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# Stochastic Models: Theory and Simulation

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# Stochastic Models: Theory and Simulation

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**Abstract**

Many problems in applied science and engineering involve physical phenomena that behave randomly in time and/or space. Examples are diverse and include turbulent flow over an aircraft wing, Earth climatology, material microstructure, and the financial markets. Mathematical models for these random phenomena are referred to as stochastic processes and/or random fields, and Monte Carlo simulation is the only general-purpose tool for solving problems of this type. The use of Monte Carlo simulation requires methods and algorithms to generate samples of the appropriate stochastic model; these samples then become inputs and/or boundary conditions to established deterministic simulation codes. While numerous algorithms and tools currently exist to generate samples of simple random variables and vectors, no cohesive simulation tool yet exists for generating samples of stochastic processes and/or random fields. There are two objectives of this report. First, we provide some theoretical background on stochastic processes and random fields that can be used to model phenomena that are random in space and/or time. Second, we provide simple algorithms that can be used to generate independent samples of general stochastic models. The theory and simulation of random variables and vectors is also reviewed for completeness.

# Acknowledgment

The author would like to acknowledge the help and guidance of Professor Mircea Grigoriu of Cornell University. Prof. Grigoriu has done pioneering work in the area of stochastic mechanics and random vibration, and I made extensive use of his books and lecture notes during the preparation of this report. I would also like to thank Mike Eldred, Tom Paez, and Laura Swiler of Sandia National Laboratories and Prof. Kurt Maute of the University of Colorado for their many useful comments and discussions on this manuscript.

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# Chapter 1

# Introduction

Stochastic models are utilized in many fields of applied science and engineering. Parzen [30] provides a nice summary of early applications of stochastic modeling in statistical physics, population growth, and communication and control. A diverse set of examples from agriculture, astronomy, dynamics, economy, environment, geotechnics, hydrology, mechanics, medicine, and transportation are provided in [20], Chapter 2. Random variables are the most common type of stochastic model; their use is limited, however, to phenomena described by a single parameter. Examples include time-invariant material properties or boundary conditions at a fixed location *e.g.*, stiffness, hardness, temperature, *etc.*

Often, phenomena can be viewed as random in space and/or time. To illustrate, we provide three examples of such phenomena of interest at Sandia National Laboratories. First consider Fig. 1.1, which provides a schematic of an aircraft encountering a storm system during flight. As the aircraft travels through a cloud, it collides with a sequence of moisture particles suspended within the atmosphere. The size and number density of these particles are described by probabilistic climate models so that the resulting force on the aircraft fuselage is random in time. A random function of time is referred to as a stochastic process; Fig. 1.2 illustrates samples from a stochastic process model used to represent the force applied to the nose of the aircraft during an in-flight storm encounter. The two plots correspond to two distinct storms of different severity [7]. We note that, in both cases, the samples are not symmetric with respect to their temporal mean.

Foam materials are used in numerous weapon systems to protect internal components from shock, thermal, and electromagnetic loads. One particular application of interest, depicted by Fig. 1.3, is the use of a particular epoxy foam for shock mitigation within a complex aerospace component. Experimental work with the foam has demonstrated that the material density can vary significantly from specimen to specimen, as well as vary spatially within a single specimen; we therefore need models for foam density to capture this behavior. A random field can be used to represent phenomena that vary randomly in space; two samples of a particular random field used for this application are illustrated by Fig. 1.4.

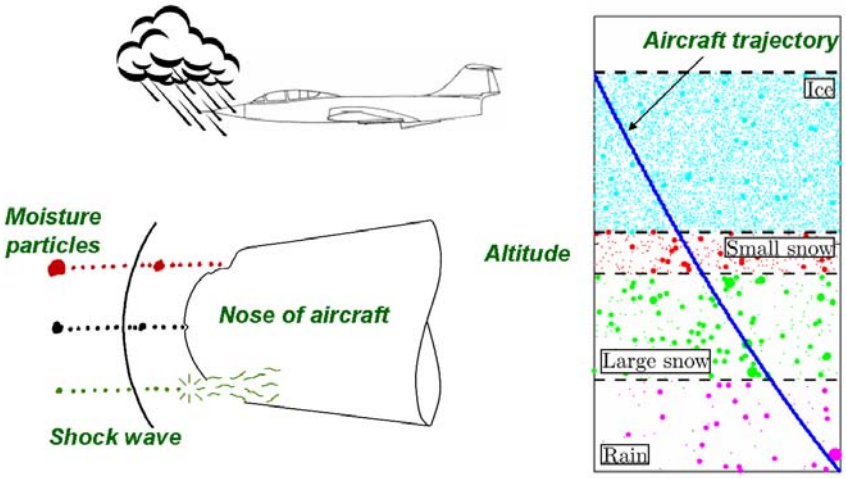


**Figure 1.1.** A schematic of an in-flight storm encounter.

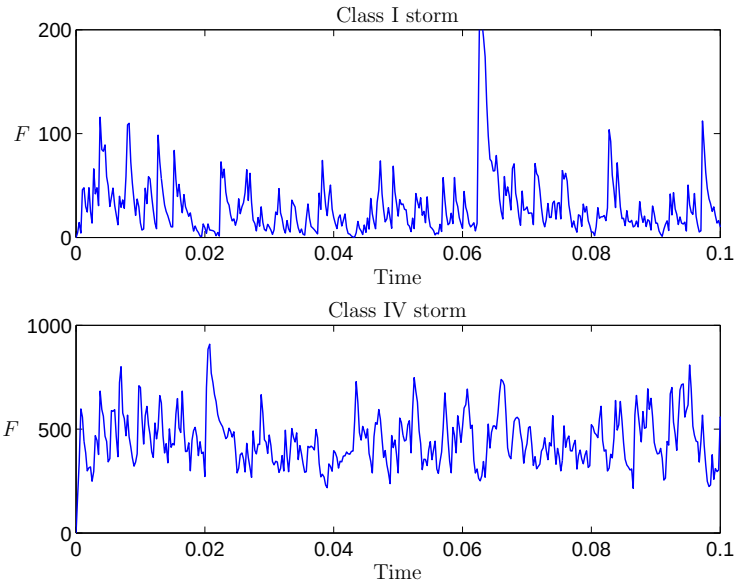


**Figure 1.2.** Stochastic model for nose force $F$ due to impact with a sequence of moisture particles during encounter with a Class I and IV storm.

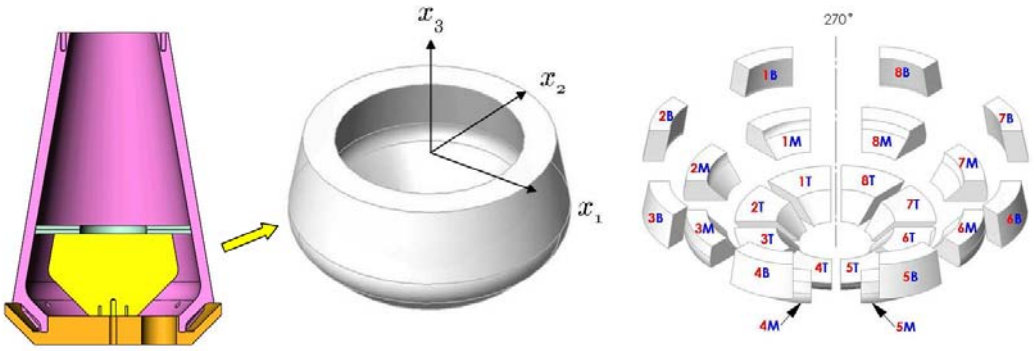


**Figure 1.3.** A schematic of one of five foam specimens split into 24 cells of equal volume for measurement of foam density (from [28]).

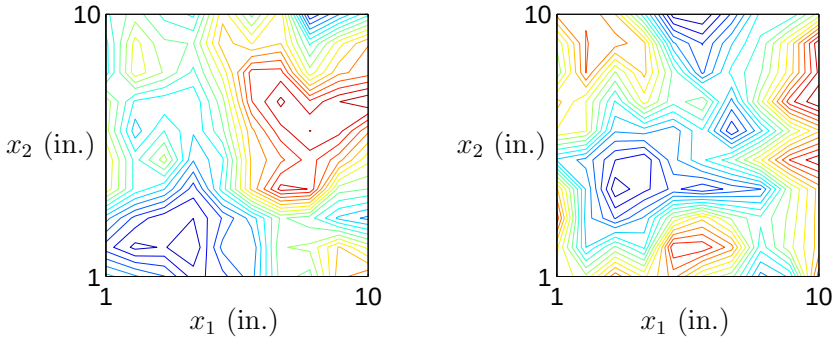


**Figure 1.4.** Two samples of a random field model for foam density along cross section of a 10 in. $\times$ 10 in. $\times$ 10 in. cube (taken from [13]). Regions of high density are shown in red; regions of low density are shown in blue.

