Quantum Entanglement of Baby Universes*

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December 2006

*This work was supported in part by the Director, Office of Science, Office of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

CALT-68-2622
NSF-KITP-06-118

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We study quantum entanglements of baby universes which appear in non-perturbative corrections to the OSV formula for the entropy of extremal black holes in Type IIA string theory compactified on the local Calabi-Yau manifold defined as a rank 2 vector bundle over an arbitrary genus G Riemann surface. This generalizes the result for $G = 1$ in hep-th/0504221. Non-perturbative terms can be organized into a sum over contributions from baby universes, and the total wave-function is their coherent superposition in the third quantized Hilbert space. We find that half of the universes preserve one set of supercharges while the other half preserve a different set, making the total universe stable but non-BPS. The parent universe generates baby universes by brane/anti-brane pair creation, and baby universes are correlated by conservation of non-normalizable D-brane charges under the process. There are no other source of entanglement of baby universes, and all possible states are superposed with the equal weight.

December, 2006

1. Introduction

What distinguishes string theory from any other approaches to quantum gravity is, among others, the fact that one can estimate non-trivial quantum gravity effects in controlled approximations. The OSV conjecture [1], for example, allows one to evaluate quantum corrections to the Bekenstein-Hawking area-entropy relation for BPS black holes in four dimensions to all orders of the expansion in powers of spacetime curvatures. The conjecture identifies each term in the expansion to the topological string partition function of a given genus, which is manifestly finite and can be computed explicitly as a function of geometry of the internal Calabi-Yau manifold. The conjecture has been tested in [2,3,4,5], and general proofs of the conjecture have been presented recently by several groups [6,7,8,9].

In some cases, one can also evaluate the entropy exactly by mapping its computation to solvable counting problems in the dual gauge theories. The large N expansion of the gauge theory results can then be used to test the OSV conjecture to all orders in the string perturbation expansion. Even better, by identifying the difference of the OSV formula and the exact gauge theory results, one can learn about non-perturbative quantum gravity effects, *e.g.* how to sum over spacetime topologies. The purpose of this paper is to apply this idea to specific examples, where computation can be done explicitly. We find that only quantum entanglement among baby universes is the one required by charge conservation, and otherwise all possible states are coherently superposed with the equal weight for a given number of baby universes.

The main ingredients of the OSV formula are topological string partition function Z^{top} of the Calabi-Yau manifold $\mathcal{M}$ and the partition function Z_{BH} of black holes obtained by wrapping D-branes on cycles of $\mathcal{M}$. The topological string partition function Z^{top} is expressed perturbatively as

$$Z^{top}(t) = \exp\left[\sum_{g=0}^{\infty} F_g(t)\; g_s^{2g-2}\right], \tag{1.1}$$

where F_g is the g-loop vacuum amplitude, and t's are the (flat) coordinates of the Calabi-Yau moduli space of $\mathcal{M}$; they are coordinates on the complexified Kähler moduli space for the A-model (in IIA) and the complex structure moduli space for the B-model (in IIB). g_s is the topological string coupling constant. The conjecture relates Z^{top} to the Laplace transform of the Witten index $\Omega(p, q)$ for supersymmetric ground states of the extremal black hole with magnetic and electric charges (p, q) realized by wrapping D branes on cycles in $\mathcal{M}$. More precisely, consider the black hole partition function

$$Z_{BH}(p, \phi) = \sum_q \Omega(p, q) e^{-q\phi}$$

where we fix the magnetic charges and sum over the electric ones. The conjecture states

$$Z_{BH}(p, \phi) = |Z^{top}(t, g_s)|^2. \tag{1.2}$$

where the parameters of the two sides are related by attractor mechanism as:

$$g_s = \frac{4\pi i}{p^0 + i\phi^0/\pi}, \qquad t^i = \frac{1}{2}\left(p^i + i\phi^i/\pi\right) g_s. \tag{1.3}$$

In the above p^0 and p^i are the magnetic charges and i runs over the number of moduli of the Calabi-Yau. In IIA, for example, p^0 is the number of D6 branes, and p^i are the D4 brane charges. The conjecture is supposed to hold to all orders in the string loop expansion. However, the relation (1.2) may be corrected by effects that are non-perturbative in the string coupling g_s.

The first concrete example of this was given in [2], where type IIA string theory on the Calabi-Yau manifold $\mathcal{M}$ which is a rank 2 vector bundle (in fact a sum of two line bundles) over an elliptic curve Σ was studied. With N D4 branes and no D6 branes on $\mathcal{M}$, the black hole partition function Z_{BH} turns out to be equal to the partition function of the 2d Yang-Mills theory on the elliptic curve, which can be computed exactly. The g_s expansion of Z_{BH} is identified with the large N expansion in the 2d Yang-Mills theory. It was shown that, to all order in the $1/N$ expansion, the 2d Yang-Mills partition function factorizes as

$$Z_{BH}(N, \phi) = \sum_{l \in \mathbf{Z}} Z^{top}(t + l g_s) \bar{Z}^{top}(\bar{t} - l g_s), \tag{1.4}$$

where $t = \frac{1}{2}(N + i\phi/\pi) g_s$. The sum over l is interpreted as a RR flux through the elliptic curve on the base. In particular, one can identify the topological A-model string theory on $\mathcal{M}$ as the large N dual of the 2d Yang-Mills theory studied earlier in [10].

In this case, non-perturbative corrections to (1.4) were evaluated in [11]. Taking into account terms non-perturbative in g_s, one finds

$$\begin{aligned} Z_{BH}(N, \phi) = &\sum_{k=1}^{\infty} (-1)^{k-1} C_k \\ &\sum_{l_1, \ldots l_k \in \mathbf{Z}} Z^{top}(t_1 + l_1 g_s) \bar{Z}^{top}(\bar{t}_1 - l_1 g_s) \cdots Z^{top}(t_k + l_k g_s) \bar{Z}^{top}(\bar{t}_k - l_k g_s), \end{aligned} \tag{1.5}$$

where C_k is the Catalan number given by

$$C_k = \frac{(2k)!}{k!(k+1)!} \tag{1.6}$$

The $k = 1$ term corresponds to the OSV formula (1.4), and the $k > 1$ terms are interpreted as having k disjoint baby universes. The Catalan number simply counts the number of ways the baby universe can be produced. In the above,

$$t_i = \frac{1}{2}\,(N_i + i\phi/\pi)g_s$$

and for each k baby universe configuration, the total D4 brane charge is N:

$$\sum_{i=1}^{k} N_k = N.$$

The topological string partition function Z^{top} has an interpretation as a wave-function in the Hilbert space $\mathcal{H}$ obtained by quantizing $H^3(\mathcal{M})$ (in the mirror B-model language), as was explained in [12,13] and studied more recently in [14,15,16]. In particular, we can invert the OSV formula (1.2) to write,

$$\Omega(p,q) = \int d\phi\; e^{q\phi}\; Z^{top}(p + i\phi/\pi)\, \bar{Z}^{top}(p - i\phi/\pi), \tag{1.7}$$

and regard the right-hand side as computing the norm-squared of the wave-function [14,17,18],

$$\psi_{p,q}(\phi) = e^{q\phi/2}\; Z^{top}(p + i\phi/\pi).$$

$\psi_{p,q}(\phi)$ was interpreted in [18] as a Hartle-Hawking type wave function for the universe defined on a $\mathcal{M} \times S^2 \times S^1$ spatial slice. Generalizing this to the baby universe expansion, (1.5) was interpreted in [11] as a norm-squared of a wave-function in the third quantized Hilbert space $\oplus_{k=1}^{\infty} \mathcal{H}^{\otimes k}$ given as a coherent superposition of baby universes. The D-brane charges are distributed among the baby universes in such a way that the total charges are conserved. Otherwise, no entanglement among the universes is found in this case, and all the states allowed by the charge conservation are summed with the same weight within a sector of a given number of universes.

The purpose of this paper is to generalize the result of [11] for a case when $\mathcal{M}$ is the sum of two line bundles over a genus G Riemann surface Σ_G for any G. We mostly focus on $G > 1$ since the $G = 0$ is similar and we do not expect any new novelty. In this case, the black hole partition function Z_{BH} is related to the partition function of a certain deformation of the 2d Yang-Mills theory on Σ_G. The precise deformation will be specified later. In [3], the partition function Z_{BH} is evaluated explicitly, and it was shown that the large N expansion of Z_{BH} reproduces the OSV formula with one important subtlety. The large N factorization of Z_{BH} produces topological string partition function on $\mathcal{M}$, but with insertions of additional "ghost" D-branes. The ghost branes generate correlations between $\bar{Z}^{top}$ and Z^{top}, *i.e.* the bra and ket vectors. Subsequently, it was shown in [19] that the insertions of ghost D branes have a dual *closed* string interpretation. Namely, because of the non-compactness of the Calabi-Yau manifold $\mathcal{M}$, there are infinitely many Kähler moduli that are not supported by compact 2-cycles. We have to treat them at the same footing as the ordinary Kähler moduli, as the wave function Z^{top} naturally depends on them. Applying the OSV prescription to these moduli, with the corresponding electric and magnetic charges set to zero, generates the correlation between $\bar{Z}^{top}$ and Z^{top}.

In this paper, we will compute non-perturbative corrections to the OSV formula for this geometry. As in the case of $G = 1$ studied in [11], the black hole partition function Z_{BH} is expressed as a sum over Young diagrams with a fixed number of rows, and this structure naturally leads to the holomorphic factorization (1.2) in the limit when the number of rows is infinite. Non-perturbative corrections arise due to the fact that the number of rows are in fact finite. We find that the non-perturbative terms can be organized into a sum over baby universes as in (1.5) with new features. We find that half of baby universes preserve one set of supercharges and the other half preserve a different set. As a consequence, the total universe is not BPS, but it is still stable since different baby universes are spatially disjoint. Another feature is presence of a new type of ghost D-branes. In addition to insertions of D-branes that correlate the bra and ket vectors, as we mentioned at the end of the previous paragraph, we find yet another set of D-branes that correlate ket vectors among themselves (and another set of D-branes for bra vectors). It turns out that these correlations also have their origin in a conserved charge: namely the charge of the non-compact D2 branes wrapping the fibers over the Riemann surface. The charges of these branes in all the universes have to add up to zero, just as in the parent universe, and this induces correlations. In the $G = 1$ case studied in [11], these branes and correlations associated to them were absent, as they would violate the toroidal symmetry of the base Riemann surface. Moreover we find no other source of correlations between different baby universes, for any G.

This paper is organized as follows. In section 2, we will summarize our result. In section 3, we will present expressions for Z_{BH} and Z^{top} for the rank 2 bundle over Σ_G computed in [3] and [20] respectively, and we will review the large N factorization of Z_{BH} discussed in [3]. In section 4, we will identify non-perturbative corrections to $|Z^{top}|^2$ by taking into account the finite size effects of Young diagrams. Various technical computations are relegated to the appendices.

2. The main result

Consider type IIA superstring theory compactified on a Calabi-Yau manifold $\mathcal{M}$ that contains a compact 2-cycle that can shrink, a Riemann surface Σ of genus G. The local region near Σ is a non-compact Calabi-Yau manifold which is the total space of the sum of two line bundles over Σ

$$\mathcal{L}_{-p} \oplus \mathcal{L}_{p+2G-2} \to \Sigma,$$

for some *positive* integer p. Now wrap N D4-branes on the divisor

$$\mathcal{D} = \mathcal{L}_{-p} \to \Sigma.$$

The D4 branes can form BPS bound states with Q_2 D2 branes wrapping Σ and Q_0 D0 branes. In [3], the Witten index $\Omega(N, Q_2, Q_0) = \mathrm{tr}'(-1)^F$ of the BPS bound states of these objects is computed by the supersymmetric path integral of the D4 brane theory on $\mathcal{D}$. (The tr' in the index refers to the fact that the zero modes corresponding to center of mass motion of the D4 branes are removed in the trace.) The black-hole partition function Z_{BH} is then defined by

$$Z_{BH} = \sum_{Q_2, Q_0} \Omega(N, Q_2, Q_0) e^{-Q_0\phi^0 - Q_2\phi^1}.$$

The large N expansion of Z_{BH} is evaluated with the following result:

$$\begin{aligned} Z_{BH}(N, \phi^0, \phi^1) = \sum_{l=-\infty}^{\infty} \int dU_1 \cdots dU_{|2G-2|} Z^{top}(t + lpg_s; U_1, \cdots, U_{|2G-2|}) \\ \times \bar{Z}^{top}(\bar{t} - lpg_s; U_1, \cdots, U_{|2G-2|}), \end{aligned} \tag{2.1}$$

where Z^{top} in the right-hand side is the topological string partition function with insertions of indefinite number of "ghost" D- branes along $|2G-2|$ Lagrangian submanifolds of $\mathcal{M}$, each of which intersects with $\mathcal{D}$ around S^1 in the fiber of $\mathcal{L}_{p+2G-2}$ over one of $|2G-2|$ points on Σ, and $U_1, \cdots, U_{|2G-2|}$ are holonomies taking value in $U(\infty)$. The sum over l is interpreted as a sum over 2-form flux through Σ. The topological string coupling g_s and the Kähler moduli t associated to the size of Σ are given by

$$g_s = \frac{4\pi i}{\phi^0}, \quad t = \frac{1}{2}(p+2G-2)Ng_s + i\frac{\phi^1}{2\pi}g_s.$$

Note that the effective D4 brane charge is larger by a factor of $p+2G-2$. The magnetic charge of a D4 brane on a divisor $\mathcal{D}$ is set by the intersection number of $\mathcal{D}$ with Σ, the two-cycle wrapped by the D2-branes. In the present context, the intersection number is

$$\#(\mathcal{D}\cap\Sigma) = p + 2G - 2,$$

and hence the above.

