A THREE-LEVEL BDDC ALGORITHM FOR SADDLE POINT PROBLEMS

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Abstract.
BDDC algorithms have previously been extended to the saddle point problems arising from mixed formulations of elliptic and incompressible Stokes problems. In these two-level BDDC algorithms, all iterates are required to be in a benign space, a subspace in which the preconditioned operators are positive definite. This requirement can lead to large coarse problems, which have to be generated and factored by a direct solver at the beginning of the computation and they can ultimately become a bottleneck. An additional level is introduced in this paper to solve the coarse problem approximately and to remove this difficulty. This three-level BDDC algorithm keeps all iterates in the benign space and the conjugate gradient methods can therefore be used to accelerate the convergence. This work is an extension of the three-level BDDC methods for standard finite element discretization of elliptic problems and the same rate of convergence is obtained for the mixed formulation of the same problems. Estimate of the condition number for this three-level BDDC methods is provided and numerical experiments are discussed.

Key words. BDDC, three-level, saddle point problem, domain decomposition, coarse problem, benign space, condition number

AMS subject classifications. 65N30, 65N55, 65F10

1. Introduction. The BDDC (Balancing Domain Decomposition by Constraints) methods, which were introduced and analyzed by Dohrmann, Mandel, and Tezaur [6, 20, 21], were originally designed for standard finite element discretization of elliptic problems. The BDDC algorithms have been extended to saddle point problems arising from mixed finite element discretization of Stokes by Li and Widlund [17], to nearly incompressible elasticity by Dohrmann [7], and to flow in porous media by the author [27, 29]. The coarse problems, in the BDDC algorithms proposed in [17, 7, 27, 29], are given in terms of a set of primal constraints chosen from each pair of adjacent subdomains. The matrices of the coarse problems are generated and factored by using a direct solver at the beginning of the computation. The number of selected primal constraints in each subdomain must be large enough to make sure that the iterates stay in the benign space, a special subspace in which the preconditioned operators are positive definite, see [23, 8, 16, 17, 27]. Therefore, the coarse component of the two-level BDDC preconditioner can become a bottleneck if there are very many subdomains, e.g., on a large parallel computing system. One way to remove this difficulty is to use inexact solvers.

Inexact solvers for iterative substructuring algorithms have been discussed in [1, 10, 9, 24, 2]. Klawonn and Widlund considered inexact solver for the one-level Finite Element Tearing and Interconnect (FETI) algorithms in [14]. In [32, 31, 30], the author introduced an additional level into the BDDC algorithms to solve the coarse problem approximately while maintaining a good convergence rate, see also [28, Chapter 3] for details. In [19], Li and Widlund considered solving the local problems in the BDDC algorithms approximately by multigrid methods. Dohrmann also has developed several versions of approximate BDDC preconditioners in [5]. Klawonn and

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Rheinbach recently provided and analyzed some inexact Dual-Primal FETI (FETI-DP) algorithms in [13]. The three-level BDDC algorithms have also been extended to mortar finite element discretization with geometrically nonconforming partition by Kim and the author, see [12]. The multi-level BDDC algorithms have also been studied by Mandel, Sousedik, and Dohrmann [22]. Most of these inexact solvers are designed for symmetric, positive definite problems.

In this paper, we extend the three-level BDDC methods of [32, 31, 30] to the saddle point problems arising from mixed formulations of elliptic and incompressible Stokes problems. For ease of the presentation, we will focus on the formulation and analysis of the mixed formulation of elliptic problems but we will provide numerical results for both cases.

It is noteworthy that Hwang and Cai have pointed out in [11] that, the two-level Schwarz preconditioners with exact coarse solvers for saddle point problems may not have optimal computing time. In their numerical experiments, the use of inexact iterative coarse solvers requires less computer time than the exact one. Our three-level BDDC algorithm provides a way to solve the coarse problem inexactly for two-level BDDC algorithms. This inexact version of BDDC algorithms requires less memory and is more efficient for parallel computation. Moreover, we can also apply several Chebyshev iterations to improve the accuracy of this inexact coarse solver, depending on the application requirement, see [32, 31].

It has been established, by Mandel, Dohrmann, and Tezaur [21], Li and Widlund [18], and Brenner and Sung [3], that a pair of the preconditioned FETI-DP and BDDC operators with the same primal constraints have the same eigenvalues, except possibly for 0 and 1. However, an additional coarse level for FETI-DP is not straightforward since the coarse problem is built into the system matrix. The inexact FETI-DP methods, developed in [13], use block-triangular preconditioners for the saddle point formulation of the FETI-DP methods. Their condition number estimate is based on the existence of a good preconditioner with optimal spectral bounds for the original FETI-DP coarse matrix. We note that an algebraic multigrid preconditioner is used in [13] and that the coarse problem will no longer be positive definite when this inexact FETI-DP algorithms are applied to saddle point problems, see [16]. It appears to be difficult to choose a good preconditioner with optimal spectral bounds for it. Here, we provide an analysis and a good preconditioner for such coarse problems. This preconditioner can also be used for the inexact FETI-DP algorithms for saddle point problems. Moreover, we can prove that our three-level BDDC algorithms will maintain all iterates in the benign space and that the conjugate gradient method can be used to accelerate the convergence.

The rest of the paper is organized as follows. We first review some mixed finite element discretizations and the two-level BDDC methods briefly in Section 2 and Section 3, respectively. We introduce our three-level BDDC method and the corresponding preconditioner $\widetilde{M}^{-1}$ in Section 4. We give some auxiliary results in Section 5. In Section 6, we provide an estimate of the condition number for the system with the preconditioner $\widetilde{M}^{-1}$ which is of the form $C\left(1+\log\frac{\hat{H}}{H}\right)^2\left(1+\log\frac{H}{h}\right)^2$, where $\hat{H}$, H, and h are typical diameters of the subregions, subdomains, and elements, respectively; we decompose the whole domain into subregions and each subregion is then partitioned into subdomains; see Section 4 for the detail. Finally, some computational results are presented in Section 7.

2. An elliptic problem discretized by mixed finite elements. We consider the following elliptic problem on a bounded polygonal domain Ω in two dimensions with a Neumann boundary condition:

$$\left\{ \begin{array}{ll} -\nabla\cdot(a\nabla p) = f & \text{in} \quad \Omega, \\ \mathbf{n}\cdot(a\nabla p) = g & \text{in} \quad \partial\Omega. \end{array} \right. \tag{2.1}$$

Here $\mathbf{n}$ is the outward normal to $\partial\Omega$ and a is a positive definite matrix function with entries in $L^\infty(\Omega)$ satisfying $\xi^T a(\mathbf{x})\xi \geq \alpha\|\xi\|^2$, for a.e. $\mathbf{x}\in\Omega$ and some positive constant α. The functions $f\in L^2(\Omega)$ and $g\in H^{-1/2}(\partial\Omega)$ satisfy the compatibility condition $\int_\Omega f d\mathbf{x} + \int_{\partial\Omega} g d\mathbf{s} = 0$.

The equation (2.1) has a solution p which is unique up to a constant. Without loss of generality, we assume that $g=0$ and that f and the solution p have mean value zero over Ω. We then have a unique solution.

We assume that we are interested in computing $-a\nabla p$ directly as is often required in flow in porous media. We then introduce the velocity $\mathbf{u} = -a\nabla p$ and call p the pressure. We obtain the following system for the velocity $\mathbf{u}$ and the pressure p:

$$\left\{ \begin{array}{ll} \mathbf{u} = -a\nabla p & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = f & \text{in} \quad \Omega, \\ \mathbf{n}\cdot\mathbf{u} = 0 & \text{in} \quad \partial\Omega. \end{array} \right. \tag{2.2}$$

Let $\rho(\mathbf{x}) = a(\mathbf{x})^{-1}$ and define a Hilbert space by

$$H(\mathrm{div}\,,\Omega) = \{\mathbf{v}\in L^2(\Omega)^2; \nabla\cdot\mathbf{v}\in L^2(\Omega)\},$$

with the norm

$$\|\mathbf{v}\|^2_{H(\mathrm{div}\,,\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)} + H_D^2\|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega)},$$

where H_D is the diameter of Ω.

Given a vector $\mathbf{u}\in H(\mathrm{div}\,,\Omega)$, it is possible to define its normal component $\mathbf{u}\cdot\mathbf{n}$ on $\partial\Omega$, as an element of $H^{-1/2}(\partial\Omega)$. Let $H_0(\mathrm{div}\,,\Omega)$ be the subspace of $H(\mathrm{div}\,,\Omega)$ defined as:

$$H_0(\mathrm{div}\,,\Omega) = \{\mathbf{v}\in L^2(\Omega)^2; \nabla\cdot\mathbf{v}\in L^2(\Omega) \text{ and } \mathbf{v}\cdot\mathbf{n} = 0 \;\text{ on }\; \partial\Omega\}.$$

The weak form of (2.2) is given as follows: find $\mathbf{u}\in H_0(\mathrm{div}\,,\Omega)$ and $p\in L_0^2(\Omega) = \{q : q\in L^2(\Omega), \int_\Omega q d\mathbf{x} = 0\}$ such that

$$\left\{ \begin{array}{lcll} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) & = & 0 & \forall \mathbf{v}\in H_0(\mathrm{div}\,,\Omega), \\ b(\mathbf{u},q) & = & -\int_\Omega f q d\mathbf{x} & \forall q\in L_0^2(\Omega), \end{array} \right.$$

where $a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}^T\rho(\mathbf{x})\mathbf{v} d\mathbf{x}$ and $b(\mathbf{u},q) = -\int_\Omega(\nabla\cdot\mathbf{u})q d\mathbf{x}$.

Let $\widehat{\mathbf{W}}$ be the lowest order Raviart-Thomas finite element space with a zero normal component on $\partial\Omega$, see [4, Chapter III, 3], and let Q be the space of piecewise constants with a zero mean value. They are finite dimensional subspaces of $H_0(\mathrm{div}\,,\Omega)$ and $L_0^2(\Omega)$, respectively. The pair $\widehat{\mathbf{W}}$, Q satisfy a uniform inf-sup condition, see [4, Chapter IV. 1.2]. The finite element discrete problem is: find $\mathbf{u}_h\in\widehat{\mathbf{W}}$ and $p_h\in Q$ such that

$$\left\{ \begin{array}{lcll} a(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) & = & 0 & \forall \mathbf{v}_h\in\widehat{\mathbf{W}}, \\ b(\mathbf{u}_h,q_h) & = & -\int_\Omega f q_h d\mathbf{x} & \forall q_h\in Q, \end{array} \right.$$

and in matrix form:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} 0 \\ F_h \end{bmatrix}. \tag{2.3}$$

The system matrix of (2.3) is symmetric indefinite with the matrix A symmetric, positive definite.

3. The two-level BDDC method. We decompose Ω into N nonoverlapping subdomains Ω_i with diameters H_i, $i = 1, \cdots, N$, and with $H = \max_i H_i$. We assume that each subdomain is a union of shape-regular coarse quadrilaterals and that the number of such quadrilaterals forming an individual subdomain is uniformly bounded. We also assume that $\rho(x)$ is a constant in each subdomain. We then introduce quasi-uniform triangulations of each subdomain. We note that the results of this paper are also valid for the finite element spaces based on triangles and the algorithm can be extended to different types of subdomains.

The global problem (2.3) is assembled from the subdomain problems

$$\begin{bmatrix} A^{(i)} & B^{(i)^T} \\ B^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_h^{(i)} \\ p_h^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ F_h^{(i)} \end{bmatrix}. \tag{3.1}$$

Let Γ be the interface between the subdomains. The set of the interface nodes Γ_h is defined as $\Gamma_h = (\cup_i \partial\Omega_{i,h}) \setminus \partial\Omega_h$, where $\partial\Omega_{i,h}$ is the set of nodes on $\partial\Omega_i$ and $\partial\Omega_h$ is the set of nodes on $\partial\Omega$. We decompose the discrete velocity and pressure spaces $\widehat{\mathbf{W}}$ and Q into

$$\widehat{\mathbf{W}} = \left(\prod_{i=1}^{N} \mathbf{W}_I^{(i)} \right) \bigoplus \widehat{\mathbf{W}}_\Gamma \quad \text{and} \quad Q = \left(\prod_{i=1}^{N} Q_I^{(i)} \right) \bigoplus Q_0.$$

Here, $\widehat{\mathbf{W}}_\Gamma$ is the space of traces on Γ of functions of $\widehat{\mathbf{W}}$. The elements of $\mathbf{W}_I^{(i)}$ are supported in the subdomain Ω_i and their normal components vanish on the subdomain interface $\Gamma_i = \Gamma \cap \partial\Omega_i$, while the elements of $Q_I^{(i)}$ are restrictions of elements in Q to Ω_i which satisfy $\int_{\Omega_i} q_I^{(i)} = 0$. Q_0 is the subspace of Q with constant values $q_0^{(i)}$ in the subdomain Ω_i that satisfy $\sum_{i=1}^{N} \ q_0^{(i)} \ m(\Omega_i) = 0$, where $m(\Omega_i)$ is the measure of the subdomain Ω_i. $R_0^{(i)}$ is the operator which maps functions in the space Q_0 to its constant component of the subdomain Ω_i.

