respectively. Note that $\mathbf{e}$ is the solenoidal component of the electric field only. The vector $\mathbf{g}$ is the port voltages. The matrix $\mathbf{S}$ is the $k \times k$ H^1 Laplace matrix, the matrices $\mathbf{M}$ and $\mathbf{Y}$ are the $m \times m$ $H(curl)$ mass and stiffness matrices, the matrix $\mathbf{D}$ is the $m \times k$ gradient matrix mapping H^1 to $H(curl)$, and the matrix $\mathbf{K}$ is the $m \times n$ curl matrix mapping $H(curl)$ to $H(div)$. Finally, $\mathbf{i}_{n+1}$ is the vector of port currents, and the matrices $\mathbf{R}$ and $\mathbf{P}$ represent the surface integral of $\vec{J} \cdot n$ over each port surface Γ_i.

As discussed in [6] this finite element algorithm is second order accurate in space, first order accurate in time, provably stable and divergence preserving. In the Introduction it was mentioned that for the problems of interest the load is dynamic, due to motion or heating or both. Coupling of the above eddy current equations with the equations of heat transfer and hydrodynamics is outlined in [7], for example, and is not discussed here. Instead, for brevity the matrices are just considered to be functions of time, with the specific time-dependence not essential to the discussion.

B. *Modified Nodal Analysis of the RLC equations*

Modified Nodal Analysis (MNA) [8] is a common approach for simulation of RLC networks. The approach is based on Kirchhoff's Current Law: the sum of all currents entering a node is zero. The Laplace-domain nodal admittance system of equations can be written compactly as

$$(\mathbf{G} + s\mathbf{C})\,\mathbf{x} = \mathbf{w}. \tag{9}$$

The matrix $\mathbf{G}$ contains the admittance of resistive elements, with $\mathbf{G}_{ii}$ being the sum of all admittances connected to node i and $\mathbf{G}_{ij}$ being the admittance between nodes i and j. Likewise, the matrix $\mathbf{C}$ contains the admittances for inductive and capacitive elements. The vector $\mathbf{x}$ represents nodal voltages, current flowing through voltage sources, and current flowing through inductors. The vector $\mathbf{w}$ contains the values of the independent current sources and independent voltage sources. An initial vector $\mathbf{x}^0$ must of course be supplied to completely specify the problem. The dimension of the system of equations is *(number of circuit nodes)* + *(number of voltage sources)* + *(number of inductors)*. Converting to the time-domain gives

$$\mathbf{C}\frac{d\mathbf{x}}{dt} = \mathbf{w} - \mathbf{G}\mathbf{x}. \tag{10}$$

Mathematicians refer to equations of the form of (10) as Differential-Algebraic system of Equations (DAE). It can be shown that for decent circuits (no loops of inductors, etc.) this

DAE has index 1 meaning that the solution exists and simple numerical methods can be employed. Note that in general the matrix $\mathbf{C}$ can be singular even for decent circuits therefore implicit methods must be used. The simplest such method is backward Euler,

$$(\mathbf{C}+\Delta t\mathbf{G})\,\mathbf{x}_{n+1} = \mathbf{C}\mathbf{x}_n + \Delta t\mathbf{w}_{n+1} \tag{11}$$

and for decent circuits the matrix $(\mathbf{C}+\Delta t\mathbf{G})$ is non-singular. While high-performance SPICE solvers use adaptive higher-order backward difference methods, backward Euler is used here to be compatible with the FEM equations discussed above.

C. Coupling the MNA equations with the FEM equations

Let's summarize all of these equations, and also define some new matrices $\mathbf{A} = (\mathbf{C}+\Delta t\mathbf{G})$ and $\mathbf{Z} = (\mathbf{M}+\Delta t\mathbf{S})$ for convenience, giving

$$\begin{aligned}
\mathbf{A}\mathbf{x}_{n+1} &= \mathbf{C}\mathbf{x}_n + \Delta t\mathbf{T}\mathbf{i}_{n+1} + \Delta t\mathbf{w}_{n+1}\\
\mathbf{S}\phi_{n+1} &= \mathbf{Q}\mathbf{x}_{n+1}\\
\mathbf{Z}\mathbf{e}_{n+1} &= (\mathbf{K})^T\mathbf{b}_n - \Delta t\mathbf{D}\phi_{n+1}\\
\mathbf{i}_{n+1} &= \mathbf{P}\phi_{n+1} + \mathbf{R}\mathbf{e}_{n+1}\\
\mathbf{b}_{n+1} &= \mathbf{b}_n - \Delta t\mathbf{K}\mathbf{e}_{n+1}
\end{aligned}$$