How is the formula (2.1), with the integral over U's, compatible with the OSV conjecture? The Calabi-Yau manifold $\mathcal{M}$ is non-compact, and there are infinitely many Kähler moduli which are not supported by compact 2-cycles. According to [19], eigenvalues of $U_1, ..., U_{|2G-2|} \in U(\infty)$, are interpreted as exponentials of these Kähler moduli. The imaginary parts of the Kähler moduli are chemical potentials for the electric charges and the integral over U's sets all the electric charges to be zero. The only nonzero magnetic charges are those induced by the D4 branes and the attractor mechanism shifts the real parts on the Kähler moduli (rescaling U's). The magnetic charge of the D4 branes under a global $U(1)$ corresponding to a non-normalizable modulus is given by their intersection number with the corresponding non-compact 2-cycle, just as in the compact 2-cycle case. This turns out to be non-zero here, giving the Kähler moduli a non-vanishing real part. The integral over the holonomies should thus be thought of as an integral over these non-normalizable Kähler moduli. It ensures that the gravity computation in the right-hand side of (2.1) matches with the gauge theory computation of Z_{BH} where we only count bound states of D4 branes on $\mathcal{D}$ with D2 branes wrapping Σ and D0 branes on Σ.

In this paper, we will follow the prescription of [11] to identify corrections to (2.1) that are non-perturbative in the topological string coupling g_s. As we will show in subsection 4.2, the first correction to (2.1) comes from configurations where the parent universe splits into two by creating a baby universe. The parent universe still ends up carrying charges corresponding to N D4 branes on $\mathcal{D}$ which it had originally. But now, it also carries charges of $-k$ branes on

$$\mathcal{D}' = \mathcal{L}_{p+2G-2} \to \Sigma,$$

with the baby universe carrying of the other k units of this charge. Let us denote by (m, m') the charge of a universe corresponding to m D4 branes on $\mathcal{D}$ and m' D4 branes on $\mathcal{D}'$. Then, this process corresponds to

$$(N, 0) \to (N, -k) \oplus (0, k).$$

Here, k is positive, as is needed[1] in order for the first universe to correspond to a mutually BPS configuration of D4 branes. This is because the effective D4 brane charge for branes wrapping $\mathcal{D}'$ is negative:

$$\#\,(\mathcal{D}' \cap \Sigma) = -p.$$

Note that, even though this looks like we have pair created k branes and anti-branes on $\mathcal{D}'$, this is true only at the level of the charges. In particular, all these universes are dual to a single theory of N D4 branes on $\mathcal{D}$. The next leading correction comes from configurations with 3 universes, where the second universes splits of a baby

$$(N, 0) \to (N, -k) \oplus (0, k) \to (N, -k) \oplus (-m, k) \oplus (m, 0)$$

while creating m units of D4 brane and anti-D4 brane charge on $\mathcal{D}$. This will be demonstrated in subsection 4.4. The configuration of branes in the second universe is now supersymmetric provided $m \geq 0$. This also gives a natural ordering of the universes so that odd (even) universes carry positive (negative) D4 brane charges. This in particular implies that the odd and even universes preserve different sets of supercharges. If odd universes are BPS, even universes are anti-BPS. The total universe is still stable since baby universes are disjoint. One consequence of this is that a ket vector for an even universe is $\bar{Z}^{top}$ while a ket vector for an odd universe is Z^{top}. This follows from the derivation of the OSV conjecture in [6] applied to the two types of baby universes. It also follows from the fact that the CPT conjugation is an anti-unitary operation, and it exchanges bra and ket vectors.

In addition to this general structure, there are two more important features. As we mentioned in the above, for a single universe studied in [3], (2.1) is imposing the constraint of vanishing electric charges for the non-normalizable Kähler moduli, and this correlates

[1] In particular, note that the charges of the branes need to be of the same sign for the branes to be mutually BPS. Only in that case does the electrostatic repulsion cancel the gravitational attraction of the particles, so the net force condition is satisfied.

the bra and ket vectors (Z^{top} and $\bar{Z}^{top}$). With more than one universe, this structure is duplicated, setting the net electric charges to zero. This induces correlations between the ket and the bra vectors of the different universes.

Furthermore, for configurations with multiple universes, another set of ghost D-branes appears that introduce correlations between ket vectors (and another copy of this for the bra vectors). In the following sections, we will show that these are also a consequence of charge conservation – this time for charges of the non-compact D2 branes that can form bound states with the D4 branes. These are also a consequence of the non-compactness of the manifold, because they correspond to having different boundary conditions at infinity of the Calabi-Yau $\mathcal{M}$.

3. q-deformed Yang-Mills, large N factorization, and topological string

In this section we will review the results of [3] computing the large N limit of partition function Z_{BH} of D-branes on $\mathcal{M}$, and relation of this to topological strings on $\mathcal{M}$[2].

3.1. Superstring and q-deformed Yang-Mills

Consider type IIA superstring theory compactified on a Calabi-Yau manifold $\mathcal{M}$ as in the previous section, where we wrap N D4-branes on the divisor $\mathcal{D} = \mathcal{L}_{-p} \to \Sigma$. The D4 branes can form BPS bound states with Q_2 D2 branes wrapping Σ and Q_0 D0 branes. The indexed degeneracies $\Omega(N, Q_2, Q_0)$ of the BPS bound states of these objects can be computed by the supersymmetric path integral of the D4 brane theory on $\mathcal{D}$, in a topological sector with

$$Q_0 = \frac{1}{8\pi^2}\int_{\mathcal{D}} \mathrm{Tr} F \wedge F, \qquad Q_2 = \frac{1}{2\pi}\int \mathrm{Tr} F \wedge k.$$

where k is the normalized Kähler form such that $\int_\Sigma k = 1$. Computing the path integral without fixing the topological sector thus automatically sums over all the D2 and the D0 brane charges. To keep track of them, we will add to the action the terms

$$\frac{1}{2g_s}\int \mathrm{tr} F \wedge F + \frac{\theta}{g_s}\int \mathrm{tr} F \wedge k.$$

Then, $\frac{4\pi^2}{g_s}$ and $\frac{2\pi\theta}{g_s}$ are the chemical potentials for D0 and the D2 brane charges. We called these ϕ^0, ϕ^1 in section 1.

[2] For related recent work see [21].

Since $\mathcal{D}$ is curved the supersymmetric theory on it automatically topologically twisted, and it corresponds to the Vafa-Witten twist of the $\mathcal{N}=4$ theory in $\mathcal{D}$. As argued in [2,3], since the manifold is non-compact and has $U(1)$ isometry corresponding to rotating the fiber, the twisted $\mathcal{N}=4$ theory is in turn localizes to a gauge theory on the Riemann surface Σ. This theory is a certain deformation of the ordinary bosonic 2d $U(N)$ Yang-Mills theory, a "quantum" Yang-Mills theory (qYM) on Σ. This is described in detail in [3] and we will only quote the results here. In particular, the difference does not spoil the solvability of the two-dimensional bosonic Yang-Mills: the sewing and gluing prescription of the amplitudes still applies. The qYM partition function on any Riemann surface can be obtained by gluing the cap $\mathcal{C}$ and the pant $\mathcal{P}$ amplitudes

$$Z(\mathcal{C})_{\mathcal{R}} = S_{0\mathcal{R}}(g_s, N),$$

$$Z(\mathcal{P})_{\mathcal{R}} = 1/S_{0\mathcal{R}}(g_s, N)$$

where $U(N)$ representations $\mathcal{R}$ correspond to the states in the Hilbert space of the 2d theory, and

$$S_{0\mathcal{R}} = \prod_{1\leq i<j\leq N} [\mathcal{R}_i - \mathcal{R}_j + j - i].$$

Here $[n]$ denotes the quantum number n, $[n] = q^{\frac{n}{2}} - q^{-\frac{n}{2}}$ with $q = e^{-g_s}$. Note that $S_{0\mathcal{Q}}$ is similar to a $U(N)$ WZW S-matrix element, but with non-integer level.

The full partition function of the modified Yang-Mills on the Riemann surface Σ of genus G is

$$Z^{qYM}(\Sigma) = z^{YM} \sum_{\mathcal{R}} (S_{0\mathcal{R}})^{2G-2}\, q^{\frac{p}{2}C_2(\mathcal{R})} e^{i\theta C_1(\mathcal{R})}, \tag{3.1}$$

where

$$C_2(\mathcal{R}) = \sum_{i=1}^{N} \mathcal{R}_i(\mathcal{R}_i - 2i + N + 1), \quad C_1(\mathcal{R}) = \sum_{i=1}^{N} \mathcal{R}_i,$$

are the $U(N)$ casimirs of the representation $\mathcal{R}$. The precise normalization factor z^{YM} will be discussed later. Note that

$$S_{0\mathcal{R}}/S_{00} = \dim_q(\mathcal{R})$$

is a quantum deformation of the dimension of a $U(N)$ representation $\mathcal{R}$. The theory here differs from the ordinary two-dimensional Yang-Mills in that the ordinary dimension of representation is replaced by its quantum deformation (for a review of ordinary 2d YM, see for example [22]). Because of this the theory at hand was termed the q-deformed

Yang-Mills theory in [3]. When the Riemann surface is a torus, as in [2,11] the difference goes away.

In the next subsection we will derive the large N limit of quantum YM amplitudes from their description in terms of free fermions.

3.2. Fermions and qYM Amplitudes

The qYM partition function on any Riemann surface can be obtained by gluing the cap $\mathcal{C}$ and the pant $\mathcal{P}$ amplitudes

$$Z(\mathcal{C})_{\mathcal{R}} = S_{0\mathcal{R}}(g_s, N),$$

$$Z(\mathcal{P})_{\mathcal{R}} = 1/S_{0\mathcal{R}}(g_s, N)$$

Knowing the large N limit of these two amplitudes, we can compute the large N limit of the YM amplitude on any Riemann surface. This was done in [3]. In the present context, we need to generalize their results to compute non-perturbative, multi-center black hole corrections. To do that, a different derivation is more adept.

There is a well known correspondence between $U(N)$ representations $\mathcal{R}$ and states in the Hilbert space of N free, non-relativistic fermions on a circle (for a review, see for example [22]). The representation $\mathcal{R}$ captures the momenta of the N fermions in the following way. If $\mathcal{R}_i$, $i = 1, \ldots N$ are the lengths of the rows of the $U(N)$ Young-tableaux corresponding to $\mathcal{R}$, than the i-th fermion momentum is

$$n_i = \mathcal{R}_i - i + \frac{N+1}{2}.$$

Recall that $U(N)$ differs from $SU(N)$ in that $\mathcal{R}_i$ are not necessarily positive. The degree of freedom corresponding to shifting all the fermion momenta and all $\mathcal{R}_i$'s by a constant, corresponds to the choice of the $U(1)$ charge of the representation. Recall that any $U(N)$ representation $\mathcal{R}$ can be obtained by tensoring an $SU(N)$ representation R by l powers of the determinant representation, and correspondingly, we have

$$\mathcal{R}_i = R_i + l.$$

The $U(1)$ charge q of the representation is then $q = Nl + |R|$.

For this fermion state, the explicit form of the master amplitude is

$$S_{O\mathcal{R}} = \prod_{1 \leq i < j \leq N} [\mathcal{R}_i - \mathcal{R}_j + j - i] \tag{3.2}$$

Note that this is independent of the $U(1)$ charge.

3.3. Large N factorization

At large N the fermion states are described by independent fluctuations about two well separated fermi surfaces. Correspondingly, the representation R can be thought of as a coupled representation $R^+\bar{R}^-$, where R^+ and R^- describe the fermion excitations about the two fermi surfaces.

