CONVERGENCE ANALYSIS OF A BALANCING DOMAIN DECOMPOSITION METHOD FOR SOLVING INTERIOR HELMHOLTZ EQUATIONS

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Abstract. A variant of balancing domain decomposition method by constraints (BDDC) is proposed for solving a class of indefinite system of linear equations, which arises from the finite element discretization of the Helmholtz equation of time-harmonic wave propagation in a bounded interior domain. The proposed BDDC algorithm is closely related to the dual-primal finite element tearing and interconnecting algorithm for solving Helmholtz equations (FETI-DPH). Under the condition that the diameters of the subdomains are small enough, the rate of convergence is established which depends polylogarithmically on the dimension of the individual subdomain problems and which improves with the decrease of the subdomain diameters. These results are supported by numerical experiments of solving a Helmholtz equation on a two-dimensional square domain.

Key words. domain decomposition, preconditioner, FETI, BDDC, indefinite, non-conforming, Helmholtz

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction.

Domain decomposition methods have been widely used and studied for solving large symmetric, positive definite linear systems arising from the finite element discretization of elliptic partial differential equations; theories on their convergence rates are well developed for such problems; see [45, 43, 41] and the references therein. Domain decomposition methods have also been applied to solving indefinite and nonsymmetric problems; cf. [1, 3, 4, 7, 8, 9, 23, 28, 29, 30, 34, 35, 39, 40, 44, 46]. A two-level overlapping Schwarz method was studied by Cai and Widlund [8] for solving indefinite elliptic problems, where they used a perturbation approach in the analysis to overcome the difficulty introduced by the indefiniteness of the problem and established that the convergence rate of the algorithm is independent of the mesh size if the coarse level mesh is fine enough. Such an approach was also used by Gopalakrishnan and Pasciak [23] and by Gopalakrishnan, Pasciak, and Demkowicz [24] in their analysis of overlapping Schwarz methods and multigrid methods for solving time harmonic Maxwell equations.

The balancing domain decomposition methods by constraints (BDDC) were introduced by Dohrmann [12] for solving symmetric positive definite problems; see also Fragakis and Papadrakakis [22], and Cros [11]. They represent an interesting redesign of the Neumann-Neumann algorithms with the coarse, global component expressed in terms of a set of primal constraints. Spectral equivalence between the BDDC algorithms and the dual-primal finite element tearing and interconnecting algorithms (FETI-DP) has been proven by Mandel, Dohrmann, and Tezaur [38]; see also Li and Widlund [36], Brenner and Sung [6]. In these papers, it is established for the symmetric positive definite case that the preconditioned operators of a pair of BDDC and

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FETI-DP algorithms, with the same primal constraints, have the same eigenvalues except possibly those equal to 0 or 1.

In this paper, we propose a type of BDDC algorithm for solving a class of indefinite system of linear equations, which arises from the finite element discretization of the Helmholtz equation in a bounded interior domain. The proposed algorithm is motivated by the dual-primal finite element tearing and interconnecting algorithm for solving the time-harmonic wave propagation problems (FETI-DPH), which was first proposed by Farhat and Li [18]; see also [19, 16]. The FETI-DPH method has been shown by extensive experiments to be parallel scalable and has been applied to the simulation of elastic waves in structural dynamics problems, and to the simulation of sound waves in acoustic scattering problems. A key component in the FETI-DPH algorithm, also used in the proposed BDDC algorithm, is some plane waves incorporated in the coarse level problem to enhance the convergence rate. These plane waves represent exact solutions of the partial differential equation in free space. This is an idea first introduced by Farhat, Macedo, and Lesoinne [20] with the FETI-H algorithm for solving the Helmholtz equations.

In our algorithms, the GMRES iteration is used to solve the preconditioned indefinite system of linear equations. Under the condition that the diameters of the subdomains are small enough, we prove that the convergence rate of the GMRES iteration depends polylogarithmically on the dimension of the individual subdomain problems and it improves with the decrease of the subdomain diameters. As in [8, 23], a perturbation approach is used in our analysis to handle the indefiniteness of the problem. An error bound for the approximation of the solution of the Helmholtz equation by a partially sub-assembled finite element problem is crucial; we view that finite element problem as a non-conforming approximation of the indefinite problem. We also establish the spectral equivalence between the proposed BDDC algorithms and the FETI-DPH algorithms for solving Helmholtz equations.

This paper is organized as follows. In Section 2, the finite element problem is given for the Helmholtz equation in a bounded polyhedral domain. The decomposition of the domain and a partially sub-assembled finite element problem are discussed in Section 3. The BDDC and FETI-DPH algorithms and their connections are discussed in Section 4. In Section 5, the convergence rate analysis of the BDDC algorithm is given; the assumptions used in the proof are verified in Section 6. To conclude, numerical experiments are given in Section 7 to demonstrate the effectiveness of our method.

2. A finite element discretization of the Helmholtz equation on bounded interior domains. We consider the solution of the following Helmholtz equation on a bounded polyhedral domain $\Omega \in \mathbf{R}^d$, d = 2, 3,

$$\left\{ \begin{array}{rcll} -\Delta u - \sigma^2 u & = & f, & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega, \end{array} \right. \tag{2.1}$$

where the real number σ is often called the wave number; we assume that σ^2 is not one of the eigenvalues of the operator $-\Delta$. Weak solution is given by: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall\ v \in H_0^1(\Omega), \tag{2.2}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v - \sigma^2 uv$, and $(f, v) = \int_\Omega fv$. Under the assumption that (2.2) has a unique solution, we can prove the following regularity result for the weak solution; cf. [26, Section 9.1] and [27, Proposition 2.24].

LEMMA 2.1. *Let $\Omega \in \mathbf{R}^d$, $d = 2, 3$, be a bounded polyhedral domain with Lipschitz continuous boundary. Given any $f \in L_2(\Omega)$, let u be the unique solution of (2.2). Then $u \in H^{1+\gamma}(\Omega) \cap H_0^1(\Omega)$, for a certain $\gamma \in (1/2, 1]$, and $\|u\|_{H^{1+\gamma}} \le C(1+\sigma^2)\|f\|_{L_2}$, where C is a constant independent of f. If Ω is convex then the result holds for $\gamma = 1$.*

Proof. Result for the case $\sigma = 0$ can be found in [25, Corollary 2.6.7]; see also [26, Section 9.1]. Here we give a proof for the case where $\sigma \neq 0$. We define an operator $\mathcal{K} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by: given any $v \in H_0^1(\Omega)$,

$$\langle \mathcal{K}v, w \rangle = \int_\Omega \nabla v \cdot \nabla w, \quad \forall w \in H_0^1(\Omega).$$

Here $\langle \mathcal{K}v, w \rangle$ is the value of the functional $\mathcal{K}v$ at w; if $\mathcal{K}v \in L_2(\Omega)$ then $\langle \cdot, \cdot \rangle$ is the L_2 inner product. Given $f \in L_2(\Omega)$, let u be the unique solution of (2.2), i.e., $u \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \nabla u \cdot \nabla w - \sigma^2 uw \; = \; \int_\Omega fw, \quad \forall\, w \in H_0^1(\Omega).$$

Then we have $\mathcal{K}u = f + \sigma^2 u$. Since $\mathcal{K}$ is invertible and its inverse $\mathcal{K}^{-1}$ is a map from the space $H^{-1}(\Omega)$ to $H_0^1(\Omega)$, we have $u = \mathcal{K}^{-1}(f + \sigma^2 u)$. Since $f \in L_2(\Omega)$ and $u \in H_0^1(\Omega)$, we know from the regularity result for the case $\sigma = 0$, cf. [26, Section 9.1], that $u \in H^{1+\gamma}(\Omega)$ for a certain $\gamma \in (1/2, 1]$, and $\|u\|_{H^{1+\gamma}} \le C(\|f\|_{L_2} + \sigma^2\|u\|_{L_2})$, where C is a constant independent of f. We also know from [15, Chapter 6, Theorem 6] that $\|u\|_{L_2} \le C\|f\|_{L_2}$. Therefore $\|u\|_{H^{1+\gamma}} \le C(1+\sigma^2)\|f\|_{L_2}$. If Ω is convex then the result holds for $\gamma = 1$. $\square$

We consider a conforming finite element approximation of the problem (2.2) and denote the continuous finite element space by $\widehat{W}$. The finite element solution $u \in \widehat{W}$ satisfies

$$a(u, v) = (f, v), \quad \forall v \in \widehat{W}. \tag{2.3}$$

The resulting system of linear equations has the form

$$Au = (K - \sigma^2 M)u = f, \tag{2.4}$$

where K is the stiffness matrix, and M the mass matrix. In this paper, we will use the same notation u to denote a finite element function and its vector of coefficients with respect to the finite element basis functions. We will also use the same notation to denote the space of the finite element functions and the space of corresponding vectors, e.g., $\widehat{W}$. We have $|u|_{H^1}^2 = u^T K u$, and $\|u\|_{L_2}^2 = u^T M u$, for all $u \in \widehat{W}$.

We assume the finite element mesh is the union of shape regular elements with a typical element diameter h. We will use the following standard approximation property of the finite element space $\widehat{W}$, cf. [45, Lemma B.6].

LEMMA 2.2. *There exists a constant C which is independent of the mesh size such that for all $u \in H^{1+\gamma}(\Omega)$, $\gamma \in (1/2, 1]$,*

$$\inf_{w \in \widehat{W}} |u - w|_{H^1(\Omega)} \le Ch^\gamma |u|_{H^{1+\gamma}(\Omega)}, \quad \textit{and} \quad \inf_{w \in \widehat{W}} \|u - w\|_{L_2(\Omega)} \le Ch^{1+\gamma} |u|_{H^{1+\gamma}(\Omega)}.$$

3. A partially sub-assembled finite element space. A partially sub-assembled finite element space was introduced by Klawonn, Widlund, and Dryja [33] in a convergence analysis of the FETI-DP algorithm. It was later used by Li and Widlund [36, 37] to give an alternative formulation of the BDDC algorithm.

The domain Ω is decomposed into N nonoverlapping polyhedral subdomains Ω_i, $i = 1, 2, ..., N$. Each subdomain is a union of shape regular elements and the nodes on the boundaries of neighboring subdomains match across the interface $\Gamma = (\cup \partial\Omega_i) \backslash \partial\Omega$. The interface Γ is composed of subdomain faces and/or edges, which are regarded as open subsets of Γ, and of the subdomain vertices, which are end points of edges. In three dimensions, the subdomain faces are shared by two subdomains, and the edges typically by more than two; in two dimensions, each edge is shared by two subdomains. The interface of the subdomain Ω_i is defined by $\Gamma_i = \partial\Omega_i \cap \Gamma$. We also denote the set of nodes on Γ_i by $\Gamma_{i,h}$. We note that all the algorithms considered here are well defined for the less regular subdomains that are obtained by mesh partitioners. When developing theory, we will assume, as is customary in domain decomposition theory, that each subdomain is the union of a bounded number of shape regular elements with diameters on the order of H; cf. [45, Section 4.2]. For recent results on the analysis for irregular subdomains in domain decomposition methods, see [13].

