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Discretizing Transient Current Densities in the Maxwell Equations

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1 Abstract

We will briefly discuss a technique for applying transient volumetric current sources in full-wave, time-domain electromagnetic simulations which avoids the need for divergence cleaning. The method involves both "edge-elements" and "face-elements" in conjunction with a particle-in-cell scheme to track the charge density. Results from a realistic, 6.7 million element, 3D simulation are shown. While the author may have a finite element bias the technique should be applicable to finite difference methods as well.

2 Introduction

The Maxwell Equations with a current density source term can be written

$$\begin{aligned} \frac{\partial}{\partial t}\vec{D} &= \nabla \times \vec{H} - \vec{J} \\ \frac{\partial}{\partial t}\vec{B} &= -\nabla \times \vec{E} \end{aligned}$$

where

$$\vec{D} = \epsilon\vec{E} \text{ and } \vec{B} = \mu\vec{H}$$

These equations are commonly discretized using "edge-elements", or discrete 1-forms, for the electric field and "face-elements", or discrete 2-forms, for the magnetic flux density. This scheme requires that $\vec{J}$ also be approximated with edge-elements, which works quite well in many situations. However, this scheme does have certain drawbacks.

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One difficulty with 1-form current densities is that they can spread through material interfaces into non-physical regions. For example, consider a vacuum region abutting a weak conductor which contains a constant current density. What value for $\vec{J}$ should be applied to the edges which are shared between these two regions? If the constant $\vec{J}$ value is used, then the conducting region will contain the correct value but the vacuum region will also contain a non-zero current density. If a value of zero is applied on these edges, then the vacuum region will correctly have zero current but the conductor will contain less current density than desired.

Another difficulty, and the one we will focus on, arises if the current density is transient and the primary interest is to determine how a cavity will resonate after a current pulse passes through it. The problem here is that the continuity equation for the electric charge is only weakly satisfied. Therefore, current densities can, and often do, leave behind non-physical charge densities after they pass through the computational mesh. These charge densities can, in turn, produce a non-physical, static, electric field which not only makes field plots appear ugly but can also reduce the accuracy of the meaningful portion of the solution.

3 Example Problem

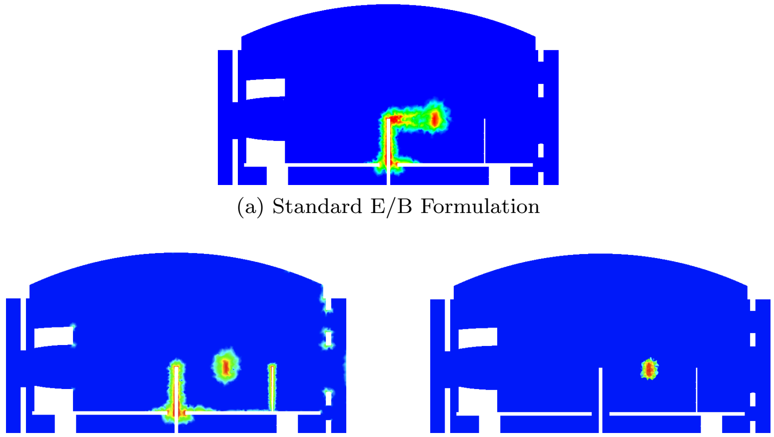
Consider a laser target chamber, which is roughly cylindrical with a height of nearly one meter and a radius of one meter. The chamber also has several port holes for diagnostic equipment as well as the input port for the laser beam. When a high power laser beam enters the chamber and strikes its target, it will partially vaporize the target and generate a flux of electrons which are propelled towards the outer walls of the chamber.

We are primarily interested in the pulse of electromagnetic waves radiated by this charge packet so we do not model the incoming laser beam or the vaporization of the target. Also, we do not currently attempt to model the charge packet as a plasma, it is simply a known charge density moving through the mesh in a prescribed fashion. This approximation is valid for this particular problem because the liberated electrons have very high energies.

4 Typical E/B Formulation

As mentioned previously a standard E/B formulation of the problem requires that $\vec{J}$ be approximated by discrete 1-forms with degrees of freedom on the edges of the mesh.