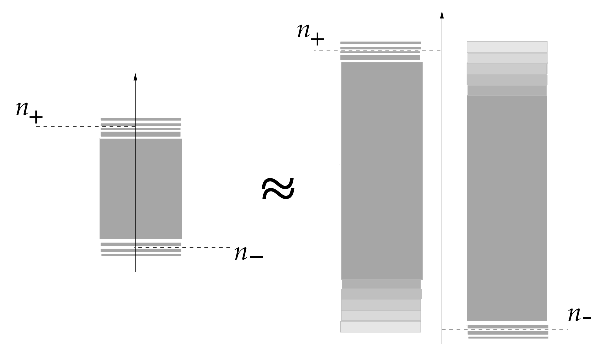

Fig.1 *Excitation about the top and the bottom fermi sea are independent at large N, although they are not uncorrelated. They are described by topological and anti-topological string amplitudes.*

The amplitude $S_{0,\mathcal{R}}/S_{0,0}$ has pairwise interactions of *excited* fermions. Thus, the N dependence of the amplitude should be localized to the interactions of the fermions at the top and bottom of the fermi seas only. So, we can guess that the amplitude factorizes at large N as:

$$S_{O\mathcal{R}}/S_{00}(N,g_s) = \prod_{1\le i<j\le\infty} \frac{[R_i^+ - R_j^+ + j - i]}{[j-i]} \prod_{1\le i<j\le\infty} \frac{[R_i^- - R_j^- + j - i]}{[j-i]} \\ \times \prod_{1\le i,j\le\infty} \frac{[R_i^+ + R_j^- + n^+ - n^- - i - j + 1]}{[n^+ - n^- - i - j + 1]} .$$

All the interactions depend only on the distances, in momentum space, of the between the excited fermions. We have denoted by n^+ and n^- the positions of the fermi surfaces (see Fig. 1), where

$$n^+ - n^- = N.$$

We will prove this formula in appendix A.

We can now easily reproduce the results of [3]. First, note that the first and the second factors are given in terms of Schur functions $s_R(q)$ (see appendix A. for a derivation)[3], which in turn correspond to topological string amplitudes:

$$W_{0,R} = s_R(q^\rho) = q^{k_R/4} \prod_{1\le i<j\le\infty} \frac{[R_i - R_j + j - i]}{[j-i]}$$

where

$$k_R = 2\sum_{\square\in R}(j(\square) - i(\square)),$$

as i runs over rows, and j over columns. Here,

$$q^\rho = (q^{-1/2}, q^{-3/2}, \dots)$$

One basic property of Schur functions is that

$$\prod_{i,j}(1 - x_i y_j) = \sum_Q (-)^{|Q|} s_Q(x) s_{Q^T}(y). \tag{3.3}$$

and

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_Q s_Q(x) s_Q(y). \tag{3.4}$$

Using (3.3), we can write the last factor above as

$$(-1)^{|R^+|+|R^-|} q^{-\frac{1}{2}N(|R^+|+|R^-|)} q^{-\frac{1}{4}(k_{R^+}+k_{R^-})} \sum_Q (-)^{|Q|} s_Q(q^{\rho+R^+}) q^{|Q|N} s_Q^T(q^{\rho+R^-}).$$

The prefactor can be obtained by carefully regularizing the infinite product, see appendix A. Finally, since

$$W_{RQ} = s_R(q^\rho) s_Q(q^{\rho+R}) \tag{3.5}$$

we reproduce the result of [3] for the large N limit of $Z(\mathcal{C}, N, g_s)_\mathcal{R} = S_{0,\mathcal{R}}(N, g_s)$, to all orders in perturbation theory:

$$\begin{aligned} S_{0,\mathcal{R}}(N, g_s) = s(N, g_s)(-1)^{|R^+|+|R^-|} q^{-\frac{1}{2}N(|R^+|+|R^-|)} q^{-\frac{1}{2}(k_{R^+}+k_{R^-})} \\ \times \sum_Q (-)^{|Q|} W_{R^+Q}(q) q^{|Q|N} W_{Q^T R^-}(q) \end{aligned} \tag{3.6}$$

[3] The Schur functions s_R are simply extensions of $U(N)$ characters to infinite rank. We will need only rudimentary properties of these functions. For more details about properties of Schur functions, see [23] .

The normalization constant $s(N, g_s)$ is independent of R. Similarly, using (3.4), we can compute the large N limit of $Z(\mathcal{P}, N, g_s) = 1/S_{0,\mathcal{R}}(N, g_s)$ of the pant amplitude:

$$S_{0,\mathcal{R}}(N, g_s)^{-1} = s(g_s, N)^{-1} q^{\frac{1}{2}N(|R^+|+|R^-|)} q^{\frac{1}{2}(k_{R^+}+k_{R^-})} \sum_Q \frac{W_{R^+Q}}{W_{R^+0}^2} q^{|Q|N} \frac{W_{QR^-}}{W_{R^-0}^2}. \quad (3.7)$$

As we saw earlier in this section, the Yang-Mills amplitude corresponding to a D4 brane wrapping a divisor $\mathcal{D} = \mathcal{L}_{p_1} \to \Sigma_g$ in X is given in (3.1)

$$Z^{YM}(N, \theta, g_s) = z^{YM}(N, \theta, g_s) \sum_{\mathcal{R} \in U(N)} S_{0\mathcal{R}}^{2-2G} \, q^{\frac{p}{2}C_2(\mathcal{R})^{U(N)}} \, e^{i\theta C_1(\mathcal{R})^{U(N)}} \quad (3.8)$$

The normalization of the partition function z^{YM} is ambiguous in the D-brane theory. In [3], motivated by the black hole physics, the normalization was chosen to be

$$z^{YM}(g_s, N) = q^{\frac{(p+2G-2)^2}{2p}\rho(N)^2} e^{\frac{N\theta^2}{2pg_s}}$$

where $\rho(N)^2$ is the norm of the Weyl vector

$$\rho(N)^2 = \frac{1}{12}(N^3 - N).$$

With the above choice of normalization and the results for the pant and cap amplitudes, the *perturbative* part of the large N expansion of the Yang-Mills theory can be written as a sum over chiral blocks:

$$Z^{YM}(N, \theta, g_s) \approx \sum_{l=-\infty}^{\infty} \sum_{R_1,\ldots R_{|2G-2|}} Z^+_{R_1,\ldots R_{|2G-2|}}(t + p\, l\, g_s) \bar{Z}^-_{R_1,\ldots,R_{|2G-2|}}(\bar{t} - p\, l\, g_s), \quad (3.9)$$

where

$$Z^+_{R_1,\ldots R_{|2G-2|}}(t, g_s) = z^+(g_s, t) e^{-\frac{t(|R_1|+\ldots|R_{|2G-2|}|)}{p+2G-2}} \sum_{R^+} \frac{W_{R_1R^+}\ldots W_{R_{|2G-2|}R^+}}{W_{R^+0}(q)^{4g-4}} \; q^{\frac{p+2G-2}{2}k_{R^+}} \; e^{-|R^+|t} \quad (3.10)$$

for $G \geq 1$ and

$$Z^+_{R_1,R_2}(t, g_s) = z^+(g_s, t) e^{-\frac{t(|R_1|+|R_2|)}{p-2}} \sum_{R^+} W_{R_1R^+} \, W_{R_2^TR^+} \, q^{\frac{p-2}{2}k_{R^+}} \; e^{-|R^+|t} \quad (3.11)$$

for $G = 0$. In either case,

$$t = \frac{p + 2G - 2}{2} N g_s - i\theta.$$

The coefficient z^{top} contains the McMahon function $M(q)^{\frac{\chi(X)}{2}}$ which counts maps to points in X, and

$$z^+(g_s, t) = M(q)^{\frac{2-2G}{2}} e^{-\frac{1}{p(p+2G-2)}\frac{t^3}{6g_s^2} + \frac{p+2G-2}{p}\frac{t}{24}}.$$

As we will see in the next subsection, (3.10) and (3.11) are precisely the topological string amplitudes on the geometry under consideration. For $G \neq 1$, these amplitudes contain contributions from open strings ending on non-compact D-branes. The integer l appearing in the sum was interpreted in [2] as the RR 2-form flux through the Riemann surface Σ.

3.4. Topological string amplitudes

In this section we review the topological string amplitudes on $\mathcal{M}$,

$$\mathcal{M} = \mathcal{L}_{p_1} \oplus \mathcal{L}_{p_2} \to \Sigma_g$$

where

$$p_1 + p_2 = 2G - 2$$

Previously, we put $p_1 = -p$, $p_2 = p + 2G - 2$. The closed topological string amplitudes on this manifold were solved recently in [20]. As explained there, the topological A-model on X is effectively a topological field theory on Σ, which is completely specified by knowing the topological string amplitudes corresponding to Σ being a cap $\mathcal{C}$ and a pant $\mathcal{P}$, with specified degrees.

The cap amplitude is given by

$$Z^{top}(\mathcal{C}; -1, 0)_R = W_{R,0}(q) e^{-|R|t}$$

where the numbers $(p_1, p_2) = (-1, 0)$ denote the degrees of the line bundles, and t is the Kähler parameter of the cap. This is nothing but the topological vertex[4] of [24] with one stack of lagrangian branes:

$$W_{R,0}(q) = C_{0,0,R}(q) q^{k_R/2}.$$

The branes are wrapping the boundary of the disk on the Riemann surface direction and extending in 2 directions of the fiber (see [24] for a more detailed discussion of this).

The pant amplitude requires specifying three representations, corresponding to boundary conditions on the three punctures. It turns out that it vanishes, unless all three representations are equal, and it is given by

$$Z^{top}(\mathcal{P}; 1, 0)_{R,R,R} = \frac{1}{W_{R,0}(q)} e^{-|R|t},$$

where t again measures the size of the pant. The change of the bundle degrees is implemented by the annulus operator

$$Z^{top}(\mathcal{A}; -1, 1) = q^{k_R/2} e^{-t|R|}$$

[4] The conventions of this paper differ from [24] : q in this paper is defined as $q = e^{-g_s}$ while q in [24] is e^{g_s}. In particular, $C_{RQP}(q)$ of this paper is defined to equal $C_{RQP}(q^{-1})$ of [24]. When reading off the topological string amplitudes, attention should be paid to this difference in conventions.

These three amplitudes completely specify the closed string theory. From these, any other amplitude can be obtained by gluing:

$$Z((\Sigma_L \cup \Sigma_R); p_1^L + p_1^R, p_2^L + p_2^R) = \sum_Q Z(\Sigma_L; p_1^L, p_2^L)_Q \; Z(\Sigma_R; p_1^R, p_2^R)_Q$$

For example, the amplitude on a genus g Riemann surface with bundle of degrees

$$(p_1, p_2) = (-p, p + 2G - 2)$$

the topological string amplitude is obtained by sewing $2G-2$ pant amplitudes with $p + 2G - 2$ annuli operators $\mathcal{A}^{(-1,1)}$ giving

$$Z(\Sigma_g;\; -p,\; p+2G-2) = \sum_R W_{R,0}(q)^{2-2G} q^{\frac{p+2G-2}{2} k_R} e^{-|R|t}$$

In addition to closed string amplitudes on $\mathcal{M}$, this allows us to compute also open topological string amplitudes corresponding to D-branes wrapping 1-cycles on Σ. Moreover, we can also compute topological string amplitudes with D-branes in the fibers over Σ. As explained in [3] the results of [20] combined with topological vertex results can be used to compute open topological string amplitudes on $\mathcal{M}$ corresponding to with Lagrangian D-branes which project to points on the Riemann surface.[5] We will need a slight refinement of the statements there, so let us review the argument of [3] in full.

The idea was to cut out a neighborhood in $\mathcal{M}$ of a point P in Σ where the D-brane is. This is a copy of $\mathbf{C}^3$, and topological string amplitudes with D-branes on $\mathbf{C}^3$ are solved by the topological vertex. Gluing the $\mathbf{C}^3$ back in, we also glue the topological string amplitudes on $\mathbf{C}^3$ and on $X - \mathbf{C}^3$ to get the amplitudes on $\mathcal{M}$ with D-branes.

For example, consider a genus G Riemann surface with a stack of D-branes over a point P. Let the amplitude $before$ inserting the branes be

$$Z(\Sigma; p_1, p_2) = \sum_R Z(\Sigma; p_1, p_2)_R$$

[5] The crucial property of $\mathcal{M}$ which [20] used in deriving their result was the invariance under the torus action that rotates the fibers over Σ. The Lagrangian branes of [24] preserve this symmetry, which is why this generalization is possible.

and define an operator $\mathcal{O}_{QR}(P)$ which inserts the brane at P with boundary conditions given by Q, so that the amplitude with the D-brane is[6]

$$Z(\Sigma, P; p_1, p_2)_Q = \sum_R \mathcal{O}(P)_{QR} Z(\Sigma; p_1, p_2)_R.$$

We can compute $\mathcal{O}(P)_{QR}$ by cutting out a neighborhood of P out of Σ:

$$Z(\Sigma; p_1, p_2) = \sum_R Z(\mathcal{C};\ -1, 0)_R Z(\Sigma - \{P\};\ p_1 + 1, p_2)_R$$

and inserting it back, with D-branes. To insert D-branes wrapping an S^1 in the $\mathcal{L}_{p_1}$ direction, we replace

$$Z(\mathcal{C};\ -1, 0)_R = C_{0,0,R}(q) q^{\frac{k_R}{2}}$$

by

$$Z(\mathcal{C};\ -1, 0)_{QR} = C_{Q^t,0,R}(q) q^{\frac{k_R}{2}} q^{-\frac{k_Q}{2}} = W_{Q^t R^t}(q) q^{\frac{k_R}{2}}.$$

For D-branes along $\mathcal{L}_{p_2}$ direction, we get

$$Z(\mathcal{C};\ -1, 0)_{QR} = C_{0,Q^t,R}(q) q^{\frac{k_R}{2}} = W_{QR}(q).$$

In writing the above, we have made a choice of framing for the D-brane, where a different framing by n units would amount to replacing $Z(\mathcal{C})_{QR} \to Z(\mathcal{C})_{QR} q^{-nk_Q/2}$, and a choice of orientation of its world volume, which replaces $Z(\mathcal{C})_{QR} \to Z(\mathcal{C})_{Q^t R}$. We have made specific choices that will be convenient for us, but for the purposes of our paper all the choices will turn out equivalent, as we will see.