The above system of equations can be written in matrix form as

$$\begin{bmatrix} \mathbf{A} & 0 & 0 & -\Delta t\mathbf{T} & 0\\ -\mathbf{Q} & \mathbf{S} & 0 & 0 & 0\\ 0 & \Delta t\mathbf{D} & \mathbf{Z} & 0 & 0\\ 0 & -\mathbf{P} & -\mathbf{R} & 1 & 0\\ 0 & 0 & -\Delta t\mathbf{K} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}\\ \phi\\ \mathbf{e}\\ \mathbf{i}\\ \mathbf{b} \end{bmatrix}^{n+1} =$$
$$\begin{bmatrix} \mathbf{C} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & (\mathbf{K})^T\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}\\ \phi\\ \mathbf{e}\\ \mathbf{i}\\ \mathbf{b} \end{bmatrix}^{n} + \begin{bmatrix} \Delta t\mathbf{w}\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}$$

This is a fully self-consistent implicit system of equations for the tightly-coupled coupling of circuit MNA equations with the eddy current FEM equations. Note that matrices $\mathbf{T}$ and $\mathbf{Q}$ are the coupling matrices. The matrix $\mathbf{T}$ maps eddy current port currents to RLC circuit port currents, and the matrix $\mathbf{Q}$ maps RLC circuit node voltages to eddy current port voltages. These are extremely sparse matrices consisting of 0's and 1's only, with the 1's denoting the coupling. Note that the large matrix does not have nice properties, it is not symmetric and not definite. In the next two sections two different approaches for simplifying the solution are examined, the operator splitting approach which is unstable, and the Sherman-Morrison-Woodbury approach which is stable.

III. Operator Splitting

The operator splitting approach is conceptually straightforward, the $\mathbf{T}$ matrix discussed above is simply moved from the left hand side of the equation to the right hand side of the equation. This means that the the RLC circuit state $\mathbf{x}_{n+1}$ is computed using the port currents $\mathbf{i}_n$, i.e.the port currents are lagged in time,

$$\begin{bmatrix} \mathbf{A} & 0 & 0 & 0 & 0\\ -\mathbf{Q} & \mathbf{S} & 0 & 0 & 0\\ 0 & \Delta t\mathbf{D} & \mathbf{Z} & 0 & 0\\ 0 & -\mathbf{P} & -\mathbf{R} & 1 & 0\\ 0 & 0 & -\Delta t\mathbf{K} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}\\ \phi\\ \mathbf{e}\\ \mathbf{i}\\ \mathbf{b} \end{bmatrix}^{n+1} =$$
$$\begin{bmatrix} \mathbf{C} & 0 & 0 & \Delta t\mathbf{T} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & (\mathbf{K})^T\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}\\ \phi\\ \mathbf{e}\\ \mathbf{i}\\ \mathbf{b} \end{bmatrix}^{n} + \begin{bmatrix} \Delta t\mathbf{w}\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}$$

Now, the large matrix on the left hand side is block lower triangular and can be solved quite simply as follows:

$$\begin{array}{llcl}
\textit{solve} & \mathbf{A}\mathbf{x}_{k+1} & = & \mathbf{C}\mathbf{x}_k + \dots\\
\textit{solve} & \mathbf{S}\phi_{k+1} & = & \mathbf{Q}\mathbf{x}_{k+1}\\
\textit{solve} & \mathbf{Z}\mathbf{e}_{k+1} & = & -\Delta t\mathbf{D}\phi_{k+1} + \dots\\
\textit{compute} & \mathbf{i}_{k+1} & = & \mathbf{P}\phi_{k+1} + \mathbf{R}\mathbf{e}_{k+1}\\
\textit{compute} & \mathbf{b}_{k+1} & = & \mathbf{b}_k - \Delta t\mathbf{K}\mathbf{e}_{k+1}
\end{array} \tag{12}$$

Each of these individual solves can be performed using optimal algorithms, i.e. multigrid, hence this approach is quite appealing. Unfortunately the method can go unstable, in fact instabilities have been observed even in the limit as $\Delta t \to 0$.