The partially sub-assembled finite element space $\widetilde{W}$ is the direct sum of a coarse level primal subspace $\widehat{W}_\Pi$, of continuous coarse level finite element functions, and a dual space W_r, which is the product of local dual subspaces, i.e.,

$$\widetilde{W} = W_r \bigoplus \widehat{W}_\Pi = \left(\prod_{i=1}^{N} W_r^{(i)} \right) \bigoplus \widehat{W}_\Pi.$$

The space $\widehat{W}_\Pi$ corresponds to a few select subdomain interface degrees of freedom for each subdomain and is typically spanned by subdomain vertex nodal basis functions, and/or interface edge and/or face basis functions with weights at the nodes of the edge or face. These basis functions will correspond to the primal interface continuity constraints enforced in the BDDC and FETI-DP algorithms. For simplicity of our analysis, we will always assume that the basis has been changed so that we have explicit primal unknowns corresponding to the primal continuity constraints of edges or faces; these coarse level primal degrees of freedom are shared by neighboring subdomains. Another way of enforcing continuity constraints over edges or faces is to introduce an additional set of Lagrange multipliers in the coarse level problem; cf. [17]. Each subdomain dual space $W_r^{(i)}$ corresponds to the subdomain interior and dual interface degrees of freedom and it is spanned by all the basis functions which vanish at the primal degrees of freedom. Thus, functions in the space $\widetilde{W}$ have a continuous coarse level, primal part and typically a discontinuous dual part across the subdomain interface.

REMARK 3.1. *As in many other papers on FETI-DP and BDDC algorithms, we talk about dual spaces. The discontinuity of elements of the dual spaces across the subdomain interface is controlled by using Lagrange multipliers in the FETI-DP algorithms.*

We define the bilinear form on the partially sub-assembled finite element space $\widetilde{W}$ by

$$\widetilde{a}(u, v) = \sum_{i=1}^{N} \int_{\Omega_i} \nabla u^{(i)} \cdot \nabla v^{(i)} - \sigma^2 u^{(i)} v^{(i)}, \quad \forall\, u, v \in \widetilde{W},$$

where $u^{(i)}$ and $v^{(i)}$ represent the restriction of u and v to the subdomain Ω_i. The matrix corresponding to the bilinear form $\widetilde{a}(\cdot,\cdot)$ is denoted by $\widetilde{A}$. $\widetilde{A} = \widetilde{K} - \sigma^2\widetilde{M}$, where $\widetilde{K}$ is the partially sub-assembled stiffness matrix and $\widetilde{M}$ is the partially sub-assembled mass matrix. We always assume that $\widetilde{A}$ is nonsingular, i.e., the following problem always has a unique solution: given any $g \in L_2(\Omega)$, find $u \in \widetilde{W}$ such that

$$\widetilde{a}(u,v) = (g,v), \quad \forall v \in \widetilde{W}. \tag{3.1}$$

We define the broken norms on the space $\widetilde{W}$, by $\|w\|^2_{L_2(\Omega)} = \sum_{i=1}^N \|w^{(i)}\|^2_{L_2(\Omega_i)}$ and $|w|^2_{H^1(\Omega)} = \sum_{i=1}^N |w^{(i)}|^2_{H^1(\Omega_i)}$. In this paper $\|w\|_{L_2(\Omega)}$ and $|w|_{H^1(\Omega)}$, for functions $w \in \widetilde{W}$, always represent the corresponding broken norms. We also have, for any $w \in \widetilde{W}$, $|w|^2_{H^1} = w^T\widetilde{K}w$, and $\|w\|^2_{L_2} = w^T\widetilde{M}w$.

In our convergence analysis of the BDDC algorithms for solving the indefinite problems, we will establish an error bound for the approximation of the solution of the Helmholtz problem by the partially sub-assembled finite element problem. For this purpose, we assume that in our decomposition of the global domain Ω, each subdomain Ω_i is of triangular or quadrilateral shape in two dimensions, and of tetrahedral or hexahedral shape in three dimensions. We also assume that the subdomains form a shape regular coarse mesh of Ω. We denote by $\widehat{W}_H$ the continuous linear, bilinear, or trilinear finite element space on the coarse subdomain mesh, and denote by I_H the finite element interpolation from the space $H^{1+\gamma}(\Omega)$, $\gamma \in (1/2, 1]$, to $\widehat{W}_H$. We have the following Bramble-Hilbert lemma; cf. [47, Theorem 2.3].

LEMMA 3.2. *There exists a constant C which is independent of the mesh size such that for all $u \in H^{1+\gamma}(\Omega)$, $\gamma \in (1/2,1]$, $\|u - I_H u\|_{H^{1+\gamma}(\Omega_i)} \leq C|u|_{H^{1+\gamma}(\Omega_i)}$, for $i = 1, 2, ..., N$.*

The problem matrix A in (2.4) can be obtained by assembling the partially sub-assembled problem matrix $\widetilde{A}$, i.e.,

$$A = \widetilde{R}^T\widetilde{A}\widetilde{R}, \tag{3.2}$$

where $\widetilde{R} : \widehat{W} \to \widetilde{W}$, is the injection operator from the space of continuous finite element functions to the space of partially sub-assembled finite element functions. In order to define a scaled injection operator, we need to introduce a positive scale factor $\delta_i^\dagger(x)$ for each node x on the interface Γ_i of the subdomain Ω_i. In applications, these scale factors will depend on the heat conduction coefficient and the first of the Lamé parameters for scalar elliptic problems and the equations of linear elasticity, respectively; see [33, 32, 42]. Here, with $\mathcal{N}_x$ the set of indices of the subdomains which have x on their boundaries, we will only need to use inverse counting functions defined by $\delta_i^\dagger(x) = 1/card(\mathcal{N}_x)$, where $card(\mathcal{N}_x)$ is the number of the subdomains in the set $\mathcal{N}_x$. It is easy to see that $\sum_{j\in\mathcal{N}_x}\delta_j^\dagger(x) = 1$. Given these scale factors at the subdomain interface nodes, we can define the scaled injection operator $\widetilde{R}_D$; each row of $\widetilde{R}$ corresponds to a degree of freedom of the space $\widetilde{W}$, and multiplying each row which corresponds to a dual interface degree of freedom by the scale factor $\delta_i^\dagger(x)$, where $x \in \Gamma_{i,h}$ is the corresponding interface node, gives us $\widetilde{R}_D$.

4. A BDDC version of the FETI-DPH method.

4.1. A review of the FETI-DPH method. The finite element tearing and interconnecting (FETI) methods [21, 17] form a family of domain decomposition methods using Lagrange multipliers and auxiliary coarse problems. Application of the early one-level FETI algorithm to exterior Helmholtz equations was studied by Farhat, Macedo, and Lesoinne [20] and the algorithm is named the FETI-H algorithm. An important feature of the FETI-H algorithm is that some plane waves are incorporated into the coarse level problem. This idea has proven very successful for improving the convergence rate for solving the Helmholtz equations. A downside of the FETI-H method is that it transforms real problems, e.g., those arising from the discritization of (2.1), into complex problems which require more memory and computational work.

FETI-DPH is the dual-primal version of the FETI-H method and was proposed by Farhat and Li [18] for solving a class of indefinite system of linear equations of the form (2.4). As in the FETI-H algorithm, plane waves are added to the coarse level problem to achieve faster convergence. But, instead of using complex regularization terms as in FETI-H to prevent the subdomain problems from being singular, the coarse level primal continuity constraints are enforced in FETI-DPH to guarantee that the subdomain problems are nonsingular. In this way no complex valued computations are required for real problems.

Before we present a formulation of the FETI-DPH algorithm for solving (2.4), we first look at some exact solutions of the homogeneous equation (2.1) (with $f = 0$) in free space. Denote by x the space coordinate vector, either in 2D or in 3D, and denote by θ any direction vector of unit length. Then all functions of the form

$$\cos(\sigma\theta \cdot x) \qquad \text{or} \qquad \sin(\sigma\theta \cdot x) \tag{4.1}$$

are solutions to the homogeneous equation (2.1).

In the FETI-DPH algorithm [18], some coarse level primal continuity constraints corresponding to plane waves are enforced across the subdomain interface, i.e., the solution at each iteration step always has the same components corresponding to the chosen plane waves across the subdomain interface. Here we discuss how to enforce a plane wave continuity constraint for two-dimensional problems; the same approach can equally well be used for three-dimensional problems; cf. [18]. Let $\mathcal{E}^{ij}$ be a subdomain interface edge, which is shared by two neighboring subdomains Ω_i and Ω_j. To define the coarse level finite element basis function corresponding to a plane wave, we denote by q the vector determined by the chosen plane wave restricted to $\mathcal{E}^{ij}$. We then choose the finite element function, which is determined by q at the nodes on $\mathcal{E}^{ij}$ and which vanishes elsewhere on the mesh, as a coarse level finite element basis function, i.e., we choose q as an element in the coarse level primal subspace $\widehat{W}_\Pi$. By sharing this common coarse level primal degree of freedom between subdomains Ω_i and Ω_j, elements in the partially sub-assembled finite element space $\widetilde{W}$ always have a common component corresponding to q across $\mathcal{E}^{ij}$. In this paper, we always assume that the basis of the finite element space has been changed and that therefore there are explicit degrees of freedom corresponding to all the coarse level primal continuity constraints. For more details on the change of basis, see [36, 32, 31].

REMARK 4.1. *We note that by choosing different directions θ in (4.1), different plane wave vectors can be obtained. Also both the cosine and sine modes of the plane waves can be used. By controlling the number of directions θ used, we control the number of the coarse level primal degrees of freedom related to the plane wave continuity constraints. As shown in [18], the larger the shift σ^2 in (2.4), the more directions need to be used to prevent a deterioration of the convergence rate.*

To derive a formulation of the FETI-DPH algorithm, the partially sub-assembled problem matrix $\widetilde{A}$ is written with blocks corresponding to the subdomain interior variables and to the subdomain interface variables as

$$\widetilde{A} = \begin{bmatrix} A_{II} & \widetilde{A}_{I\Gamma} \\ \widetilde{A}_{\Gamma I} & \widetilde{A}_{\Gamma\Gamma} \end{bmatrix},$$

where A_{II} is block diagonal with one block for each subdomain, and $\widetilde{A}_{\Gamma\Gamma}$ is assembled across the subdomain interface Γ only with respect to the coarse level primal degrees of freedom.