In summary, we found that operator inserting the branes along $\mathcal{L}_{p_1}$ direction is

$$\mathcal{O}_{QR}(P) = \frac{W_{Q^t R^t}}{W_{0 R^t}}$$

[6] More precisely, fixing Q is related to fixing a particular topological sector for worldsheet instantons with boundaries on the brane. In string theory one naturally sums over these, so the quantity which appears is

$$\sum_Q Z(\Sigma, P)_Q e^{-t_C |Q|} \mathrm{Tr}_Q U.$$

Above, t_C is the size of the holomorphic disk ending on the brane – the opens string Kähler modulus. Moreover U is the holonomy of the D-brane gauge field.

and the operator inserting a brane along $\mathcal{L}_{p_2}$ is

$$\mathcal{O}_{QR}(P) = \frac{W_{QR}}{W_{0R}}$$

where we used $W_{R0}(q) = q^{\frac{k_R}{2}} W_{R^t0}(q)$.

Using the above operators, we can compute the amplitudes corresponding to inserting branes at arbitrary points $P_1, \ldots, P_k$ on Σ. As explained in [3] the positions of the branes are complex structure parameters, and correspondingly, the amplitudes are *indepentent* of the location of the points. More precisely, this is so as long as the points P_i do not coincide, that is away from the boundaries of the moduli space of punctured Riemann surfaces.

When P_i and P_j coincide, new contributions appear. These are the amplitudes for a new holomorphic annulus and its multi-coverings, connecting the two stacks of branes at P_i and P_j, without wrapping Σ.[7] If the two stacks are along the same fiber at a single base point, the annulus amplitudes can be obtained from the one-stack case by taking the holonomy matrix to be block-diagonal and then by expanding in two unitary matrices. If the two stacks lie on different fibers at a single base point, we need to insert an operator constructed from the full topological vertex $C_{Q_iQ_jR}$ depending on the boundary conditions Q_i and Q_j on the two stacks of branes. In this paper, we will not encounter these kinds of annulus amplitudes.

3.5. Relation to the conjecture of OSV

Combining the results of the previous two subsections, we now see that counting of BPS states of N D4 branes with D2 and D0 branes in this geometry precisely realize the OSV conjecture. In particular, at large N, the D-brane partition function factorizes into sum over chiral blocks, with chiral amplitudes (3.10) and (3.11) which are precisely the topological string amplitudes in this geometry [3]. Namely,

$$Z^{YM}(N,\theta,g_s) \approx \sum_{l=-\infty}^{\infty} \sum_{R_1,\ldots R_{|2G-2|}} Z^+_{R_1,\ldots R_{|2G-2|}}(t + p\,l\,g_s) \bar{Z}^-_{R_1,\ldots,R_{|2G-2|}}(\bar{t} - p\,l\,g_s),$$

where

$$Z^+_{R_1,\ldots R_{|2G-2|}}(t,g_s) = Z^{top}_{R_1,\ldots R_{|2G-2|}}(t,g_s)$$

[7] Even when P_i and P_j are distinct points, there are holomorphic annuli connecting the two stacks, but wrapping Σ, if both stacks of branes are in the same fiber direction.

corresponds to topological string amplitudes with insertions of $|2G-2|$ "ghost" D-branes along the normal bundle to the divisor $\mathcal{D}$. The Kähler modulus t associated to the size of the base Riemann surface Σ_G can be read off from (3.10) and (3.11) to be

$$t = \frac{1}{2}(p+2G-2)\, N g_s + i\theta.$$

As explained in [3] this is exactly as predicted by the attractor mechanism and the OSV conjecture. Namely, the attractor mechanism sets

$$Re(t) = \frac{1}{2}\#(\mathcal{D}\cap\Sigma)\, N g_s$$

where the intersection number enters because the effective magnetic charge of a D4 brane on $\mathcal{D}$ is set by the intersection number of $\mathcal{D}$ with Σ, the two-cycle wrapped by the D2-branes. In the present context, the intersection number can be computed by deforming $\mathcal{D}$ along a section of its normal bundle $O(p+2G-2)$ in $\mathcal{M}$. As this has $p+2G-2$ zeros, this gives

$$\#(\mathcal{D}\cap\Sigma) = p + 2G - 2,$$

exactly as expected.

Now consider the open string moduli associated to insertions of ghost D-branes along the normal bundle. These would enter the topological string amplitude as

$$Z^{+}_{R_1 \ldots R_{|2G-2|}} \sim e^{-\sum_{i=1}^{|2G-2|} t_{C_i} |R_i|}$$

where the t_{C_i} is the complexified size of the holomorphic disk C_i ending on the i-th stack of branes (see footnote 4). From (3.10) and (3.11) we can read off:

$$Re(t_{C_i}) = \frac{1}{2} N g_s,$$

As was explained in [19] this value is in fact exactly what is expected. Namely, while the open string moduli are not supported by compact 2-cycles, they still can feel the charge of the D4 brane. The attractor mechanism fixes their values according to the intersection number of the open 2-cycle ending on the Lagrangian branes with the divisor D wrapped by the D4 brane,

$$Re(t_{C_i}) = \frac{1}{2}\#(\mathcal{D}\cap C_i) N g_s.$$

Since the C_i lies simply along the line bundle normal to $\mathcal{D}$, the intersection numbers are canonical,

$$\#(\mathcal{D}\cap C_i) = 1,$$

which is exactly what we see in (3.10) and (3.11). More precisely, at each puncture so for each i, we will get as many open string moduli as there are D branes there, in this case infinitely many. However, only the "center" of mass degree of freedom will get fixed by the D4 brane charge, as the differences between the holomorphic disks ending on various branes have zero intersection with $\mathcal{D}$.

As explained in [19], the open string moduli have an interpretation as non-normalizable *closed* Kähler moduli in the dual picture where the ghost branes deform the geometry. The duality in question is nothing but the open-closed string duality, which can be understood very precisely in the topological string context. The insertions of "ghost" branes which correspond to the amplitudes above are dual to certain non-normalizable deformations of the A-model geometry. Because of this, the fact that the D4 brane charges affected the open string moduli in the way they did, is in fact forced on us![8] To make the fact that the normalizable and non-normalizable Kähler moduli are at the same footing manifest, define

$$Z^{top}(t, g_s, U_1 \ldots U_{|2G-2|}) = \sum_{R_1,\ldots R_{|2G-2|}} Z_{R_1,\ldots R_{|2G-2|}} \mathrm{Tr}_{R_1} U_1 \ldots \mathrm{Tr}_{R_{|2G-2|}} U_{|2G-2|},$$

where U_i are the $U(\infty)$ valued holonomies on the ghost branes and parameterize non-normalizable deformations of the geometry in the closed string language. Note that we can write (3.9) as

$$Z^{YM}(N, \theta, g_s) \approx \sum_{l=-\infty}^{\infty} \int dU_1 \ldots dU_{|2G-2|} Z^{top}(t+lpg_s; U_1 \ldots U_{|2G-2|}) \bar{Z}^{top}(\bar{t}-lpg_s; U_1^\dagger, \ldots U_{|2G-2|}^\dagger), \tag{3.12}$$

This has a very natural interpretation. The above expresses the fact that we are explicitly working with an ensemble where the electric charges of the non-compact modes are set to zero! The magnetic charges are also set to zero, apart from those induced by the N D4 branes, which we discussed above.[9]

[8] Note that, while in the topological string theory we can use either language, in the physical superstring we do not have this freedom – since there are no Ramond-Ramond fluxes, the only available interpretation is the closed string one.

[9] The more general charges that one can turn on have a physical interpretation, as explained in [19] in terms of counting BPS states in *two* dimensions.

4. Non-perturbative corrections and baby universes

As reviewed in section two, in the large N limit, the D-brane partition function factorizes, schematically as

$$Z^{YM}(N, g_s) \approx Z^{top}(N, g_s)\bar{Z}^{top}(N, g_s).$$

As explained in [11] the right hand side over-counts fermion states which appear in Z^{YM}. The configurations which are over-counted are those in the figure below.

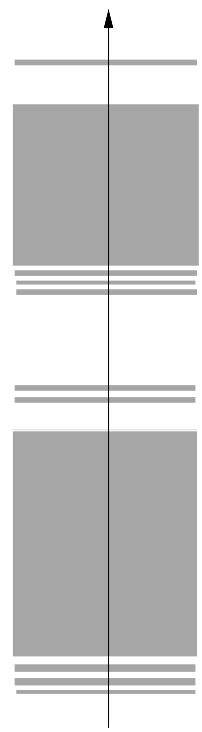

Fig.2 *Topological string amplitudes describe excitations about the top and the bottom fermi seas, as in the figure* 1, *without restriction on the size of excitation. Since N is finite, this will over-count the configurations like those in the figure. Here we cannot decide whether to include this in Z^{top} or $\bar{Z}^{top}$, and the configurations is thus counted twice in $|Z^{top}|^2$.*

This is a non-perturbative correction to the topological string amplitude. There will be others, corresponding to a large number N of fermions splitting in arbitrary ways.

In this section we will first study the 2-black hole correction in the large N limit. Then we will generalize the computations to the $M(\geq 3)$-black hole case.

4.1. Non-perturbative corrections to cap and pant amplitudes

As explained in [11], the fermion fluctuations around the two fermi surfaces are independent only in perturbative $1/N$ expansion. Non-perturbatively, the two fermi surfaces are entangled. The non-perturbative corrections to the factorization are described by [11] splitting of N fermions to groups with smaller numbers of fermions. In the gravity theory, this corresponds to having multiple black holes. For example, consider the 2 black hole case:

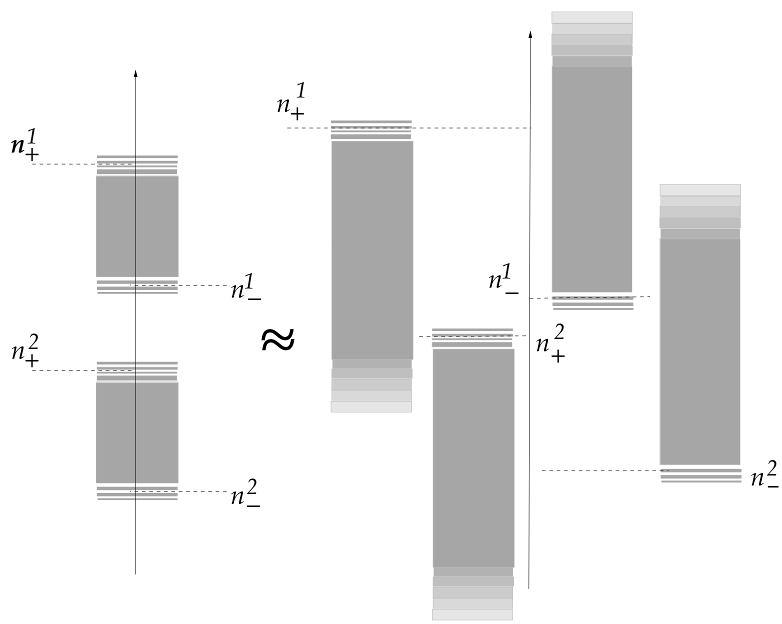

Fig.3 *At large $N_{1,2}$ the configurations in Fig. 2 are described by excitations of fermions about each of the 4 fermi sea surfaces. Each set of excitations is described by a topological string partition function.*

The N fermions split into two groups of $N_1 = n_1^+ - n_1^-$ and $N_2 = n_2^+ - n_2^-$ fermions where $n_i^\pm$ denote fermi-surface momenta. Following the same logic as before, we can immediately write down the corresponding amplitude. Let $\mathcal{R}$ and $\mathcal{Q}$ be representations of $U(N_1)$ and $U(N_2)$ describing the fermion states. We can of course interpret $\mathcal{R}\mathcal{Q}$ as an irreducible representation of $U(N)$. The amplitude we want to evaluate at large N, N_1 and N_2 is

$$S_{0,\mathcal{R}\mathcal{Q}}(N, g_s).$$

At large $N_{1,2}$, $\mathcal{R}$ and $\mathcal{Q}$ factorize as $\mathcal{R}_+\bar{\mathcal{R}}_-$ and $\mathcal{Q}_+\bar{\mathcal{Q}}_-$ with $U(1)$ charges determined by the locations of the fermi seas: If the corresponding $SU(N)$ representations factorize as