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\epsilon\vec{E}\right) &= \nabla\times\frac{1}{\mu}\vec{B}-\vec{J} \\
\frac{\partial}{\partial t}\vec{B} &= -\nabla\times\vec{E}
\end{aligned}
$$



(a) Standard E/B Formulation

(b) E/B with Divergence Cleaning

(c) D/H Formulation with PIC Source

Figure 1: The Divergence of the vector field $\vec{D}$ plotted on a logarithmic scale.

If we take the divergence of Ampère's law and make use of the fact that the charge density is related to the electric displacement via $\nabla \cdot \vec{D} = \rho$, we find the following:

$$\nabla \cdot \frac{\partial}{\partial t}\left(\epsilon \vec{E}\right) = -\nabla \cdot \vec{J}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0.$$

The divergence of a 1-form can only be defined in a weak sense, i.e. as a type of least squares best fit. Hence the continuity equation for the electric charge may not be locally satisfied everywhere although it should be nearly satisfied globally.

Figure 1a shows an example of a charge density plot for our model problem. The image clearly shows the charge packet itself just to the right of center. Unfortunately, it also shows a large non-physical charge buildup left behind in the wake of the packet. The boundary of the computational domain is assumed to be a perfect electrical conductor so the charge near the boundary can be interpreted as being related to the surface charge density. This is actually another oddity of the E/B formulation, surface charges appear smeared into the volume elements which touch the surface. This may not be an attractive feature of the image but at least it has a reasonable physical interpretation.

5 Divergence Cleaning

The non-physical charge buildup can be removed by performing divergence cleaning when deemed necessary or perhaps at every time step. This is the

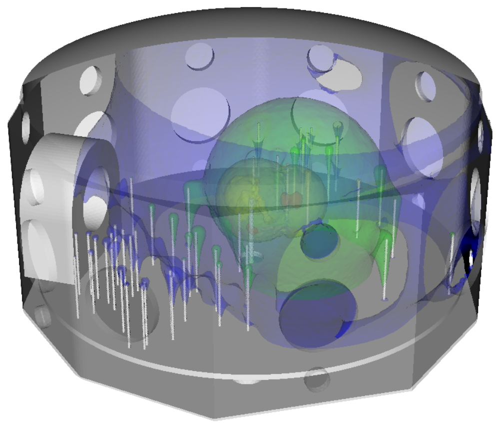


Figure 2: Logarithmically scaled contour plot of the magnitude of the electric field computed using the E/B formulation with divergence cleaning.

process of adding something to the field so that its divergence has a desired value but its curl remains unchanged. For the model problem we can add a $\vec{\tilde{J}}$ to the source so that the divergence will match the desired change in charge density given by $\tilde{\dot{\rho}}$. We assume that the correction to $\vec{J}$ is the gradient of a scalar field ψ so that it will have zero curl.

$$
\begin{aligned}
\dot{\rho} &= -\nabla \cdot \vec{J} \text{ (the \textit{computed} change in } \rho) \\
\tilde{\dot{\rho}} &= \dot{\rho} + \dot{\tilde{\rho}} = -\nabla \cdot \vec{J} - \nabla \cdot \vec{\tilde{J}} \text{ (the \textit{desired} change)} \\
\nabla \cdot \vec{\tilde{J}} &= -\tilde{\dot{\rho}} - \nabla \cdot \vec{J} \text{ (the necessary correction to } \vec{J}) \\
\nabla^2 \tilde{\psi} &= -\tilde{\dot{\rho}} - \nabla \cdot \vec{J}
\end{aligned}
$$

Each divergence cleaning operation then requires an additional linear solve to compute the scalar field ψ.

With this correction we see that the divergence of $\vec{D}$, shown in figure 1b, now matches the desired charge density. Again, note that the charge density near the surfaces is due to the presence of a surface charge density.

Unfortunately, this method has a drawback when the charge density has a velocity near the speed of light. The correction introduces a small quasi-static field centered on the charge density, which appears to propagate faster than the speed of light. Figure 2 shows a logarithmically scaled contour plot of the electric field magnitude which clearly shows contours well beyond the charge packet, which is located near the innermost contour. In figure 1b this component of the field can also be seen because it introduces surface charge densities on the metal object ahead of the charge packet and on several sharp corners farther away. These non-physical charge densities are obviously due to the global solve necessary to compute ψ.