The solution of the original system of linear equations (2.4) can be obtained as the solution to the following system of linear equations

$$\begin{bmatrix} A_{II} & \widetilde{A}_{I\Gamma} & 0 \\ \widetilde{A}_{\Gamma I} & \widetilde{A}_{\Gamma\Gamma} & B_\Gamma^T \\ 0 & B_\Gamma & 0 \end{bmatrix} \begin{bmatrix} u_I \\ \widetilde{u}_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} f_I \\ \widetilde{f}_\Gamma \\ 0 \end{bmatrix}, \tag{4.2}$$

where $\widetilde{f}_\Gamma$ on the right hand side is assembled only with respect to the coarse level primal degrees of freedom across the subdomain interface. The matrix B_Γ has elements from the set $\{0, 1, -1\}$ and is used to enforce the continuity of the solution across the subdomain interface. Eliminating the variables u_I and $\widetilde{u}_\Gamma$ from (4.2), the following equation for the Lagrange multipliers λ is obtained,

$$B_\Gamma \widetilde{S}_\Gamma^{-1} B_\Gamma^T \lambda = B_\Gamma \widetilde{S}_\Gamma^{-1} (\widetilde{f}_\Gamma - \widetilde{A}_{\Gamma I} A_{II}^{-1} f_I), \tag{4.3}$$

where $\widetilde{S}_\Gamma = \widetilde{A}_{\Gamma\Gamma} - \widetilde{A}_{\Gamma I} A_{II}^{-1} \widetilde{A}_{I\Gamma}$.

In the FETI-DPH algorithm, a preconditioned GMRES iteration is used to solve the equation (4.3); after obtaining the Lagrange multipliers λ, we find u_I and $\widetilde{u}_\Gamma$ by back solving. Two types of preconditioners have been used: the Dirichlet preconditioner $B_{D,\Gamma} \widetilde{S}_\Gamma B_{D,\Gamma}^T$, and the lumped preconditioner $B_{D,\Gamma} \widetilde{A}_{\Gamma\Gamma} B_{D,\Gamma}^T$; cf. [21, 17]. Here $B_{D,\Gamma}$ is obtained from B_Γ by an appropriate scaling across the subdomain interface. In each GMRES iteration of the FETI-DPH algorithm, to multiply $\widetilde{S}_\Gamma^{-1}$ by a vector, a coarse level problem and subdomain problems with Neumann boundary conditions and with fixed primal values need be solved; to multiply $\widetilde{S}_\Gamma$ by a vector, subdomain problems with Dirichlet boundary conditions need be solved; cf. [17, 33].

When $\widetilde{S}_\Gamma$ is applied to a vector in the Dirichlet preconditioner, we need to multiply $\widetilde{A}_{\Gamma\Gamma} - \widetilde{A}_{\Gamma I} A_{II}^{-1} \widetilde{A}_{I\Gamma}$ by the vector. An alternative used in [18] is to multiply by $\widetilde{A}_{\Gamma\Gamma} - \widetilde{K}_{\Gamma I} K_{II}^{-1} \widetilde{K}_{I\Gamma}$ in the Dirichlet preconditioner. This corresponds to the use of discrete harmonic extensions to the interior of subdomains. We denote this alternative Dirichlet preconditioner in the FETI-DPH algorithm by $B_{D,\Gamma} \widetilde{S}_\Gamma^{\mathcal{H}} B_{D,\Gamma}^T$. The numerical experiments in Section 7 will show that using either $B_{D,\Gamma} \widetilde{S}_\Gamma B_{D,\Gamma}^T$ or $B_{D,\Gamma} \widetilde{S}_\Gamma^{\mathcal{H}} B_{D,\Gamma}^T$ gives almost the same convergence rate.

4.2. A BDDC version of the FETI-DPH method. We now present a BDDC version of the FETI-DPH method. The BDDC algorithms and the closely related primal versions of the FETI algorithms were proposed by Dohrmann [12], Fragakis and Papadrakakis [22], and Cros [11], for solving symmetric positive definite problems. Here we follow an alternative presentation of the BDDC algorithm given by Li and Widlund [37]. The formulation of BDDC preconditioners for the indefinite problems are in fact the same as for the symmetric positive definite case, except that the corresponding blocks are now indefinite matrices determined by $K - \sigma^2 M$ in (2.4).

A BDDC preconditioner for solving the indefinite problem (2.4) can be written as

$$B_1^{-1} = \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D, \tag{4.4}$$

where $\widetilde{R}_D : \widehat{W} \to \widetilde{W}$, is the scaled restriction introduced in the end of Section 3. To multiply $\widetilde{A}^{-1}$ by a vector $\widetilde{g}$, the following partially sub-assembled problem needs be solved,

$$\widetilde{A}\widetilde{u} = \begin{bmatrix} A_{rr}^{(1)} & & & \widetilde{A}_{r\Pi}^{(1)} \\ & \ddots & & \vdots \\ & & A_{rr}^{(N)} & \widetilde{A}_{r\Pi}^{(N)} \\ \widetilde{A}_{\Pi r}^{(1)} & \dots & \widetilde{A}_{\Pi r}^{(N)} & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_r^{(1)} \\ \vdots \\ u_r^{(N)} \\ u_\Pi \end{bmatrix} = \begin{bmatrix} g_r^{(1)} \\ \vdots \\ g_r^{(N)} \\ g_\Pi \end{bmatrix} = \widetilde{g}. \tag{4.5}$$

The leading diagonal blocks correspond to subdomain Neumann problems with given coarse level primal values. $\widetilde{A}_{\Pi\Pi}$ corresponds to the coarse level primal degrees of freedom and is assembled across the subdomain interface. The inverse of $\widetilde{A}$ can be written as

$$\widetilde{A}^{-1} = \begin{bmatrix} A_{rr}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1}\widetilde{A}_{r\Pi} \\ I \end{bmatrix} \widetilde{S}_\Pi^{-1} \begin{bmatrix} -\widetilde{A}_{\Pi r}A_{rr}^{-1} & I \end{bmatrix}, \tag{4.6}$$

where A_{rr}, $\widetilde{A}_{r\Pi}$, and $\widetilde{A}_{\Pi r}$ represent the corresponding blocks of $\widetilde{A}$ in (4.5), and $\widetilde{S}_\Pi = \widetilde{A}_{\Pi\Pi} - \sum_{i=1}^N \widetilde{A}_{\Pi r}^{(i)} A_{rr}^{(i)^{-1}} \widetilde{A}_{r\Pi}^{(i)}$.

From (4.6), we see that the BDDC preconditioner (4.4) can be regarded as the summation of subdomain corrections and a coarse level correction. Let us denote

$$\Psi = \begin{bmatrix} -A_{rr}^{-1}\widetilde{A}_{r\Pi} \\ I \end{bmatrix}.$$

Then we can see that $\widetilde{S}_\Pi = \Psi^T \widetilde{A} \Psi$. Therefore the first term in the right hand of (4.6) corresponds to subdomain corrections for which all the coarse level primal variables vanish, and the second term corresponds to a projection onto the coarse space determined by Ψ; cf. [38, 36]. Ψ represents extensions of the chosen coarse level finite element basis functions to the interior of the subdomains, and these extensions are waves for the Helmholtz problems. In Figure 4.1, we plot the extensions of a subdomain corner basis function and a subdomain edge average basis function to the interior of the subdomain.

Another BDDC preconditioner for solving (2.4) is of the form

$$B_2^{-1} = (\widetilde{R}_D^T - \mathcal{H}J_D)\widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T). \tag{4.7}$$

Here $J_D : \widetilde{W} \to \widetilde{W}$. For any $w \in \widetilde{W}$, the component of $J_D w$, for the subdomain Ω_i, is defined by

$$(J_D w(x))^{(i)} = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x)(w^{(i)}(x) - w^{(j)}(x)), \quad \forall x \in \Gamma_{i,h},$$

where $J_D w$ vanishes in the interior of the subdomain and for the coarse level primal component. For a matrix form of the operator J_D, see Toselli and Widlund [45, Section 6.3]. The component of $J_D^T w$ for the subdomain Ω_i is then given by

$$(J_D^T w(x))^{(i)} = \sum_{j \in \mathcal{N}_x} (\delta_j^\dagger(x) w^{(i)}(x) - \delta_i^\dagger(x) w^{(j)}(x)), \quad \forall x \in \Gamma_{i,h}.$$

FIG. 4.1. *Extension of the coarse level primal finite element basis functions to the interior of a subdomain: extension of a corner basis function (left); extension of an edge average basis function (right).*

The subdomain interior and the coarse level primal components of $J_D^T w$ also vanish. The operator $\mathcal{H}$ in (4.7) is direct sum of the subdomain discrete harmonic extensions $\mathcal{H}^{(i)}$, where $\mathcal{H}^{(i)} = -K_{II}^{(i)-1} K_{I\Gamma}^{(i)}$, $i = 1, 2, ..., N$. $\mathcal{H} J_D$ represents the discrete harmonic extension of the jump of the dual interface variables to the interior of the subdomains.

An alternative to the discrete harmonic extension $\mathcal{H}$ used in the preconditioner B_2^{-1} is an extension based on solving the indefinite subdomain Dirichlet problems. Let $\mathcal{H}_A^{(i)} = -A_{II}^{(i)-1} A_{I\Gamma}^{(i)}$ and denote the direct sum of the $\mathcal{H}_A^{(i)}$ by $\mathcal{H}_A$. Then the corresponding preconditioner is defined by

$$B_3^{-1} = (\widetilde{R}_D^T - \mathcal{H}_A J_D) \widetilde{A}^{-1} (\widetilde{R}_D - J_D^T \mathcal{H}_A^T). \tag{4.8}$$

4.3. Spectral equivalence between FETI-DPH method and its BDDC counterpart. Spectral equivalence results for FETI-DP and BDDC methods for symmetric positive definite problems were first proven by Mandel, Dohrmann, and Tezaur [38]; see also Fragakis and Papadrakakis [22], Li and Widlund [36, 37], and Brenner and Sung [6]. These arguments do not depend on the positive definiteness of the problem, and are also valid for indefinite problems; cf. [37, 36]. We have

THEOREM 4.2. *1. The preconditioned operator $B_1^{-1} A$ has the same eigenvalues as the preconditioned FETI-DPH operator with the lumped preconditioner $B_{D,\Gamma} \widetilde{A}_{\Gamma\Gamma} B_{D,\Gamma}^T$, except for possible eigenvalues equal to 0 and 1.*

2. The preconditioned operator $B_3^{-1} A$ has the same eigenvalues as the preconditioned FETI-DPH operator with the Dirichlet preconditioner $B_{D,\Gamma} \widetilde{S}_\Gamma B_{D,\Gamma}^T$, except for possible eigenvalues equal to 0 and 1.

We will demonstrate the spectral connection between the BDDC algorithms and the FETI-DPH algorithms in Section 7. The spectral equivalence between the preconditioned BDDC operator $B_2^{-1} A$ and the preconditioned FETI-DPH operator with preconditioner $B_{D,\Gamma} \widetilde{S}_\Gamma^{\mathcal{H}} B_{D,\Gamma}$ is not clear, even though their convergence rates are also quite similar in our numerical experiments.