$R_+\bar{R}_-$ and $Q_+\bar{Q}_-$ [22,3], then we define:

$$\mathcal{R}_i^+ := R_i^+ + n_1^+, \quad \mathcal{R}_i^- := R_i^- - n_1^-,$$

$$\mathcal{Q}_i^+ := Q_i^+ + n_2^+, \quad \mathcal{Q}_i^- := Q_i^- - n_2^-,$$

It is also useful to define the $U(1)$ charges of $\mathcal{R}$ and $\mathcal{Q}$:

$$l_1 := \frac{n_1^+ + n_1^-}{2} - \frac{N_2}{2}, \quad l_2 := \frac{n_2^+ + n_2^-}{2} + \frac{N_1}{2}.$$

Only the states near the fermi surfaces interact in the amplitude we are computing, and the interaction depends only on the distance between the fermions. The normalized amplitude thus consists of self-interaction pieces of the two groups of fermions, whose large N limit we have just described:

$$S_{0,\mathcal{R}}/S_{0,0}(N_1, g_s)\; S_{0,\mathcal{Q}}/S_{0,0}(N_2, g_s)$$

and an interaction piece between them, consisting of

$$\prod_{i,j=1}^{\infty} \frac{[R_i^+ - Q_j^+ + n_1^+ - n_2^+ - i + j]}{[n_1^+ - n_2^+ - i + j]} \frac{[-R_i^- + Q_j^- + n_1^- - n_2^- + i - j]}{[n_1^- - n_2^- + i - j]} \tag{4.1}$$

and

$$\prod_{i,j=1}^{\infty} \frac{[R_i^+ + Q_j^- + n_1^+ - n_2^- - i - j + 1]}{[n_1^+ - n_2^- - i - j + 1]} \frac{[-R_i^- - Q_j^+ + n_1^- - n_2^+ + i + j - 1]}{[n_1^- - n_2^+ + i + j - 1]} \tag{4.2}$$

The contributions of fermi excitations around the bottom fermi seas at $p = n_{1,2}^-$ come with a minus relative sign, since there fermion excitations have negative momenta relative to the fermi seas in their ground states. While this derivation may appear heuristic, we will verify in the appendix, via a careful calculation, that it is indeed correct.

It is easy now to rewrite the large N limit of the this in terms of topological string objects. We will do so in a way which will be convenient for us later. We can write

$$S_{0,\mathcal{R}\mathcal{Q}}(g_s, N) = \mathrm{pr}_{\mathcal{R}\mathcal{Q}}\; W_{0R^+}\, W_{0R^-}\, W_{0Q^+}\, W_{0Q^-}\; \mathrm{cr}_{\mathcal{R}\mathcal{Q}}.$$

Here the prefactor pr is given by[10]

$$\begin{aligned}\mathrm{pr}_{\mathcal{R}\mathcal{Q}} = &\pm M(q)^2 \eta(q)^N q^{-\frac{1}{2}(k_{R^+} + k_{R^-} + k_{Q^+} + k_{Q^-})} q^{-\frac{N_1}{2}(|R^+| + |R^-|) - \frac{N_2}{2}(|Q^+| + |Q^-|)} \\ &\times q^{-\frac{1}{2}N_1 N_2 (l_1 - l_2)} q^{-\frac{N_2}{2}(|R^+| - |R^-|) + \frac{N_1}{2}(|Q^+| - |Q^-|)},\end{aligned} \tag{4.3}$$

[10] We omit specifying the overall sign that will drop out when we take the even power of the whole expression.

and the correction part cr by[11]

$$
\begin{aligned}
\mathrm{cr}_{\mathcal{RQ}} = \prod_{i,j=1}^{\infty} &(1-q^{n_1^+-n_1^-+R_i^++R_j^--i-j+1})(1-q^{n_2^+-n_2^-+Q_i^++Q_j^--i-j+1}) \\
&\times (1-q^{n_1^+-n_2^++R_i^+-Q_j^+-i+j})(1-q^{n_1^--n_2^--R_i^-+Q_j^-+i-j}) \\
&\times (1-q^{n_1^+-n_2^-+R_i^++Q_j^--i-j+1})(1-q^{n_1^--n_2^+-R_i^--Q_j^++i+j-1}).
\end{aligned}
\tag{4.4}
$$

${\mathrm{cr}_{\mathcal{RQ}}}^{-1}$, which enters the pant amplitude $S_{0,\mathcal{RQ}}(g_s,N)^{-1}$, can be expanded by Schur functions with summations over Young diagrams. This is done in Appendix C. As in the single black hole case, these Young diagrams represent interactions among different Fermi surfaces. From the topological string point of view, the Young diagrams are ghost branes. We only need the pant amplitude as we will focus on the $G \geq 1$ case, though the corresponding expression for the cap amplitude is easy to obtain. We will put off dealing with the correction part until subsection 3.3.

4.2. Leading block amplitudes in the two-universe case

We now want to evaluate the two-universe contribution. This can be obtained by evaluating the contribution to YM amplitude of representations in Figure 2:

$$
z^{YM}(N,\theta,g_s)\sum_{\mathcal{R},\mathcal{Q}} S_{0,\mathcal{RQ}}^{2-2G}(N,g_s)\; q^{\frac{p}{2}C_2(\mathcal{RQ})^{U(N)}}\;\; e^{iC_1(\mathcal{RQ})^{U(N)}\theta}
\tag{4.5}
$$

To simplify the algebra, let's first consider the leading chiral block, where the interactions between the fermi seas are turned off. That is, consider the piece of the amplitude where all ghost contributions are set to zero. The pant amplitude in this sector is

$$
S_{0,\mathcal{RQ}}(g_s,N)^{-1} = \mathrm{pr}_{\mathcal{RQ}}^{-1}\; W_{0R^+}^{-1} W_{0R^-}^{-1} W_{0Q^+}^{-1} W_{0Q^-}^{-1},
$$

where $\mathrm{pr}_{\mathcal{RQ}}$ is defined in (4.3). The Casimir $C_2(\mathcal{RQ})$ can be explicitly evaluated, see appendix (C.2). We will show in the appendix C, that (4.5) can be expressed in terms of topological string partition functions on $\mathcal{M}$:

$$
\begin{aligned}
\sum_{l_1',l_2'=-\infty}^{\infty} & Z^{top}(t_1+pl_1'g_s)\; Z^{top}(t_2-pl_1'g_s+(p+2G-2)l_2'g_s) \\
&\times \bar{Z}^{top}(\bar{t}_2+pl_1'g_s-(p+2G-2)l_2'g_s)\; Z^{\bar{t}op}(\bar{t}_1-pl_1'g_s),
\end{aligned}
\tag{4.6}
$$

[11] The infinite product, as it appears, is not convergent and contains vanishing factors, but it makes sense in terms of the finite products given in the appendix.

where $Z^{top}(t)$ is the closed topological string amplitude on $\mathcal{M}$ given in (3.10), corresponding to setting all the ghost representations to zero. The Kähler moduli take the form

$$t_1 = \frac{p+2G-2}{2} N g_s + \frac{p}{2} k g_s - i\theta, \qquad t_2 = -\frac{p}{2} k g_s + i\theta. \tag{4.7}$$

Here $k = n_1^- - n_2^+$.[12]

What is the interpretation of this? To begin with, note that in addition to the divisor $\mathcal{D}$ that we wrapped D4 branes on,

$$\mathcal{D} = \mathcal{L}_{-p} \to \Sigma$$

there is another divisor $\mathcal{D}'$ in $\mathcal{M}$

$$\mathcal{D}' = \mathcal{L}_{p+2G-2} \to \Sigma,$$

that the branes could wrap.[13]

Now while

$$\#(\mathcal{D} \cap \Sigma) = p + 2G - 2$$

we have

$$\#(\mathcal{D}' \cap \Sigma) = -p.$$

This implies that (4.7) is precisely consistent with attractor mechanism in the two universes if we had

$$(N, -k), \qquad (0, k)$$

branes along $(\mathcal{D}, \mathcal{D}')$ in the first and the second universes, respectively.

Note first of all that the net D4 brane charge is conserved: when the baby universes split, we pair-create k D4 branes and k anti-D4 branes along $\mathcal{D}'$. Moreover, even though in the first universe we have both N D4 branes on $\mathcal{D}$ and $-k$ D4 branes on $\mathcal{D}'$, the effective D4 brane charge of either is positive. Correspondingly, the two kinds of branes are mutually

[12] The summed discrete parameters that shift the Kähler moduli in (4.6) are defined by

$$l_1' = \frac{n_1^+ + n_2^-}{2}, \quad l_2' = \frac{n_1^+ - n_1^- - n_2^+ + n_2^-}{2}.$$

[13] There are other divisors in $\mathcal{M}$, but only $\mathcal{D}$ and $\mathcal{D}'$ respect the toric symmetry of the Calabi-Yau, that we used heavily throughout.

BPS, and the first universe is supersymmetric. However, by the same token, the second universe preserves opposite supersymmetry: it is anti-BPS. Since the baby universes are disconnected, the entire configuration is still stable.

Note that in writing (4.7), we have flipped the sign of the theta-angle in the second universe. Correspondingly, relative to the BPS universe what we mean by the kets and the bra's get exchanged. The assignment is natural as CPT that exchanges the BPS and the anti-BPS universes is an anti-linear transformation. It is also natural in the context of the proof of the OSV conjecture given in [6]. There factorization of Z_{BH} was a consequence of localization of membranes and anti-membranes to the north and the south poles of the S^2 in a near horizon geometry of the black holes in M-theory compactification on Calabi-Yau $\mathcal{M}$. The localization is due to supersymmetry: the supersymmetry preserved by either branes depends on their position on the S^2, and the anti-membranes on one pole preserve the *same* supersymmetry as the membranes on the north pole. In a BPS universe, the topological string partition function came about from summing over membrane states, and the anti-topological one over the anti-membrane states. Going from a BPS to an anti-BPS universe these assignments should naturally flip, as the membranes and anti-membranes get exchanged. This exactly leads to the exchange of the kets and the bras we have seen above.

4.3. Ghost brane amplitudes in the two-universe case

We now turn on ghost brane contributions and the subleading chiral blocks. From (C.1), we get that the full two-universe amplitude can be written as follows:

$$\begin{aligned}\sum_{l_1',l_2'=-\infty}^{\infty} \sum_{\{P\},\{P_1^{\pm}\},\{P_2^{\pm}\},\{S^{\pm}\}} \prod_{a=1}^{2G-2} & N_{P_{1,a}^{+}P_{1,a}^{-}}^{P_a} N_{P_{2,a}^{+}P_{2,a}^{-}}^{P_a} \\ & \times Z_1^{top}(t_1 - p_1 l_1' g_s)_{\{S^+\},\{P_1^+\}}\, Z_2^{top}(t_2 + p_1 l_1' g_s + p_2 l_2' g_s)_{\{S^+\},\{P_1^-\}} \\ & \times \bar{Z}_2^{top}(\bar{t}_2 - p_1 l_1' g_s - p_2 l_2' g_s)_{\{S^-\},\{P_2^+\}}\, \bar{Z}_1^{top}(\bar{t}_1 + p_1 l_1' g_s)_{\{S^-\},\{P_2^-\}},\end{aligned} \tag{4.8}$$

where

$$p_1 = -p, \qquad p_2 = p + 2G - 2.$$

Here, for example, $\{P_1^+\}$ denotes collectively a set of Young diagrams $(P^+_{1,1}, ..., P^+_{1,|2G-2|})$. Explicitly,

$$
\begin{aligned}
Z_1^{top}(t_1)_{\{S^+\},\{P_1^+\}} =& z^{\rm top}(g_s,t_1)\sum_{R^+} e^{-|R^+|t_1}\, q^{\frac{p+2G-2}{2}k_{R^+}}\, W_{0R^+}^{2-2G}\prod_a \frac{W_{R^+S_a^+}}{W_{0R^+}}\frac{W_{R^+P_{1,a}^+}}{W_{0R^+}}\\
&\times e^{-g_s n_1^+|\{P_1^+\}|}e^{-\frac{t_1}{p+2G-2}|\{S^+\}|},\\
Z_2^{top}(t_2)_{\{S^+\},\{P_1^-\}} =& z^{\rm top}(g_s,t_2)\sum_{R^-} e^{-|R^-|t_2}\, q^{\frac{p+2G-2}{2}k_{R^-}}\, W_{0R^-}^{2-2G}\prod_a \frac{W_{R^-S_a^+}}{W_{R^-}}\frac{W_{R^{-T}P_{1,a}^{-\ T}}}{W_{R^{-T}}}\\
&\times e^{-g_s n_1^-|P_1^-|}(-1)^{|P_1^-|}e^{-\frac{t_2}{p+2G-2}|S^+|},\\
\bar{Z}_2^{top}(\bar{t}_2)_{\{S^-\},\{P_2^+\}} =& z^{\rm top}(g_s,\bar{t}_2)\sum_{Q^+} e^{-|Q^+|\bar{t}_2} q^{\frac{p+2G-2}{2}k_{Q^+}} W_{Q^+}^{2-2G}\prod_a \frac{W_{Q^+S_a^-}}{W_{Q^+}}\frac{W_{Q^{+T}P_{2,a}^{+\ T}}}{W_{Q^{+T}}}\\
&\times e^{g_s n_2^+|P_2^+|}(-1)^{|P_2^+|}e^{-\frac{\bar{t}_2}{p+2G-2}|S^-|},\\
\bar{Z}_1^{top}(\bar{t}_1)_{\{S^-\},\{P_2^-\}} =& z^{\rm top}(g_s,\bar{t}_1)\sum_{Q^-} e^{-|Q^-|\bar{t}_1} q^{\frac{p+2G-2}{2}k_{Q^-}} W_{Q^-}^{2-2G}\prod_a \frac{W_{Q^-S_a^-}}{W_{Q^-}}\frac{W_{Q^-P_{2,a}^-}}{W_{Q^-}}\\
&\times e^{g_s n_2^-|P_2^-|}e^{-\frac{\bar{t}_1}{p+2G-2}|S^-|},
\end{aligned}
\tag{4.9}
$$

where a runs from 1 to $2G-2$. All of the Z_i^{top}, in the above are topological string amplitudes on $\mathcal{M}$, but with different configuration of branes. The two sets of ghost branes corresponding to representations labeled by $\{S\}$ and $\{P\}$ that appear, are qualitatively different.