We denote the subdomain interface velocity variables by $\mathbf{W}_\Gamma^{(i)}$ and the associate product space by $\mathbf{W}_\Gamma = \prod_{i=1}^{N} \mathbf{W}_\Gamma^{(i)}$.

In order to define the BDDC algorithm, we also need to introduce a partially assembled interface velocity space $\widetilde{\mathbf{W}}_\Gamma$ by

$$\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta = \widehat{\mathbf{W}}_\Pi \bigoplus \left(\prod_{i=1}^{N} \mathbf{W}_\Delta^{(i)} \right).$$

Here, $\widehat{\mathbf{W}}_\Pi$ is the coarse level, primal interface velocity space which is spanned by subdomain interface edge basis functions with constant values at the nodes of the edge. We change the variables so that the degree of freedom of each primal constraint is explicit, see [18] and [15]. The space $\mathbf{W}_\Delta$ is the direct sum of the $\mathbf{W}_\Delta^{(i)}$, which is spanned by the remaining interface velocity degrees of freedom, which have a zero average over each edge. In the space $\widetilde{\mathbf{W}}_\Gamma$, we have relaxed most continuity constraints on the velocity across the interface but have retained all primal continuity constraints, which has the important advantage that all the linear systems are nonsingular in the computation.

We also need to introduce several restriction, injection, and scaling operators between different spaces.

The restriction operators are:

$$\widehat{\mathbf{W}}_\Gamma \xrightarrow{R_\Gamma^{(i)}} \mathbf{W}_\Gamma^{(i)}, \quad \widehat{\mathbf{W}}_\Pi \xrightarrow{R_\Pi^{(i)}} \mathbf{W}_\Pi^{(i)}, \quad \mathbf{W}_\Delta \xrightarrow{R_\Delta^{(i)}} \mathbf{W}_\Delta^{(i)}, \quad \widetilde{\mathbf{W}}_\Gamma \xrightarrow{R_{\Gamma\Delta}} \mathbf{W}_\Delta, \quad \text{and} \quad \widetilde{\mathbf{W}}_\Gamma \xrightarrow{R_{\Gamma\Pi}} \widehat{\mathbf{W}}_\Pi.$$

We also introduce two injection operators:

$$\widehat{\mathbf{W}}_\Gamma \xrightarrow{\widetilde{R}_\Gamma} \widetilde{\mathbf{W}}_\Gamma \xrightarrow{\overline{R}_\Gamma} \mathbf{W}_\Gamma.$$

The scaled injection operator $\widetilde{R}_{D,\Gamma}$ can be written as $\widetilde{R}_{D,\Gamma} = D\widetilde{R}_\Gamma$, where D is a diagonal scaling matrix. The diagonal elements of D, corresponding to the primal variables, are 1, and all others are given by $\delta_i^\dagger(x)$. Here, we define the scale factor $\delta_i^\dagger(x)$ as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x)}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h, \tag{3.2}$$

where $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_j$. We then note that $\delta_i^\dagger(x)$ is constant on an edge since the nodes on an edge are shared by the same pair of subdomains.

We also use the notations

$$\overline{R} = \begin{bmatrix} \overline{R}_\Gamma & \\ & I \end{bmatrix}, \quad \widetilde{R} = \begin{bmatrix} \widetilde{R}_\Gamma & \\ & I \end{bmatrix} \quad \text{and} \quad \widetilde{R}_D = \begin{bmatrix} \widetilde{R}_{D,\Gamma} & \\ & I \end{bmatrix}.$$

The subdomain saddle point problems (3.1) can be written as

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{\Delta I}^{(i)^T} & A_{\Pi I}^{(i)^T} & 0 \\ B_{II}^{(i)} & 0 & B_{I\Delta}^{(i)} & B_{I\Pi}^{(i)} & 0 \\ A_{\Delta I}^{(i)} & B_{I\Delta}^{(i)^T} & A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)^T} & B_{0\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & B_{I\Pi}^{(i)^T} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} & B_{0\Pi}^{(i)^T} \\ 0 & 0 & B_{0\Delta}^{(i)} & B_{0\Pi}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{h,I}^{(i)} \\ p_{h,I}^{(i)} \\ \mathbf{u}_{h,\Delta}^{(i)} \\ \mathbf{u}_{h,\Pi}^{(i)} \\ p_{h,0}^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ F_{h,I}^{(i)} \\ 0 \\ 0 \\ F_{h,0}^{(i)} \end{bmatrix}, \tag{3.3}$$

where $(\mathbf{u}_{h,I}^{(i)}, p_{h,I}^{(i)}, \mathbf{u}_{h,\Delta}^{(i)}, \mathbf{u}_{h,\Pi}^{(i)}, p_{h,0}^{(i)}) \in (\mathbf{W}_I^{(i)}, Q_I^{(i)}, \mathbf{W}_\Delta^{(i)}, \mathbf{W}_\Pi^{(i)}, Q_0^{(i)})$. We note that, by the divergence theorem, the lower left block of the matrix of (3.3) is zero since the bi-linear form $b(\mathbf{v}_I^{(i)}, q_0^{(i)})$ always vanishes for any $\mathbf{v}_I^{(i)} \in \mathbf{W}_I^{(i)}$ and any constant $q_0^{(i)}$ in the subdomain Ω_i.

We now reduce the global problem (2.3) to an interface problem. We first introduce the subdomain Schur complements $S_\Gamma^{(i)}$ by eliminating the subdomain interior variables $u_{h,I}^{(i)}$ and $p_{h,I}^{(i)}$ in (3.3):

$$S_\Gamma^{(i)} = \begin{bmatrix} A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)^T} \\ A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix} - \begin{bmatrix} A_{\Delta I}^{(i)} & B_{I\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & B_{I\Pi}^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} \\ B_{II}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\Delta I}^{(i)^T} & A_{\Pi I}^{(i)^T} \\ B_{I\Delta}^{(i)} & B_{I\Pi}^{(i)} \end{bmatrix}. \tag{3.4}$$

Given the definition of $S_\Gamma^{(i)}$, the subdomain problems (3.3) are reduced to the subdomain interface problems

$$\begin{bmatrix} S_\Gamma^{(i)} & B_{0\Gamma}^{(i)^T} \\ B_{0\Gamma}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{h,\Gamma}^{(i)} \\ p_{h,0}^{(i)} \end{bmatrix} = \mathbf{g}^{(i)}, \quad i = 1, 2, ..., N,$$

where

$$\mathbf{u}_{h,\Gamma}^{(i)} = \begin{bmatrix} \mathbf{u}_{h,\Delta}^{(i)} \\ \mathbf{u}_{h,\Pi}^{(i)} \end{bmatrix}, \quad B_{0\Gamma}^{(i)} = \begin{bmatrix} B_{0\Delta}^{(i)} & B_{0\Pi}^{(i)} \end{bmatrix},$$

and

$$\mathbf{g}^{(i)} = \begin{bmatrix} 0 \\ 0 \\ F_{h,0}^{(i)} \end{bmatrix} - \begin{bmatrix} A_{\Delta I}^{(i)} & B_{I\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & B_{I\Pi}^{(i)^T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} \\ B_{II}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ F_{h,I}^{(i)} \end{bmatrix}.$$

Let

$$S_\Gamma = \begin{bmatrix} S_\Gamma^{(1)} & & \\ & \ddots & \\ & & S_\Gamma^{(N)} \end{bmatrix}, \quad B_{0\Gamma} = \begin{bmatrix} B_{0\Gamma}^{(1)} & & \\ & \ddots & \\ & & B_{0\Gamma}^{(N)} \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} S_\Gamma & B_{0\Gamma}^T \\ B_{0\Gamma} & 0 \end{bmatrix}.$$

The partially assembled Schur complement $\widetilde{S}$ is obtained from S by assembling the primal variables on the subdomain interface, i.e.

$$\widetilde{S} = \overline{R}^T S \overline{R} = \begin{bmatrix} \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma & \overline{R}_\Gamma^T B_{0\Gamma}^T \\ B_{0\Gamma} \overline{R}_\Gamma & 0 \end{bmatrix} := \begin{bmatrix} \widetilde{S}_\Gamma & \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} & 0 \end{bmatrix}. \tag{3.5}$$

$\widetilde{S}$ can be further assembled with respect to the variables of the $\mathbf{W}_\Delta$. The fully assembled Schur complement

$$\widehat{S} = \widetilde{R}^T \widetilde{S} \widetilde{R} = \begin{bmatrix} \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma & \widetilde{R}_\Gamma^T \widetilde{B}_{0\Gamma}^T \\ \widetilde{B}_{0\Gamma} \widetilde{R}_\Gamma & 0 \end{bmatrix} := \begin{bmatrix} \widehat{S}_\Gamma & \widehat{B}_{0\Gamma}^T \\ \widehat{B}_{0\Gamma} & 0 \end{bmatrix}. \tag{3.6}$$

and the reduced interface problem can be written as: find $(\mathbf{u}_{h,\Gamma}, p_{h,0}) \in \widehat{\mathbf{W}}_\Gamma \times Q_0$, such that

$$\widehat{S} \begin{bmatrix} \mathbf{u}_{h,\Gamma} \\ p_{h,0} \end{bmatrix} = \mathbf{g} \ , \tag{3.7}$$

where $\mathbf{g} = \sum_{i=1}^N \begin{bmatrix} R_\Gamma^{(i)^T} & 0 \\ 0 & R_0^{(i)^T} \end{bmatrix} \mathbf{g}^{(i)}$.

The two-level preconditioned BDDC algorithm is of the form: find $(\mathbf{u}_{h,\Gamma}, p_{h,0}) \in \widehat{\mathbf{W}}_\Gamma \times Q_0$, such that

$$M^{-1} \widehat{S} \begin{bmatrix} \mathbf{u}_{h,\Gamma} \\ p_{h,0} \end{bmatrix} = M^{-1} \mathbf{g}. \tag{3.8}$$

where the preconditioner $M^{-1} = \widetilde{R}_D^T \widetilde{S}^{-1} \widetilde{R}_D$ has the following form:
(3.9)

$$\widetilde{R}_D^T \left\{ \begin{bmatrix} R_{\Gamma\Delta}^T \\ 0 \end{bmatrix} \left(\sum_{i=1}^N \begin{bmatrix} 0 \\ R_\Delta^{(i)} \\ 0 \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & B_{II}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & B_{I\Delta}^{(i)^T} \\ B_{II}^{(i)} & B_{I\Delta}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R_\Delta^{(i)} \\ 0 \end{bmatrix} \right) [R_{\Gamma\Delta}\ 0] + \Phi S_\Pi^{-1} \Phi^T \right\} \widetilde{R}_D.$$

Here Φ is the matrix given by
(3.10)

$$\begin{bmatrix} R_{\Gamma\Pi}^T & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} R_{\Gamma\Delta}^T \\ 0 \end{bmatrix} \sum_{i=1}^N \begin{bmatrix} 0 \\ R_\Delta^{(i)} \\ 0 \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & B_{II}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & B_{I\Delta}^{(i)^T} \\ B_{II}^{(i)} & B_{I\Delta}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} & 0 \\ A_{\Pi\Delta}^{(i)^T} & B_{0\Delta}^{(i)^T} \\ B_{I\Pi}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} R_\Pi^{(i)} & 0 \\ 0 & R_0^{(i)} \end{bmatrix}.$$

The coarse level problem matrix S_Π is determined by

$$\begin{aligned} S_\Pi = \sum_{i=1}^N \begin{bmatrix} R_\Pi^{(i)^T} & 0 \\ 0 & R_0^{(i)^T} \end{bmatrix} \Bigg\{ & \begin{bmatrix} A_{\Pi\Pi}^{(i)} & B_{0\Pi}^{(i)^T} \\ B_{0\Pi}^{(i)} & 0 \end{bmatrix} \\ & - \begin{bmatrix} A_{\Pi I}^{(i)^T} & 0 \\ A_{\Pi\Delta}^{(i)^T} & B_{0\Delta}^{(i)^T} \\ B_{I\Pi}^{(i)} & 0 \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & B_{II}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & B_{I\Delta}^{(i)^T} \\ B_{II}^{(i)} & B_{I\Delta}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} & 0 \\ A_{\Pi\Delta}^{(i)^T} & B_{0\Delta}^{(i)^T} \\ B_{I\Pi}^{(i)} & 0 \end{bmatrix} \Bigg\} \begin{bmatrix} R_\Pi^{(i)} & 0 \\ 0 & R_0^{(i)} \end{bmatrix}, \end{aligned} \tag{3.11}$$

which is obtained by assembling subdomain matrices; for additional details, cf. [6, 20, 18, 27].