6 D/H Formulation

Obviously, the difficulties discussed in this paper stem from the treatment of $\vec{J}$ as a 1-form vector field. Current density is, however, a flux vector, i.e. the amount of charge crossing a given area per unit time. Flux vector fields are more naturally described using 2-forms so we should have more luck if we approximate Ampère's law using discrete 2-forms.

$$\begin{aligned} \frac{\partial}{\partial t}\vec{D} &= \nabla \times \vec{H} - \vec{J} \\ \frac{\partial}{\partial t}\left(\mu \vec{H}\right) &= -\nabla \times \frac{1}{\epsilon}\vec{D} \end{aligned}$$

In this formulation the curl of $\vec{D}$ must be computed in the weak sense. This weak form requires the solution of a linear system to update $\vec{H}$ using Faraday's law. In the standard E/B formulation it is the curl of $\vec{B}$ that must be computed in the weak sense, requiring a linear solve in Ampère's law to update $\vec{E}$. Normally this linear solve allows us to apply voltage boundary conditions on $\vec{E}$ where we can specify that the tangential component of $\vec{E}$ is zero on perfect electrical conductors. In the D/H formulation this constraint becomes unnecessary because the natural boundary condition is that the tangential component of $\nabla \times \vec{H} = 0$ but, of course, this equation is equivalent to $\vec{E} = 0$ on the boundary.

Simply treating $\vec{J}$ as a 2-form does not magically solve all of our problems. What it does is convert our charge buildup problem from a global least-squares fit into much more simple local charge conservation problem. One way to solve this problem is to use a particle-in-cell (PIC) technique. We don't have the space to describe this procedure in detail but the essential idea is simple enough. Split up the trajectory of the charge packet into a group of rays and imagine the charges themselves as beads moving along these rays. Each time a bead crosses a cell boundary a small flux is applied to the corresponding face in the mesh. If enough rays are used and there are enough beads strung along each ray, then the source will appear reasonably smooth. We should emphasize that we are not performing a self-consistent PIC simulation. The fields do not effect the motion of the charge packet in any way. We are simply using the PIC concept as a bookkeeping scheme to maintain charge conservation.

Figure 1c shows a charge density plot produced using this scheme. Clearly the image shows no sign of non-physical charge buildup. Additionally, the surface charge density does not appear. A further advantage of this method is that the charge density and current density on the surface of perfect electrical conductors can be more accurately computed, if desired. These surface fields

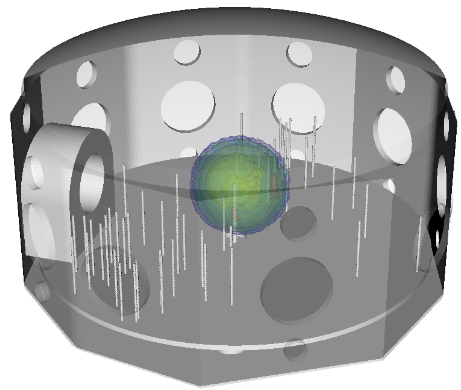


Figure 3: Logarithmically scaled contour plot of the magnitude of the electric field computed using the D/H formulation with a PIC source.

can be directly computed from the surface degrees of freedom for $\vec{D}$ and $\vec{H}$ respectively.

Figure 3 again shows a logarithmically scaled contour plot of the electric field magnitude, analogous to that shown in figure 2. However, in the new plot the non-physical, quasi-static field contours are no longer present. The fields now properly propagate within a spherical shell which expands at the speed of light.

7 Conclusion

We have presented an outline for a charge conserving method of applying transient current sources to the Maxwell Equations in the time-domain. Some of the advantages of using a D/H formulation of the coupled first order wave equation have been discussed. The ability to run charge conserving simulations of transient current densities, while optionally computing accurate representations of surface currents and charge densities, is very appealing. The added benefit of more easily coupling to a PIC simulation, capable of more accurately modeling the motion of the charge packet itself, provides numerous avenues for enhancing the modeling of similar problems.

It should also be noted that the standard E/B formulation and the PIC method placed essentially equivalent demands on computing resources. Each simulation was performed using the same number processors and ran for virtually the same length of time. Conversely, the divergence cleaning procedure, using an algebraic multigrid solver, increased the run time by a factor of roughly 2.8.