The ghost branes associated to $\{P\}$ type representations correspond to the kinds of the non-normalizable Kähler moduli that we had before. The sum over $\{P\}$ type representations can be efficiently replaced by an integral,[14] and we can write (4.8) as

$$
\begin{aligned}
\sum_{l_1',\,l_2'=-\infty}^{\infty}\int\{dU\}\sum_{\{S_+\}} & Z_1^{top}(t_1-p_1l_1'g_s;\{U\})_{\{S_+\}}Z_2^{top}(t_2+p_1l_1'g_s+p_2l_2'g_s;\{U\})_{\{S_+\}}\\
\times\sum_{\{S_-\}} & \bar{Z}_2^{top}(\bar{t}_2-p_1l_1'g_s-p_2l_2'g_s;\{U^{-1}\})_{\{S_-\}}\bar{Z}_1^{top}(\bar{t}_1+p_1l_1'g_s;\{U\})_{\{S_-\}}
\end{aligned},
$$

[14] This uses the properties of the tensor product coefficients

$$\mathrm{Tr}_{P_1}U\mathrm{Tr}_{P_2}U=\sum_P N^P_{P_1P_2}\mathrm{Tr}_P U.$$

were $\{dU\} = \prod_{a=1}^{|2G-2|} dU_a$. This is in precise agreement with expectations of [11]: the above simply expresses the fact that there are no net electric charges turned on for the non-normalizable modes! This had to be the case, since we have not turned them on in the theory on the D4 branes.

The only magnetic charges turned on for these modes are those induced by the D4 branes. To see this, we need to consider the above amplitudes in more detail. Consider first the anti-BPS universe. This has negative D4 brane charge $(0, k)$ corresponding to k D4 banes on $\mathcal{D}' = \mathcal{L}_{p+2G-2}$. The $\{P\}$ ghost branes in this universe are along the opposite line bundle, i.e. along fibers of $\mathcal{L}_{-p}$. The intersection number of $\mathcal{D}'$ with the disks C' ending on the ghost branes of that universe are unambiguous and canonical

$$\#(\mathcal{D}' \cap C') = +1,$$

and compute the effective magnetic charge of the k D4 branes: This implies, as we discussed, that the sizes of the disks are fixed to

$$\mathrm{Re}(t'_C) = \frac{1}{2} k g_s.$$

Since $(n_1^- - n_2^+)g_s = kg_s$, this is in precise agreement with (4.8).

In the universe with positive D4 brane charge $(N, -k)$ we have N D4 branes along $\mathcal{D}$ and $-k$ along $\mathcal{D}'$. The $\{P\}$ ghost branes lie along the fibers of $\mathcal{L}_{p+2G-2}$ bundle. The intersections of disks along $\mathcal{D}'$ (ending on the $\{P\}$ ghost branes of that universe) with $\mathcal{D}$ are unambiguous and equal to 1, as we saw before. However, the intersections with $\mathcal{D}'$ are ambiguous. As as $(n_1^+ - n_2^-)g_s = (N + k)g_s$, the result in (4.8) is consistent with

$$\#(\mathcal{D}' \cap C) = 1, \tag{4.10}$$

implying

$$\mathrm{Re}(t_C) = \frac{1}{2}(N + k)g_s.$$

The $\{S\}$ ghost branes are a new phenomenon, as they appear only when we have 2 or more universes. Before we turn to discussing them, let's consider the general pattern of the baby universe creation.

4.4. Multi-universe amplitudes

The analysis of the previous subsections can be straightforwardly generalized to the case where the parent universe splits into M baby universes. The detailed computations are relegated to Appendix C. The YM amplitude in the M-universe case is

$$\begin{aligned}
\sum_{\{S^{\pm}_{ij}\}} \int \{dU\} \times Z^{top}(t_1;\{U\})_{\{S^+_{12}\},\{S^+_{13}\},\ldots,\{S^+_{1M}\}} \\
\times Z^{top}(t_2,\{U\})_{\{S^+_{12}\},\{S^+_{23}\},\{S^+_{24}\},\ldots,\{S^+_{2,M}\}} \\
\ldots \\
\times Z^{top}(t_M;\{U\})_{\{S^+_{1,M}\},\{S^+_{2,M}\},\ldots,\{S^+_{(M-1),M}\}} \\
\times \bar{Z}^{top}(\bar{t}_M,\{U^{-1}\})_{\{S^-_{1,M}\},\{S^-_{2,M}\},\ldots,\{S^-_{(M-1),M}\}} \\
\ldots \\
\times \bar{Z}^{top}(\bar{t}_2,\{U^{-1}\})_{\{S^-_{1,2}\};\{S^-_{2,3}\},\ldots,\{S^-_{2,M}\}} \\
\times \bar{Z}^{top}(\bar{t}_1;\{U^{-1}\})_{\{S^-_{12}\},\{S^-_{13}\},\ldots,\{S^-_{1,M}\}}
\end{aligned} \tag{4.11}$$

The Kähler moduli are given by

$$\begin{aligned}
t_1 =& \frac{1}{2}(p+2G-2)Ng_s + \frac{1}{2}pk_1g_s + i\theta, \\
t_2 =& -\frac{1}{2}(p+2G-2)k_1g_s - \frac{1}{2}pk_2g_s - i\theta, \\
t_3 =& \frac{1}{2}(p+2G-2)k_2g_s + \frac{1}{2}pk_3g_s + i\theta, \\
\ldots& \\
t_M =& \frac{1}{2}(p+2G-2)k_{M-1}g_s + i\theta,\ for\ M\ odd
\end{aligned}$$

and

$$t_M = -\frac{1}{2}pk_{M-1}g_s - i\theta,\ for\ M\ even$$

Here, we have omitted the discrete shifts in the Kähler moduli for simplicity. It is easy to see that this corresponds to the following pattern of baby universe creation: Initially we have one BPS universe with $(N,0)$ branes along $(\mathcal{D},\mathcal{D}')$ respectively. Then, this splits into two universes, a BPS one with $(N,-k_1)$ branes and another anti-BPS with $(0,k_1)$ branes. In creating a new, third baby universe, the universe with charge $(0,k_1)$ just created splits into two, one with $(-k_2, k_1)$ branes which is anti-BPS, and another, BPS one with $(k_2, 0)$ branes, and so on.

$$(N,0) \to (N,-k_1) \,\oplus\, (0,k_1) \,\to (N,-k_1) \oplus\, (-k_2,k_1) \,\oplus\, (k_2,0)$$

It is easy to see that this pattern and attractor mechanism exactly reproduces the Kähler moduli above, with a sign of θ correlated with whether the universe is BPS or anti-BPS. Moreover, throughout this process, the net electric and the magnetic charges for the non-normalizable Kähler moduli stay turned off, apart from the magnetic charges induced by the D4 branes. This explains (4.11) apart from the $\{S\}$ ghost branes which we turn to next.

4.5. The $\{S\}$ ghost branes

The $\{S\}$-ghost branes are pairwise interactions between the tops (bottoms) of the Fermi seas, i.e. between the chiral (anti-chiral) amplitudes. Suppose we just created a baby universe, so M increased by one. If the new baby universe is BPS and effective D4 brane charge in it is of the form $(k, 0)$, the $\{S\}$ ghost branes in it are along the fibers of $\mathcal{L}_{-p}$. Conversely, if it is anti-BPS with charge $(0, k)$, the $\{S\}$ ghost branes are along $\mathcal{L}_{p+2G-2}$. Thus, the $\{S\}$-ghost branes are along the line bundle where the D4 branes would have been in that universe. Its interaction with the parent universes depends only on whether they are BPS or anti-BPS. A parent BPS universe (one of the M original ones) has ghost branes along $\mathcal{L}_{-p}$ just like the baby. If the parent is anti-BPS, the $\{S\}$ ghost branes are along $\mathcal{L}_{p+2G-2}$.

This suggests the following interpretation. Before we considered baby universes, and in defining the partition function of the D4 branes, we had a choice of the boundary conditions at infinity on the D4 branes. In section two, we picked the bundle along the fibers to be flat, the only singularities of the bundle coming from D2 branes wrapping the Riemann surface and D0 branes bound to them. We can in principle pick any choice of boundary conditions at infinity that is consistent with toric symmetry we used to compute the partition function. In particular we can pick boundary conditions corresponding to having non-compact D2 branes at wrapping the fibers above $|2G-2|$ points on the Riemann surface.

Choosing the boundary conditions means instead of computing

$$\sum_{\mathcal{R}} S_{0\mathcal{R}}^{2-2G} q^{C_2(\mathcal{R})} e^{i\theta C_1(\mathcal{R})}$$

we compute

$$\sum_{\mathcal{R}} S_{\mathcal{Q}_1\mathcal{R}} \dots S_{\mathcal{Q}_{|2G-2|}\mathcal{R}} / S_{0\mathcal{R}}^{|4G-4|} q^{C_2(\mathcal{R})} e^{i\theta C_1(\mathcal{R})}. \tag{4.12}$$

This as corresponds to having D2 branes wrapping the fibers at $|2G-2|$ points on the Riemann surface, where $\mathcal{Q}_i^{(a)}$, $a = 1, ...N$ of them bind to the a-th D4 brane. This is because adding the D2 branes with charges $\mathcal{Q}$ above a point P on the Riemann surface corresponds to inserting an operator

$$\mathrm{Tr}_{\mathcal{Q}} e^{i\Phi(P)}$$

in the path integral. This shifts the expectation value of the flux F through the disk C_P centered at P by

$$\int_{C_P} F^{(a)} = \mathcal{Q}^{(a)} g_s, \qquad a = 1, \dots N$$

since

$$\int_{C_P} F^{(a)} = \oint_P A^{(a)},$$

and A and Φ are canonically conjugate [4]:

$$\langle \oint_P A^{(a)} \, \mathrm{Tr}_{\mathcal{Q}} \, e^{i\Phi(P)} \rangle = \mathcal{Q}^{(a)} g_s \, \langle \mathrm{Tr}_{\mathcal{Q}} \, e^{i\Phi(P)} \rangle.$$

This has exactly the same effect as placing D-branes there of charges $\mathcal{Q}^a$, as claimed. At large N, a $U(N)$ representation $\mathcal{Q}$ would factorize into two independent representations Q_+ and Q_-, so this does not introduce any additional correlations between the chiral and the anti-chiral amplitude. However, the Q_+ and Q_- dependence of this would look like introducing ghost branes along the fibers wrapped by the D4 branes – it would look precisely like the insertions of the $\{S\}$ ghost branes! From the closed string perspective, these ghost branes also correspond to turning on non-normalizable deformations of the closed string geometry, but these have their origin already in the non-normalizable deformations of the D4 branes – namely in turning on charges of non-compact D2 branes.

The explanation of the $\{S\}$ ghost brane correlations is then as follows. The original BPS universe corresponds to N branes along $\mathcal{D}$ with trivial boundary conditions at infinity, and no non-compact 2-brane charge. When it fractionates as

$$(N, 0) \to (N, -k) \;\oplus\; (0, k).$$

a baby anti-BPS universe is produced. However, the 2 baby universes that result have non-compact D2 brane charges turned on, but in such a way that the net brane charge is zero. At the next step

$$(N,0) \to (N,-k) \ \oplus \ (0,k) \ \to (N,-k) \oplus \ (-m,k) \ \oplus \ (m,0)$$

a new BPS universe is created with m units of D4 brane charge along $\mathcal{L}_{p_1}$, but also carrying some non-compact D2 brane charge. Their charge is canceled by creating non-compact 2-branes in the two older universes.[15]

Acknowledgments

We would like to thank Nori Iizuka, Marcos Marino, Natalia Saulina and Cumrun Vafa, for valuable discussions. The research of M.A. is supported in part by the UC Berkeley Center for Theoretical Physics, a DOI OJI Award, the Alfred P. Sloan Fellowship and the NSF grant PHY-0457317. The research of T.O. is supported in part by the NSF grants PHY-9907949 and PHY-0456556. H.O. is supported in part by the DOE grant DE-FG03-92-ER40701 and the NSF grant OISE-0403366.