We define two subspaces $\widehat{\mathbf{W}}_{\Gamma,B}$ and $\widetilde{\mathbf{W}}_{\Gamma,B}$ of $\widehat{\mathbf{W}}_\Gamma$ and $\widetilde{\mathbf{W}}_\Gamma$, respectively, as in [17, Definition 1]:

$$\begin{aligned} \widehat{\mathbf{W}}_{\Gamma,B} &= \{\mathbf{w}_\Gamma \in \widehat{\mathbf{W}}_\Gamma \mid \widehat{B}_{0\Gamma}\mathbf{w}_\Gamma = 0\}, \\ \widetilde{\mathbf{W}}_{\Gamma,B} &= \{\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma \mid \widetilde{B}_{0\Gamma}\mathbf{w}_\Gamma = 0\}. \end{aligned} \tag{3.12}$$

We call $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ the *benign subspaces* of $\widehat{\mathbf{W}}_\Gamma \times Q_0$ and $\widetilde{\mathbf{W}}_\Gamma \times Q_0$, respectively. It is easy to check that both operators $\widehat{S}$ and $\widetilde{S}$, given in (3.6) and (3.5), are symmetric, positive definite when restricted to the benign subspaces $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, respectively.

We note that the solution of (3.7) is not in the benign subspace. However, we can find a special discrete velocity $\mathbf{u}_{h,\Gamma}^* \in \widehat{\mathbf{W}}_\Gamma$ and define $\mathbf{u}_\Gamma = \mathbf{u}_{h,\Gamma} - \mathbf{u}_{h,\Gamma}^*$ such that the correction $\mathbf{u}_\Gamma$ belongs to the benign subspace.

Let $p = p_h$ and $\begin{bmatrix} \mathbf{u}_{h,\Gamma}^* \\ p_0^* \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ F_{h,\Gamma}^{(1)} \\ \vdots \\ F_{h,\Gamma}^{(N)} \end{bmatrix}$. We obtain $\mathbf{u}_h^*$ from $\mathbf{u}_{h,\Gamma}^*$ by solving subdomain saddle point problem with Dirichlet boundary condition in each subdomain and let $\mathbf{u} = \mathbf{u}_h - \mathbf{u}_h^*$. The correction $(\mathbf{u}, p)^T$ satisfies

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} -A\mathbf{u}_h^* \\ F_h - B\mathbf{u}_h^* \end{bmatrix}. \tag{3.13}$$

With this special $\mathbf{u}_h^*$ we choose here, the divergence of the correction $\mathbf{u}$ is not zero, but $\mathbf{u}_\Gamma$ is in the benign subspace $\widehat{\mathbf{W}}_{\Gamma,B}$, for details, see [28, Section 4.8].

The rest of the algorithm is the same as before except that

$$\mathbf{g}^{(i)} = \begin{bmatrix} -(A\mathbf{u}_h^*)_\Delta^{(i)} \\ -(A\mathbf{u}_h^*)_\Pi^{(i)} \\ F_{h,0}^{(i)} - (B\mathbf{u}_h^*)_0 \end{bmatrix} - \begin{bmatrix} A_{\Delta I}^{(i)} & B_{I\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & B_{I\Pi}^{(i)^T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} \\ B_{II}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} -(A\mathbf{u}_h^*)_I^{(i)} \\ F_{h,I}^{(i)} - (B\mathbf{u}_h^*)_I^{(i)} \end{bmatrix}.$$

Because of the choice of our primal subspace $\widehat{\mathbf{W}}_\Pi$, we have the following results, see [27, Lemma 4.1], [17, Lemmas 6.1 and 6.2]:

LEMMA 3.1. *$B_{0\Delta}^{(i)} = 0$, for i=1, $\cdots, N$. For $\mathbf{w}_\Gamma \in \widetilde{\mathbf{W}}_\Gamma$, we have $\widetilde{B}_{0\Gamma}\mathbf{w}_\Gamma = \widetilde{B}_{0\Pi}\mathbf{w}_\Pi$, where $\widetilde{B}_{0\Pi} = \sum_{i=1}^N R_0^{(i)^T} B_{0\Pi}^{(i)} R_\Pi^{(i)}$ and $\mathbf{w}_\Pi$ is the primal part of $\mathbf{w}_\Gamma$.*

LEMMA 3.2. *For any $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, $\widetilde{R}_D^T\mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$.*

Since the correction $(\mathbf{u}_\Gamma, p)^T$ lies in this benign subspace, we choose the initial guess in the benign subspace and the preconditioned operator defined in (3.8) will keep all the iterates in this benign subspace by Lemma 3.2, in which the preconditioned operator is positive definite and a preconditioned conjugate gradient method can be applied.

We have the following result for the two-level BDDC algorithm, see [27, Theorem 6.1]:

THEOREM 1. *The preconditioned operator $M^{-1}\widehat{S}$ is symmetric, positive definite with respect to the bi-linear form $\langle\cdot,\cdot\rangle_{\widehat{S}}$ on the benign space $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ and*

$$\langle \mathbf{u},\mathbf{u}\rangle_{\widehat{S}} \le \left\langle M^{-1}\widehat{S}\mathbf{u},\mathbf{u}\right\rangle_{\widehat{S}} \le C\left(1+\log\frac{H}{h}\right)^2 \langle \mathbf{u},\mathbf{u}\rangle_{\widehat{S}}, \ \forall \mathbf{u} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0. \tag{3.14}$$

Here, C is a constant which is independent of h and H.

4. A three-level BDDC method. For the three-level cases, as in [32, 31, 12], the coarse problem matrix S_Π defined in (3.11) will not be factored exactly. Instead, an additional level is introduced and the coarse problem is solved approximately. Call the new level the subregion level. To distinguish the spaces and operators for the subregion level from those for the subdomain level, we use the subscript c for the former.

We decompose Ω into N_c subregions Ω^j with diameters $\hat{H}^j$, $j = 1, \cdots, N_c$. Each subregion Ω^j is the union of N_j subdomains Ω_i^j with diameters H_i^j. Let $\hat{H} = \max_j \hat{H}^j$ and $H = \max_{i,j} H_i^j$, for $j = 1, \ \cdots, \ N_c$, and $i = 1, \cdots, N_j$. Then N, the total number of subdomains, is given by $N = N_1 + \cdots + N_{N_c}$. We assume that $\rho(x)$ is a constant in each subregion.

We introduce the subregional Schur complement, which is assembled only from the subdomains in subregion Ω^j,

$$\begin{aligned} S_\Pi^{(j)} = \sum_{i=1}^{N_j} \begin{bmatrix} R_\Pi^{(i)^T} & 0 \\ 0 & R_0^{(i)^T} \end{bmatrix} \Bigg\{ &\begin{bmatrix} A_{\Pi\Pi}^{(i)} & B_{0\Pi}^{(i)^T} \\ B_{0\Pi}^{(i)} & 0 \end{bmatrix} \\ &- \begin{bmatrix} A_{\Pi I}^{(i)^T} & 0 \\ A_{\Pi\Delta}^{(i)^T} & B_{0\Delta}^{(i)^T} \\ B_{I\Pi}^{(i)} & 0 \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & B_{II}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & B_{I\Delta}^{(i)^T} \\ B_{II}^{(i)} & B_{I\Delta}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} & 0 \\ A_{\Pi\Delta}^{(i)^T} & B_{0\Delta}^{(i)^T} \\ B_{I\Pi}^{(i)} & 0 \end{bmatrix} \Bigg\} \begin{bmatrix} R_\Pi^{(i)} & 0 \\ 0 & R_0^{(i)} \end{bmatrix}. \end{aligned} \tag{4.1}$$

By Lemma 3.1 and (4.1), we have $B_{0\Delta}^{(i)} = 0$ and

$$S_{\Pi}^{(j)} := \begin{bmatrix} A_{\Pi}^{(j)} & B_{\Pi}^{(j)^T} \\ B_{\Pi}^{(j)} & 0 \end{bmatrix},$$

where

(4.2)

$$A_{\Pi}^{(j)} = \sum_{i=1}^{N_j} R_{\Pi}^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \\ B_{I\Pi}^{(i)} \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & B_{II}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & B_{I\Delta}^{(i)^T} \\ B_{II}^{(i)} & B_{I\Delta}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \\ B_{I\Pi}^{(i)} \end{bmatrix} \right\} R_{\Pi}^{(i)},$$

$$B_{\Pi}^{(j)} = \sum_{i=1}^{N_j} R_0^{(i)^T} B_{0\Pi}^{(i)} R_{\Pi}^{(i)}. \tag{4.3}$$

We note that the coarse problem matrix S_{Π} can be assembled from the $S_{\Pi}^{(j)}$. Therefore, we can write S_{Π} as

$$S_{\Pi} = \begin{bmatrix} A_{\Pi} & B_{\Pi}^T \\ B_{\Pi} & 0 \end{bmatrix}, \tag{4.4}$$

where A_{Π} and B_{Π} are assembled from $A_{\Pi}^{(j)}$ and $B_{\Pi}^{(j)}$, respectively.

Recall that $\widetilde{B}_{0\Pi} = \sum_{i=1}^{N} R_0^{(i)^T} B_{0\Pi}^{(i)} R_{\Pi}^{(i)}$, is defined in Lemma 3.1. By the definition of B_{Π} and $\widetilde{B}_{0\Pi}$, we have

LEMMA 4.1. $B_{\Pi} = \widetilde{B}_{0\Pi}$.

In the two-level case, S_{Π} is factored by a direct solver at the beginning of the computation, cf. (3.9). Here, we build M_{Π}^{-1} to approximate S_{Π}^{-1}.

Replacing S_{Π}^{-1} in (3.9) with M_{Π}^{-1} gives us the three-level preconditioner $\widetilde{M}^{-1}$:

$$\widetilde{R}_D^T \left\{ \begin{bmatrix} R_{\Gamma\Delta}^T \\ 0 \end{bmatrix} \left(\sum_{i=1}^{N} \begin{bmatrix} 0 \\ R_{\Delta}^{(i)} \\ 0 \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & B_{II}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & B_{I\Delta}^{(i)^T} \\ B_{II}^{(i)} & B_{I\Delta}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R_{\Delta}^{(i)} \\ 0 \end{bmatrix} \right) [R_{\Gamma\Delta} \ 0] + \Phi M_{\Pi}^{-1} \Phi^T \right\} \widetilde{R}_D.$$

To define M_{Π}^{-1} in detail, we need to introduce several spaces and operators.

Let Γ_c be the interface between the subregions; note that $\Gamma_c \subset \Gamma$. For each subregion Ω^i, we denote the space corresponding to the subdomain edge average variables in this subregion, by $\mathbf{W}_c^{(i)}$. Let $\mathbf{W}_c = \prod_{i=1}^{N_c} \mathbf{W}_c^{(i)}$ and let $\widehat{\mathbf{W}}_c$ be the subspace of $\mathbf{W}_c$ of elements that are continuous across Γ_c. $\mathbf{W}_c^{(i)}$ can be decomposed into a subregion interior part $\mathbf{W}_{I_c}^{(i)}$ and a subregion interface part $\mathbf{W}_{\Gamma_c}^{(i)}$. We further decompose the subregion interface part $\mathbf{W}_{\Gamma_c}^{(i)}$ into a primal subspace $\mathbf{W}_{\Pi_c}^{(i)}$ and a dual subspace $\mathbf{W}_{\Delta_c}^{(i)}$. Here, we will only consider the averages over subregion edges as the primal variables. Again, we should change the variables for all local coarse matrices corresponding to these edge average primal variables. We will assume that all matrices are written in term of the new variables.

We denote the associated subregion interface product space by $\mathbf{W}_{\Gamma_c} := \prod_{i=1}^{N_c} \mathbf{W}_{\Gamma_c}^{(i)}$. We note that the elements in $\mathbf{W}_{\Gamma_c}$ can be discontinuous across the subregion interface

Γ_c. $\widehat{\mathbf{W}}_{\Gamma_c}$ and $\widetilde{\mathbf{W}}_{\Gamma_c}$ are two subspaces of $\mathbf{W}_{\Gamma_c}$. The elements of $\widehat{\mathbf{W}}_{\Gamma_c}$ are continuous across Γ_c, whereas only the primal variables are continuous across Γ_c in $\widetilde{\mathbf{W}}_{\Gamma_c}$. Thus, we have $\widehat{\mathbf{W}}_{\Gamma_c} \subset \widetilde{\mathbf{W}}_{\Gamma_c} \subset \mathbf{W}_{\Gamma_c}$. We also need two injection operators $\widetilde{R}_{\Gamma_c}$ and $\overline{R}_{\Gamma_c}$:

$$\widehat{\mathbf{W}}_{\Gamma_c} \xrightarrow{\widetilde{R}_{\Gamma_c}} \widetilde{\mathbf{W}}_{\Gamma_c} \xrightarrow{\overline{R}_{\Gamma_c}} \mathbf{W}_{\Gamma_c},$$

which are similar to $\widetilde{R}_\Gamma$ and $\overline{R}_\Gamma$, respectively.