IV. Sherman-Morrison-Woodbury Formula

For simplicity denote the large fully coupled system of equations as

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 & 0 & -\Delta t\mathbf{T} & 0\\ -\mathbf{Q} & \mathbf{S} & 0 & 0 & 0\\ 0 & \Delta t\mathbf{D} & \mathbf{Z} & 0 & 0\\ 0 & -\mathbf{P} & -\mathbf{R} & 1 & 0\\ 0 & 0 & -\Delta t\mathbf{K} & 0 & 1 \end{bmatrix} \tag{13}$$

and denote the same matrix, but without the $\mathbf{T}$ submatrix, as

$$\bar{\mathbf{M}} = \begin{bmatrix} \mathbf{A} & 0 & 0 & 0 & 0\\ -\mathbf{Q} & \mathbf{S} & 0 & 0 & 0\\ 0 & \Delta t\mathbf{D} & \mathbf{Z} & 0 & 0\\ 0 & -\mathbf{P} & -\mathbf{R} & 1 & 0\\ 0 & 0 & -\Delta t\mathbf{K} & 0 & 1 \end{bmatrix} \tag{14}$$

Two new vectors $\mathbf{z}$ and $\mathbf{y}$ are defined as

$$\mathbf{z} = \begin{bmatrix} \mathbf{x}\\ \phi\\ \mathbf{e}\\ \mathbf{i}\\ \mathbf{b} \end{bmatrix}^{n+1} \tag{15}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & 0 & 0 & \Delta t\mathbf{T} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & (\mathbf{K})^T\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}\\ \phi\\ \mathbf{e}\\ \mathbf{i}\\ \mathbf{b} \end{bmatrix}^{n} + \begin{bmatrix} \Delta t\mathbf{w}\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}. \tag{16}$$

The large system of equations can then be written compactly as $\bar{\mathbf{A}}\mathbf{z} = \mathbf{y}$. The matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{M}}$ are related by the low rank modification

$$\bar{\mathbf{A}} = \mathbf{M} + \mathbf{u}\mathbf{v}^T \tag{17}$$

where

$$\mathbf{u} = \begin{bmatrix} -\Delta t\mathbf{T} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{18}$$

The vectors $\mathbf{u}, \mathbf{v}$ have one column for each port. Therefore the matrix $\mathbf{u}\mathbf{v}^T$, which is the difference between the big global system matrix $\bar{\mathbf{A}}$ and the preconditioner $\bar{\mathbf{M}}$, is of rank number-of-ports, which will be denoted as np.

Consider the Sherman-Morrison-Woodbury (SMW) formula

$$\left(\mathbf{M} + \mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{M}^{-1} - \mathbf{M}^{-1}\mathbf{u}\left(\mathbf{1} + \mathbf{v}^T\mathbf{M}^{-1}\mathbf{u}\right)^{-1}\mathbf{v}^T\mathbf{M}^{-1} \tag{19}$$

This formula is exact. The SMW formula is applied as follows,

$$\begin{aligned} \mathbf{z} &= \bar{\mathbf{A}}^{-1}\mathbf{y} \\ &= \left(\mathbf{M} + \mathbf{u}\mathbf{v}^T\right)^{-1}\mathbf{y} \\ &= \mathbf{M}^{-1}\mathbf{y} - \mathbf{M}^{-1}\mathbf{u}\left(\mathbf{1} + \mathbf{v}^T\mathbf{M}^{-1}\mathbf{u}\right)^{-1}\mathbf{v}^T\mathbf{M}^{-1}\mathbf{y} \\ &= \mathbf{g} - \mathbf{f}\alpha\mathbf{v}^T\mathbf{g} \end{aligned}$$

where the vectors $\mathbf{g}$ and $\mathbf{g}$ are defined as solutions of

$$\begin{aligned} \mathbf{M}\mathbf{g} &= \mathbf{y} \\ \mathbf{M}\mathbf{f} &= \mathbf{u} \\ \alpha &= \left(1 + \mathbf{v}^T\mathbf{f}\right)^{-1} \end{aligned}$$