Appendix A. Infinite products as finite products

In section two we explained how to take the large N limit of our amplitudes, based on the free fermion description. The free fermion description was easy to derive and convenient in terms of making contact to topological strings. Our derivation of the free fermion amplitudes was however heuristic.

[15] There are two subtleties we should mention. If we took (4.12) on the nose, the $\{S\}$-ghost branes and the $\{P\}$ ghost branes would be inserted at the same points on the Riemann surface. In the large N expansion of the amplitude at hand, the branes are not inserted at the same points. Moreover, from (4.11) it seems to follow that the non-compact 2-brane charge is conserved pairwise, because the correlations between the universes are pairwise. This appears rather odd, because all other charges have only net charge conserved. The two technical issues should be related. Namely, the reason the charges appear pairwise conserved is that the ghost branes connecting one universe with two others are not inserted at coincident points, but at different points. Since they are inserted at different points it is natural that the charges appear conserved only pairwise. In general, the amplitudes are independent of the location of the ghost-branes on the Riemann surface branes, as long as these do not coincide.

Here we will show how to rewrite the infinite product forms of the amplitudes – as appear naturally when we think of them as amplitudes of $N \to \infty$ of fermions, in terms of finite products. The finite product form of the amplitudes is easy to derive by direct algebraic manipulations (see Appendix B.) We will begin by making contact with derivation in [3] for the one universe case, as a warmup, and then proceed to more complicated ones.

A.1. The case of one universe

In section 3 we argued, based on the free fermion description of the states that

$$\begin{aligned} S_{O\mathcal{R}}/S_{00}(N, g_s) = &\prod_{1\leq i<j\leq\infty} \frac{[R_i^+ - R_j^+ + j - i]}{[j-i]} \prod_{1\leq i<j\leq\infty} \frac{[R_i^- - R_j^- + j - i]}{[j-i]} \\ &\times \prod_{1\leq i,j\leq\infty} \frac{[R_i^+ + R_j^- + N - i - j + 1]}{[N-i-j+1]} \end{aligned} \tag{A.1}$$

In this appendix we will explicitly regulate the infinite sums and show that above is equal to the finite product form of the amplitude derived in [3].

Let's start with the first factor:

$$\prod_{1\leq i<j\leq\infty} \frac{[R_i^+ - R_j^+ + j - i]}{[j-i]}.$$

Let c_{R^+} denote the number of rows in R^+. Then, we can write the above as

$$\prod_{1\leq i<j\leq c_{R^+}} \frac{[R_i^+ - R_j^+ + j - i]}{[j-i]}. \prod_{1\leq i\leq c_{R^+};1\leq j\leq\infty} \frac{[R_i^+ + c_{R^+} + j - i]}{[c_R^+ + j - i]}.$$

The infinite product factor can be written as a finite product

$$\prod_{1\leq i\leq c_{R^+};1\leq j\leq\infty} \frac{[R_i^+ + c_{R^+} + j - i]}{[c_{R^+} + j - i]} = \prod_{1\leq i\leq c_{R^+}} \prod_{1\leq \mu_i\leq R_i} \frac{1}{[c_{R^+} + \mu_i - i]}$$

Putting the contributions together (see for example, [24]),

$$\begin{aligned} \prod_{1\leq i<j\leq\infty} \frac{[R_i^+ - R_j^+ + j - i]}{[j-i]} &= \prod_{1\leq i<j\leq c_{R^+}} \frac{[R_i^+ - R_j^+ + j - i]}{[j-i]} \prod_{1\leq i\leq c_{R^+}} \prod_{1\leq \mu_i\leq R_i} \frac{1}{[c_{R^+} + \mu_i - i]} \\ &= d_q(R^+). \end{aligned}$$

The answer above is the quantum dimension $d_q(R)$ of the symmetric group representation corresponding to R. This can be rewritten in a perhaps more familiar form:

$$d_q(R) = \prod_{\square \in R} \frac{1}{[h(\square)]},$$

where $h(\square)$ is the hook length of the corresponding box.

There are different ways to think about the answer above, however, as either the topological string amplitude

$$d_q(R) = C_{R,0,0} q^{k_R/4} = W_{R,0}(q) q^{k_R/4}$$

or in terms of representation theory, as a Schur function.

$$W_{R,0}(q) = s_R(q)$$

Now, lets consider the last factor in (A.1):

$$\prod_{1 \le i < j < \infty} \frac{[R_i^+ + R_j^- + N - i - j + 1]}{[N - i - j + 1]}$$

We can again break it to finite products: denoting by c_{R^-} the number of non-zero rows in R^-,

$$\begin{aligned}
\prod_{1 \le i,j < \infty} \frac{[R_i^+ + R_j^- + N - i - j + 1]}{[N - i - j + 1]} &= \prod_{1 \le i < c_{R^+}; 1 \le j \le c_{R^-}} \frac{[R_i^+ + R_j^- + N - i - j + 1]}{[N - i - j + 1]} \\
&\times \prod_{1 \le i \le c_{R^+}; 1 \le j \le \infty} \frac{[R_i^+ + N - c_{R^-} - i - j + 1]}{[N - c_{R^-} - i - j + 1]} \\
&\times \prod_{1 \le i < \infty; 1 \le j \le c_{R^-}} \frac{[R_i^- + N - c_{R^+} - i - j + 1]}{[N - c_{R^+} - i - j + 1]}
\end{aligned}$$

The infinite products can be rewritten as follows:

$$\begin{aligned}
\prod_{1 \le i \le c_{R^+}; 1 \le j < \infty} \frac{[R_i^+ + N - c_{R^-} - i - j + 1]}{[N - c_{R^-} - i - j + 1]} &= \prod_{1 \le i \le c_{R^+}; 1 \le j < \infty} \frac{[R_i^+ + N - i - j + 1]}{[N - i - j + 1]} \\
&\times \prod_{1 \le i \le c_{R^+}; 1 \le j \le c_{R^-}} \frac{[N - i - j + 1]}{[R_i^+ + N - i - j + 1]} \\
&= \prod_{1 \le i \le c_{R^+}; 1 \le \mu_i \le R_i} [N + \mu_i - i] \\
&\times \prod_{1 \le i \le c_{R^+}; 1 \le j \le c_{R^-}} \frac{[N - i - j + 1]}{[R_i^+ + N - i - j + 1]}
\end{aligned}$$

In the last step we rewrote the one infinite product as a finite product.

Now, we will put everything together. First of all, recall that the quantum dimension $dim_q(R)$ of a representation R of $U(N)$ can be written as

$$dim_q(R) = \prod_{\square \in R} \frac{[N + j(\square) - i(\square)]}{h(\square)} = \prod_{i=1}^{c_{R^+}} \prod_{\mu_i=1}^{R_i} [N + \mu_i - i] d_q(R).$$

where j counts the columns and i the rows of the tableau.

Then, collecting all the factors, we have

$$S_{O\mathcal{R}}/S_{00}(N, g_s) = dim_q(R_+) dim_q(R_-) \prod_{i=1}^{c_{R^+}} \prod_{i=1}^{c_{R^-}} \frac{[R_i^+ + R_j^- + N - i - j + 1][N - i - j + 1]}{[R_i^+ + N - i - j + 1][R_i^- + N - i - j + 1]}$$

This is the result in [3].

A.2. The case of two universes

In a similar way to the one-universe case, we will here rewrite the infinite products of (4.1) and (4.2) as finite products. In appendix B we will obtain them from an honest computation involving only finite products.

Let us take the first factor in (4.1) and write it as

$$\begin{aligned}
\prod_{1 \le i,j < \infty} \frac{[R_i^+ - Q_j^+ + n_1^+ - n_2^+ - i + j]}{[n_1^+ - n_2^+ - i + j]} &= \prod_{1 \le i \le c_{R^+}; 1 \le j \le c_{Q^+}} \frac{[R_i^+ - Q_j^+ + n_1^+ - n_2^+ - i + j]}{[n_1^+ - n_2^+ - i + j]} \\
&\times \prod_{1 \le i \le c_{R^+}; 1 \le j < \infty} \frac{[R_i^+ + n_1^+ - n_2^+ + c_{Q^+} - i + j]}{[n_1^+ - n_2^+ + c_{Q^+} - i + j]} \\
&\times \prod_{1 \le i < \infty; 1 \le j \le c_{Q^+}} \frac{[-Q_j^+ + n_1^+ - n_2^+ - c_{R^+} - i + j]}{[n_1^+ - n_2^+ - c_{R^+} - i + j]}.
\end{aligned}$$

The first infinite product on the r.h.s. can be rewritten as follows:

$$\begin{aligned}
\prod_{1 \le i \le c_{R^+}; 1 \le j < \infty} \frac{[R_i^+ + n_1^+ - n_2^+ + c_{Q^+} - i + j]}{[n_1^+ - n_2^+ + c_{Q^+} - i + j]} &= \prod_{1 \le i \le c_{R^+}; 1 \le j < \infty} \frac{[R_i^+ + n_1^+ - n_2^+ - i + j]}{[n_1^+ - n_2^+ - i + j]} \\
&\times \prod_{1 \le i \le c_{R^+}; 1 \le j \le c_{Q^+}} \frac{[n_1^+ - n_2^+ - i + j]}{[R_i^+ + n_1^+ - n_2^+ - i + j]} \\
&= \prod_{1 \le i \le c_{R^+}; 1 \le \mu_i \le R_i^+} \frac{1}{[n_1^+ - n_2^+ - i + \mu_i]} \\
&\times \prod_{1 \le i \le c_{R^+}; 1 \le j \le c_{Q^+}} \frac{[n_1^+ - n_2^+ - i + j]}{[R_i^+ + n_1^+ - n_2^+ - i + j]}
\end{aligned}$$

Other infinite products in (4.1) and (4.2) can be regularized in the same way. Combining everything, we find that the interaction piece between two universes is

$$\begin{aligned}&\prod_{1\le i\le c_{R^+};1\le j\le c_{Q^+}}\frac{[R_i^+-Q_j^++n_1^+-n_2^+-i+j][n_1^+-n_2^+-i+j]}{[R_i^++n_1^+-n_2^+-i+j][-Q_j^++n_1^+-n_2^+-i+j]}\\&\times\prod_{1\le i\le c_{R^-};1\le j\le c_{Q^-}}\frac{[-R_i^-+Q_j^-+n_1^--n_2^-+i-j][n_1^--n_2^-+i-j]}{[-R_i^-+n_1^--n_2^-+i-j][Q_j^-+n_1^--n_2^-+i-j]}\\&\times\prod_{1\le i\le c_{R^+};1\le j\le c_{Q^-}}\frac{[R_i^++Q_j^-+n_1^+-n_2^--i-j+1][n_1^+-n_2^--i-j+1]}{[R_i^++n_1^+-n_2^--i-j+1][Q_j^-+n_1^+-n_2^--i-j+1]}\\&\times\prod_{1\le i\le c_{R^-};1\le j\le c_{Q^+}}\frac{[-R_i^--Q_j^++n_1^--n_2^++i+j-1][n_1^--n_2^++i+j-1]}{[-R_i^-+n_1^--n_2^++i+j-1][-Q_j^++n_1^--n_2^++i+j-1]}\\&\times\prod_{1\le i\le c_{R^+};1\le\mu_i\le R_i^+}\frac{[n_1^+-n_2^--i+\mu_i]}{[n_1^+-n_2^+-i+\mu_i]}\cdot\prod_{1\le j\le c_{Q^+};1\le\mu_j\le Q_j^+}\frac{[n_1^--n_2^+-\mu_j+j]}{[n_1^+-n_2^+-\mu_j+j]}\\&\times\prod_{1\le i\le c_{R^-};1\le\mu_i\le R_i^-}\frac{[n_1^--n_2^++i-\mu_i]}{[n_1^--n_2^-+i-\mu_i]}\cdot\prod_{1\le j\le c_{Q^-};1\le\mu_j\le Q_j^-}\frac{[n_1^+-n_2^-+\mu_j-j]}{[n_1^--n_2^-+\mu_j-j]}\end{aligned}\tag{A.2}$$