Similarly, we also denote by $Q_c^{(i)}$, the pressure space of piecewise constant on each subdomain of the subregion Ω^i by $Q_c^{(i)}$. Let $Q_c = \Pi_{i=1}^{N_c} Q_c^{(i)}$. Q_c can be decomposed into $\Pi_{i=1}^{N_c} Q_{I_c}^{(i)}$ and Q_{0_c}, where the elements of $Q_{I_c}^{(i)}$ are restrictions of elements in Q_c to Ω^i which satisfy $\int_{\Omega^i} q_{I_c}^{(i)} = 0$. Q_{0_c} is the subspace of Q_c with constant values $q_{0_c}^{(i)}$ in the subregion Ω^i that satisfy $\sum_{i=1}^{N_c} \ q_{0_c}^{(i)} \ m(\Omega^i) = 0$, where $m(\Omega^i)$ is the measure of the subregion Ω^i. $R_{0_c}^{(i)}$ is the operator which maps functions in the space Q_{0_c} to its constant component of the subregion Ω^i.

We also use the notation

$$\overline{R}_c = \begin{bmatrix} \overline{R}_{\Gamma_c} & \\ & I \end{bmatrix} \quad \text{and} \quad \widetilde{R}_c = \begin{bmatrix} \widetilde{R}_{\Gamma_c} & \\ & I \end{bmatrix}.$$

We denote by $\widehat{\mathbf{F}}_c$ and $\widehat{\mathbf{F}}_{\Gamma_c}$ the dual spaces of $\widehat{\mathbf{W}}_c$ and $\widehat{\mathbf{W}}_{\Gamma_c}$, respectively. We also denote by $\mathbf{G}_c$ and $\mathbf{G}_{0_c}$ the dual spaces of Q_c and Q_{0_c}, respectively.

We are now ready to explain how M_Π^{-1} works on a vector in $\widehat{\mathbf{F}}_c \times \mathbf{G}_c$. Given a vector $\begin{bmatrix} \mathbf{\Psi} \\ \mathbf{\Theta} \end{bmatrix} \in \widehat{\mathbf{F}}_c \times \mathbf{G}_c$, let $\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = S_\Pi^{-1} \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{\Theta} \end{bmatrix}$ and $\begin{bmatrix} \widehat{\mathbf{y}} \\ \widehat{\mathbf{z}} \end{bmatrix} = M_\Pi^{-1} \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{\Theta} \end{bmatrix}$. We write $\mathbf{\Psi}$, $\mathbf{y}$, and $\widehat{\mathbf{y}}$ in terms of subregion interior and interface parts, i.e., $\mathbf{\Psi} = \left(\mathbf{\Psi}_{I_c}^{(1)}, \cdots, \mathbf{\Psi}_{I_c}^{(N_c)}, \mathbf{\Psi}_{\Gamma_c}\right)^T$, $\mathbf{y} = \left(\mathbf{y}_{I_c}^{(1)}, \cdots, \mathbf{y}_{I_c}^{(N_c)}, \mathbf{y}_{\Gamma_c}\right)^T$, and $\widehat{\mathbf{y}} = \left(\widehat{\mathbf{y}}_{I_c}^{(1)}, \cdots, \widehat{\mathbf{y}}_{I_c}^{(N_c)}, \widehat{\mathbf{y}}_{\Gamma_c}\right)^T$. We also write $\mathbf{\Theta}$, $\mathbf{z}$, and $\widehat{\mathbf{z}}$ as $\mathbf{\Theta} = \left(\mathbf{\Theta}_{I_c}^{(1)}, \cdots, \mathbf{\Theta}_{I_c}^{(N_c)}, \mathbf{\Theta}_{0_c}\right)^T$, $\mathbf{z} = \left(\mathbf{z}_{I_c}^{(1)}, \cdots, \mathbf{z}_{I_c}^{(N_c)}, \mathbf{z}_{0_c}\right)^T$, and $\widehat{\mathbf{z}} = \left(\widehat{\mathbf{z}}_{I_c}^{(1)}, \cdots, \widehat{\mathbf{z}}_{I_c}^{(N_c)}, \widehat{\mathbf{z}}_{0_c}\right)^T$.

To obtain $\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix}$, we solve $S_\Pi \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{\Theta} \end{bmatrix}$ by a block factorization. Let $R_{\Gamma_c}^{(i)} : \widehat{\mathbf{W}}_{\Gamma_c} \to \mathbf{W}_{\Gamma_c}^{(i)}$ be a restriction operator. We write A_Π and B_Π given in (4.4) using subregion interior and interface blocks:

$$A_\Pi = \begin{bmatrix} A_{\Pi_{I_c I_c}}^{(1)} & 0 & 0 & A_{\Pi_{\Gamma_c I_c}}^{(1)^T} R_{\Gamma_c}^{(1)} \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & A_{\Pi_{I_c I_c}}^{(N_c)} & A_{\Pi_{\Gamma_c I_c}}^{(N_c)^T} R_{\Gamma_c}^{(N_c)} \\ R_{\Gamma_c}^{(1)^T} A_{\Pi_{\Gamma_c I_c}}^{(1)} & \cdots & R_{\Gamma_c}^{(N_c)^T} A_{\Pi_{\Gamma_c I_c}}^{(N_c)} & \widehat{A}_{\Pi_{\Gamma_c \Gamma_c}} \end{bmatrix}$$

and

$$B_\Pi = \begin{bmatrix} B_{\Pi_{I_c I_c}}^{(1)} & 0 & 0 & B_{\Pi_{I_c \Gamma_c}}^{(1)} R_{\Gamma_c}^{(1)} \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & B_{\Pi_{I_c I_c}}^{(N_c)} & B_{\Pi_{I_c \Gamma_c}}^{(N_c)} R_{\Gamma_c}^{(N_c)} \\ 0 & \cdots & 0 & \widehat{B}_{\Pi_{0_c \Gamma_c}} \end{bmatrix}, \tag{4.5}$$

where $\widehat{A}_{\Pi_{\Gamma_c\Gamma_c}} = \sum_{i=1}^{N_c} R_{\Gamma_c}^{(i)^T} A_{\Pi_{\Gamma_c\Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)}$ and $\widehat{B}_{\Pi_{0_c\Gamma_c}} = \sum_{i=1}^{N_c} R_{0_c}^{(i)^T} B_{\Pi_{0_c\Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)}$.

We solve $\begin{bmatrix} \mathbf{y}_{I_c}^{(i)} \\ \mathbf{z}_{I_c}^{(i)} \end{bmatrix}$ in terms of $\begin{bmatrix} \mathbf{y}_{\Gamma_c} \\ \mathbf{z}_{0_c} \end{bmatrix}$ and have

(4.6)
$$\begin{bmatrix} \mathbf{y}_{I_c}^{(i)} \\ \mathbf{z}_{I_c}^{(i)} \end{bmatrix} = \begin{bmatrix} A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T} \\ B_{\Pi_{I_cI_c}}^{(i)} & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{\Psi}_{I_c}^{(i)} \\ \mathbf{\Theta}_{I_c}^{(i)} \end{bmatrix} - \begin{bmatrix} A_{\Pi_{I_c\Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)} & 0 \\ B_{\Pi_{I_c\Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\Gamma_c} \\ \mathbf{z}_{0_c} \end{bmatrix} \right).$$

Let

$$T_{\Gamma_c}^{(i)} = A_{\Pi_{\Gamma_c\Gamma_c}}^{(i)} - \begin{bmatrix} A_{\Pi_{\Gamma_cI_c}}^{(i)} & B_{\Pi_{I_c\Gamma_c}}^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T} \\ B_{\Pi_{I_cI_c}}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi_{I_c\Gamma_c}}^{(i)} \\ B_{\Pi_{I_c\Gamma_c}}^{(i)} \end{bmatrix}$$

and let

$$T^{(i)} = \begin{bmatrix} T_{\Gamma_c}^{(i)} & B_{\Pi_{0_c\Gamma_c}}^{(i)^T} \\ B_{\Pi_{0_c\Gamma_c}}^{(i)} & 0 \end{bmatrix},$$

be the subregion Schur complement. We then obtain the subregion interface problem:

$$\left(\sum_{i=1}^{N_c} \begin{bmatrix} R_{\Gamma_c}^{(i)^T} & 0 \\ 0 & R_{0_c}^{(i)^T} \end{bmatrix} T^{(i)} \begin{bmatrix} R_{\Gamma_c}^{(i)} & 0 \\ 0 & R_{0_c}^{(i)} \end{bmatrix} \right) \begin{bmatrix} \mathbf{y}_{\Gamma_c} \\ \mathbf{z}_{0_c} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{\Psi}_{\Gamma_c} \\ \mathbf{\Theta}_{0_c} \end{bmatrix} - \sum_{i=1}^{N_c} \begin{bmatrix} R_{\Gamma_c}^{(i)^T} & 0 \\ 0 & R_{0_c}^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{\Pi_{\Gamma_cI_c}}^{(i)} & B_{\Pi_{I_c\Gamma_c}}^{(i)^T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T} \\ B_{\Pi_{I_cI_c}}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{\Psi}_{I_c}^{(i)} \\ \mathbf{\Theta}_{I_c}^{(i)} \end{bmatrix}.$$
(4.7)

Denote

$$T_{\Gamma_c} = \begin{bmatrix} T_{\Gamma_c}^{(1)} & & \\ & \ddots & \\ & & T_{\Gamma_c}^{(N_c)} \end{bmatrix}, \quad B_{\Pi_{0_c,\Gamma_c}} = \begin{bmatrix} B_{\Pi_{0_c\Gamma_c}}^{(1)} & & \\ & \ddots & \\ & & B_{\Pi_{0_c\Gamma_c}}^{(N_c)} \end{bmatrix},$$

and

$$T = \begin{bmatrix} T_{\Gamma_c} & B_{\Pi_{0_c\Gamma_c}}^T \\ B_{\Pi_{0_c\Gamma_c}} & 0 \end{bmatrix}.$$

As on the subdomain level case, we introduce a partially assembled Schur complement of S_Π, and denote it by $\widetilde{T}$. $\widetilde{T}$ can be written as:

$$\widetilde{T} = \overline{R}_c^T T \overline{R}_c := \begin{bmatrix} \widetilde{T}_{\Gamma_c} & \widetilde{B}_{\Pi_{0_c\Gamma_c}}^T \\ \widetilde{B}_{\Pi_{0_c\Gamma_c}} & 0 \end{bmatrix}. \tag{4.8}$$

$\widetilde{T}$ can be further assembled with respect to the variables of $\mathbf{W}_{\Delta_c}^{(i)}$. $\widehat{T}$, the fully assembled subregion Schur complement of S_Π, can be written as:

$$\widehat{T} = \widetilde{R}_c^T \widetilde{T} \widetilde{R}_c := \begin{bmatrix} \widehat{T}_{\Gamma_c} & \widehat{B}_{\Pi_{0_c\Gamma_c}}^T \\ \widehat{B}_{\Pi_{0_c\Gamma_c}} & 0 \end{bmatrix}. \tag{4.9}$$

We define $\begin{bmatrix} \mathbf{h}_{\Gamma_c} \\ \mathbf{h}_{0_c} \end{bmatrix} \in \widehat{\mathbf{F}}_{\Gamma_c} \times \mathbf{G}_c$ by

(4.10)
$$\begin{bmatrix} \mathbf{\Psi}_{\Gamma_c} \\ \mathbf{\Theta}_{0_c} \end{bmatrix} - \sum_{i=1}^{N_c} \begin{bmatrix} R_{\Gamma_c}^{(i)^T} & 0 \\ 0 & R_{0_c}^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{\Pi_{\Gamma_c I_c}}^{(i)} & B_{\Pi_{I_c \Gamma_c}}^{(i)^T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{\Pi_{I_c I_c}}^{(i)} & B_{\Pi_{I_c I_c}}^{(i)^T} \\ B_{\Pi_{I_c I_c}}^{(i)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{\Psi}_{I_c}^{(i)} \\ \mathbf{\Theta}_{I_c}^{(i)} \end{bmatrix}.$$

The reduced subregion interface problem (4.7) can be written as: find $\begin{bmatrix} \mathbf{y}_{\Gamma_c} \\ \mathbf{z}_{0_c} \end{bmatrix} \in \widehat{\mathbf{W}}_{\Gamma_c} \bigoplus Q_{0_c}$, such that

$$\widehat{T} \begin{bmatrix} \mathbf{y}_{\Gamma_c} \\ \mathbf{z}_{0_c} \end{bmatrix} = \widetilde{R}_c^T \widetilde{T} \widetilde{R}_c \begin{bmatrix} \mathbf{y}_{\Gamma_c} \\ \mathbf{z}_{0_c} \end{bmatrix} = \begin{bmatrix} \mathbf{h}_{\Gamma_c} \\ \mathbf{h}_{0_c} \end{bmatrix}. \tag{4.11}$$

To obtain the approximation $\begin{bmatrix} \widehat{\mathbf{y}} \\ \widehat{\mathbf{z}} \end{bmatrix} = M_\Pi^{-1} \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{\Theta} \end{bmatrix}$, we do not solve (4.11) exactly. Instead, we compute $\begin{bmatrix} \widehat{\mathbf{y}}_{\Gamma_c} \\ \widehat{\mathbf{z}}_{0_c} \end{bmatrix}$ as

$$\begin{bmatrix} \widehat{\mathbf{y}}_{\Gamma_c} \\ \widehat{\mathbf{z}}_{0_c} \end{bmatrix} = \widetilde{R}_{D_c}^T \widetilde{T}^{-1} \widetilde{R}_{D_c} \begin{bmatrix} \mathbf{h}_{\Gamma_c} \\ \mathbf{h}_{0_c} \end{bmatrix}. \tag{4.12}$$

Here $\widetilde{R}_{D_c}$ is a scaled injection operator which is similar to $\widetilde{R}_D$; we can write $\widetilde{R}_{D_c} = \begin{bmatrix} D_c \widetilde{R}_{\Gamma_c} & 0 \\ 0 & I \end{bmatrix}$, where D_c is a diagonal scaling matrix. The diagonal elements of D_c, corresponding to the primal variables, are 1, and all others are given by $\delta_{c,i}^\dagger(x)$. Here $\delta_{c,i}^\dagger(x)$ is similar to $\delta_i^\dagger(x)$, which is defined in (3.2), except that $\delta_{c,i}^\dagger(x)$ is defined for the subregion interface instead of subdomain interface nodes. For an x on the subregion interface, $\delta_{c,i}^\dagger(x)$ is defined as follows: for $\gamma \in [1/2, \infty)$,

$$\delta_{c,i}^\dagger(x) = \frac{\rho_i^\gamma(x)}{\sum_{j \in \mathcal{N}_x} \rho_j^\gamma(x)},$$

where $\mathcal{N}_x$ is the set of indices j of the subregions such that $x \in \partial\Omega^j$. Recall that in our theory, we assume the $\rho(x)$ are constant in the subregions.