5. Convergence rate analysis. The GMRES iteration is used in our BDDC algorithm to solve the preconditioned system of linear equations. For the convenience of our analysis, we use the inner product defined by the matrix $K + \sigma^2 M$ in the GMRES iteration. We define $\Lambda = K + \sigma^2 M$ and $\widetilde{\Lambda} = \widetilde{K} + \sigma^2 \widetilde{M}$, respectively. To estimate the convergence rate of the GMRES iteration, we use the following result due to Eisenstat, Elman, and Schultz [14].

THEOREM 5.1. *Let c_1 and C_2 be two parameters such that, for all $u \in \widehat{W}$,*

$$c_1 \langle u, u \rangle_\Lambda \le \langle u, Tu \rangle_\Lambda, \tag{5.1}$$

$$\langle Tu, Tu \rangle_\Lambda \le C_2^2 \langle u, u \rangle_\Lambda. \tag{5.2}$$

If $c_1 > 0$, then

$$\frac{\|r_m\|_\Lambda}{\|r_0\|_\Lambda} \le \left(1 - \frac{c_1^2}{C_2^2}\right)^{m/2},$$

where r_m is the residual at step m of the GMRES iteration applied to the operator T.

REMARK 5.2. *The convergence rate of the GMRES iteration using the standard L_2 inner product will not be estimated in this paper. In our numerical experiments, we have found that using the $K+\sigma^2 M$ inner product or the standard L_2 inner product gives the same convergence rate.*

In Theorem 5.11, we will estimate c_1 and C_2 in (5.1) and (5.2), for the preconditioned BDDC operators $B_1^{-1}A$ and $B_2^{-1}A$. The analysis for $B_3^{-1}A$ is not available yet.

We first make an assumption on the coarse level primal subspace $\widehat{W}_\Pi$ in our analysis. We denote the subdomain interface edges by $\mathcal{E}^k$ and, for three dimensions, the subdomain faces by $\mathcal{F}^l$. For each edge $\mathcal{E}^k$, we denote the set of all subdomains that share $\mathcal{E}^k$ by $\mathcal{N}_{\mathcal{E}^k}$; for each face $\mathcal{F}^l$, the set of all subdomains that share $\mathcal{F}^l$ is denoted by by $\mathcal{N}_{\mathcal{F}^l}$. We assume the coarse level primal subspace $\widehat{W}_\Pi$ is chosen such that it satisfies the following assumption.

ASSUMPTION 5.3. *The coarse level primal subspace $\widehat{W}_\Pi$ contains all subdomain corner degrees of freedom, one degree of freedom for each edge $\mathcal{E}^k$, and one for each face $\mathcal{F}^l$ (for three-dimensional problems), such that for any $w \in \widetilde{W}$, the values $\int_{\mathcal{E}^k} w^{(i)}$ are the same for all $i \in \mathcal{N}_{\mathcal{E}^k}$, and, for three dimensions, the values $\int_{\mathcal{F}^l} w^{(i)}$ are the same for all $i \in \mathcal{N}_{\mathcal{F}^l}$.*

Assumption 5.3 requires one coarse level primal degree of freedom for each edge and one for each face, respectively. Those constant edge or face average constraints correspond to the restriction of a cosine plane wave in (4.1) with the chosen angle θ perpendicular to the edge or to the face. From Figure 4.1, we see that the extension of the edge constant basis function to the interior of the subdomains represents a plane wave. When more than one plane wave continuity constraints are enforced on the same edge or face, it can easily happen that the coarse level primal basis vectors are linearly dependent on that edge or face. In order to make sure that the primal basis functions maintain linear independence, we can use a singular value decomposition on each edge and face, in a preprocessing step of the algorithm, to single out those that are numerically linearly independent and should be retained in the coarse level primal subspace. This device for eliminating linearly dependent coarse level primal constraints has previously been applied for FETI-DPH algorithms; see [18].

Using Assumption 5.3, we have the following lemma, which is essentially a Poincaré-Friedrichs inequality proven by Brenner in [5, (1.3)].

LEMMA 5.4. *Let Assumption 5.3 hold. There exists a constant C, which is independent of H and h, such that $\langle u, u\rangle_{\widetilde{M}} \le C\,\langle u, u\rangle_{\widetilde{K}}$, $\forall u \in \widetilde{W}$.*

¿From Assumption 5.3, we also obtain a result on the stability of certain average operators, which are defined by $E_{D,1} = \widetilde{R}\widetilde{R}_D^T$ and $E_{D,2} = \widetilde{R}(\widetilde{R}_D^T - \mathcal{H}J_D)$, corresponding to the preconditioned BDDC operators $B_1^{-1}A$ and $B_2^{-1}A$, respectively. The following lemma can be found in [33, 32, 37].

LEMMA 5.5. *There exist functions $\Phi_i(H, h)$, $i = 1, 2$, such that*

$$|E_{D,i}w|^2_{H^1(\Omega)} \le \Phi_i(H,h)|w|^2_{H^1(\Omega)}, \quad \forall\ w \in \widetilde{W}.$$

If Assumption 5.3 holds, then for two-dimensional problems, $\Phi_1(H,h) = CH/h$ *and* $\Phi_2(H,h) = C\left(1+\log(H/h)\right)^2$*; for three-dimensional problems,* $\Phi_1(H,h) = C(H/h)(1+\log(H/h))$ *and* $\Phi_2(H,h) = C\left(1+\log(H/h)\right)^2$*. Here* C *is a positive constant independent of* H *and* h*.*

Using Lemma 5.5, we can prove the following lemma.

LEMMA 5.6. *Let Assumption 5.3 hold. Then,*

$$\|E_{D,i}w\|_{\widetilde{\Lambda}}^2 \le (1 + C\sigma^2H^2)\Phi_i(H,h)\|w\|_{\widetilde{\Lambda}}^2, \quad \forall\, w \in \widetilde{W}, \quad i = 1,2,$$

where $\Phi_i(H,h)$ *are determined as in Lemma 5.5, and* C *is a positive constant independent of* H *and* h*.*

Proof. We know that

$$\|E_{D,i}w\|_{\widetilde{\Lambda}}^2 = \|E_{D,i}w\|_{\widetilde{K}}^2 + \sigma^2\|E_{D,i}w\|_{\widetilde{M}}^2 = |E_{D,i}w|_{H^1}^2 + \sigma^2\|E_{D,i}w\|_{L_2}^2.$$

Using Lemma 5.5 for the first term on the right side, and writing the second term as $E_{D,i}w = w - P_{D,i}w$, where $P_{D,i}$ represents a jump operator, cf. [37, 45, Lemma 6.10], we have

$$\|E_{D,i}w\|_{\widetilde{\Lambda}}^2 \le \Phi_i(H,h)|w|_{H^1}^2 + \sigma^2(\|w\|_{L_2}^2 + \|P_{D,i}w\|_{L_2}^2).$$

¿From Assumption 5.3, we know that $P_{D,i}w$ has zero averages on the subdomain interfaces. Using a Poincaré-Friedrichs inequality and then a result similar to Lemma 5.5 on the stability of the jump operator $P_{D,i}$, cf. [33, Lemma 9], we have

$$\|P_{D,i}w\|_{L_2}^2 \le CH^2|P_{D,i}w|_{H^1}^2 \le CH^2\Phi_i(H,h)|w|_{H^1}^2.$$

Therefore, we have

$$\begin{aligned}\|E_{D,i}w\|_{\widetilde{\Lambda}}^2 &\le \Phi_i(H,h)|w|_{H^1}^2 + \sigma^2\|w\|_{L_2}^2 + C\sigma^2H^2\Phi_i(H,h)|w|_{H^1}^2 \\ &\le (1 + C\sigma^2H^2)\Phi_i(H,h)(|w|_{H^1}^2 + \sigma^2\|w\|_{L_2}^2).\end{aligned}$$

□

The next assumption will be verified in Section 6.

ASSUMPTION 5.7. *There exists a positive constant* C*, which is independent of* H *and* h*, such that if* $\sigma(1+\sigma^3)H^\gamma$ *is sufficiently small, then for all* $u \in \widehat{W}$*,* $i = 1,2$*,*

$$\begin{aligned}\left|\left\langle w_i - \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{M}}\right| &\le C(1+\sigma^3)H^\gamma(1+\sqrt{\Phi_i(H,h)})\left\langle u,u\right\rangle_\Lambda, \\ \left|\left\langle w_i - \widetilde{R}u, w_i\right\rangle_{\widetilde{M}}\right| &\le C(1+\sigma^3)H^\gamma\left(1+\Phi_i(H,h)\right)\left\langle u,u\right\rangle_\Lambda, \\ \left|\left\langle z_i - \widetilde{R}u, w_i\right\rangle_{\widetilde{M}}\right| &\le CH\left(1+\Phi_i(H,h)\right)\left\langle u,u\right\rangle_\Lambda, \\ \|w_i\|_{\widetilde{K}} &\le C(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda.\end{aligned}$$

Here $w_1 = \widetilde{A}^{-1}\widetilde{R}_DAu$*,* $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au$*,* $z_1 = \widetilde{M}^{-1}\widetilde{R}_DMu$*,* $z_2 = \widetilde{M}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Mu$*, and* $\Phi_i(H,h)$ *are determined as in Lemma 5.5.*

The following lemma can be found in [45, Lemma B.31].

LEMMA 5.8. *The mass matrix* $\widetilde{M}$ *is spectrally equivalent to a diagonal matrix with diagonal entries on the order of* h^d*, where* h *is the mesh size and* $d = 2,3$*, i.e., there exist positive constants* c *and* C *which are independent of the mesh size such that* $ch^d \le \lambda_{min}(\widetilde{M}) \le \lambda_{max}(\widetilde{M}) \le Ch^d$*.*

LEMMA 5.9. *There exists a positive constant C, which is independent of H and h, such that for all $u, v \in \widetilde{W}$, $|u^T \widetilde{A} v| \leq C |u|_{\widetilde{\Lambda}} \, |v|_{\widetilde{\Lambda}}$, and $|u|_{\widetilde{\Lambda}} \leq |u|_{\widetilde{K}} + \sigma |u|_{\widetilde{M}} \leq \sqrt{2} |u|_{\widetilde{\Lambda}}$.*

Proof. To prove the first inequality, we have, for all $u, v \in \widetilde{W}$,

$$\begin{aligned} |u^T \widetilde{A} v| &= |u^T \widetilde{K} v - \sigma^2 u^T \widetilde{M} v| \leq (u^T \widetilde{K} u)^{1/2} (v^T \widetilde{K} v)^{1/2} + \sigma^2 (u^T \widetilde{M} u)^{1/2} (v^T \widetilde{M} v)^{1/2} \\ &\leq C (u^T \widetilde{\Lambda} u)^{1/2} (v^T \widetilde{\Lambda} v)^{1/2}. \end{aligned}$$

The second inequality can be derived by using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ and $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$, for any positive a and b. $\square$

LEMMA 5.10. *Let Assumption 5.7 hold. Then there exists a constant C, which is independent of H and h, such that if $\sigma(1+\sigma^3)H^\gamma$ is sufficiently small, then*

$$\langle w_i, w_i \rangle_{\widetilde{\Lambda}} \leq \left\langle u, B_i^{-1} A u \right\rangle_\Lambda + C \ \sigma^2 (1+\sigma^3) H^\gamma \left(1 + \Phi_i(H,h)\right) \langle u, u \rangle_\Lambda, \ \forall \ u \in \widehat{W}, \ i = 1, 2,$$

where $w_1 = \widetilde{A}^{-1} \widetilde{R}_D A u$ and $w_2 = \widetilde{A}^{-1} (\widetilde{R}_D - J_D^T \mathcal{H}^T) A u$.