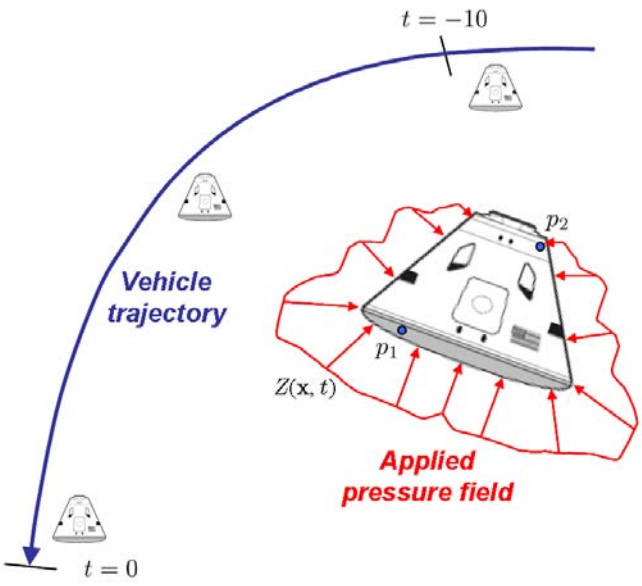


**Figure 1.5.** Turbulent flow over space vehicle during re-entry.

Lastly, as depicted by Fig. 1.5, we consider the case of fully turbulent airflow over a space vehicle. Pressure fluctuations in the turbulent boundary layer provide dynamic excitation to the vehicle and its internal components; this dynamic excitation is random in both time and space. Figure 1.6 shows one sample of a space-time stochastic process model used to represent the applied pressure field; two plots are shown, corresponding to the pressure applied at two distinct spatial locations on the surface of the space vehicle. We note that the scale of the fluctuations is time-dependent.

There are two objectives of this report. First, we provide some theoretical background on stochastic processes and random fields that can be used to model phenomena that are random in space and/or time. As motivated by the above examples, special emphasis is placed on models that are non-Gaussian and/or non-stationary to capture phenomena that is not symmetric (Fig. 1.2) or where the scale of the uncertainty may be space/time dependent (Figs. 1.4 and 1.6). Second, we provide simple algorithms that can be used to generate independent samples of general stochastic models. It is hoped that, in the future, these simple algorithms will be incorporated into an efficient, cohesive toolkit for use with the current modeling and simulation environment at Sandia National Laboratories.

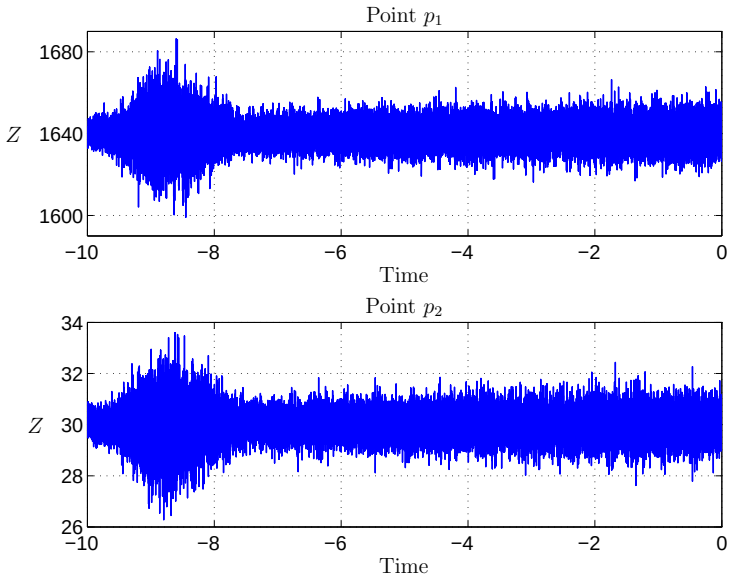


**Figure 1.6.** Space-time stochastic process model for turbulent flow at points $p_1$ and $p_2$ on SV surface.

# Chapter 2

# Essentials of random variables and vectors

Consider a random experiment, that is, an experiment with a random or unpredictable outcome. It is assumed that all possible distinct outcomes of a random experiment are known, and they are elements of a fundamental set $\Omega$ known as the sample space. If we assign a real number $X(\omega)$ to each outcome $\omega \in \Omega$ following a certain set of rules, the result is called a random variable, and the value of $X(\omega)$ for fixed $\omega$ is referred to as a sample of random variable $X$. It should be obvious that $X$ is therefore a function that maps each outcome of the experiment to the real line. We note that it is common practice to omit the functional dependence of random variable $X(\omega)$ on $\omega$ and simply write $X$.

The above discussion is a very brief overview of some of the underlying fundamentals of probability theory; more complete discussions are presented in [29] and [42]. In this Chapter, we will present essentials on the theory of random variables and random vectors in Sections 2.1 and 2.2, respectively; the simulation of random variables and vectors will be discussed in Chapter 3. Our convention throughout this report will be to use capital letters to denote random quantities, and lower-case letters for deterministic quantities.

## 2.1 Random variables

Random variables are defined by a cumulative distribution function, briefly reviewed in Section 2.1.1. Special properties of random variables, referred to as statistical moments, are discussed in Section 2.1.2. The Gaussian random variable, presented in Section 2.1.3 is a particular type of random variable often used in practice.

### 2.1.1 Distribution and density

As mentioned above, a random variable is a function that maps elements of the sample space to values on the real line. The most direct way to define a random variable is therefore to define this mapping. However, this approach is not very useful in practice. An alternative approach is to instead define the probability law, denoted by $\Pr(X \leq x)$, for all fixed values $x$ on the real line. This expression is interpreted as the "probability that random variable $X$ takes values less than or equal to $x$."

Let $X$ be a random variable; $X$ is defined by its **cumulative distribution function** (CDF), *i.e.*,

$$F(x) = \Pr(X \leq x), \ -\infty < x < \infty. \tag{2.1}$$

We can show that: (i) $F$ is a right continuous, non-decreasing function of $x$ with range $[0, 1]$, *i.e.*, a probability is between zero and one, (ii) $\lim_{x\to\infty} F(x) = 1$, (iii) $\lim_{x\to-\infty} F(x) = 0$, and (iv) $\Pr(a < X \leq b) = F(b) - F(a) \geq 0$ for $a \leq b$. A random variable $X$ is called discrete if it takes discrete values on the real line; in this case, it is possible to enumerate all the values $X$ may assume. In contrast, a continuous random variable takes values over continuous intervals of the real line. In the discussion that follows, we will assume continuous random variables. For discussion on discrete random variables, see [1], [29], or [35].

Assuming $F(x)$ is differentiable,

$$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x} \tag{2.2}$$

is the **probability density function** (PDF) of $X$, where: (i) $f(x) \geq 0$ because $F(x)$ is non-decreasing, (ii) $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = \Pr(a < X \leq b)$, and (iii) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$. Further, we note that while $F$ is a probability, $f$ is not. In practice, it is more common to define random variables using the PDF, $f(x)$, rather than the CDF, $F(x)$. Some examples of common random variables follow.

**Example 2.1**: $C$ is a Cauchy random variable if $f(c) = a/[\pi(c^2 + a^2)]$, $a > 0$, $-\infty < c < \infty$. The corresponding CDF is $F(c) = 1/\pi \arctan(c/a) + 1/2$. The PDF and CDF of the Cauchy random variable with $a = 1/2$ are illustrated by Fig. 2.1(a) and (d), respectively. $\lozenge$

**Example 2.2**: $X$ is an Exponential random variable if $f(x) = \lambda \exp(-\lambda x)$, $\lambda > 0$, $x \geq 0$. The corresponding CDF is $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$. The PDF and CDF of the Exponential random variable with $\lambda = 1$ are illustrated by Fig. 2.1(b) and (e), respectively. $\lozenge$

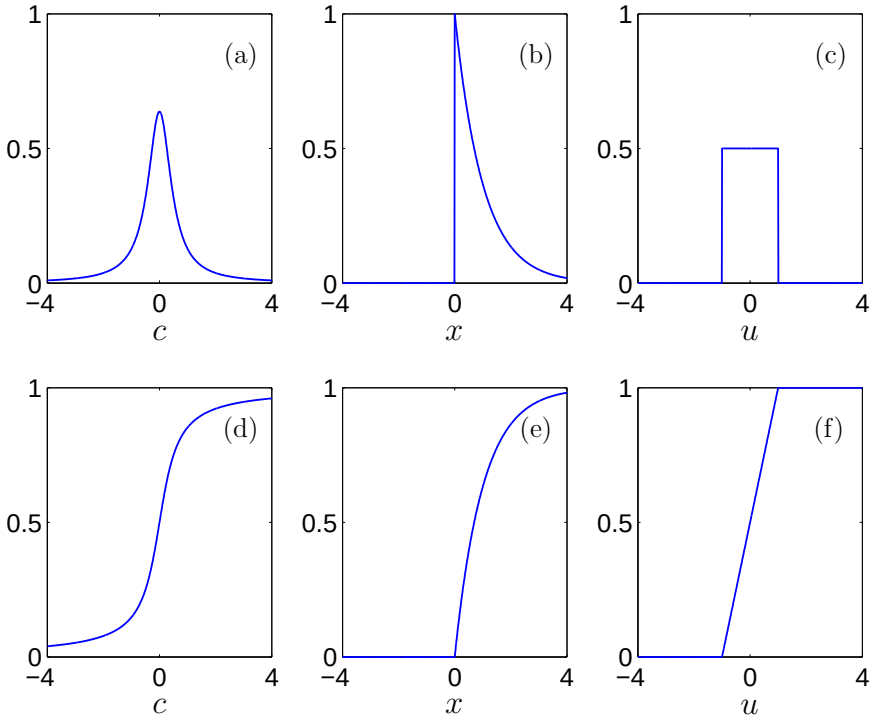


**Figure 2.1.** PDFs (panels (a), (b), and (c)) and CDFs (panels (d), (e), and (f)) of the Cauchy, Exponential, and Uniform random variables.