We will maintain the same relation between $\begin{bmatrix} \widehat{\mathbf{y}}_{I_c}^{(i)} \\ \widehat{\mathbf{z}}_{I_c}^{(i)} \end{bmatrix}$ and $\begin{bmatrix} \widehat{\mathbf{y}}_{\Gamma_c} \\ \widehat{\mathbf{z}}_{0_c} \end{bmatrix}$ as for $\begin{bmatrix} \mathbf{y}_{I_c}^{(i)} \\ \mathbf{z}_{I_c}^{(i)} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{y}_{\Gamma_c} \\ \mathbf{z}_{0_c} \end{bmatrix}$ in (4.6), i.e.,

(4.13)
$$\begin{bmatrix} \widehat{\mathbf{y}}_{I_c}^{(i)} \\ \widehat{\mathbf{z}}_{I_c}^{(i)} \end{bmatrix} = \begin{bmatrix} A_{\Pi_{I_c I_c}}^{(i)} & B_{\Pi_{I_c I_c}}^{(i)^T} \\ B_{\Pi_{I_c I_c}}^{(i)} & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{\Psi}_{I_c}^{(i)} \\ \mathbf{\Theta}_{I_c}^{(i)} \end{bmatrix} - \begin{bmatrix} A_{\Pi_{I_c \Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)} & 0 \\ B_{\Pi_{I_c \Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{y}}_{\Gamma_c} \\ \widehat{\mathbf{z}}_{0_c} \end{bmatrix} \right).$$

We also define the subspaces $\widehat{\mathbf{W}}_{\Gamma_c, B_c}$ and $\widetilde{\mathbf{W}}_{\Gamma_c, B_c}$ of $\widehat{\mathbf{W}}_{\Gamma_c}$ and $\widetilde{\mathbf{W}}_{\Gamma_c}$ respectively as in (3.12):

$$\widehat{\mathbf{W}}_{\Gamma_c, B_c} = \{\mathbf{w}_{\Gamma_c} \in \widehat{\mathbf{W}}_{\Gamma_c} \;|\; \widehat{B}_{\Pi_{0_c \Gamma_c}} \mathbf{w}_\Gamma = 0\},$$
$$\widetilde{\mathbf{W}}_{\Gamma_c, B_c} = \{\mathbf{w}_{\Gamma_c} \in \widetilde{\mathbf{W}}_{\Gamma_c} \;|\; \widetilde{B}_{\Pi_{0_c \Gamma_c}} \mathbf{w}_{\Gamma_c} = 0\}.$$

We note that the operators $\widetilde{T}$ and $\widehat{T}$ are positive definite in the space $\widehat{\mathbf{W}}_{\Gamma_c,B_c} \times Q_{0_c}$ and $\widetilde{\mathbf{W}}_{\Gamma_c,B_c} \times Q_{0_c}$, respectively.

Now we prove that our new three-level preconditioner $\widetilde{M}^{-1}$ can keep all the iterates in the benign space $\widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$.

LEMMA 4.2. *For any* $\mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$*, we have* $\widetilde{M}^{-1}\widehat{S}\mathbf{w} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$.

Proof: By Lemmas 3.1, 3.2 and the definition of $\widetilde{M}^{-1}$, we only need to prove $\Phi M_{\Pi}^{-1}\Phi^T \widetilde{R}_D \widehat{S}\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$.

Denote $\mathbf{w} = \begin{bmatrix} \mathbf{w}_\Gamma \\ p \end{bmatrix}$. Since $\mathbf{w}_\Gamma \in \widehat{\mathbf{W}}_{\Gamma,B}$ and $B_{0\Delta}^{(i)} = 0$, we have

$$\Phi^T \widetilde{R}_D \widehat{S}\mathbf{w} = \begin{bmatrix} \mathbf{f}_\Pi \\ 0 \end{bmatrix},$$

where $\mathbf{f}_\Pi \in \widehat{\mathbf{F}}_c$ and recall that Φ is defined in (3.10). Let $\begin{bmatrix} \widehat{\mathbf{y}} \\ \widehat{\mathbf{z}} \end{bmatrix} = M_{\Pi}^{-1} \begin{bmatrix} \mathbf{f}_\Pi \\ 0 \end{bmatrix}$. We obtain $\begin{bmatrix} \widehat{\mathbf{y}} \\ \widehat{\mathbf{z}} \end{bmatrix}$ by solving (4.12) and (4.13) with $\mathbf{\Psi} = \mathbf{f}_\Pi$ and $\mathbf{\Theta} = 0$.

By (4.10), we know that $\mathbf{h}_{0_c} = 0$ in (4.12) and that $\widetilde{T}^{-1}\widetilde{R}_{D_c} \begin{bmatrix} \mathbf{h}_{\Gamma_c} \\ 0 \end{bmatrix} \in \widetilde{\mathbf{W}}_{\Gamma_c,B_c} \times Q_{0_c}$. Similarly as the proof of Lemma 3.2, see [27, Lemma 4.1] for the details, we have $\widetilde{R}_{D_c}^T \widetilde{T}^{-1}\widetilde{R}_{D_c} \begin{bmatrix} \mathbf{h}_{\Gamma_c} \\ 0 \end{bmatrix} \in \widehat{\mathbf{W}}_{\Gamma_c,B_c} \times Q_{0_c}$, i.e.,

$$\widehat{B}_{\Pi_{0_c}\Gamma_c}\widehat{\mathbf{y}}_{\Gamma_c} = 0. \tag{4.14}$$

Since $\mathbf{\Theta} = 0$, by (4.13), we have

$$B_{\Pi_{I_c I_c}}^{(i)} \widehat{\mathbf{y}}_{I_c}^{(i)} + B_{\Pi_{I_c\Gamma_c}}^{(i)} R_{\Gamma_c}^{(i)} \widehat{\mathbf{y}}_{\Gamma_c} = 0. \tag{4.15}$$

Therefore, by (4.5), (4.14), (4.15), and Lemma 4.1, we have $B_\Pi \widehat{\mathbf{y}} = 0$, i.e., $\widetilde{B}_{0\Pi}\widehat{\mathbf{y}} = 0$. Finally, we have

$$\Phi M_{\Pi}^{-1}\Phi^T \widetilde{R}_D \widehat{S}\mathbf{w} = \begin{bmatrix} * \\ \widehat{\mathbf{y}} \\ \widehat{\mathbf{z}} \end{bmatrix}$$

where $*$ denote the dual part of the velocity. By Lemma 3.1, we have

$$\widetilde{B}_{0\Gamma} \begin{bmatrix} * \\ \widehat{\mathbf{y}} \end{bmatrix} = \widetilde{B}_{0\Pi}\widehat{\mathbf{y}} = 0$$

and we conclude that $\Phi M_{\Pi}^{-1}\Phi^T \widetilde{R}_D \widehat{S}\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$.

□

Remark: In the proof of Lemma 4.2, $B_{0\Delta}^{(i)} = 0$ *is the key ingredient, which is obtained by choosing enough primal constraints in each subdomain. Similarly, for the Stokes problem, such a result is given in [17, Lemma 6.1]; it is based on [17, Assumption 1]. Therefore, Lemma 4.2 is valid for Stokes problem with [17, Assumption 1] held. Our numerical experiments for Stokes in Section 7 are also consistent with this.*

As for the two-level BDDC method, we choose the initial guess in the benign subspace and the three-level BDDC preconditioned operator $\widetilde{M}^{-1}\widehat{S}$ will by Lemma 4.2 keep all the iterates in this benign subspace, in which the preconditioned operator is positive definite and a preconditioned conjugate gradient method can be applied.

5. Some auxiliary results. In this section, we will collect a number of results which are needed in our theory. In order to avoid a proliferation of constants, we will use the notation $A \approx B$. This means that there are two constants c and C, independent of any parameters, such that $cA \leq B \leq CA$, where $C < \infty$ and $c > 0$.

In our theory, we make an assumption for our decomposition of the global domain Ω.

ASSUMPTION 5.1. *Each subdomain Ω_i is quadrilateral. The subdomains form a quasi-uniform coarse mesh of Ω with mesh size H.*

We list some results for Raviart-Thomas finite element function spaces needed in our analysis. These results were originally given in [26, 34, 25, 33].

We define an the interpolation operator Π_{RT}^H from $\widehat{\mathbf{W}}$ onto $\widehat{\mathbf{W}}^H$, where $\widehat{\mathbf{W}}^H$ is the Raviart-Thomas finite element space on the coarse mesh with mesh size H. $\widehat{\mathbf{W}}^H$ is defined in terms of the degrees of freedom $\lambda_{\mathcal{E}}$ over the edges $\mathcal{E}$ in the coarse mesh, by

$$\lambda_{\mathcal{E}}(\Pi_{RT}^H \mathbf{u}) := \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \mathbf{u} \cdot \mathbf{n} ds.$$

We consider the stability of the interpolant Π_{RT}^H in the next lemma.

LEMMA 5.2. *There exists a constant C, which depends only on the aspect ratios of $K \in \mathcal{T}_H$ and of the elements of $\mathcal{T}_h$, such that, for all $\mathbf{u} \in \widehat{\mathbf{W}}$,*

$$\|\mathrm{div}\,(\Pi_{RT}^H \mathbf{u})\|_{L^2(K)}^2 \leq \|\mathrm{div}\,\mathbf{u}\|_{L^2(K)}^2,$$

$$\|\Pi_{RT}^H \mathbf{u}\|_{L^2(K)}^2 \leq C \left(1 + \log \frac{H}{h}\right) \left(\|\mathbf{u}\|_{L^2(K)}^2 + H_K^2 \|\mathrm{div}\,\mathbf{u}\|_{L^2(K)}^2\right).$$

Proof: See [26, Lemma 3.2].

□

We next introduce three useful extension operators.

We define $N(\partial\Omega_i)$ as the the space of functions that are constant on each element of the edges of the boundary of Ω_i and its subspace $N_0(\partial\Omega_i)$, of functions that have mean value zero on $\partial\Omega_i$.

DEFINITION 1. *Given the boundary data with zero mean value on $\partial\Omega_i$, define a discrete divergence free extension operator by solving the local saddle point problem (3.3) with $\begin{bmatrix} \mathbf{u}_{h,\Delta}^{(i)} \\ \mathbf{u}_{h,\Pi}^{(i)} \end{bmatrix}$ given and zero right hand side. The resulting $\mathbf{u}_h^{(i)}$ has the minimal L^2-norm of all discrete divergence free velocities with the given boundary values, i.e., $\tilde{\mathcal{H}}_i : N_0(\partial\Omega_i) \longrightarrow \mathbf{W}^{(i)}$ satisfies*

$$B^{(i)} \tilde{\mathcal{H}}_i \mu = 0 \quad \textit{and} \quad \|\tilde{\mathcal{H}}_i \mu\|_{L^2(\Omega_i)} = \min_{v \in \mathbf{W}^{(i)},\ B^{(i)} v = 0,\ v|_{\partial\Omega_i} = \mu} \|v\|_{L^2(\Omega_i)},$$

for any $\mu \in N_0(\partial\Omega_i)$.