Note that any solve involving $\mathbf{M}$ is straightforward, it is just the step-by-step sequential solution (12) outlined above. Note that if there are np ports, then $\mathbf{M}\mathbf{f} = \mathbf{u}$ represents np separate solves. Note that the matrix $\alpha = \left(\mathbf{1} + \mathbf{v}^T\mathbf{f}\right)$ is a tiny $np \times np$ matrix that is simply inverted. For the case of just one port, there are just two solves. The solve for $\mathbf{g}$ is is an update of the MNA+FEM equations, taking into account the previous MNA+FEM state (i.e. the $\mathbf{y}$ vector) but ignoring the effect of the current $\mathbf{i}$ on the circuit. The solve for $\mathbf{f}$ is an update of the MNA+FEM equations using a unit current $\mathbf{i} = 1$ supplied to the circuit equations, but ignoring the previous MNA+FEM state and ignoring any independent voltage and current sources, i.e. independent of the $\mathbf{y}$ vector. This latter solve gives the *impulse response* of the MNA+FEM system of equations. Note that if the FEM problem is such that there is no motion and no change of material properties, than this latter solve need only be computed once, since the impulse is response is then time-invariant. To summarize, the procedure is

$$\begin{array}{llcll} \textit{solve} & \mathbf{M}\mathbf{f} & = & u & \textit{using}\ (12) \\ \textit{solve} & \mathbf{M}\mathbf{g} & = & y & \textit{using}\ (12) \\ \textit{compute} & \alpha & = & \left(1 + \mathbf{v}^T\mathbf{f}\right)^{-1} & \\ \textit{compute} & \mathbf{z} & = & \mathbf{g} - \mathbf{f}\alpha\mathbf{v}^T\mathbf{g} & \end{array} \tag{20}$$

The vector $\mathbf{z}$ is the final solution of the fully coupled MNA+FEM equations. Note that the matrix $\mathbf{M}$ is the same for each solve, so it is most computationally efficient to solve $\mathbf{M}(\mathbf{f}, \mathbf{g}) = (\mathbf{u}, \mathbf{y})$ if possible. For time-independent loads, the impulse response vector $\mathbf{f}$ need only be computed once. For time dependent loads, all matrices and vectors need to computed at each time step.

V. Conclusions

Many important low-frequency electromagnetic problems can be analyzed by decomposing the problem into a power supply coupled to a complicated load. The power supply is modeled as a collection of RLC circuits, and the numerical solution of the circuit equations is formulated using Modified Nodal Analysis (MNA). The load is modeled as a Partial Differential Equation, i.e. the Ampere-Faraday equations of low-frequency electromagnetics, and the numerical solution of this PDE is formulated using the Finite Element Method (FEM). Both the MNA equations and the FEM equations are integrated in time using backward Euler. The MNA equations are coupled with the FEM equations by matching port voltages and currents at each port. For stability, the MNA equations and the FEM equations must be tightly-coupled, i.e. solved simultaneously. A brute-force assembly of the MNA+FEM equations results in a large, asymmetric, indefinite system of equations that is difficult to solve. In this paper a novel method based on the Sherman-Morrison-Woodbury formula is developed. The Sherman-Morrison-Woodbury formula is an exact linear algebra identity that relates the solution of a system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ to the solution of a similar set of equations $\mathbf{M}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{M}$ differ by a rank np matrix. In our case, np is the number of external RLC circuits. The key advantage of this approach is that the FEM matrices are not modified, hence fast algorithms such as multigrid can be used. A second advantage is that an arbitrary number of external circuits can be coupled with the FEM equations.

References

[1] G. Bedrosian. A new method or coupling finite element solutions with external circuits. *IEEE Trans. Mag.*, 29(2):1664–1669, 1993.

[2] I. A. Tsukerman, A. Konrad, and J. Lavers. A method for circuit connections in time dependent eddy current problems. *IEEE Trans. Mag.*, 28(2):1299–1302, 1992.

[3] I. A. Tsukerman, A. Konrad, G. Meuner, and J. C. Sabonmadiere. Coupled field-circuit problems: Trends and accomplishments. *IEEE Trans. Mag.*, 29(2):1701–1704, 1993.

[4] F. Piriou and A. Razek. Finite element analysis in electromagnetic systems accounting for electric circuits. *IEEE Trans. Mag.*, 29(2):1669–1675, 1993.

[5] H. DeGersem, R. Mertens, D. Lahaye, S. Vanderwalle, and K. Hameyer. Solution strategies for transient, field-circuit coupled systems. *IEEE Trans. Mag.*, 36(4).

[6] R. Rieben and D. White. Verification of high-order mixed finite element solution of transient magnetic diffusion problems. *IEEE Trans. Mag.*, 42(1):25–39, 2006.

[7] R. Rieben, D. White, B. Wallin, and J. Solberg. An arbitrary Lagrangian Eulerian discretization of MHD on 3D unstructured grids. *J. Comput. Phys.*, 226:534–570, 2007.

[8] J. Vlack and K. Singhal. *Computer Methods for Circuit Analysis and Design*. Von Nostrand Reinhold, 1983.

This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.