**Example 2.3**: $U$ is a Uniform random variable if $f(u) = 1/(\beta - \alpha)$, $\alpha \leq u \leq \beta$. The corresponding CDF is

$$F(u) = \begin{cases} 0 & u < \alpha \\ \dfrac{u - \alpha}{\beta - \alpha} & \alpha \leq u \leq \beta \\ 1 & u > \beta \end{cases}$$

The PDF and CDF of the Uniform random variable with $\alpha = -1$ and $\beta = 1$ are illustrated by Fig. 2.1(c) and (f), respectively. $\Diamond$

### 2.1.2 Moments

The study of weighted averages of random variables is very useful in applications. To do so, we make use of the operator of mathematical expectation, denoted by $\mathrm{E}[\,\cdot\,]$. The expected

value of a function of random variable, $h(X)$, is given by

$$\mathrm{E}[h(X)] = \int_{-\infty}^{\infty} h(x)\,\mathrm{d}F(x) = \int_{-\infty}^{\infty} h(x)\,f(x)\,\mathrm{d}x, \tag{2.3}$$

where the latter follows assuming $F$ is differentiable. We can compute the **moments** of random variable $X$ (assuming they exist) via

$$\mathrm{E}[X^p] = \int_{-\infty}^{\infty} x^p\,f(x)\,\mathrm{d}x. \tag{2.4}$$

For $p = 1$ we get the **mean** of $X$, commonly denoted by symbol $\mu = \mathrm{E}[X]$. The **mean-square** value of $X$ is obtained by Eq. (2.4) with $p = 2$; the **root-mean-square (RMS)** value of $X$, commonly used in engineering applications, is the square root of the mean-square value of $X$, *i.e.*, $\sqrt{\mathrm{E}[X^2]}$.

An alternative definition, referred to as the **central moments** of $X$, is

$$\mathrm{E}[(X-\mu)^p] = \int_{-\infty}^{\infty} (x-\mu)^p\,f(x)\,\mathrm{d}x, \tag{2.5}$$

where $\mu$ is the mean of $X$ described above. For $p = 2$, we get the **variance** of $X$, commonly denoted by $\sigma^2 = \mathrm{E}[(X-\mu)^2]$; the **standard deviation** of $X$ is $\sigma = \sqrt{\mathrm{E}[(X-\mu)^2]}$. Note that, by Eqs. (2.4) and (2.5), the RMS value and standard deviation of $X$ are not the same in general; they are identical if, and only if, $\mu = 0$.

We remark that if $X$ is a random variable with zero mean, unit variance, and PDF $f$, then the PDF of random variable $Y = \mu + \sigma\,X$ is a shifted and scaled version of $f$ with the same shape. Further, $Y$ has mean $\mu$ and variance $\sigma^2$; these properties hold for both Gaussian and non-Gaussian random variables. Because of this, it is common to work with zero-mean, unit-variance random variables.

**Example 2.4**: Let $X$ be an exponential random variable. The mean and variance of $X$ are given by

$$\mu = \mathrm{E}[X] = \int_0^{\infty} x\,\lambda\,e^{\lambda\,x}\,\mathrm{d}x = \frac{1}{\lambda}$$
$$\sigma^2 = \mathrm{E}[(X-\mu)^2] = \int_0^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda\,e^{\lambda\,x}\,\mathrm{d}x = \frac{1}{\lambda^2}$$

Random variable $Y = (X-\mu)/\sigma = \lambda\,X - 1$ is also an exponential random variable, but with zero mean and unit variance. $\Diamond$

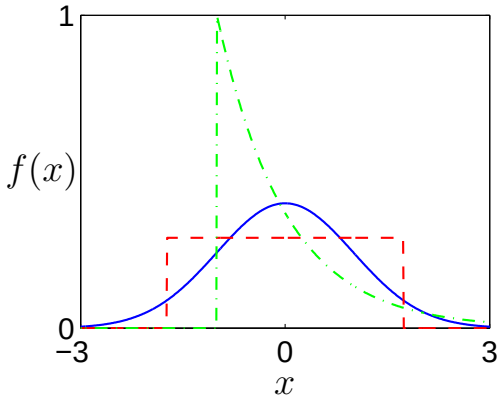


**Figure 2.2.** Three PDFs for $X$ such that $\mu = 0$ and $\sigma^2 = 1$.

The **coefficients of skewness** and **kurtosis**, given by

$$\gamma_3 = \frac{\mathrm{E}[(X-\mu)^3]}{\sigma^3} \quad \text{and} \quad \gamma_4 = \frac{\mathrm{E}[(X-\mu)^4]}{\sigma^4} \tag{2.6}$$

are other commonly used moments. They provide a measure of the symmetry and rate of decay in the tails of the PDF of $X$, respectively. If $\gamma_3 = 0$, the distribution of a random variable is symmetric about its mean value.

**Example 2.5**: The moments, $\mathrm{E}[C^p]$, of the Cauchy random variable $C$ are undefined for $p \geq 1$, demonstrating that the mean, variance, skewness, kurtosis, or other moments of a random variable may not always exist. ♢

**Example 2.6**: It is typical in applications to have a finite collection of samples of $X$ from which we can estimate the first few moments of $X$. It is important to realize that having these estimates is much less information about $X$ than is the PDF or CDF. In other words, one cannot uniquely define the CDF $F(x)$ given a collection $\{\mathrm{E}[X^p], p = 1, 2, \ldots, n\}$. To illustrate, consider the case where $\mu = 0$ and $\sigma^2 = 1$, but no additional information on moments or the distribution of $X$ is available. Three possible PDFs for $X$ are illustrated by Fig. 2.2; many more distributions are possible. While a random variable is completely defined by its PDF or CDF, it is only partially defined by its moments. ♢

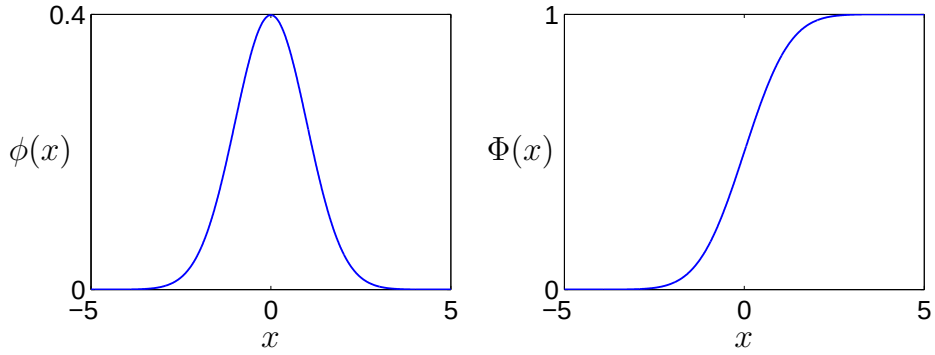


**Figure 2.3.** The PDF and CDF of a standard Gaussian random variable.

### 2.1.3 The Gaussian random variable

The Gaussian random variable is probably the most commonly used model for random phenomena. $X$ is a **Gaussian random variable** (also called a **normal random variable**) with mean $\mu$ and variance $\sigma^2$ if, and only if, the PDF of $X$ is

$$f(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma}\,\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right],\ -\infty < x < \infty, \tag{2.7}$$

or, equivalently, if and only if the CDF of $X$ is

$$F(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma}\int_{-\infty}^{x}\exp\left[-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right]\mathrm{d}u,\ -\infty < x < \infty. \tag{2.8}$$

It is common to use $X \sim N(\mu, \sigma^2)$ to denote a Gaussian random variable with mean $\mu$ and variance $\sigma^2$. We can show that the coefficients of skewness and kurtosis defined by Eq. (2.6) are $\gamma_3 = 0$ and $\gamma_4 = 3$, respectively, for a Gaussian random variable. For the special case when $\mu = 0$ and $\sigma = 1$, we say that $X \sim N(0, 1)$ is a **standard Gaussian random variable** and use the notation $f(x) = \phi(x)$ and $F(x) = \Phi(x)$ to denote the PDF and CDF, respectively; $\phi(x)$ and $\Phi(x)$ are illustrated by Fig. 2.3.