Proof. For $w_1 = \widetilde{A}^{-1} \widetilde{R}_D A u$, we have,

$$\begin{aligned} &\langle w_1, w_1 \rangle_{\widetilde{\Lambda}} = \langle w_1, w_1 \rangle_{\widetilde{A}} + 2\sigma^2 \langle w_1, w_1 \rangle_{\widetilde{M}} \\ &= u^T A \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{A} \widetilde{A}^{-1} \widetilde{R}_D A u + 2\sigma^2 \langle w_1, w_1 \rangle_{\widetilde{M}} = \left\langle u, B_1^{-1} A u \right\rangle_A + 2\sigma^2 \langle w_1, w_1 \rangle_{\widetilde{M}} \\ &= \left\langle u, B_1^{-1} A u \right\rangle_\Lambda - 2\sigma^2 \left\langle u, B_1^{-1} A u \right\rangle_M + 2\sigma^2 \langle w_1, w_1 \rangle_{\widetilde{M}} \\ &= \left\langle u, B_1^{-1} A u \right\rangle_\Lambda - 2\sigma^2 (u^T M \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D A u - u^T A \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{M} \widetilde{A}^{-1} \widetilde{R}_D A u) \\ &= \left\langle u, B_1^{-1} A u \right\rangle_\Lambda - 2\sigma^2 (u^T M \widetilde{R}_D^T \widetilde{M}^{-1} \widetilde{M} \widetilde{A}^{-1} \widetilde{R}_D A u - u^T A \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{M} \widetilde{A}^{-1} \widetilde{R}_D A u) \\ &= \left\langle u, B_1^{-1} A u \right\rangle_\Lambda - 2\sigma^2 \langle z_1 - w_1, w_1 \rangle_{\widetilde{M}} \\ &= \left\langle u, B_1^{-1} A u \right\rangle_\Lambda - 2\sigma^2 \left\langle z_1 - \widetilde{R} u, w_1 \right\rangle_{\widetilde{M}} + 2\sigma^2 \left\langle w_1 - \widetilde{R} u, w_1 \right\rangle_{\widetilde{M}}, \end{aligned}$$

where $z_1 = \widetilde{M}^{-1} \widetilde{R}_D M u$.

For $w_2 = \widetilde{A}^{-1} (\widetilde{R}_D - J_D^T \mathcal{H}^T) A u$, we have, cf. [37, Theorem 3],

$$\langle w_2, w_2 \rangle_{\widetilde{\Lambda}} = \left\langle u, B_2^{-1} A u \right\rangle_\Lambda - 2\sigma^2 \left\langle z_2 - \widetilde{R} u, w_2 \right\rangle_{\widetilde{M}} + 2\sigma^2 \left\langle w_2 - \widetilde{R} u, w_2 \right\rangle_{\widetilde{M}}.$$

Then, using Assumption 5.7 for both cases, the lemma is proven. $\square$

THEOREM 5.11. *Let Assumptions 5.3 and 5.7 hold. If $\sigma^2(1+\sigma^3)\left(1 + \Phi_i(H,h)\right) H^\gamma$ is sufficiently small, then, for $i = 1, 2$,*

$$c \langle u, u \rangle_\Lambda \leq \langle u, T_i u \rangle_\Lambda \tag{5.3}$$

$$\langle T_i u, T_i u \rangle_\Lambda \leq C_1 (1+\sigma^2)(1 + C_2 \sigma^2 H^2)(1 + \Phi_i(H,h)^2) \langle u, u \rangle_\Lambda. \tag{5.4}$$

Here $T_i = B_i^{-1} A$, $\Phi_i(H,h)$ are determined as in Lemma 5.5, c, C_1 and C_2 are positive constants independent of H and h.

Proof. We only prove the result for the preconditioned operator $T_1 = B_1^{-1} A$. The few modifications in the proof for $B_2^{-1} A$ can be found in [37, Theorem 3].

We first prove the upper bound (5.4). Given any $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1} \widetilde{R}_D A u$. We have,

$$\begin{aligned} &\left\langle B_1^{-1} A u, B_1^{-1} A u \right\rangle_\Lambda = \left\langle \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D A u, \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D A u \right\rangle_\Lambda \\ &= \left\langle \widetilde{R} \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D A u, \widetilde{R} \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D A u \right\rangle_{\widetilde{\Lambda}} = \left\langle \widetilde{R} \widetilde{R}_D^T w_1, \widetilde{R} \widetilde{R}_D^T w_1 \right\rangle_{\widetilde{\Lambda}} = \|E_{D,1} w_1\|_{\widetilde{\Lambda}}^2 \\ &\leq (1 + C\sigma^2 H^2) \Phi_1(H,h) \|w_1\|_{\widetilde{\Lambda}}^2 \leq (1 + C\sigma^2 H^2) \Phi_1(H,h)(1+\sigma^2) \|w_1\|_{\widetilde{K}}^2 \\ &\leq C_1 (1+\sigma^2)(1 + C_2 \sigma^2 H^2)(1 + \Phi_1(H,h)^2) \langle u, u \rangle_\Lambda, \end{aligned}$$

where we have used Lemmas 5.6, 5.4, and the last inequality of Assumption 5.7.

To prove the lower bound (5.3), we have, from $\widetilde{R}^T\widetilde{R}_D = I$ and by using the Cauchy-Schwarz inequality, that

$$\begin{aligned}
\langle u,u\rangle_\Lambda &= \langle u,u\rangle_A + 2\sigma^2\langle u,u\rangle_M = u^TAu + 2\sigma^2\langle u,u\rangle_M \\
&= u^T\widetilde{R}^T\widetilde{A}\widetilde{A}^{-1}\widetilde{R}_DAu + 2\sigma^2\langle u,u\rangle_M = \left\langle w_1,\widetilde{R}u\right\rangle_{\widetilde{A}} + 2\sigma^2\langle u,u\rangle_M \\
&= \left\langle w_1,\widetilde{R}u\right\rangle_{\widetilde{\Lambda}} - 2\sigma^2\left\langle w_1,\widetilde{R}u\right\rangle_{\widetilde{M}} + 2\sigma^2\langle u,u\rangle_M \\
&= \left\langle w_1,\widetilde{R}u\right\rangle_{\widetilde{\Lambda}} - 2\sigma^2\left\langle w_1-\widetilde{R}u,\widetilde{R}u\right\rangle_{\widetilde{M}} \\
&\le \langle w_1,w_1\rangle_{\widetilde{\Lambda}}^{1/2}\left\langle \widetilde{R}u,\widetilde{R}u\right\rangle_{\widetilde{\Lambda}}^{1/2} - 2\sigma^2\left\langle w_1-\widetilde{R}u,\widetilde{R}u\right\rangle_{\widetilde{M}} \\
&= \langle w_1,w_1\rangle_{\widetilde{\Lambda}}^{1/2}\langle u,u\rangle_\Lambda^{1/2} - 2\sigma^2\left\langle w_1-\widetilde{R}u,\widetilde{R}u\right\rangle_{\widetilde{M}}.
\end{aligned}$$

Then, from Assumption 5.7, we have

$$\langle u,u\rangle_\Lambda \le \langle w_1,w_1\rangle_{\widetilde{\Lambda}}^{1/2}\langle u,u\rangle_\Lambda^{1/2} + C\ \sigma^2(1+\sigma^3)H^\gamma(1+\sqrt{\Phi_1(H,h)})\langle u,u\rangle_\Lambda .$$

If $\sigma^2(1+\sigma^3)H^\gamma(1+\sqrt{\Phi_i(H,h)})$ is sufficiently small, then $\langle u,u\rangle_\Lambda \le C\langle w_1,w_1\rangle_{\widetilde{\Lambda}}$, where C is independent of H and h. Therefore, using Lemma 5.10, we have

$$\langle u,u\rangle_\Lambda \le C\left(\left\langle u,B_1^{-1}Au\right\rangle_\Lambda + \sigma^2(1+\sigma^3)H^\gamma(1+\Phi_1(H,h))\langle u,u\rangle_\Lambda\right).$$

If $\sigma^2(1+\sigma^3)H^\gamma(1+\Phi_1(H,h))$ is sufficiently small, then (5.3) is proven with c independent of H and h. ∎

Theorem 5.11 provides an estimate of the convergence rate of the BDDC algorithm for solving indefinite problems of the form (2.4). We see that the convergence rate depends on $\Phi_i(H,h)$ in Assumption 5.5, the shift σ^2, and the product σH. For a fixed σ, the upper bound in (5.4) improves with the decrease of H.

6. Verifying Assumption 5.7. In this section, we give a proof of Assumption 5.7. We first prove an error bound in Lemmas 6.1-6.4 for the solution of the partially sub-assembled finite element problem.

Given $g \in L_2(\Omega)$, we define $\varphi_g \in H_0^1(\Omega)$ and $\widetilde{\varphi}_g \in \widetilde{W}$ as the solutions of the following problems,

$$a(u,\varphi_g) = (u,g), \quad \forall u \in H_0^1(\Omega), \tag{6.1}$$

$$\widetilde{a}(w,\widetilde{\varphi}_g) = (w,g), \quad \forall w \in \widetilde{W}, \tag{6.2}$$

respectively. From Lemma 2.1, we know that $\varphi_g \in H_0^1(\Omega)\cap H^{1+\gamma}(\Omega)$, for some $\gamma \in (1/2,1]$.