DEFINITION 2. *Given edge average values over the subdomain edges with a zero mean value over $\partial\Omega_i$, define a discrete divergence free extension operator $\hat{\mathcal{H}}_i$ by solving the local saddle point problem (3.3) with given $\mathbf{u}_{h,\Pi}^{(i)}$ and zero right hand side. The resulting velocity $\mathbf{u}_h^{(i)}$ has minimal L^2-norm of all discrete divergence free velocities with given edge averages, i.e., $\hat{\mathcal{H}}_i : N_0(\partial\Omega_i) \longrightarrow \mathbf{W}^{(i)}$ satisfies*

$$B^{(i)} \hat{\mathcal{H}}_i \mu = 0 \quad \textit{and} \quad \|\hat{\mathcal{H}}_i \mu\|_{L^2(\Omega_i)} = \min_{v \in \mathbf{W}^{(i)},\ B^{(i)} v = 0,\ \overline{v}_{\mathcal{E}} = \mu} \|v\|_{L^2(\Omega_i)},$$

for any $\mu \in N_0(\partial\Omega_i)$,

DEFINITION 3. *Given the values* $\mathbf{w}_{\Gamma_c}^{(i)} \in \widetilde{\mathbf{W}}_{\Gamma_c,B_c}^{(i)}$, *define the extension operator* $\mathcal{H}^i$ *by obtaining* $\mathbf{w}_{I_c}^{(i)}$ *using (4.6) with* $R_{\Gamma_c}^{(i)}\mathbf{y}_{\Gamma_c} = \mathbf{w}_{\Gamma_c}^{(i)}$ *and with* $\mathbf{w}_{I_c}^{(i)} = \mathbf{y}_{I_c}^{(i)}$, *i.e.,* $\mathcal{H}^i : \widetilde{\mathbf{W}}_{\Gamma_c,B_c}^{(i)} \longrightarrow \mathbf{W}_c^{(i)}$ *satisfies:*

$$B_\Pi^{(i)}\mathcal{H}^i\mathbf{w}_{\Gamma_c} = 0, \quad \text{and} \quad \|\mathcal{H}^i\mathbf{w}_{\Gamma_c}\|_{A_\Pi^{(i)}} = \min_{\mathbf{v}\in\mathbf{W}_c^{(i)},\ B_\Pi\mathbf{v}=0,\ \mathbf{v}|_{\partial\Omega^i}=\mathbf{w}_{\Gamma_c}^{(i)}} \|v\|_{A_\Pi^{(i)}}.$$

From the definitions of $\tilde{\mathcal{H}}_i$, $\hat{\mathcal{H}}_i$, and $\mathcal{H}^i$, we have the following lemma:

LEMMA 5.3.

$$\|\mu\|_{S_\Gamma^{(i)}}^2 = \|\tilde{\mathcal{H}}_i\mu\|_{L^2(\Omega_i)}^2, \ \ \|\mu\|_{A_\Pi^{(i)}}^2 = \sum_{j=1}^{N_i}\|\hat{\mathcal{H}}_j(R_\Pi^{(i)}\mu)\|_{L^2(\Omega_j^i)}^2, \ \ \|\mathbf{w}_{\Gamma_c}^{(i)}\|_{T_{\Gamma_c}^{(i)}}^2 = \|\mathcal{H}^i\mathbf{w}_{\Gamma_c}\|_{A_\Pi^{(i)}}^2.$$

In next two lemmas, we establish the relation of the L_2-norm of the functions in $\widehat{\mathbf{W}}^H$ and their $\hat{\mathcal{H}}$ extensions.

LEMMA 5.4. *Let* $\mathcal{D}$ *be a square with vertices* $A = (0,0)$, $B = (H,0)$, $C = (H,H)$, *and* $D = (0,H)$, *with a quasi-uniform triangulation of mesh size* h. *Then, there exists a divergence free Raviart Thomas finite element function* $\mathbf{v}$ *defined on* $\mathcal{D}$ *such that* $\|\mathbf{v}\|_{L^2(\mathcal{D})}^2 \leq C(1+\log\frac{H}{h})$ *and* $\Pi_{RT}^H\mathbf{v}\cdot\mathbf{n}_{AB} \approx \frac{1}{H}\left(1+\log\frac{H}{h}\right)$, $\Pi_{RT}^H\mathbf{v}\cdot\mathbf{n}_{AD} \approx -\frac{1}{H}\left(1+\log\frac{H}{h}\right)$, $\Pi_{RT}^H\mathbf{v}\cdot\mathbf{n}_{BC} = \Pi_{RT}^H\mathbf{v}\cdot\mathbf{n}_{CD} = 0$.

Proof: Using [32, Lemma 4.1], we can construct a discrete harmonic piecewise bilinear finite element function φ in $\mathcal{D}$ with the following properties:

$$|\varphi|_{H^1(\mathcal{D})}^2 \approx 1+\log\frac{H}{h}, \quad \|\varphi\|_{L^\infty(\mathcal{D})} = \varphi(A) \approx 1+\log\frac{H}{h}, \quad \varphi(B)=\varphi(C)=\varphi(D)=0.$$

Let $\mathbf{v} = (-\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial x})$. We have $\operatorname{div}\mathbf{v} = 0$ and

$$\|\mathbf{v}\|_{L^2(\mathcal{D})}^2 \leq |\varphi|_{H^1(\mathcal{D})}^2 \approx 1+\log\frac{H}{h},$$

$$\Pi_{RT}^H\mathbf{v}\cdot\mathbf{n}_{AB} = -\frac{1}{H}\int_0^H \frac{\partial\varphi}{\partial x}(x,0)\,dx \approx \frac{1}{H}\left(1+\log\frac{H}{h}\right),$$

and

$$\Pi_{RT}^H\mathbf{v}\cdot\mathbf{n}_{DA} = -\frac{1}{H}\int_0^H -\frac{\partial\varphi}{\partial y}(0,y)\,dy \approx -\frac{1}{H}\left(1+\log\frac{H}{h}\right).$$

□

Remark: In Lemma 5.4, we have constructed the function $\mathbf{v}$ *for the square* $\mathcal{D}$. *By using similar ideas, we can easily construct a function* $\mathbf{v}$, *which satisfies the same properties, for other shape-regular quadrilaterals which can be obtained from the reference square by a sufficiently benign mapping.*

LEMMA 5.5. *Under Assumption 5.1, let* W_i^H *be the lowest order Raviart-Thomas finite element function space on a subregion* Ω^i *with a quasi-uniform coarse mesh with mesh size* H. *And let* $W_{i,j}^h$, $j = 1,\cdots,N_i$, *be the lowest order Raviart-Thomas finite element function space on a subdomain* Ω_j^i *with a quasi-uniform fine mesh with mesh* *size h. Given a discrete divergence free $\mathbf{u} \in W_i^H$, i.e., $\mathbf{u} \cdot \mathbf{n}$ with a zero mean value on $\partial\Omega_j^i$, there exist two positive constants C_1 and C_2, which are independent of $\hat{H}$, H, and h, such that*

$$C_1(1+\log\frac{H}{h})\left(\sum_{j=1}^{N_i}\|\hat{\mathcal{H}}_j(\mathbf{u})\|^2_{L^2(\Omega_j^i)}\right) \le \|\mathbf{u}\|^2_{L^2(\Omega^i)} \le C_2(1+\log\frac{H}{h})\left(\sum_{j=1}^{N_i}\|\hat{\mathcal{H}}_j(\mathbf{u})\|^2_{L^2(\Omega_j^i)}\right),$$

where $\hat{\mathcal{H}}_j$ is defined in Definition 2 for each subdomain Ω_j^i, $j = 1, \cdots, N_i$.

Proof: Without loss of generality, we assume that the subdomains are squares. Denote the edges of the subdomain Ω_j^i by a_j, b_j, c_j, and d_j, and denote the normal components of $\mathbf{u}$ at these four edges by $\mathbf{u}\cdot\mathbf{n}_{a_j} = u(a_j)$, $\mathbf{u}\cdot\mathbf{n}_{b_j} = u(b_j)$, $\mathbf{u}\cdot\mathbf{n}_{c_j} = u(c_j)$, and $\mathbf{u}\cdot\mathbf{n}_{d_j} = u(d_j)$, respectively. We have $u(a_j) = -(u(b_j)+u(c_j)+u(d_j)$ since $\mathbf{u}$ has zero mean value over $\partial\Omega_j^i$. $\mathbf{u}$ is a lowest order Raviart-Thomas finite element function and we have, $\|\mathbf{u}\|^2_{L^2(\Omega^i)} = \sum_{j=1}^{N_i}\|\mathbf{u}\|^2_{L^2(\Omega_j^i)}$ and

$$\|\mathbf{u}\|^2_{L^2(\Omega_j^i)} \approx CH^2\left(u^2(a_j)+u^2(b_j)+u^2(c_j)+u^2(d_j))\right). \tag{5.1}$$

According to Lemma 5.4, we can construct divergence free functions ϕ_b, ϕ_c, and ϕ_d on Ω_j^i such that

$$\begin{aligned}
&\Pi_{RT}^H\phi_b\cdot\mathbf{n}_{b_j} = u(b_j)\frac{1}{H}(1+\log\frac{H}{h}),\\
&\Pi_{RT}^H\phi_b\cdot\mathbf{n}_{c_j} = -\Pi_{RT}^H\phi_b\cdot\mathbf{n}_{b_j}, \quad \Pi_{RT}^H\phi_b\cdot\mathbf{n}_{d_j} = \Pi_{RT}^H\phi_b\cdot\mathbf{n}_{a_j} = 0;\\
&\Pi_{RT}^H\phi_c\cdot\mathbf{n}_{c_j} = (u(b_j)+u(c_j))\frac{1}{H}(1+\log\frac{H}{h}),\\
&\Pi_{RT}^H\phi_c\cdot\mathbf{n}_{d_j} = -\Pi_{RT}^H\phi_c\cdot\mathbf{n}_{c_j}, \quad \Pi_{RT}^H\phi_c\cdot\mathbf{n}_{a_j} = \Pi_{RT}^H\phi_c(b_j)\cdot\mathbf{n}_{b_j} = 0;\\
&\Pi_{RT}^H\phi_d\cdot\mathbf{n}_{d_j} = (u(b_j)+u(c_j)+u(d_j))\frac{1}{H}(1+\log\frac{H}{h}),\\
&\Pi_{RT}^H\phi_d\cdot\mathbf{n}_{a_j} = -\Pi_{RT}^H\phi_d\cdot\mathbf{n}_{d_j}, \quad \Pi_{RT}^H\phi_d\cdot\mathbf{n}_{b_j} = \Pi_{RT}^H\phi_d\cdot\mathbf{n}_{c_j} = 0;
\end{aligned}$$

and with

$$\begin{aligned}
&\|\phi_b\|^2_{L^2(\Omega_j^i)} \le u(b_j)^2(1+\log\frac{H}{h}),\\
&\|\phi_c\|^2_{L^2(\Omega_j^i)} \le (u(b_j)+u(c_j))^2(1+\log\frac{H}{h}),\\
&\|\phi_d\|^2_{L^2(\Omega_j^i)} \le (u(b_j)+u(c_j)+u(d_j))^2(1+\log\frac{H}{h}).
\end{aligned} \tag{5.2}$$

Let $\mathbf{v}_j = \frac{H}{1+\log\frac{H}{h}}(\phi_b+\phi_c+\phi_d)$; we then have $\Pi_{RT}^H v_j(m_j) = u(m_j)$, $m = a, b, c, d$, and

$$\begin{aligned}
\|\mathbf{v}_j\|^2_{L^2(\Omega^i_j)} &= \left(\frac{H}{1+\log\frac{H}{h}}\right)^2 \|\phi_b+\phi_c+\phi_d\|^2_{L^2(\Omega^i_j)} \\
&\le 3\left(\frac{H}{1+\log\frac{H}{h}}\right)^2 \sum_{m=b,c,d} \|\phi_m\|^2_{L^2(\Omega^i_j)} \\
&\le C\left(\frac{1}{c^{1/2}(1+\log\frac{H}{h})}\right)^2 \left(1+\log\frac{H}{h}\right) H^2 \sum_{m=a,b,c,d} (u(m_j))^2 \\
&\le \frac{1}{C_1(1+\log\frac{H}{h})}\|\mathbf{u}\|^2_{L^2(\Omega^i_j)}. \qquad (5.3)
\end{aligned}$$

Here, we have used (5.1) and (5.2) for the last two inequalities.

By the definition of $\hat{\mathcal{H}}_j(\mathbf{u})$, we have,

$$\|\hat{\mathcal{H}}_j(\mathbf{u})\|^2_{L^2(\Omega^i_j)} \le \|\mathbf{v}_j\|^2_{L^2(\Omega^i_j)} \le \frac{1}{C_1(1+\log\frac{H}{h})}\|\mathbf{u}\|^2_{L^2(\Omega^i_j)}.$$

Summing over all the subdomains of the subregion Ω^i, we have,

$$C_1\left(1+\log\frac{H}{h}\right)\left(\sum_{j=1}^{N_i}\|\hat{\mathcal{H}}_j(\mathbf{u})\|^2_{L^2(\Omega^i_j)}\right) \le \sum_{j=1}^{N_i}\|\mathbf{u}\|^2_{L^2(\Omega^i_j)} = \|\mathbf{u}\|^2_{L^2(\Omega^i)}.$$

This proves the first inequality.