If the PDF and CDF of $X$ are different from Eqs. (2.7) and (2.8), respectively, we say that $X$ is a **non-Gaussian random variable**. Non-Gaussian models are useful when, for example, we know that the range of $X$ is bounded or its probability law is not symmetric with respect to its mean (non-zero skewness). The Exponential, Cauchy, and Uniform cases discussed above are all examples of non-Gaussian random variables. Let $X$ be a non-Gaussian

random variable with CDF $F$; then random variable

$$Y = F^{-1}(U), \tag{2.9}$$

has the same distribution as $X$, where $U$ denotes a random variable uniformly distributed on $[0, 1]$ (see [33], Section 8.3). Note that because $F$ is monotonic, $F^{-1}$ always exists. Hence, any random variable can be written as a deterministic mapping of a Uniform random variable. This result is very useful for generating samples of non-Gaussian random variables.

## 2.2 Random vectors

Let $\mathbf{X} = (X_1, \ldots, X_d)^T$ be a vector with $d \geq 1$ coordinates; $\mathbf{X}$ is a random vector if, and only if, each coordinate $X_i$ is a random variable. The concepts of distributions, densities, and moments discussed in Section 2.1 extend to random vectors; these extensions are presented in Sections 2.2.1 and 2.2.2, respectively. The Gaussian random vector is presented in Section 2.2.3. A brief overview of the polynomial chaos approximation for non-Gaussian random vectors, a popular technique at Sandia National Laboratories and the probabilistic methods community at large, is given in Section 2.2.4.

### 2.2.1 Joint distribution and density

Random vector $\mathbf{X}$ is defined by its **joint cumulative distribution function**, *i.e.*,

$$F(\mathbf{x}) = \Pr(X_1 \leq x_1 \cap \cdots \cap X_d \leq x_d),\ \mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d, \tag{2.10}$$

where $\Pr(A_1 \cap \cdots \cap A_n)$ is the probability that events $A_1, \ldots, A_n$ are all true. If $F$ is such that

$$f(\mathbf{x}) = \frac{\partial^d F(\mathbf{x})}{\partial x_1 \cdots \partial x_d} \tag{2.11}$$

exists, then $f$ is called the **joint probability density function** of $\mathbf{X}$. As for the scalar case, $F$ is monotonic non-decreasing and takes values on $[0, 1]$, and $f$ is non-negative and satisfies

$$\int_{\mathbb{R}^d} f(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1. \tag{2.12}$$

We note that the word "joint" in the above definitions is important since it implies that all possible dependencies among the coordinates of $\mathbf{X}$ are completely described by $f(\mathbf{x})$ or,

equivalently, by $F(\mathbf{x})$. For applications, we usually do not know all the interdependencies between the coordinates of $\mathbf{X}$ so that the joint CDF and/or joint PDF are only partially known.

The distribution of one or more coordinates of $\mathbf{X}$ can be obtained from the joint distribution or the joint density of $\mathbf{X}$. For example, the **marginal distribution** and **marginal density** of $X_1$ are

$$F_1(x_1) = F(x_1, \infty, \ldots, \infty) \text{ and} \tag{2.13}$$

$$f_1(x_1) = \int_{\mathbb{R}^{d-1}} f(\mathbf{x})\, \mathrm{d}x_2 \cdots \mathrm{d}x_d = \frac{\mathrm{d}F_1(x_1)}{\mathrm{d}x_1} \tag{2.14}$$

respectively. Knowledge of marginal distributions $F_1, \ldots, F_d$ is not, in general, equivalent to knowledge of the complete joint CDF of $\mathbf{X}$ defined by Eq. (2.10).

### 2.2.2 Second-moment properties

We next apply the expectation operator used in Section 2.1.2 to the case of random vectors. Define

$$\begin{aligned} \mu_i &= \mathrm{E}[X_i], \\ r_{i,j} &= \mathrm{E}[X_i\, X_j], \quad \text{and} \\ c_{i,j} &= \mathrm{E}[(X_i - \mu_i)\,(X_j - \mu_j)], \end{aligned} \tag{2.15}$$

for $i, j = 1, \ldots, d$. The $d \times d$ **correlation** and **covariance matrices** of random vector $\mathbf{X}$ are $\mathbf{r} = \{r_{i,j}\} = \mathrm{E}\left[\mathbf{X}\,\mathbf{X}^T\right]$ and $\mathbf{c} = \{c_{i,j}\} = \mathrm{E}\left[(\mathbf{X} - \boldsymbol{\mu})\,(\mathbf{X} - \boldsymbol{\mu})^T\right]$, respectively, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)^T$ is the **mean vector**. It can be shown that both $\mathbf{r}$ and $\mathbf{c}$ are symmetric and positive-definite, and $\mathbf{c} = \mathbf{r} - \boldsymbol{\mu}\,\boldsymbol{\mu}^T$. For the special case where each coordinate of $\mathbf{X}$ has zero mean, $\boldsymbol{\mu} = \mathbf{0}$ and $\mathbf{r} = \mathbf{c}$. The variance of $\mathbf{X}$ is given by the diagonal elements of $\mathbf{c}$, *i.e.*, $\mathrm{Var}[X_i] = c_{i,i}$, $i = 1, \ldots, d$.

We remark that there is some ambiguity on the definition of correlation. The standard deviation in probability theory is $\mathbf{r} = \mathrm{E}\left[\mathbf{X}\,\mathbf{X}^T\right]$ defined by Eq. (2.15). An alternative definition used often in statistics is $\boldsymbol{\rho} = \{\rho_{i,j}\}$, where

$$\rho_{i,j} = \frac{c_{i,j}}{\sqrt{c_{i,i}\, c_{j,j}}}, \tag{2.16}$$

is commonly referred to as the (Pearson) **correlation coefficient** between coordinates $X_i$ and $X_j$; we note that by Eq. (2.16), $-1 \leq \rho_{i,j} \leq 1$.

Quantities $\boldsymbol{\mu}$, $\mathbf{r}$, $\mathbf{c}$, and $\boldsymbol{\rho}$ are collectively referred to as the **second-moment properties** of $\mathbf{X}$. Knowing the second-moment properties is much less information about $\mathbf{X}$

than is knowing the joint PDF, $f(\mathbf{x})$ defined by Eq. (2.11), or even the marginal PDFs, $f_1(x_1), \ldots, f_d(x_d)$ defined by Eq. (2.14).

Two special cases related to the interdependencies of the coordinates of $\mathbf{X}$ are of interest and they are often confused. Random variables $X_1, \ldots, X_d$ are **independent** if, and only if,

$$F(x_1, \ldots, x_d) = \prod_{i=1}^{d} F_i(x_i), \text{ or, equivalently } f(x_1, \ldots, x_d) = \prod_{i=1}^{d} f_i(x_i), \tag{2.17}$$

where $F_i$ and $f_i$ denote the marginal CDF and marginal PDF of random variable $X_i$, respectively. The coordinates of $\mathbf{X}$ are said to be **uncorrelated** if, and only if, the covariance matrix, $\mathbf{c}$, is diagonal, meaning that $\mathbf{c}_{i,j} = \mathrm{E}[(X_i - \mu_i)(X_j - \mu_j)] = 0$, $i \neq j$. In summary, independence implies requirements on the joint CDF and joint PDF, while requirements on the second-moment properties are sufficient for random variables to be uncorrelated. The latter is a much weaker condition than the former. It follows that if $X_1, \ldots, X_d$ are independent, then they are uncorrelated; the converse is not true in general.

### 2.2.3 The Gaussian random vector

We say $\mathbf{X} = (X_1, \ldots, X_d)^T$ is a **Gaussian random vector** with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{c}$, in short $\mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{c})$ if, and only if, it has joint PDF

$$f(\mathbf{x}) = \left[(2\,\pi)^d \det(\mathbf{c})\right]^{-1/2} \exp\left[-\frac{1}{2}\left((\mathbf{x} - \boldsymbol{\mu})^T \mathbf{c}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)\right], \ \mathbf{x} \in \mathbb{R}^d, \tag{2.18}$$

or, equivalently, if it has joint CDF

$$F(\mathbf{x}) = \left[(2\,\pi)^d \det(\mathbf{c})\right]^{-1/2} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \exp\left[-\frac{1}{2}\left((\mathbf{u} - \boldsymbol{\mu})^T \mathbf{c}^{-1} (\mathbf{u} - \boldsymbol{\mu})\right)\right] \mathrm{d}u_1 \cdots \mathrm{d}u_d \tag{2.19}$$

where $\det(\mathbf{c}) > 0$ denotes the determinant of $\mathbf{c}$.