LEMMA 6.1. *Let Assumption 5.3 hold. Given $g \in L_2(\Omega)$, let φ_g be the solution of the problem (6.1). Let $L_h(\varphi_g,q) = (g,q) - \widetilde{a}(\varphi_g,q)$ for any $q \in \widetilde{W} \cup \left(H_0^1(\Omega)\cap H^{1+\gamma}(\Omega)\right)$, $\gamma \in (1/2,1]$. Then $|L_h(\varphi_g,q)| \le CH^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}}|q|_{H^1(\Omega)}$.*

Proof. Given any $q \in \widetilde{W}\cup\left(H_0^1(\Omega)\cap H^{1+\gamma}(\Omega)\right)$, we have

$$L_h(\varphi_g,q) = (g,q) - \widetilde{a}(\varphi_g,q) = -\sum_{i=1}^N\int_{\Omega_i}\left(\nabla\varphi_g\nabla q - \sigma^2\varphi_g\ q\right)dx + \int_\Omega g\ qdx$$

$$= -\sum_{i=1}^{N}\left(\int_{\partial\Omega_i}\partial_\nu\varphi_g q ds + \int_{\Omega_i}(-\Delta\varphi_g - \sigma^2\varphi_g)q dx\right) + \int_\Omega g q dx$$

$$= -\sum_{i=1}^{N}\int_{\partial\Omega_i}\partial_\nu\varphi_g q ds, \tag{6.3}$$

where we use the fact that $-\Delta\phi_g - \sigma^2\phi_g = g$ holds in the weak sense, cf. (6.1).

Let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, which can be either a face or an edge. For any function $q \in \widetilde{W} \cup \left(H_0^1(\Omega) \cap H^{1+\gamma}(\Omega)\right)$, we denote its common edge or face average over Γ_{ij} by $\overline{q}_{\Gamma_{ij}}$; cf. Assumption 5.3. We have, from (6.3), that

$$\begin{aligned}L_h(\varphi_g, q) &= -\sum_{i=1}^{N}\int_{\partial\Omega_i}\partial_\nu\varphi_g\ q ds = -\sum_{i=1}^{N}\sum_{\Gamma_{ij}\subset\partial\Omega_i}\int_{\Gamma_{ij}}\partial_\nu\varphi_g(q-\overline{q}_{ij})ds \\ &= -\sum_{i=1}^{N}\sum_{\Gamma_{ij}\subset\partial\Omega_i}\int_{\Gamma_{ij}}\partial_\nu(\varphi_g - I_H\varphi_g)(q-\overline{q}_{ij})ds,\end{aligned} \tag{6.4}$$

where $I_H\varphi_g$ is the linear interpolant of φ_g using its values at the subdomain corners. It then follows from the Cauchy-Schwarz inequality that

$$|L_h(\varphi_g, q)| \le \sum_{i=1}^{N}\sum_{\Gamma_{ij}\subset\partial\Omega_i}\left(\int_{\Gamma_{ij}}|\nabla(\varphi_g - I_H\varphi_g)|^2 ds\int_{\Gamma_{ij}}|q-\overline{q}_i|^2 ds\right)^{1/2}. \tag{6.5}$$

By using a trace theorem and Lemma 3.2, the first factor on the right hand side can be estimated as follows,

$$\begin{aligned}\int_{\Gamma_{ij}}|\nabla(\varphi_g - I_H\varphi_g)|^2 ds &\le CH^\gamma\|\nabla(\varphi_g - I_H\varphi_g)\|^2_{H^\gamma(\Omega_i)} \\ &\le CH^\gamma\|\varphi_g - I_H\varphi_g\|^2_{H^{1+\gamma}(\Omega_i)} \le CH^\gamma|\varphi_g|^2_{H^{1+\gamma}(\Omega_i)}.\end{aligned} \tag{6.6}$$

By using a trace theorem and a Poincaré-Friedrichs inequality, we have an estimate for the second factor,

$$\int_{\Gamma_{ij}}|q-\overline{q}_{ij}|^2 ds \le CH\|q-\overline{q}_{ij}\|^2_{H^1(\Omega_i)} \le CH|q|^2_{H^1(\Omega_i)}. \tag{6.7}$$

Combining (6.5), (6.6), and (6.7), we have

$$\begin{aligned}|L_h(\varphi_g, q)| &\le C\sum_{i=1}^{N}H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}(\Omega_i)}|q|_{H^1(\Omega_i)} \\ &\le CH^{(1+\gamma)/2}\left(\sum_{i=1}^{N}|\varphi_g|^2_{H^{1+\gamma}(\Omega_i)}\right)^{1/2}\left(\sum_{i=1}^{N}|q|^2_{H^1(\Omega_i)}\right)^{1/2} \\ &= CH^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}(\Omega)}|q|_{H^1(\Omega)}.\end{aligned}$$

□

The following lemma is established by using Lemma 6.1.

LEMMA 6.2. *Let Assumption 5.3 hold. Given $g \in L_2(\Omega)$, let φ_g and $\widetilde{\varphi}_g$ be the solutions of the problems (6.1) and (6.2), respectively. If $\sigma(1+\sigma^2)h^\gamma$ is sufficiently small, then*

$$\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} \le C(1+\sigma^2)H^\gamma \left(|\varphi_g - \widetilde{\varphi}_g|_{H^1(\Omega)} + H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}(\Omega)} \right),$$

where C is a positive constant independent of H and h.

Proof. Given any $q \in L_2(\Omega)$, let $z_q \in H_0^1(\Omega)$ be the solution of

$$a(z_q, v) = (q, v), \quad \forall v \in H_0^1(\Omega). \tag{6.8}$$

We know, from Lemma 2.1, that $z_q \in H^{1+\gamma}(\Omega)$ for some $\gamma \in (1/2, 1]$, and $\|z_q\|_{H^{1+\gamma}} \le C(1+\sigma^2)\|q\|_{L_2}$. From a Strang lemma, cf. [10, Remark 31.1], we have

$$\begin{aligned}\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} \le \sup_{q \in L_2(\Omega)} \frac{1}{\|q\|_{L_2}} \Big\{ \inf_{\widetilde{z} \in \widetilde{W}} \big(&|a_h(\varphi_g - \widetilde{\varphi}_g, z_q - \widetilde{z})| \\ &+ |L_h(\varphi_g, z_q - \widetilde{z})| + |L_h(z_q, \varphi_g - \widetilde{\varphi}_g)| \ \big) \Big\}.\end{aligned}$$

Then, using Lemmas 5.9 and 6.1, we have

$$\begin{aligned}\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} \le C \sup_{q \in L_2(\Omega)} \frac{1}{\|q\|_{L_2}} \inf_{\widetilde{z} \in \widetilde{W}} \{ &(|\varphi_g - \widetilde{\varphi}_g|_{H^1} + \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2})(|z_q - \widetilde{z}|_{H^1} \\ +\sigma\|z_q - \widetilde{z}\|_{L_2}) + H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}}|z_q - \widetilde{z}|_{H^1} &+ H^{(1+\gamma)/2}|z_q|_{H^{1+\gamma}}|\varphi_g - \widetilde{\varphi}_g|_{H^1}\}.\end{aligned}$$

¿From Lemma 2.2, we know that $\inf_{\widetilde{z} \in \widetilde{W}} |z_q - \widetilde{z}|_{H^1(\Omega)} \le Ch^\gamma \|z_q\|_{H^{1+\gamma}}$, and $\inf_{\widetilde{z} \in \widetilde{W}} \|z_q - \widetilde{z}\|_{L_2(\Omega)} \le Ch^{1+\gamma}\|z_q\|_{H^{1+\gamma}}$. Then from $\|z_q\|_{H^{1+\gamma}} \le C(1+\sigma^2)\|q\|_{L_2}$, we have

$$\begin{aligned}\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} \le C(1+\sigma^2)H^\gamma &\left((1 + H^{(1-\gamma)/2} + \sigma h)|\varphi_g - \widetilde{\varphi}_g|_{H^1} + H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}} \right) \\ &+ C\sigma(1+\sigma^2)h^\gamma(1+\sigma h)\|\varphi_g - \widetilde{\varphi}_g\|_{L_2}.\end{aligned}$$

If $\sigma(1+\sigma^2)h^\gamma$ is sufficiently small and therefore σh is less than a certain constant, then we have $\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} \le C(1+\sigma^2)H^\gamma(|\varphi_g - \widetilde{\varphi}_g|_{H^1} + H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}})$. ∎

The proof of the following lemma is essentially an extension of the proof in [2, Chapter III, Lemma 1.2] to the indefinite case.

LEMMA 6.3. *Let Assumption 5.3 hold. Given $g \in L_2(\Omega)$, let φ_g and $\widetilde{\varphi}_g$ be the solutions of the problems (6.1) and (6.2), respectively. If σh is less than a certain constant, then*

$$|\varphi_g - \widetilde{\varphi}_g|_{H^1} \le \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} + CH^\gamma|\varphi_g|_{H^{1+\gamma}},$$

where C is a positive constant independent of H and h.

Proof. For any given $\widetilde{\psi} \in \widetilde{W}$, we have

$$\begin{aligned}|\widetilde{\varphi}_g - \widetilde{\psi}|_{H^1}^2 - \sigma^2\|\widetilde{\varphi}_g - \widetilde{\psi}\|_{L_2}^2 &= \widetilde{a}(\widetilde{\varphi}_g - \widetilde{\psi}, \widetilde{\varphi}_g - \widetilde{\psi}) \\ &= \widetilde{a}(\varphi_g - \widetilde{\psi}, \widetilde{\varphi}_g - \widetilde{\psi}) + ((g, \widetilde{\varphi}_g - \widetilde{\psi}) - \widetilde{a}(\varphi_g, \widetilde{\varphi}_g - \widetilde{\psi})).\end{aligned}$$

Dividing by $|\widetilde{\varphi}_g - \widetilde{\psi}|_{H^1} + \sigma\|\widetilde{\varphi}_g - \widetilde{\psi}\|_{L_2}$ on both sides and denoting $\widetilde{\varphi}_g - \widetilde{\psi}$ by q_h, we have, from Lemmas 5.9 and 6.1, that

$$\begin{aligned}|\widetilde{\varphi}_g - \widetilde{\psi}|_{H^1} - \sigma\|\widetilde{\varphi}_g - \widetilde{\psi}\|_{L_2} &\le (|\varphi_g - \widetilde{\psi}|_{H^1} + \sigma\|\varphi_g - \widetilde{\psi}\|_{L_2}) + \frac{\left|(g, q_h) - \widetilde{a}(\varphi_g, q_h)\right|}{|q_h|_{H^1}} \\ &\le |\varphi_g - \widetilde{\psi}|_{H^1} + \sigma\|\varphi_g - \widetilde{\psi}\|_{L_2} + CH^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}}.\end{aligned}$$

Using the triangle inequality, we have

$$\begin{aligned}|\varphi_g - \widetilde{\varphi}_g|_{H^1} &\le |\widetilde{\varphi}_g - \widetilde{\psi}|_{H^1} + |\varphi_g - \widetilde{\psi}|_{H^1} \\ &\le \sigma\|\widetilde{\varphi}_g - \widetilde{\psi}\|_{L_2} + 2|\varphi_g - \widetilde{\psi}|_{H^1} + \sigma\|\varphi_g - \widetilde{\psi}\|_{L_2} + CH^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}} \\ &\le \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} + 2|\varphi_g - \widetilde{\psi}|_{H^1} + 2\sigma\|\varphi_g - \widetilde{\psi}\|_{L_2} + CH^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|\varphi_g - \widetilde{\varphi}_g|_{H^1} &\le \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} + 2\inf_{\widetilde{\psi}\in\widetilde{W}}(|\varphi_g - \widetilde{\psi}|_{H^1} + \sigma\|\varphi_g - \widetilde{\psi}\|_{L_2}) + CH^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}} \\ &\le \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} + CH^{\gamma}|\varphi_g|_{H^{1+\gamma}},\end{aligned}$$

where in the last step, we have used the approximation property in Lemma 2.2 and that σh is less than a certain constant. ▯

The following lemma follows from Lemmas 6.2 and 6.3.