We prove the second inequality by noting that $\mathbf{u} = \Pi_{RT}^H(\hat{\mathcal{H}}(\mathbf{u}))$. Since $\operatorname{div}\hat{\mathcal{H}}(\mathbf{u}) = 0$, we have the second inequality by Lemma 5.2.

□

We next list several results for the two-level BDDC methods. To be fully rigorous, we assume that each subregion is a union of shape-regular coarse quadrilaterals and that the number of such quadrilaterals forming an individual subregion is uniformly bounded. Moreover the fine triangulation of each subdomain is quasi uniform. Under Assumption 5.1, we can then get uniform constants C_1 and C_2 in Lemma 5.5, which hold for all the subregions.

We define subregion interface averaging operator $E_{D_c} = \widetilde{R}_c\widetilde{R}^T_{D_c}$ and $E_{D_c,\Gamma_c} = \widetilde{R}_{\Gamma_c}\widetilde{R}^T_{D_c,\Gamma_c}$, which computes the velocity averages across the subregion interface Γ_c and then distributes the averages to the boundary points of the subregions. We note that the estimate of E_{D_c} is the key to the analysis of our three-level BDDC algorithms.

We have for any vector $\mathbf{w}_c = (\mathbf{w}_{\Gamma_c}, q_{0_c}) \in \widetilde{\mathbf{W}}_{\Gamma_c} \times Q_{0_c}$,

$$E_{D_c}\begin{bmatrix}\mathbf{w}_{\Gamma_c}\\ q_{0_c}\end{bmatrix} = \begin{bmatrix}E_{D_c,\Gamma_c}\mathbf{w}_{\Gamma_c}\\ q_{0_c}\end{bmatrix}, \qquad (5.4)$$

see [27, (5.3)] for more details.

The interface averaging operator E_{D_c} satisfies the following bound, see [27, Lemma 5.6]:

Lemma 5.6. *For the two-level BDDC, we have*

$$|E_{D_c}\mathbf{u}_c|^2_{\widetilde{S}_H} \le C\left(1+\log\frac{\hat{H}}{H}\right)^2 |\mathbf{u}_c|^2_{\widetilde{S}_H}, \quad \forall \mathbf{u}_c \in \widetilde{\mathbf{W}}_{\Gamma_H,B_H} \times Q_{0_H},$$

where $\widetilde{S}_H$ and $\widetilde{W}_{\Gamma_H,B_H} \times Q_{0_H}$, which corresponds to a mesh size H, are analogous to $\widetilde{S}$ and $\widetilde{W}_{\Gamma,B} \times Q_0$, which corresponds to a mesh size h, respectively.

We next list some results for the subspace $\widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ and $\widetilde{\mathbf{W}}_{\Gamma_c,B_c} \times Q_{0_c}$.

Let $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$, $\|\mathbf{w}\|^2_{\widetilde{S}} = \mathbf{w}^T \widetilde{S} \mathbf{w}$, and $\|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma} = \mathbf{w}_\Gamma^T \widetilde{S}_\Gamma \mathbf{w}_\Gamma$. Similarly, let $\mathbf{w}_c \in \widetilde{\mathbf{W}}_{\Gamma_c,B_c} \times Q_{0_c}$, $\|\mathbf{w}_c\|^2_{\widetilde{T}} = \mathbf{w}^T \widetilde{T} \mathbf{w}$, and $\|\mathbf{w}_{\Gamma_c}\|^2_{\widetilde{T}_{\Gamma_c}} = \mathbf{w}_{\Gamma_c}^T \widetilde{T}_{\Gamma_c} \mathbf{w}_{\Gamma_c}$. As in [27, Lemma 5.5], we have the following result:

Lemma 5.7. *Given any $\mathbf{w} \in \widetilde{\mathbf{W}}_{\Gamma,B} \times Q_0$ and $\mathbf{w}_c \in \widetilde{\mathbf{W}}_{\Gamma_c,B_c} \times Q_{0_c}$, we have*

$$\|\mathbf{w}\|^2_{\widetilde{S}} = \|\mathbf{w}_\Gamma\|^2_{\widetilde{S}_\Gamma} \quad \text{and} \quad \|\mathbf{w}_c\|^2_{\widetilde{T}} = \|\mathbf{w}_{\Gamma_c}\|^2_{\widetilde{T}_{\Gamma_c}}.$$

In addition, we have, with $\widetilde{T}$ defined in (4.8):

Lemma 5.8.

$$\|E_{D_c}\mathbf{w}_c\|^2_{\widetilde{T}} \le C\left(1+\log\frac{\hat{H}}{H}\right)^2 \|\mathbf{w}_c\|^2_{\widetilde{T}},$$

for any $\mathbf{w}_c = \begin{bmatrix} w_{\Gamma_c} \\ q_{0_c} \end{bmatrix} \in \widetilde{\mathbf{W}}_{\Gamma_c,B_c} \times Q_{0_c}$. Here C is a constant independent of $\hat{H}$, H, h, and ρ.

Proof: The proof is similar to that of [32, Lemma 4.3].

Denote by $\mathcal{H}^i$ and $\tilde{\mathcal{H}}^i$ (on the mesh size H), the extensions in each subregion Ω^i, and by $\hat{\mathcal{H}}^i_j$ the extension in each subdomain Ω^i_j, where $i = 1, \cdots, N$, and $j = 1, \cdots, N_i$. We recall that $\mathcal{H}$, $\tilde{\mathcal{H}}$, and $\hat{\mathcal{H}}$ are defined in Definitions 3, 1, and 2, respectively.

Let $\overline{R}^{(i)}_{\Gamma_c}$ be a restriction operator from $\widetilde{\mathbf{W}}_{\Gamma_c}$ to $\mathbf{W}^{(i)}_{\Gamma_c}$. Using the definitions of $\mathcal{H}$, $\hat{\mathcal{H}}$, $\tilde{\mathcal{H}}$, and Lemma 5.3, we have

$$\begin{aligned}\|E_{D_c}\mathbf{w}_c\|^2_{\widetilde{T}} = \|E_{D_c,\Gamma_c}\mathbf{w}_{\Gamma_c}\|^2_{\widetilde{T}_{\Gamma_c}} &= \sum_{i=1}^{N_c} \|\mathcal{H}^i(\overline{R}^{(i)}_{\Gamma_c} E_{D_c,\Gamma_c}\mathbf{w}_{\Gamma_c})\|^2_{A^{(i)}_\Pi} \le \sum_{i=1}^{N_c} \|\tilde{\mathcal{H}}^i(\overline{R}^{(i)}_{\Gamma_c} E_{D_c,\Gamma_c}\mathbf{w}_{\Gamma_c})\|^2_{A^{(i)}_\Pi} \\ &= \sum_{i=1}^{N_c} \rho_i \left(\sum_{j=1}^{N_i} \|\hat{\mathcal{H}}^i_j\left(\tilde{\mathcal{H}}^i(\overline{R}^{(i)}_{\Gamma_c} E_{D_c,\Gamma_c}\mathbf{w}_{\Gamma_c})\right)\|^2_{L^2(\Omega^i_j)}\right).\end{aligned}$$

By Lemmas 5.5, 5.3, and 5.7,

$$\begin{aligned}|E_{D_c}\mathbf{w}_c|^2_{\widetilde{T}} &\le \sum_{i=1}^{N_c} \rho_i \left(\sum_{j=1}^{N_i} \|\hat{\mathcal{H}}^i_j\left(\tilde{\mathcal{H}}^i(\overline{R}^{(i)}_{\Gamma_c} E_{D_c,\Gamma_c}\mathbf{w}_{\Gamma_c})\right)\|^2_{L^2(\Omega^i_j)}\right) \\ &\le \frac{1}{C_1(1+\log\frac{H}{h})} \sum_{i=1}^{N_c} \rho_i \left(\|\left(\tilde{\mathcal{H}}^i(\overline{R}^{(i)}_{\Gamma_c} E_{D_c,\Gamma_c}\mathbf{w}_{\Gamma_c})\right)\|^2_{L^2(\Omega^i)}\right) \\ &= \frac{1}{C_1(1+\log\frac{H}{h})} \|E_{D_c}\mathbf{w}_c\|^2_{\widetilde{S}_H}.\end{aligned}$$

Using Lemmas 5.6, 5.7, and 5.3, we obtain

$$
\begin{aligned}
\|E_{D_c}\mathbf{w}_c\|_{\widetilde{T}}^2 &\le \frac{1}{C_1(1+\log\frac{H}{h})}\|E_{D_c}\mathbf{w}_c\|_{\widetilde{S}_H}^2 \\
&\le \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 \|\mathbf{w}_c\|_{\widetilde{S}_H}^2 \\
&= \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\sum_{i=1}^{N_c}\rho_i\|\left(\tilde{\mathcal{H}}^i(\overline{R}_{\Gamma_c}^{(i)}\mathbf{w}_{\Gamma_c})\right)\|_{L^2(\Omega^i)}^2\right) \\
&\le \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\sum_{i=1}^{N_c}\rho_i\|\left(\mathcal{H}^i(\overline{R}_{\Gamma_c}^{(i)}\mathbf{w}_{\Gamma_c})\right)\|_{L^2(\Omega^i)}^2\right).
\end{aligned}
$$

By Lemmas 5.5, 5.3, and 5.7, we have

$$
\begin{aligned}
\|E_{D_c}\mathbf{w}_c\|_{\widetilde{T}}^2 &\le \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 \left(\sum_{i=1}^{N_c}\rho_i\|\left(\mathcal{H}^i(\overline{R}_{\Gamma_c}^{(i)}\mathbf{w}_{\Gamma_c})\right)\|_{L^2(\Omega^i)}^2\right) \\
&\le \frac{C}{C_1(1+\log\frac{H}{h})}\left(1+\log\frac{\hat{H}}{H}\right)^2 C_2\left(1+\log\frac{H}{h}\right)\cdot \\
&\quad \left(\sum_{i=1}^{N_c}\rho_i\sum_{j=1}^{N_i}\left(\|\hat{\mathcal{H}}_j^i\left(\mathcal{H}^i(R_{\Gamma_c}^{(i)}\mathbf{w}_{\Gamma_c})\right)\|_{L^2(\Omega_j^i)}^2\right)\right) \\
&= \frac{CC_2}{C_1}\left(1+\log\frac{\hat{H}}{H}\right)^2\left(\sum_{i=1}^{N_c}\|\mathcal{H}^i(\overline{R}_{\Gamma_c}^{(i)}\mathbf{w}_{\Gamma_c})\|_{A_\Pi^{(i)}}^2\right) \\
&= \frac{CC_2}{C_1}\left(1+\log\frac{\hat{H}}{H}\right)^2\|\mathbf{w}_{\Gamma_c}\|_{\widetilde{T}_{\Gamma_c}}^2 = \frac{CC_2}{C_1}\left(1+\log\frac{\hat{H}}{H}\right)^2\|\mathbf{w}_c\|_{\widetilde{T}}^2.
\end{aligned}
$$

□

Lemma 5.9. *Given any* $\mathbf{u}\in\widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$, *let* $\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{\Theta}\end{bmatrix}=\Phi^T\widetilde{R}_D\widehat{S}\mathbf{u}$. *We have,*

$$
\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{\Theta}\end{bmatrix}^T S_\Pi^{-1}\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{\Theta}\end{bmatrix} \le \begin{bmatrix}\mathbf{\Psi}\\ \mathbf{\Theta}\end{bmatrix}^T M_\Pi^{-1}\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{\Theta}\end{bmatrix} \le C\left(1+\log\frac{\hat{H}}{H}\right)^2\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{\Theta}\end{bmatrix}^T S_\Pi^{-1}\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{\Theta}\end{bmatrix}
$$

Proof: Since $\mathbf{u}\in\widehat{\mathbf{W}}_{\Gamma,B}\times Q_0$, we have $\mathbf{\Theta}\equiv 0$ as in Lemma 4.2.