**Example 2.7**: Consider the case of $d = 2$, $\boldsymbol{\mu} = \mathbf{0}$, $c_{1,1} = c_{2,2} = 1$, and $c_{1,2} = c_{2,1} = \rho$, where $|\rho| \leq 1$; this is referred to as the standard bivariate Gaussian vector. The joint PDF of $\mathbf{X} = (X_1, X_2)^T$ is illustrated by Fig. 2.4. Contours of $f(\mathbf{x})$ are also plotted for $\rho = 0$, $-1/3$, and $3/4$; when $\rho = 0$, the coordinates of $\mathbf{X}$ are uncorrelated. $\Diamond$

The Gaussian random vector has two important properties that prove very useful for applications. First, uncorrelated Gaussian vectors are independent. To illustrate the second property, let $\mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{c})$ be a Gaussian random vector and let $\mathbf{a}$ and $\mathbf{b}$ be $q \times d$ and $q \times 1$

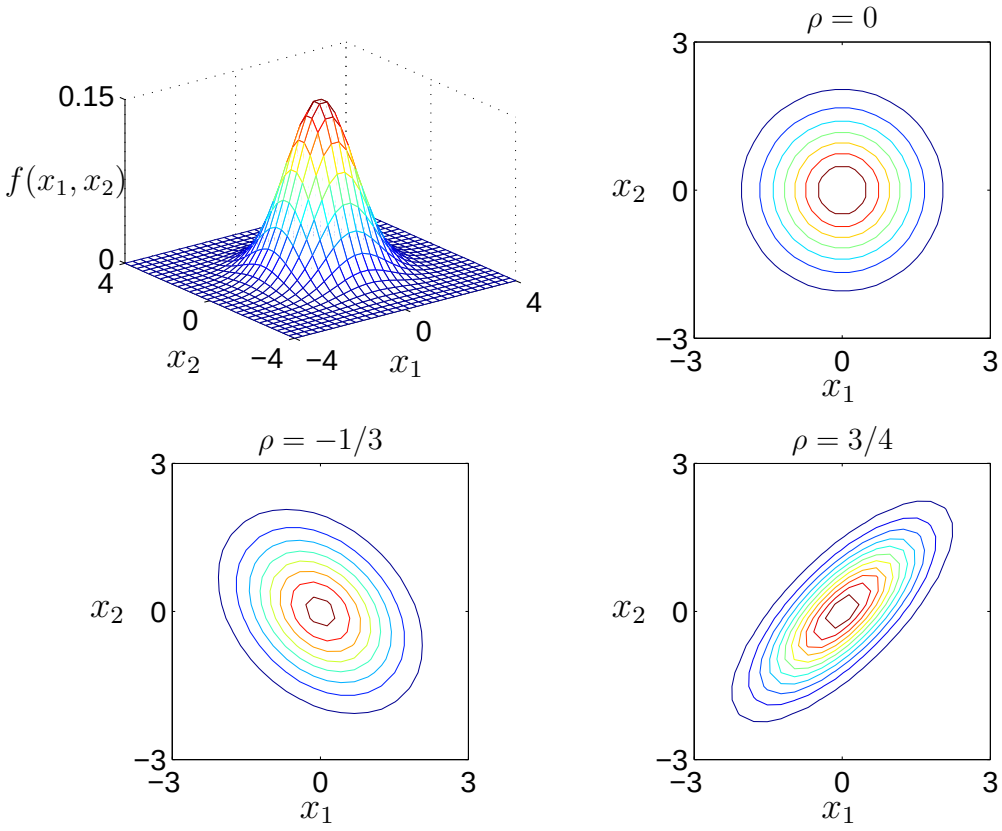


**Figure 2.4.** Joint PDF of the bivariate Gaussian vector.

constant matrices. It can be shown that $\mathbf{Y} = \mathbf{a}\,\mathbf{X} + \mathbf{b}$ is a Gaussian random vector with mean $\mathbf{a}\,\boldsymbol{\mu} + \mathbf{b}$ and covariance matrix $\mathbf{a}^T\mathbf{c}\,\mathbf{a}$. Hence, linear transformations of a Gaussian vector are Gaussian vectors; a proof of this result is provided by [21], p. 67. Non-Gaussian random vectors do not satisfy either property. For example, if $X_1$ and $X_2$ are independent random variables distributed uniformly on intervals $[a, b]$ and $[c, d]$, respectively, such that $b - a = d - c$, then the density of $Y = X_1 + X_2$ has a triangular shape (see [29], p. 136) so that $Y$ is not a uniform random variable.

### 2.2.4 Approximation by polynomial chaos

Polynomial chaos (PC) representations for non-Gaussian random variables are infinite series of Hermite polynomials of standard Gaussian random variables with deterministic coefficients. They can be viewed as an extension to the classical use of infinite series of orthogonal functions to represent certain classes of deterministic functions on the real line, *e.g.*, the Fourier series. For calculations, the infinite PC representations are truncated at a finite number of terms, creating what are herein referred to as PC approximations.

As we will demonstrate, the use of PC approximations requires us to calculate the collection of deterministic coefficients that defines the approximation, and this can prove difficult for some problems. However, once the coefficients are obtained, the use of PC approximations for representing non-Gaussian random variables is straightforward because they are expressed simply as sums and products of independent standard Gaussian random variables. PC approximations have been applied to a diverse set of problems in fluid, structural, and thermal mechanics.

Consider the class of non-Gaussian, $\mathbb{R}^d-$valued random variables, $\mathbf{Y}$, that can be written as a function of a standard Gaussian random vector, *i.e.*,

$$\mathbf{Y} = \mathbf{g}(\mathbf{X}), \tag{2.20}$$

where $\mathbf{g} : \mathbb{R}^k \to \mathbb{R}^d$ is a deterministic mapping, $\mathbf{X} \sim N(\mathbf{0}, \mathbf{i})$ is an $\mathbb{R}^k-$ valued vector of independent, identically distributed (iid), zero mean, unit variance Gaussian random variables, and $\mathbf{i}$ denotes the $k \times k$ identity matrix. It is assumed that mapping $\mathbf{g}$ is such that all coordinates of $\mathbf{Y}$ have finite variance.

The series [17, 23]

$$Y_j = g_j(\mathbf{X}) = \sum_{i_1, i_2, \ldots, i_k \geq 0} a^{(j)}_{i_1, i_2, \ldots, i_k}\, h_{i_1, i_2, \ldots, i_k}(\mathbf{X}),\ j = 1, 2, \ldots, d, \tag{2.21}$$

is convergent in $L_2$ and constitutes the **polynomial chaos (PC) representation** for coordinate $Y_j$ of $\mathbf{Y}$. Here, $a^{(j)}_{i_1, i_2, \ldots, i_k}$, $j = 1, 2, \ldots, d$, are deterministic coefficients that must be

determined, and $h_{i_1,i_2,\ldots,i_k}$ are $k-$dimensional orthogonal Hermite polynomials given by [20]

$$h_{\alpha,\ldots,\omega}(\mathbf{X}) = e^{\frac{1}{2}\mathbf{X}^T\mathbf{X}}\left(-\frac{\partial}{\partial X_\alpha}\right)\cdots\left(-\frac{\partial}{\partial X_\omega}\right)e^{-\frac{1}{2}\mathbf{X}^T\mathbf{X}}, \tag{2.22}$$

where $\mathbf{X} = (X_1,\ldots,X_k)^T$ and $\alpha,\ldots,\omega$ are indices in $\{0,\ldots,k\}$ that need not be different. The Hermite polynomials have the properties

$$\begin{aligned}\mathrm{E}\left[h_{\alpha_i,\ldots,\omega_i}(\mathbf{X})\,h_{\alpha_j,\ldots,\omega_j}(\mathbf{X})\right] &= \mathrm{E}\left[h_{\alpha_i,\ldots,\omega_i}(\mathbf{X})^2\right]\,\delta_{ij},\\ \mathrm{E}\left[h_{\alpha_i,\ldots,\omega_i}(\mathbf{X})\right] &= \begin{cases}1, & i=0,\\ 0, & \forall i\geq 1,\end{cases}\end{aligned} \tag{2.23}$$

and define an orthogonal basis in $L_2$ [16]. The coefficients of the series in Eq. (2.21) can be calculated by exploiting the properties of Eq. (2.23). For example, the PC representation in Eq. (2.21) simplifies to

$$Y = g(X_1,X_2) = \sum_{i_1,i_2=0}^{\infty} a_{i_1,i_2}\,h_{i_1,i_2}(X_1,X_2) \tag{2.24}$$

for $d=1, k=2$, where $a^{(1)}_{i_1,i_2} = a_{i_1,i_2}$ and the first few Hermite polynomials of Eq. (2.22) are

$$\begin{aligned}h_{0,0}(X_1,X_2) &= 1,\\ h_{1,0}(X_1,X_2) &= X_1,\\ h_{0,1}(X_1,X_2) &= X_2,\\ h_{2,0}(X_1,X_2) &= X_1^2-1,\\ h_{1,1}(X_1,X_2) &= X_1\,X_2,\\ h_{0,2}(X_1,X_2) &= X_2^2-1.\end{aligned} \tag{2.25}$$