Appendix B. Derivation in terms of finite products

In this appendix, we will re-derive the interaction (A.2) between two universes by an honest computation involving only finite products. Let M_1, M_2 be integers such that $c_{R^+} \le M_1 \le N_1 - c_{R^-}$, $c_{Q^+} \le M_2 \le N_1 - c_{Q^-}$. The final result will not depend on M_1 or M_2. The interaction piece, up to the overall factor independent of tableaux, is

$$\begin{aligned}&\prod_{i=1}^{M_1}\prod_{j=1}^{M_2}\frac{[R_i^+-Q_j^++n_1^+-n_2^++j-i]}{[n_1^+-n_2^++j-i]}\\&\times\prod_{i=1}^{N_1-M_1}\prod_{j=1}^{N_2-M_2}\frac{[-R_i^-+Q_j^-+n_1^--n_2^--j+i]}{[n_1^--n_2^--j+i]}\\&\times\prod_{i=1}^{M_1}\prod_{j=1}^{N_2-M_2}\frac{[R_i^++Q_j^-+n_1^+-n_2^--i-j+1]}{[n_1^+-n_2^--i-j+1]}\\&\times\prod_{i=1}^{N_1-M_1}\prod_{j=1}^{M_2}\frac{[-R_i^--Q_j^++n_1^--n_2^++i+j-1]}{[n_1^--n_2^++i+j-1]},\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\prod_{i=1}^{M_1}\prod_{j=1}^{M_2}\frac{[R_i^+ - Q_j^+ + n_1^+ - n_2^+ + j - i][n_1^+ - n_2^+ + j - i]}{[R_i^+ + n_1^+ - n_2^+ + j - i][-Q_j^+ + n_1^+ - n_2^+ + j - i]}\\
&\times \prod_{i=1}^{N_1-M_1}\prod_{j=1}^{N_2-M_2}\frac{[-R_i^- + Q_j^- + n_1^- - n_2^- - j + i][n_1^- - n_2^- - j + i]}{[-R_i^- + n_1^- - n_2^- - j + i][Q_j^- + n_1^- - n_2^- - j + i]}\\
&\times \prod_{i=1}^{M_1}\prod_{j=1}^{N_2-M_2}\frac{[R_i^+ + Q_j^- + n_1^+ - n_2^- - i - j + 1][n_1^+ - n_2^- - i - j + 1]}{[R_i^+ + n_1^+ - n_2^- - i - j + 1][Q_j^- + n_1^+ - n_2^- - i - j + 1]}\\
&\times \prod_{i=1}^{N_1-M_1}\prod_{j=1}^{M_2}\frac{[-R_i^- - Q_j^+ + n_1^- - n_2^+ + i + j - 1][n_1^- - n_2^+ + i + j - 1]}{[-R_i^- + n_1^- - n_2^+ + i + j - 1][-Q_j^+ + n_1^- - n_2^+ + i + j - 1]}
\end{aligned} \tag{B.1}$$

times

$$\begin{aligned}
&\prod_{i=1}^{M_1}\prod_{j=1}^{M_2}\frac{[R_i^+ + n_1^+ - n_2^+ + j - i][-Q_j^+ + n_1^+ - n_2^+ + j - i]}{[n_1^+ - n_2^+ + j - i]^2}\\
&\times \prod_{i=1}^{N_1-M_1}\prod_{j=1}^{N_2-M_2}\frac{[-R_i^- + n_1^- - n_2^- - j + i][Q_j^- + n_1^- - n_2^- - j + i]}{[n_1^- - n_2^- - j + i]^2}\\
&\times \prod_{i=1}^{M_1}\prod_{j=1}^{N_2-M_2}\frac{[R_i^+ + n_1^+ - n_2^- - i - j + 1][Q_j^- + n_1^+ - n_2^- - i - j + 1]}{[n_1^+ - n_2^- - i - j + 1]^2}\\
&\times \prod_{i=1}^{N_1-M_1}\prod_{j=1}^{M_2}\frac{[-R_i^- + n_1^- - n_2^+ + i + j - 1][-Q_j^+ + n_1^- - n_2^+ + i + j - 1]}{[n_1^- - n_2^+ + i + j - 1]^2}.
\end{aligned} \tag{B.2}$$

Due to cancelations in the ratios, we can change the index ranges and rewrite (B.1) as the first four products in (A.2). (B.2) reduces to

$$\begin{aligned}
&\prod_{i=1}^{N_1}\prod_{j=1}^{N_2}\frac{[R_i^+ + n_1^+ - n_2^+ + j - i][-Q_j^+ + n_1^+ - n_2^+ + j - i]}{[n_1^+ - n_2^+ + j - i]^2}\\
&\times \frac{[-R_i^- + n_1^- - n_2^- - j + i][Q_j^- + n_1^- - n_2^- - j + i]}{[n_1^- - n_2^- - j + i]^2}.
\end{aligned}$$

It is easy to check that these agree with the last four products in (A.2).

Appendix C. Expanding $S^{-1}_{0,\mathcal{RQ}}$ and $S^{-1}_{0,\mathcal{R}_1\ldots\mathcal{R}_M}$

We first aim to rewrite the ghost brane correction part $\mathrm{cr}_{\mathcal{RQ}}{}^{-1}$ as defined in (4.4) in a convenient form. To do this recall the "coproduct" property of Schur functions:

$$s_R(x,y) = \sum_{P,Q} N^R_{PQ}\, s_P(x) s_Q(y)$$

where N_{PQ}^R are tensor product coefficients.[16] Using this, we will write

$$\prod_{i,j=1}^{\infty}(1-q^{n_1^+-n_2^++R_i^+-Q_j^+-i+j})^{-1}(1-q^{n_1^--n_2^--R_i^-+Q_j^-+i-j})^{-1}$$
$$\times(1-q^{n_1^+-n_2^-+R_i^++Q_j^--i-j+1})^{-1}(1-q^{n_1^--n_2^+-R_i^--Q_j^++i+j-1})^{-1}$$

as

$$\begin{aligned}&\sum_P s_P(q^{n_1^+}q^{R^++\rho},q^{n_1^-}q^{-R^--\rho})\ s_P(q^{-n_2^+}q^{-Q^+-\rho},q^{-n_2^-}q^{Q^-+\rho})\\ =&\sum_{P,P_1^\pm,P_2^\pm}N_{P_1^+P_1^-}^P\ s_{P_1^+}(q^{n_1^+}q^{R^++\rho})\ s_{P_1^-}(q^{n_1^-}q^{-R^--\rho})\\ &\quad\times N_{P_2^+P_2^-}^P\ s_{P_2^+}(q^{-n_2^+}q^{-Q^+-\rho})\ s_{P_2^-}(q^{-n_2^-}q^{Q^-+\rho}).\end{aligned}$$

We apply (3.5) to rewrite the Schur functions in terms of the ratios of W's. The result of the calculation is

$$\begin{aligned}\mathrm{cr}_{\mathcal{RQ}}{}^{-1}=&\sum_{P,P_1^\pm,P_2^\pm}N_{P_1^+P_1^-}^P N_{P_2^+P_2^-}^P\frac{W_{R^+S^+}}{W_{0R^+}}\frac{W_{R^+P_1^+}}{W_{R^+}}\frac{W_{R^-S^+}}{W_{0R^-}}\frac{W_{R^{-T}P_1^{-T}}}{W_{R^{-T}}}\\ &\times\frac{W_{Q^+S^-}}{W_{0Q^+}}\frac{W_{Q^{+T}P_2^{+T}}}{W_{Q^{+T}}}\frac{W_{Q^-S^-}}{W_{0Q^-}}\frac{W_{Q^-P_2^-}}{W_{Q^-}}\\ &\times(-1)^{|P_1^-|}(-1)^{|P_2^+|}\ q^{N_1|S^+|+N_2|S^-|+n_1^+|P_1^+|+n_1^-|P_1^-|-n_2^+|P_2^+|-n_2^-|P_2^-|}.\end{aligned}\tag{C.1}$$

We now compute (4.5). The dependence of the quadratic Casimir on $(R^\pm,Q^\pm)$ can be read off from its relation to the fermion energy $C_2(\mathcal{RQ})=2E(\mathcal{RQ})+\text{const}$. Also note that $C_2(R^\pm=Q^\pm=0)=N_1l_1^2+N_2l_2^2+N_1N_2(l_1-l_2)$. We then have

$$\begin{aligned}C_2(\mathcal{RQ})=&N_1l_1^2+N_2l_2^2+N_1N_2(l_1-l_2)+k_{R^+}+k_{R^-}+k_{Q^+}+k_{Q^-}\\ &+2n_1^+|R^+|-2n_1^-|R^-|+2n_2^+|Q^+|-2n_2^-|Q^-|.\end{aligned}\tag{C.2}$$

[16] When $x=(x_1,\ldots,x_{N_1})$ and $y=(y_1,\ldots y_{N_2})$ this describes branching rules of $U(N_1+N_2)\to U(N_1)\times U(N_2)$, as long as R is a representation obtained by tensoring copies of fundamental representation of $U(N_1+N_2)$.

In the sector where the ghost branes are turned off, the YM amplitude (4.5) becomes:

$$\sum_{l_1,l_2,R_\pm,Q_\pm} q^{\frac{(p+2G-2)^2}{2p}\rho(N)^2}\ e^{\frac{N\theta^2}{2pg_s}}\ M(q)^{4-4g}$$
$$\times\, q^{\frac{p}{2}N_1l_1^2+\frac{p}{2}N_2l_2^2}\ q^{pl_1(|R_+|-|R_-|)+pl_2(|Q_+|-|Q_-|)}\ q^{\frac{(p+2G-2)}{2}N_1N_2(l_1-l_2)}$$
$$\times\, q^{\frac{(p+2G-2)N_2}{2}(|R_+|-|R_-|)}\ q^{-\frac{(p+2G-2)N_1}{2}(|Q_+|-|Q_-|)} e^{i\theta(N_1l_1+|R_+|-|R_-|)} e^{i\theta(N_2l_2+|Q_+|-|Q_-|)}\ .$$
$$\times\ q^{\frac{(p+2G-2)}{2}(k_{R_+}+k_{R_-})}\ q^{\frac{(p+2G-2)}{2}(k_{Q_+}+k_{Q_-})}\ q^{\frac{(p+2G-2)N_1}{2}(|R_+|+|R_-|)}\ q^{\frac{(p+2G-2)N_2}{2}(|Q_+|+|Q_-|)}$$
$$\times\, W_{R^+}^{2-2G}\ W_{R^-}^{2-2G}\ W_{Q^+}^{2-2G}\ W_{Q^-}^{2-2G} \tag{C.3}$$

This expression can be rewritten in terms of closed topological string partition functions as in (4.6). The YM amplitude with ghost brane contributions is given in (4.9).

In the M-universe case, the pant amplitude is

$$\begin{aligned}
&S^{-1}_{0\mathcal{R}_1\ldots\mathcal{R}_M}(g_s,N)\\
&=\pm M(q)^{-M}\prod_{I=1}^{M}\eta(q)^{-N_I}q^{\frac{1}{2}(k_{R_I^+}+k_{R_I^-})}q^{\frac{N_I}{2}(|R_I^+|-|R_I^-|)}W^{-1}_{R_I^+}W^{-1}_{R_I^-}\\
&\times\prod_{1\le I<J\le M} q^{\frac{1}{2}N_IN_J(l_I-l_J)}q^{\frac{N_J}{2}(|R_I^+|-|R_I^-|)-\frac{N_I}{2}(|R_J^+|-|R_J^-|)}\\
&\times\prod_I\prod_{i,j=1}^{\infty}(1-q^{n_I^+-n_I^-+R^+_{I,i}+R^-_{I,j}-i-j+1})^{-1}\\
&\times\prod_{I<J}\prod_{i,j=1}^{\infty}(1-q^{n_I^+-n_J^++R^+_{I,i}-R^+_{J,j}-i+j})^{-1}(1-q^{n_I^--n_J^--R^-_{I,i}+R^-_{J,j}+i-j})^{-1}\\
&\times(1-q^{n_I^+-n_J^-+R^+_{I,i}+R^-_{J,j}-i-j+1})^{-1}(1-q^{n_I^--n_J^+-R^-_{I,i}-R^+_{J,j}+i+j-1})^{-1}.
\end{aligned}$$

The infinite products in the last three lines can be written in terms of the W-functions as in the two-universe case. We can again use the co-product property of Schur functions, by putting the $2M$ Fermi surfaces into two groups, the upper ones (with fluctuations $R_1^+,R_1^-,\ldots,R^-_{(M+1)/2}$ if M is odd) and the lower ones (with fluctuations $R^-_{(M+1)/2},\ldots,R_M^-,R_M^+$). The quadratic Casimir in the M-universe case decomposes as

$$\begin{aligned}
&C_2(\mathcal{R}_1\ldots\mathcal{R}_M)\\
&=\sum_{I=1}^{M}\left(l_IN_I(l_I-\sum_{J<I}N_J+\sum_{J>I}N_J)+k_{R_I^+}+k_{R_I^-}+2n_I^+|R_I^+|-2n_I^-|R_I^-|\right),
\end{aligned}$$

where we have defined the $U(1)$ charge of $\mathcal{R}_I$ by

$$l_I := \frac{1}{2}\left(\sum_{J<I} N_J + n_I^+ + n_I^- - \sum_{J>I} N_J\right) .$$

These ingredients are used to express the YM partition function in the M-universe case in terms of topological string partition functions, as in (4.11) .

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