Using (4.6), (4.10), and (4.11), we have

$$\left[\begin{array}{c}\mathbf{\Psi}\\ \mathbf{\Theta}\end{array}\right]^T S_\Pi^{-1}\left[\begin{array}{c}\mathbf{\Psi}\\ \mathbf{\Theta}\end{array}\right]=\sum_{i=1}^{N_c}\mathbf{\Psi}_{I_c}^{(i)^T}\mathbf{y}_{I_c}^{(i)}+\mathbf{\Psi}_{\Gamma_c}^T\mathbf{y}_{\Gamma_c}$$

$$=\sum_{i=1}^{N_c}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]^T\left[\begin{array}{cc}A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T}\\ B_{\Pi_{I_cI_c}}^{(i)} & 0\end{array}\right]^{-1}\left(\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]-\left[\begin{array}{c}A_{\Pi_{I_c\Gamma_c}}^{(i)}\\ B_{\Pi_{I_c\Gamma_c}}^{(i)}\end{array}\right]R_{\Gamma_c}^{(i)}\mathbf{y}_{\Gamma_c}\right)$$

$$+\left(\mathbf{h}_{\Gamma_c}+\sum_{i=1}^{N_c}R_{\Gamma_c}^{(i)^T}\left[\begin{array}{cc}A_{\Pi_{\Gamma_cI_c}}^{(i)} & B_{\Pi_{I_c\Gamma_c}}^{(i)^T}\end{array}\right]\left[\begin{array}{cc}A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T}\\ B_{\Pi_{I_cI_c}}^{(i)} & 0\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]\right)^T\mathbf{y}_{\Gamma_c}$$

$$=\sum_{i=1}^{N_c}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)^T}\\ 0\end{array}\right]\left[\begin{array}{cc}A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T}\\ B_{\Pi_{I_cI_c}}^{(i)} & 0\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]+\mathbf{h}_{\Gamma_c}^T\mathbf{y}_{\Gamma_c}$$

$$=\sum_{i=1}^{N_c}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)^T}\\ 0\end{array}\right]\left[\begin{array}{cc}A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T}\\ B_{\Pi_{I_cI_c}}^{(i)} & 0\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]+\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right]^T\left(\widetilde{R}_c^T\widetilde{T}\widetilde{R}_c\right)^{-1}\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right].$$

Using (4.13), (4.10), and (4.12), we also have

$$\left[\begin{array}{c}\mathbf{\Psi}\\ \mathbf{\Theta}\end{array}\right]^T M_\Pi^{-1}\left[\begin{array}{c}\mathbf{\Psi}\\ \mathbf{\Theta}\end{array}\right]=\sum_{i=1}^{N_c}\mathbf{\Psi}_{I_c}^{(i)^T}\widehat{\mathbf{y}}_{I_c}^{(i)}+\mathbf{\Psi}_{\Gamma_c}^T\widehat{\mathbf{y}}_{\Gamma_c}$$

$$=\sum_{i=1}^{N_c}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]^T\left[\begin{array}{cc}A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T}\\ B_{\Pi_{I_cI_c}}^{(i)} & 0\end{array}\right]^{-1}\left(\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]-\left[\begin{array}{c}A_{\Pi_{I_c\Gamma_c}}^{(i)}\\ B_{\Pi_{I_c\Gamma_c}}^{(i)}\end{array}\right]R_{\Gamma_c}^{(i)}\widehat{\mathbf{y}}_{\Gamma_c}\right)$$

$$+\left(\mathbf{h}_{\Gamma_c}+\sum_{i=1}^{N_c}R_{\Gamma_c}^{(i)^T}\left[\begin{array}{cc}A_{\Pi_{\Gamma_cI_c}}^{(i)} & B_{\Pi_{I_c\Gamma_c}}^{(i)^T}\end{array}\right]\left[\begin{array}{cc}A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T}\\ B_{\Pi_{I_cI_c}}^{(i)} & 0\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]\right)^T\widehat{\mathbf{y}}_{\Gamma_c}$$

$$=\sum_{i=1}^{N_c}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)^T}\\ 0\end{array}\right]\left[\begin{array}{cc}A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T}\\ B_{\Pi_{I_cI_c}}^{(i)} & 0\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]+\mathbf{h}_{\Gamma_c}^T\widehat{\mathbf{y}}_{\Gamma_c}$$

$$=\sum_{i=1}^{N_c}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)^T}\\ 0\end{array}\right]\left[\begin{array}{cc}A_{\Pi_{I_cI_c}}^{(i)} & B_{\Pi_{I_cI_c}}^{(i)^T}\\ B_{\Pi_{I_cI_c}}^{(i)} & 0\end{array}\right]^{-1}\left[\begin{array}{c}\mathbf{\Psi}_{I_c}^{(i)}\\ 0\end{array}\right]+\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right]^T\widetilde{R}_{D_c}^T\widetilde{T}^{-1}\widetilde{R}_{D_c}\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right].$$

We only need to compare $\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right]^T\left(\widetilde{R}_c^T\widetilde{T}\widetilde{R}_c\right)^{-1}\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right]$ and $\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right]^T\widetilde{R}_{D_c}^T\widetilde{T}^{-1}\widetilde{R}_{D_c}\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right]$ for any $\mathbf{h}_{\Gamma_c}\in\widehat{\mathbf{F}}_{\Gamma_c}$. Let $\mathbf{h}=\left[\begin{array}{c}\mathbf{h}_{\Gamma_c}\\ 0\end{array}\right]$ and let

$$\mathbf{w}_c=\left(\widetilde{R}_c^T\widetilde{T}\widetilde{R}_c\right)^{-1}\mathbf{h}\in\widehat{\mathbf{W}}_{\Gamma_c,B_c}\times Q_{0_c}\text{ and }\mathbf{v}_c=\widetilde{T}^{-1}\widetilde{R}_{D_c}\mathbf{h}\in\widetilde{\mathbf{W}}_{\Gamma_c,B_c}\times Q_{0_c}. \tag{5.5}$$

Following the proofs for [32, Lemma 4.6] and [31, Lemma 4.7], we can prove:

$$\mathbf{h}^T\left(\widetilde{R}_{D_c}^T\widetilde{T}^{-1}\widetilde{R}_{D_c}\right)\mathbf{h}\le C\left(1+\log\frac{\hat{H}}{H}\right)^2\left(\mathbf{h}^T\left(\widetilde{R}_c^T\widetilde{T}\widetilde{R}_c\right)^{-1}\mathbf{h}\right).$$

□

TABLE 1

Condition number bounds and iteration counts with the two-level preconditioner M^{-1} and the three-level preconditioner $\widetilde{M}^{-1}$. N is the number of the subdomains used for M^{-1} and N_c is the number of the subregions used for $\widetilde{M}^{-1}$. $\frac{\hat{H}}{H}=4$ and $\frac{H}{h}=4$. We keep the same numbers of the subdomains and the same subdomain local problem size for these two preconditioners. $\rho \equiv 1$

Exact			Inexact		
N	Cond	Iter	N_c	Cond	Iter
16×16	2.32	8	4×4	3.45	10
32×32	2.34	8	8×8	4.20	12
48×48	2.34	7	12×12	4.39	11
64×64	2.35	6	16×16	4.45	11
80×80	Out of Memory	-	20×20	4.48	11

TABLE 2

Condition number bounds and iteration counts with the preconditioner $\widetilde{M}^{-1}$ with a change of the number of subdomains and the size of subdomain problems with $N_c = 4 \times 4$ subregions. $\rho \equiv 1$

4×4 subregions, $\frac{H}{h}$ fixed			4×4 subregions, $\widehat{N}$ fixed		
$\frac{\widehat{H}}{H}$	Cond	Iter	$\frac{H}{h}$	Cond	Iter
4	3.45	10	4	3.45	10
8	4.75	11	8	4.58	12
12	5.66	12	12	5.34	13
16	6.37	13	16	5.93	13
20	6.96	14	20	6.41	14

6. Condition number estimate for the new preconditioner. In order to estimate the condition number for the system with the new preconditioner $\widetilde{M}^{-1}$, we compare it to the system with the preconditioner M^{-1}.

LEMMA 6.1. *Given any $\mathbf{u} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$,*

$$\mathbf{u}^T M^{-1} \widehat{S} \mathbf{u} \leq \mathbf{u}^T \widetilde{M}^{-1} \widehat{S} \mathbf{u} \leq C \left(1 + \log \frac{\hat{H}}{H}\right)^2 \mathbf{u}^T M^{-1} \widehat{S} \mathbf{u}. \tag{6.1}$$

Proof: We obtain our result by calculating $\mathbf{u}^T M^{-1} \widehat{S} \mathbf{u}$ and $\mathbf{u}^T \widetilde{M}^{-1} \mathbf{u}$ for any $\mathbf{u} \in \widehat{\mathbf{W}}_{\Gamma,B} \times Q_0$ and using Lemma 5.9.

□

THEOREM 6.2. *The condition number for the system with the three-level preconditioner $\widetilde{M}^{-1}$ is bounded by $C(1+\log \frac{\hat{H}}{H})^2(1+\log \frac{H}{h})^2$.*

Proof: Combining the condition number bound, given in (3.14), for the two-level BDDC method, and Lemma 6.1, we find that the condition number for the three-level method is bounded by $C(1+\log \frac{\hat{H}}{H})^2(1+\log \frac{H}{h})^2$.

□

7. Numerical experiments. We have applied our three-level BDDC algorithm to the model problem (2.1), where $\Omega = [0,1]^2$. We decompose the unit square into $\widehat{N} \times \widehat{N}$ subregions with the side-length $\hat{H} = 1/\widehat{N}$ and each subregion into $N_h \times N_h$ subdomains with the side-length $H = \hat{H}/N_h$. Equation (2.1) is discretized, in each subdomain, by the lowest order Raviart-Thomas finite elements and the space of piecewise constants with a finite element diameter h, for the velocity and pressure,

Table 3

Condition number bounds and iteration counts with the two-level preconditioner M^{-1} and the three-level preconditioner $\widetilde{M}^{-1}$. N is the number of the subdomains used for M^{-1} and N_c is the number of the subregions used for $\widetilde{M}^{-1}$. $\frac{\hat{H}}{H} = 4$ and $\frac{H}{h} = 4$. We keep the same numbers of the subdomains and the same subdomain local problem size for these two preconditioners. ρ is in a checkerboard pattern.

Exact			Inexact		
N	Cond	Iter	N_c	Cond	Iter
16×16	2.30	9	4×4	2.31	9
32×32	2.30	9	8×8	2.32	9
48×48	2.30	8	12×12	2.33	9
64×64	2.30	8	16×16	2.33	8
80×80	Out of Memory	-	20×20	2.33	8

Table 4

Condition number bounds and iteration counts with the preconditioner $\widetilde{M}^{-1}$ with a change of the number of subdomains and the size of subdomain problems with $N_c = 4 \times 4$ subregions. ρ is in a checkerboard pattern.

4×4 subregions, $\frac{H}{h}$ fixed			4×4 subregions, $\widehat{N}$ fixed		
$\frac{\widehat{H}}{H}$	Cond	Iter	$\frac{H}{h}$	Cond	Iter
4	2.31	9	4	2.31	9
8	2.36	9	8	3.15	11
12	2.37	9	12	3.70	12
16	2.43	10	16	4.13	13
20	2.45	10	20	4.48	13

Table 5

(Stokes) Condition number bounds and iteration counts with the two-level preconditioner M^{-1} and the three-level preconditioner $\widetilde{M}^{-1}$. N is the number of the subdomains used for M^{-1} and N_c is the number of the subregions used for $\widetilde{M}^{-1}$. $\frac{\hat{H}}{H} = 4$ and $\frac{H}{h} = 4$. We keep the same numbers of the subdomains and the same subdomain local problem size for these two preconditioners.

Exact			Inexact		
N	Cond	Iter	N_c	Cond	Iter
16×16	2.54	11	4×4	4.67	14
32×32	2.54	11	8×8	5.91	18
48×48	2.53	11	12×12	6.19	19
64×64	2.55	12	16×16	6.27	20
80×80	Out of Memory	-	20×20	6.33	21

Table 6

(Stokes) Condition number bounds and iteration counts with the preconditioner $\widetilde{M}^{-1}$ with a change of the number of subdomains and the size of subdomain problems with $N_c = 4 \times 4$ subregions.

4×4 subregions, $\frac{H}{h}$ fixed			4×4 subregions, $\widehat{N}$ fixed		
$\frac{\widehat{H}}{H}$	Cond	Iter	$\frac{H}{h}$	Cond	Iter
4	4.67	14	4	4.67	14
8	6.01	16	8	6.70	18
12	6.89	18	12	8.11	20
16	7.54	19	16	9.23	22
20	8.06	20	20	10.15	23

respectively. The preconditioned conjugate gradient iteration is stopped when the l_2-norm of the residual has been reduced by a factor 10^{-6}.

We have carried out two different sets of experiments to obtain iteration counts and condition number estimates. All the experimental results are fully consistent with our theory.

In the first set of experiments, we take the coefficient $\rho \equiv 1$. Table 1 gives the iteration counts and condition number estimates for the two-level and the three-level BDDC algorithms. We use the same numbers of subdomains for both two and three levels with a change of the number of subregions for the three level case. We find that the condition numbers are independent of the number of subregions. And when the number of the subdomains is large, the two-level algorithms is out of memory, while the three-level algorithm is still fine. Table 2 gives results with a change of the number of the subdomains in each subregion and a change of the size of the subdomain problem.

In the second set of experiments, we take the coefficient $\rho = 1$ in half of the subregions and $\rho = 100$ in the neighboring subregions, in a checkerboard pattern. Everything else is the same as in the first set of experiments. The results are reported in Table 3 and 4.

Finally, we have applied our three-level BDDC algorithm to incompressible Stokes equations. We use the same discretization as in [17], where the two-level BDDC algorithm was applied for Stokes. Table 5 gives the iteration counts and condition number estimates for the two-level and the three-level BDDC algorithms. We use the same numbers of the subdomains for both two and three levels with a change of the number of subregions for the three level case. The similar results are obtained for Stokes as for the elliptic problems. Table 6 gives results with a change of the number of the subdomains in each subregion and a change of the size of the subdomain problem. These numerical results are also consistent with our condition number estimates.

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