The coefficients of the series in Eq. (2.24) result from

$$\begin{aligned}\mathrm{E}\left[Y_j\,h_{j_1,j_2}(X_1,X_2)\right] &= \mathrm{E}\left[\sum_{i_1,i_2=0}^{\infty} a_{i_1,i_2}\,h_{i_1,i_2}(X_1,X_2)\,h_{j_1,j_2}(X_1,X_2)\right]\\ &= \sum_{i_1,i_2=0}^{\infty} a_{i_1,i_2}\mathrm{E}\left[h_{i_1,i_2}(X_1,X_2)\,h_{j_1,j_2}(X_1,X_2)\right]\\ &= a_{j_1,j_2}\,\mathrm{E}\left[h_{j_1,j_2}(X_1,X_2)^2\right],\end{aligned} \tag{2.26}$$

provided that the expectation can be calculated term by term. Under this condition, the coefficients in Eq. (2.21) are given by

$$a^{(j)}_{i_1,i_2,\ldots,i_k} = \frac{\mathrm{E}\left[Y_j\,h_{j_1,j_2,\ldots,j_k}(\mathbf{X})\right]}{\mathrm{E}\left[h_{j_1,j_2,\ldots,j_k}(\mathbf{X})^2\right]},\ j=1,2,\ldots,d. \tag{2.27}$$

**Example 2.8**: Let

$$Y = g(X) = e^X,\ X \sim N(0,1),$$

so that $Y$ is a lognormal random variable with CDF $F(y) = \Phi(\ln y),\ y > 0$, where $\Phi(\,\cdot\,)$ denotes the CDF of a $N(0,1)$ random variable. It follows that $Y = \sum_{k\geq 0} a_k\, h_k(X)$ is the PC representation for $Y$, where (see [11])

$$a_k = \frac{1}{k!}\,\mathrm{E}\left[e^X\, h_k(X)\right] = \frac{1}{\sqrt{2\pi}\,k!}\int_{-\infty}^{\infty}\exp\left(u - \frac{u^2}{2}\right) h_k(u)\,\mathrm{d}u = \frac{1}{k!}\,e^{1/2}, k = 0, 1, \ldots.$$

$\Diamond$

**Example 2.9**: Let $Y$ have a uniform distribution over $[\alpha, \beta]$, meaning that it can be expressed as

$$Y = g(X) = \alpha + (\beta - \alpha)\,\Phi(X),\ X \sim N(0,1).$$

The coefficients of the PC representation for $Y$ are given by (see [12], Appendix B)

$$\begin{aligned} a_0 &= \frac{\alpha + \beta}{2}, \quad a_{2k} = 0,\ k = 1, \ldots \\ a_{2k+1} &= (-1)^k\,\frac{(\beta - \alpha)\,(2k)!}{2^{2k+1}\,\sqrt{\pi}\,(2k+1)!\,k!},\ k = 0, 1, \ldots \end{aligned}$$

$\Diamond$

For calculations, the infinite series discussed above must be truncated at a finite number of terms. Let

$$Y_j^{(p)} = g_j^{(p)}(\mathbf{X}) = \sum_{\substack{i_1, i_2, \ldots, i_k \geq 0 \\ i_1 + i_2 + \cdots + i_k \leq q}} a_{i_1, i_2, \ldots, i_k}^{(j)}\, h_{i_1, i_2, \ldots, i_k}(\mathbf{X}),\ j = 1, 2, \ldots, d, \tag{2.28}$$

be the **PC approximation** for $Y_j$ in Eq. (2.21), which consists of Hermite polynomials up to and including order $q$. The PC approximation for $Y_j$ has $p + 1$ terms, where [17]

$$p = \sum_{s=1}^{q}\frac{1}{s!}\left\{\prod_{r=0}^{s-1}(k + r)\right\}. \tag{2.29}$$

For example, the PC approximation for the series in Eq. (2.24) is

$$Y^{(p)} = \sum_{\substack{i_1, i_2 \geq 0 \\ i_1 + i_2 \leq q}} a_{i_1, i_2}\, h_{i_1, i_2}(X_1, X_2). \tag{2.30}$$

It can be shown that the PC approximation of Eq. (2.28) approaches the PC representation of Eq. (2.21) in mean square as the number of terms retained, $p$, increases. However, this property can be of limited use for some applications since, generally, $p$ is small so that asymptotic properties of $Y_j^{(p)}$ do not apply. More discussion on the accuracy and convergence properties of PC approximations is presented in [11, 12].

# Chapter 3

# Simulation of random variables and vectors

Essentials of random variables and vectors were presented in Chapter 2; we next present methods for generating samples of these random quantities. Simple MATLAB algorithms are presented to generate samples of Gaussian and non-Gaussian random variables and vectors. As mentioned, these methods are well-established so our discussion will be brief. Methods to verify that the generated samples achieve the desired properties are also presented.

## 3.1 Gaussian random variables and vectors

Let $X$ be a standard Gaussian random variable, *i.e.*, a Gaussian random variable with zero mean and unit variance. In MATLAB, we can generate $n$ independent samples of $X$ using:

```
x=randn(1,n);
```

Further, we note that:

```
y = m + sqrt(v)*x;
```

will give $n$ independent samples of $Y$, a Gaussian random variable with mean `m` and variance `v`.

Next let $\mathbf{X} = (X_1, \ldots, X_d)^T$ denote a vector of $d$ uncorrelated Gaussian random variables with zero mean and unit variance. In MATLAB, we can generate $n$ independent samples of $\mathbf{X}$ using:

```
x=randn(d,n);
```

where each column of `x` is one sample of vector $\mathbf{X}$. These samples can be mapped to samples of $\mathbf{Y}$, a Gaussian random vector with mean vector `m` and covariance matrix `c`, by the use of

the Cholesky or eigenvalue decomposition of `c`:

```
b=transpose(chol(c));
[v,w]=eig(c);
for i=1:n,
   y1(:,i) = m + b*x(:,i);          % Cholesky decomposition
   y2(:,i) = m + v*w.^(1/2)*x(:,i); % eigenvalue decomposition
end
```

where `b` is a lower triangular matrix of Cholesky factors for `c`, while `w` and `v` denote $n \times n$ matrices of eigenvalues and eigenvectors, respectively, such that `c = v*w*v`$^T$. Arrays `y1` and `y2` are both collections of samples of $\mathbf{Y}$; it may be advantageous to use the eigenvalue decomposition (`y2`) if `n` is large or `c` is not well-conditioned.

## 3.2 Non-Gaussian random variables and vectors

Let $Y$ be a non-Gaussian random variable with CDF $F$; we can make use of Eq. (2.9) to generate independent samples of $Y$. This approach, referred to as the Inverse Transform Method, always applies but can be inefficient if $F^{-1}$ must be calculated numerically. Alternative techniques include the methods of Rejection, Decomposition, and Acceptance-Complement (see [6], Chapter 2). The polynomial chaos approximation discussed in Section 2.2.4 can also be used. However, this approach may prove infeasible since calculating the coefficients of the approximation can be quite difficult (see Eq. (2.27)).

**Example 3.1**: Let $X$ be an exponential random variable with mean $1/\lambda$, $\lambda > 0$, so that $F_X(x) = 1 - \exp(-\lambda x)$ and $F_X^{-1}(u) = -\ln(1-u)/\lambda$. Let $C$ be a Cauchy random variable with parameter $a > 0$ so that $F_C(c) = 1/\pi \arctan(c/a) + 1/2$ and $F_C^{-1}(u) = a \tan(\pi(u - 1/2))$. In MATLAB, we can generate $n$ independent samples of $X$ and $C$ by:

```
u=rand(1,n);
x=-log(1-u)/lambda;
c=a*tan(pi*(u-1/2));
```

A histogram of 1000 independent samples of $U$ is illustrated by Fig. 3.1(a). The histogram of the corresponding samples of $X$ and $C$ are illustrated by Fig. 3.1(b) and (c) assuming $\lambda = a = 1$. $\Diamond$

The simulation of non-Gaussian random vectors with arbitrary distribution requires knowledge of the joint CDF which, as mentioned, is usually unknown for problems of practical interest. One class of non-Gaussian random vectors useful in practice are the so-called