LEMMA 6.4. *Let Assumption 5.3 hold. Given $g \in L_2(\Omega)$, let φ_g and $\widetilde{\varphi}_g$ be the solutions of problems (6.1) and (6.2), respectively. If $\sigma(1+\sigma^2)H^{\gamma}$ is sufficiently small, then*

$$|\varphi_g - \widetilde{\varphi}_g|_{H^1} + \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} \le CH^{\gamma}|\varphi_g|_{H^{1+\gamma}} \le C(1+\sigma^2)H^{\gamma}\|g\|_{L_2},$$

where C is a positive constant independent of H and h.

Proof. ¿From Lemmas 6.3 and 6.2, we have

$$\begin{aligned}&|\varphi_g - \widetilde{\varphi}_g|_{H^1} + \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} \le 2\sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} + CH^{\gamma}|\varphi_g|_{H^{1+\gamma}} \\ &\le C_1\sigma(1+\sigma^2)H^{\gamma}(|\varphi_g - \widetilde{\varphi}_g|_{H^1} + H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}}) + CH^{\gamma}|\varphi_g|_{H^{1+\gamma}}.\end{aligned}$$

Therefore, if $\sigma(1+\sigma^2)H^{\gamma}$ is sufficiently small, then we have

$$|\varphi_g - \widetilde{\varphi}_g|_{H^1} + \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L_2} \le CH^{\gamma}|\varphi_g|_{H^{1+\gamma}} \le C(1+\sigma^2)H^{\gamma}\|g\|_{L_2}.$$

▯

LEMMA 6.5. *Let Assumption 5.3 hold. Given $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1}\widetilde{R}_D A u$, and $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au$. If $\sigma(1+\sigma^2)H^{\gamma}$ is sufficiently small, then*

$$\|w_i - u\|_{L_2} \le C(1+\sigma^2)H^{\gamma}(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_{\Lambda}), \quad i = 1, 2,$$

where C is a positive constant which is independent of H and h.

Proof. Given $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1}\widetilde{R}_D A u$. We have, for any $v \in \widetilde{W}$,

$$v^T\widetilde{A}w_1 = v^T\widetilde{A}\widetilde{A}^{-1}\widetilde{R}_D A u = v^T\widetilde{A}\widetilde{A}^{-1}\widetilde{R}_D\widetilde{R}^T\widetilde{A}\widetilde{R}u = \left\langle \widetilde{R}u, \widetilde{R}\widetilde{R}_D^T v\right\rangle_{\widetilde{A}}.$$

Let $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au$. We have, for any $v \in \widetilde{W}$,

$$v^T\widetilde{A}w_2 = v^T\widetilde{A}\widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au = \left\langle \widetilde{R}u, \widetilde{R}(\widetilde{R}_D^T - \mathcal{H}J_D)v\right\rangle_{\widetilde{A}}.$$

This shows that for any $u \in \widehat{W}$,

$$\widetilde{a}(w_i, v) = \widetilde{a}(\widetilde{R}u, E_{D,i}v), \quad \forall v \in \widetilde{W},\ i = 1, 2, \tag{6.9}$$

where $E_{D,1} = \widetilde{R}\widetilde{R}_D^T$ and $E_{D,2} = \widetilde{R}(\widetilde{R}_D^T - \mathcal{H}J_D)$. Therefore,

$$\widetilde{a}(w_i - \widetilde{R}u, v) = 0, \quad \forall v \in \widetilde{R}(\widehat{W}),\ i = 1, 2. \tag{6.10}$$

For any $g \in L_2(\Omega)$, let φ_g and $\widetilde{\varphi}_g$ be the solutions of problems (6.1) and (6.2), respectively. We denote by $I_h\varphi_g$ the finite element interpolation of φ_g onto the space $\widehat{W}$. From (6.1) and (6.2), we have,

$$\begin{aligned}
(w_i - u, g) = (w_i, g) - (u, g) &= \widetilde{a}(w_i, \widetilde{\varphi}_g) - a(u, \varphi_g)\\
&= \widetilde{a}(w_i, \widetilde{\varphi}_g) - a(u, I_h\varphi_g) - a(u, \varphi_g - I_h\varphi_g)\\
&= \widetilde{a}(w_i, \widetilde{\varphi}_g) - \widetilde{a}(\widetilde{R}u, \widetilde{R}I_h\varphi_g) - a(u, \varphi_g - I_h\varphi_g)\\
&= \widetilde{a}(w_i - \widetilde{R}u, \widetilde{\varphi}_g) - \widetilde{a}(\widetilde{R}u, \widetilde{R}I_h\varphi_g - \widetilde{\varphi}_g) - a(u, \varphi_g - I_h\varphi_g).
\end{aligned}$$

¿From (6.10), we know that $\widetilde{a}(w_i - \widetilde{R}u, \widetilde{R}I_h\varphi_g) = 0$. Therefore,

$$\begin{aligned}
|(w_i - u, g)| &= |\widetilde{a}(w_i - \widetilde{R}u, \widetilde{\varphi}_g - \widetilde{R}I_h\varphi_g) - \widetilde{a}(\widetilde{R}u, \widetilde{R}I_h\varphi_g - \widetilde{\varphi}_g) - a(u, \varphi_g - I_h\varphi_g)|\\
&\le C(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|\widetilde{R}u\|_{\widetilde{\Lambda}})(\|\widetilde{\varphi}_g - \widetilde{R}I_h\varphi_g\|_{\widetilde{\Lambda}} + \|\varphi_g - I_h\varphi_g\|_\Lambda)\\
&\le C(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|\widetilde{R}u\|_{\widetilde{\Lambda}})(|\widetilde{\varphi}_g - I_h\varphi_g|_{H^1} + \sigma\|\widetilde{\varphi}_g - I_h\varphi_g\|_{L_2} +\\
&\quad |\varphi_g - I_h\varphi_g|_{H^1} + \sigma\|\varphi_g - I_h\varphi_g\|_{L_2})\\
&\le C(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|\widetilde{R}u\|_{\widetilde{\Lambda}})(|\widetilde{\varphi}_g - \varphi_g|_{H^1} + \sigma\|\widetilde{\varphi}_g - \varphi_g\|_{L_2} +\\
&\quad 2|\varphi_g - I_h\varphi_g|_{H^1} + 2\sigma\|\varphi_g - I_h\varphi_g\|_{L_2})
\end{aligned}$$

where we have used Lemma 5.9 in the middle. Then, using Lemmas 6.4 and 2.2, we have that if $\sigma(1+\sigma^2)H^\gamma$ is sufficiently small, then

$$\big|(w_i - u, g)\big| \le C(1+\sigma^2)H^\gamma(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda)\|g\|_{L_2}.$$

Therefore,

$$\|w_i - u\|_{L_2} = \sup_{g \in L_2(\Omega)} \frac{\big|(w_i - u, g)\big|}{\|g\|_{L_2}} \le C(1+\sigma^2)H^\gamma(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda).$$

□

LEMMA 6.6. *Let Assumption 5.3 hold. Given $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1}\widetilde{R}_D Au$, and $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au$. If $\sigma(1+\sigma^3)H^\gamma$ is sufficiently small, then*

$$\|w_i\|_{\widetilde{K}} \le C(1 + \sqrt{\Phi_i(H,h)})\|u\|_\Lambda, \quad i = 1, 2,$$

where $\Phi_i(H,h)$ are determined as in Lemma 5.5, and C is a positive constant independent of H and h.

Proof. We have, for any $w \in \widetilde{W}$, $\|w\|^2_{\widetilde{K}} - \sigma^2\|w\|^2_{\widetilde{M}} = \widetilde{a}(w, w)$. Then, from (6.9) and Lemma 5.9, we have, for $i = 1, 2$,

$$\|w_i\|^2_{\widetilde{K}} - \sigma^2\|w_i\|^2_{\widetilde{M}} = \widetilde{a}(w_i, w_i) = \widetilde{a}(\widetilde{R}u, E_{D,i}w_i) \le \|u\|_\Lambda \|E_{D,i}w_i\|_{\widetilde{\Lambda}}.$$

Using Lemma 5.6, we have that if σH is less than a certain constant, then

$$\|w_i\|^2_{\widetilde{K}} - \sigma^2\|w_i\|^2_{\widetilde{M}} \le C\sqrt{\Phi_i(H,h)}\|u\|_\Lambda\|w_i\|_{\widetilde{\Lambda}}, \quad i = 1, 2.$$

Dividing by $\|w_i\|_{\widetilde{K}} + \sigma\|w_i\|_{\widetilde{M}}$ on both sides and using Lemma 5.9, we have,

$$\|w_i\|_{\widetilde{K}} - \sigma\|w_i\|_{\widetilde{M}} \leq C\sqrt{\Phi_i(H,h)}\|u\|_{\Lambda}.$$

Using this inequality and the triangle inequality, in particular that $-\|w_i - \widetilde{R}u\|_{\widetilde{M}} \leq -\|w_i\|_{\widetilde{M}} + \|\widetilde{R}u\|_{\widetilde{M}}$, we have

$$\begin{aligned}\|w_i - \widetilde{R}u\|_{\widetilde{K}} - \sigma\|w_i - \widetilde{R}u\|_{\widetilde{M}} &\leq \|w_i\|_{\widetilde{K}} + \|\widetilde{R}u\|_{\widetilde{K}} - \sigma\|w_i\|_{\widetilde{M}} + \sigma\|\widetilde{R}u\|_{\widetilde{M}} \\ &\leq C\sqrt{\Phi_i(H,h)}\|u\|_{\Lambda},\end{aligned}$$

where we have also used Lemma 5.9 in the last step. From Lemmas 6.5 and 5.4, we know that if $\sigma(1+\sigma^2)H^\gamma$ is sufficiently small, then

$$\begin{aligned}\|w_i - \widetilde{R}u\|_{\widetilde{M}} &\leq C(1+\sigma^2)H^\gamma(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_{\Lambda}) \\ &\leq C(1+\sigma^2)H^\gamma\left((1+\sigma)\|w_i - \widetilde{R}u\|_{\widetilde{K}} + \|u\|_{\Lambda}\right).\end{aligned}$$

Therefore,

$$\|w_i - \widetilde{R}u\|_{\widetilde{K}} \leq C\sigma(1+\sigma^2)H^\gamma\left((1+\sigma)\|w_i - \widetilde{R}u\|_{\widetilde{K}} + \|u\|_{\Lambda}\right) + C\sqrt{\Phi_i(H,h)}\|u\|_{\Lambda}.$$

If $\sigma(1+\sigma^3)H^\gamma$ is small enough, then we have

$$\|w_i - \widetilde{R}u\|_{\widetilde{K}} \leq C(1+\sqrt{\Phi_i(H,h)})\|u\|_{\Lambda}.$$

□

In order to confirm Assumption 5.7, we also need the following lemma.