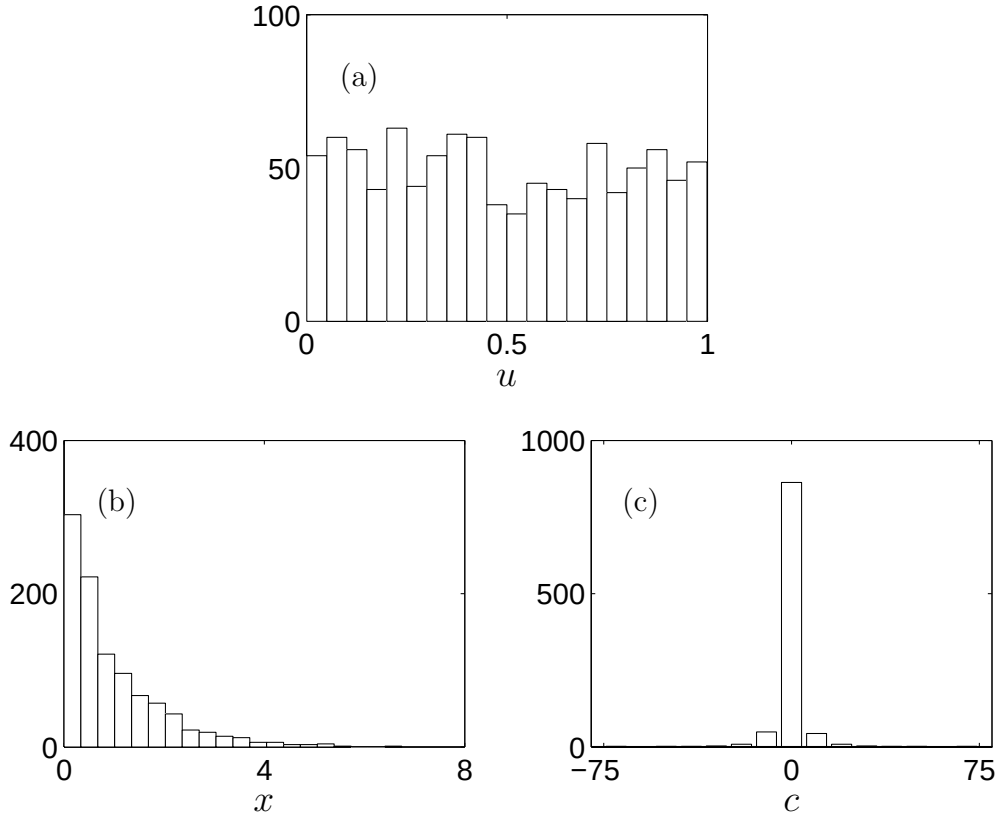


**Figure 3.1.** Histograms of 1000 independent samples of: (a) uniform random variable $U$, (b) exponential random variable $X$, and (c) Cauchy random variable $C$.

translation vectors, defined by $\mathbf{Y} = \mathbf{h}(\mathbf{X})$, where $\mathbf{X} = (X_1, \ldots, X_d)^T$ is a Gaussian vector with zero mean and covariance matrix $\mathbf{c} = \{c_{i,j} = \mathrm{E}[X_i\, X_j]\}$ such that $c_{i,i} = 1$, $i = 1, \ldots, d$, and

$$Y_i = h_i(X_i) = F_i^{-1}\left[\Phi(X_i)\right]. \tag{3.1}$$

In this case, samples of translation random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)^T$ can be generated from samples of Gaussian random vector $\mathbf{X}$ (Section 3.1) and the mapping defined by Eq. (3.1). The method is attractive since all that is required are the marginal distributions for each coordinate of $\mathbf{Y}$, and the second-moment properties of $\mathbf{X}$. One shortcoming of this approach is that the user must specify the covariance of the Gaussian vector $\mathbf{X}$ instead of the non-Gaussian vector $\mathbf{Y}$. Further, $\mathrm{E}[Y_i\, Y_j] = \mathrm{E}[h_i(X_i)\, h_j(X_j)] \neq c_{i,j}$ so that the correlation of $\mathbf{Y}$ and $\mathbf{X}$ are, in general, different. Approximate methods to address this issue have been developed (see, for example, [5]).

## 3.3 Verification

Various checks are available to verify that samples of a random variable or vector created by the above methods achieve the desired properties. In general, it is useful to check the mean, variance, covariance, and marginal distribution of the samples. In MATLAB, we can make use of functions `mean`, `var`, `cov`, and `hist`, respectively, to do this. More sophisticated verification checks for the marginal distribution are available; examples include the Chi-Square Goodness of Fit and Kolmogorov-Smirnov Tests (see [24], Chapter 10).

**Example 3.2**: Let $\mathbf{X} = (X_1, \ldots, X_d)^T$ be a zero-mean Gaussian random vector with $d = 20$ coordinates such that $\mathrm{E}[X_i\, X_j] = e^{-|i-j|/5}$, $i, j = 1, \ldots, 20$. Suppose we have used the methods from Section 3.1 to generate $n = 1000$ samples of $\mathbf{X}$, denoted by $d \times n$ array `x`. In MATLAB:

```
% estimate and plot mean vector
m=mean(x');
stairs([1:d],m)

% estimate and plot covariance matrix
c=cov(x');
mesh([1:d],[1:d],c)
```

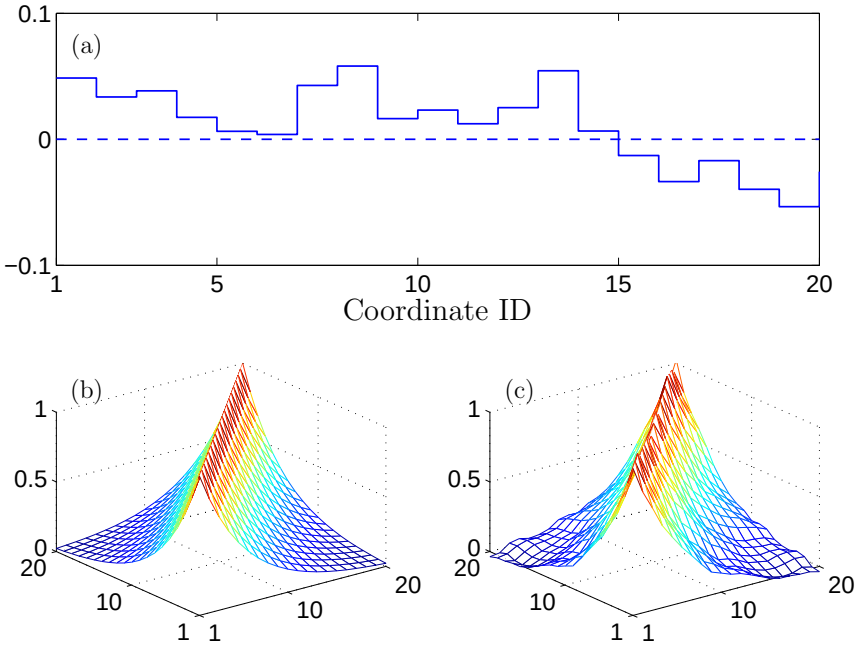


**Figure 3.2.** Second-moment properties of Gaussian random vector: (a) sample mean (dashed line) and exact mean (solid line), (b) exact covariance matrix, and (c) estimated covariance matrix.

```
% estimate and plot marginal PDF of coordinate 1
nb=20;x1=x(1,:);
[nx,xx]=hist(x1,nb);
dx=(max(x1)-min(x1))/nb;
f1=nx/(n*dx);
bar(xx,f1,'w')
```

can be used to check the mean, covariance, and marginal histograms of the samples. These results are illustrated by Figs. 3.2 and 3.3. Estimates are based on 1,000 Monte Carlo samples. ◊

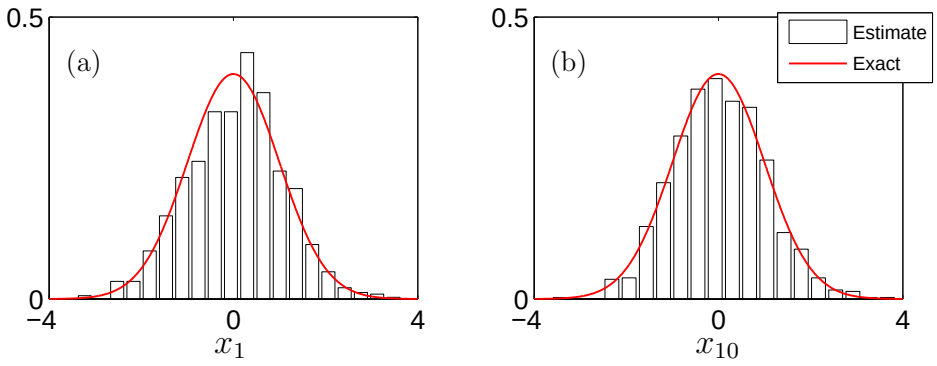


**Figure 3.3.** Marginal density of Gaussian random vector: (a) coordinate $X_1$, and (b) coordinate $X_{10}$. Estimates are based on 1,000 Monte Carlo samples.

# Chapter 4

# Essentials of stochastic processes

Recall random vector $\mathbf{X} = (X_1, \ldots, X_d)^T$ discussed in Section 2.2. We next consider the case where each coordinate of $\mathbf{X}$ is indexed by $t$ so that we write $\mathbf{X} = \mathbf{X}(t)$. Typically, $t$ is a time coordinate defined on $t \in [0, \infty)$, an uncountable index set. If $d = 1$, we call $\mathbf{X}(t) = X(t)$ a **stochastic process**, and if $d > 1$, we call $\mathbf{X}(t)$ a **vector stochastic process**. Four samples of a stochastic process are illustrated by Fig. 4.1 showing that the value for $X$ changes in time $t$ as well as from sample to sample. If $\mathbf{X}$ is indexed by a space coordinate $\mathbf{u} \in D \subset \mathbb{R}^q$ rather than time $t$, then $\mathbf{X}(\mathbf{u})$ is called a **random field**. There are also situations where $\mathbf{X}$ can be indexed by coordinates in both time and space; in this case, we refer to $\mathbf{X}(t, \mathbf{u})$ as a **space-time stochastic process**. The remainder of this report is limited to scalar and vector stochastic processes; many of the definitions, concepts, and numerical algorithms can be extended to consider random fields and/or space-time stochastic processes.