LEMMA 6.7. *Given $u \in \widehat{W}$, let $z_1 = \widetilde{M}^{-1}\widetilde{R}_D Mu$ and $z_2 = \widetilde{M}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Mu$. Then,*

$$\|z_i - u\|_{L_2} \leq CH\sqrt{\Phi_i(H,h)}\ |u|_{H^1}, \quad i = 1,2,$$

where $\Phi_i(H,h)$ are determined as in Lemma 5.5, and C is a positive constant independent of H and h.

Proof. We only give the proof for z_2 in the following. Essentially the same arguments applies to z_1. We have

$$\begin{aligned}\|z_2 - u\|_{L_2} &= \|\widetilde{M}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)\widetilde{R}^T\widetilde{M}\widetilde{R}u - \widetilde{R}u\|_{L_2} \\ &= \|\widetilde{M}^{-1}\left(E_{D,2}^T - I\right)\widetilde{M}\widetilde{R}u\|_{L_2} = \|\widetilde{M}^{-1}P_{D,2}^T\widetilde{M}\widetilde{R}u\|_{L_2} \\ &\leq Ch^{-d}\|P_{D,2}^T\widetilde{M}\widetilde{R}u\|_{L_2},\end{aligned}$$

where we have used Lemma 5.8 in the last step, and that $P_{D,2}^T = I - E_{D,2}^T = I - (\widetilde{R}_D - J_D^T\mathcal{H}^T)\widetilde{R}^T$. Since $P_{D,2}^T\widetilde{M}\widetilde{R}u$ has a zero average over each subdomain interface edge, then by using the Poincaré-Friedrichs inequality, and a result similar to Lemma 5.5, cf. [33, 37, 45, Lemma 4.26], we have

$$\|z_2 - u\|_{L_2} \leq C\frac{H}{h^d}|P_{D,2}^T\widetilde{M}\widetilde{R}u|_{H^1} \leq C\frac{H}{h^d}\sqrt{\Phi_2(H,h)}\ |\widetilde{M}\widetilde{R}u|_{H^1}.$$

Then using Lemma 5.8 again, we have $\|z_2 - u\|_{L_2} \leq CH\sqrt{\Phi_2(H,h)}\ |\widetilde{R}u|_{H^1}$. □

Using Lemmas 6.5, 6.6, and 6.7, we can establish Assumption 5.7.

LEMMA 6.8. *Let Assumption 5.3 hold. Then Assumption 5.7 also holds.*

Proof. Lemma 6.6 proves the last inequality in Assumption 5.7.

To prove the first inequality in Assumption 5.7, we have, by using Lemmas 6.5 and 6.6, that if $\sigma(1+\sigma^3)H^\gamma$ is sufficiently small, then

$$\begin{aligned}\left|\left\langle w_i - \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{M}}\right| &\leq \|w_i - \widetilde{R}u\|_{\widetilde{M}}\|u\|_M \leq C(1+\sigma^2)H^\gamma(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda)\|u\|_M \\ &\leq C(1+\sigma^3)H^\gamma(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda^2,\end{aligned}$$

which proves the first inequality in Assumption 5.7. Similarly, to prove the second inequality in Assumption 5.7, we have, from Lemmas 6.5 and 6.6, that

$$\begin{aligned}\left|\left\langle w_i - \widetilde{R}u, w_i\right\rangle_{\widetilde{M}}\right| &\leq C(1+\sigma^2)H^\gamma(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda)\|w_i\|_{\widetilde{M}} \\ &\leq C(1+\sigma^3)H^\gamma\left(1+\Phi_i(H,h)\right)\|u\|_\Lambda^2.\end{aligned}$$

To prove the third inequality, we have, from Lemmas 6.7, 6.6, and 5.4, that

$$\left|\left\langle z_i - \widetilde{R}u, w_i\right\rangle_{\widetilde{M}}\right| \leq C\|z_i - \widetilde{R}u\|_{\widetilde{M}}\|w_i\|_{\widetilde{M}} \leq CH\left(1+\Phi_i(H,h)\right)\|u\|_\Lambda^2.$$

□

7. Numerical experiments. FETI-DPH methods have been proven successful and parallel scalable for solving a large class of indefinite problems of the form (2.4). Applications of FETI-DPH methods include structural dynamics problems, acoustic problems, etc.; cf. [18, 16, 19].

Here we use the solution of the problem (2.1) to demonstrate the algorithmic scalability of the BDDC algorithms discussed in this paper, and also demonstrate their spectral equivalence with the FETI-DPH methods. The problem (2.1) is solved on a 2π by 2π square domain with Dirichlet boundary conditions $u = 1$ on the four sides of the square. Q1 finite elements are used and the original square domain is decomposed uniformly into square subdomains. In the GMRES iteration, the $\langle\cdot,\cdot\rangle_{K+\sigma^2 M}$ inner product is used; using L_2 inner product gives the same convergence rates. The iteration is stopped when the residual is reduced by 10^{-6}. To have an idea how the shift σ^2 in (2.4) affects the eigenvalues of the matrix $K-\sigma^2 M$, we give the number of negative eigenvalues of $K-\sigma^2 M$ in Table 7.1, for different meshes with 1089, 10201, and 20449 degrees of freedom, respectively, and for different shifts $\sigma^2 = 100$, $\sigma^2 = 200$, and $\sigma^2 = 400$.

TABLE 7.1
Number of negative eigenvalues of $K-\sigma^2 M$ for different meshes and different σ^2.

# of degrees of freedom	$\sigma^2 = 100$	$\sigma^2 = 200$	$\sigma^2 = 400$
1089	243	445	843
10201	290	575	1109
20449	290	585	1161

In our experiments, we test three different choices of the coarse level primal space in our BDDC algorithm. In our first test, the coarse level primal variables are only those at the subdomain corners. No plane wave continuity constraints are enforced

TABLE 7.2
Iteration counts for $B_2^{-1}A$ for $H/h = 8$ and changing number of subdomains.

σ^2	# subdomains	Iteration Count		
		0-pwa	1-pwa	2-pwa
	16×16	183	37	14
100	24×24	205	20	7
	32×32	> 300	13	6
	16×16	> 300	143	112
200	24×24	> 300	85	39
	32×32	> 300	47	28
	16×16	> 300	> 300	236
400	24×24	> 300	> 300	75
	32×32	> 300	192	49

TABLE 7.3
Iteration counts for $B_2^{-1}A$ for 24×24 subdomains and changing H/h.

σ^2	H/h	Iteration Count		
		0-pwa	1-pwa	2-pwa
	8	205	20	7
100	12	188	25	8
	16	182	27	8
	8	> 300	85	39
200	12	> 300	108	60
	16	> 300	114	68
	8	> 300	> 300	75
400	12	> 300	> 300	108
	16	> 300	> 300	111

across the subdomain edges; this choice of the coarse level primal space does not satisfy Assumption 5.3. In our second test, in addition to the subdomain corner variables, we also include one edge average degree of freedom for each subdomain edge, as required in Assumption 5.3, in the coarse level primal variable space. This edge average degree of freedom corresponds to the vector determined by the cosine plane wave in (4.1) with the angle θ chosen perpendicular to the edge. In our last test, we further add to the coarse level primal space another plane wave continuity constraint on each edge corresponding to the cosine plane wave in (4.1) with the angle θ chosen tangential to the edge. In the following tables, we represent these three different choices of the coarse level primal space by 0-pwa, 1-pwa, and 2-pwa, respectively.

Tables 7.2 and 7.3 show the GMRES iteration counts for the preconditioned operator $B_2^{-1}A$, corresponding to different number of subdomains, different subdomain problem sizes, and the three different choices of the coarse level primal space. With only subdomain corner variables in the coarse level primal space, the convergence cannot be achieved within 300 iterations in most cases. With the inclusion of the edge plane wave augmentations in the coarse level primal space, we see from Table 7.2 that

TABLE 7.4
Iteration counts for BDDC operators $B_1^{-1}A$, $B_2^{-1}A$, and $B_3^{-1}A$, and for FETI-DPH with lumped preconditioner $B_{D,\Gamma}\widetilde{A}_{\Gamma\Gamma}B_{D,\Gamma}^T$, with Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_\Gamma^{\mathcal{H}}B_{D,\Gamma}^T$, and with Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_\Gamma B_{D,\Gamma}^T$, for the case $\sigma^2 = 200$ and 2-pwa.

# Subs	H/h	$B_1^{-1}A$	DPH ($\widetilde{A}_{\Gamma\Gamma}$)	$B_2^{-1}A$	DPH ($\widetilde{S}_\Gamma^{\mathcal{H}}$)	$B_3^{-1}A$	DPH ($\widetilde{S}_\Gamma$)
16×16		114	97	112	108	115	106
24×24	8	40	40	39	39	39	39
32×32		29	30	28	28	28	28
	8	40	40	39	39	39	39
24×24	12	55	54	60	57	58	56
	16	70	70	68	68	67	66

the iteration counts decrease with the increase of the number of subdomains for a fixed subdomain problem size. We see from Table 7.3 that when the number of subdomains is fixed and H/h increases, the iteration counts increase slowly, seemingly in a logarithmic pattern of H/h. Tables 7.2 and 7.3 also show that the convergence becomes slower with the increase of the shift σ^2 and that the convergence rate is improved by including more plane wave continuity constraints in the coarse level primal subspace.

In Table 7.4, we compare the GMRES iteration counts of the BDDC operators $B_1^{-1}A$, $B_2^{-1}A$, and $B_3^{-1}A$ with those of the FETI-DPH operators with the lumped preconditioner $B_{D,\Gamma}\widetilde{A}_{\Gamma\Gamma}B_{D,\Gamma}^T$, with the Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_\Gamma^{\mathcal{H}}B_{D,\Gamma}^T$, and with the Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_\Gamma B_{D,\Gamma}^T$, respectively. We see that the corresponding BDDC and FETI-DPH algorithms have similar convergence rates. We also see that using either subdomain discrete harmonic extension or the extension based on the shifted operator in the BDDC and FETI-DPH algorithms gives almost the same convergence rates.

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