**Example 4.1**: Let $X(t) = A\cos(10t)$, $t \geq 0$, where $A$ is a random variable uniformly distributed on $[0, 1]$. $X(t)$ is a stochastic process because at any fixed time $t_0$, $X(t_0)$ is a uniform random variable. Three samples of $X(t)$ are illustrated by Fig. 4.2 for $0 \leq t \leq 1$. $\Diamond$

**Example 4.2**: Let $B(t)$, $t \geq 0$, be a real-valued stochastic process such that it: (i) starts at zero, *i.e.*, $B(0) = 0$, and (ii) has increments $\Delta B(t) = B(t + \Delta t) - B(t)$ over non-overlapping time intervals $\Delta t > 0$ that form a sequence of independent, Gaussian random variables with zero mean and variance $\Delta t$. By (ii), the increments of this process satisfy $\Delta B(t) \sim N(0, \Delta t)$. Process $B(t)$ is called a **Brownian motion** (or Wiener process); three independent samples of $B(t)$ are illustrated by Fig. 4.3. This model for Brownian motion was first developed by Einstein in 1905 to represent the random movement of particles suspended in a fluid. $\Diamond$

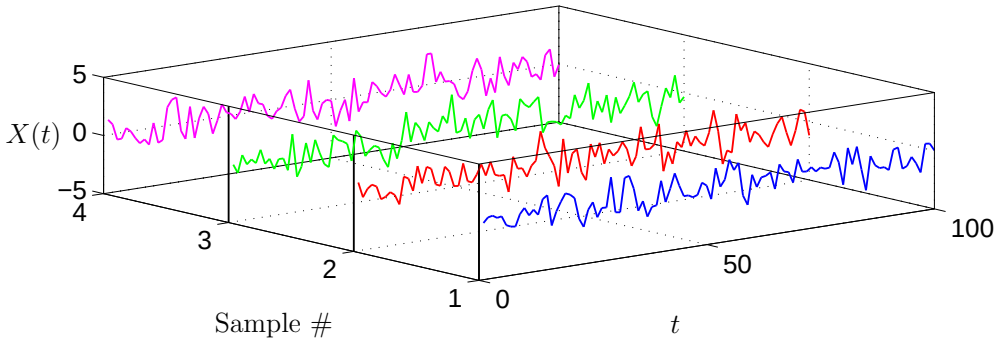


**Figure 4.1.** Four samples of a stochastic process.

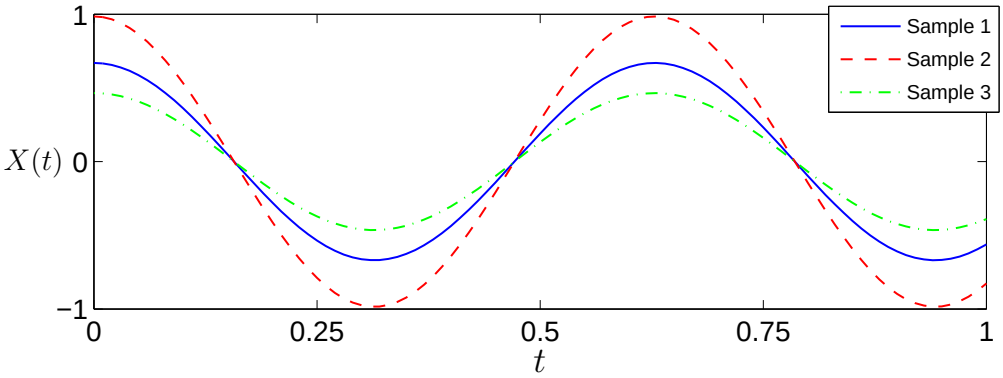


**Figure 4.2.** Three samples of stochastic process $X(t) = A\cos(10t)$.

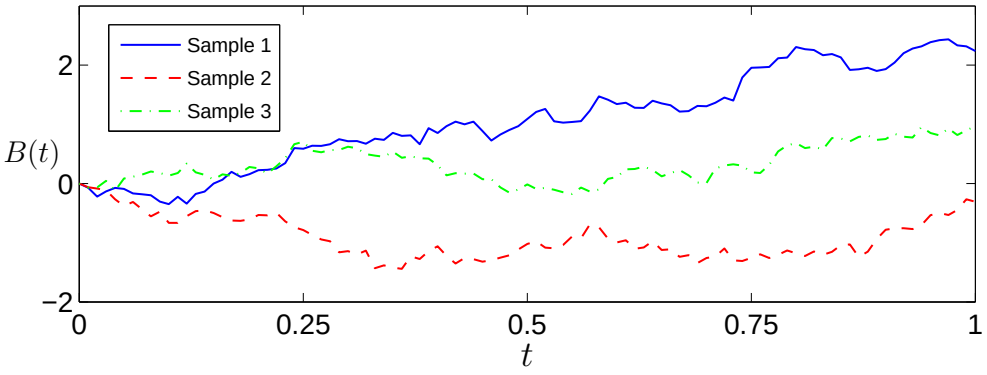


**Figure 4.3.** Three samples of the Brownian motion, $B(t)$.

## 4.1 Finite dimensional distributions and densities

Let $X(t)$, $t \geq 0$, be a stochastic process, and let $t_i \geq 0$, $i = 1, \ldots, n$ denote arbitrary distinct times in $[0, \infty)$. Stochastic process $X(t)$ is defined by the following *collection* of CDFs

$$\begin{aligned} F_n\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right) &= \Pr\left(\{X(t_1) \leq x_1\} \cap \cdots \cap \{X(t_n) \leq x_n\}\right) \\ &= \Pr\left(\cap_{i=1}^n \{X(t_i) \leq x_i\}\right), \end{aligned} \tag{4.1}$$

where each $x_i$, $i = 1, \ldots, n$, is a point on the real line, and $\Pr(A_1 \cap \cdots \cap A_n)$ is the probability that events $A_1, \ldots, A_n$ are all true. The collection defined by Eq. (4.1) are called the **finite dimensional distributions** of order $n$ of $X(t)$. The complete probability law of process $X(t)$ requires this collection be defined for all times $t_1, \ldots, t_n \in [0, \infty)$, all intervals $\{(-\infty, x_i], i = 1, \ldots, n\}$ on the real line, and all integers $n \geq 1$. Hence, there is an infinite number of CDFs we must know in order to specify a stochastic process. This is in contrast to the case of random variables and vectors which are completely defined by a single CDF (see Eqs. (2.1) and (2.10), respectively).

The definition for the collection of finite dimensional distributions for scalar process $X(t)$ given by Eq. (4.1) can be extended to the case when $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))^T$ is a vector stochastic process with $d > 1$ coordinates, *i.e.*,

$$F_n(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}; t_1, \ldots, t_n) = \Pr\left(\cap_{i=1}^n \left\{\mathbf{X}(t_i) \in \times_{k=1}^d (-\infty, x_{i,k}]\right\}\right), \tag{4.2}$$

where $n \geq 1$ is an integer, each $t_i \geq 0$, $i = 1, \ldots, n$, are arbitrary, distinct times, and each $\mathbf{x}^{(i)} \in \mathbb{R}^d$, $i = 1, \ldots, n$. A corresponding collection of **finite dimensional densities**, provided they exist, can be computed by differentiating Eq. (4.2) with respect to $\mathbf{x}$. For

example, for scalar stochastic process $\mathbf{X}(t) = X(t)$ we have

$$f_n(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_n(x_1, \ldots, x_n; t_1, \ldots, t_n) \tag{4.3}$$

for all $n \geq 1$ and all partitions $t_1, \ldots, t_n$.

The **marginal CDF / PDF** of stochastic process $X(t)$ are the distribution / density of random variable $X(t')$, where $t'$ is a fixed point in time, and are denoted by $F_1(x_1, t')$ and $f_1(x_1, t')$, respectively. Generally, the information available on a stochastic process is sufficient to estimate at most its first and second order finite dimensional distributions.

## 4.2 Classes of stochastic processes

Stochastic processes can be organized into classes, where each member of the class satisfies certain properties. In the following sections, we provide brief summaries of certain classes of stochastic processes, including the class of stationary, ergodic, Gaussian, translation, diffusion, and random point processes. These particular classes are frequently used for applications.

### 4.2.1 Stationary processes

Consider the special case where the collection of CDFs defined by Eq. (4.2) are shift-invariant, *i.e.*,

$$F_n(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}; t_1, \ldots, t_n) = F_n(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}; t_1 + \tau, \ldots, t_n + \tau), \tag{4.4}$$

for any $n \geq 1$, any distinct times $t_i$, $i = 1, \ldots, n$, and any shift $\tau$. In this case, we say that $\mathbf{X}(t)$ is a **stationary** or **strictly stationary** stochastic process. It can be shown that the marginal distribution of a stationary process is time-invariant.

**Example 4.3**: Let $X(t) = \alpha \cos(\omega t + \Theta)$, $t \geq 0$, be a stochastic process, where $\Theta$ is a random variable taking a uniform distribution over $[0, 2\pi]$, and $\alpha$ and $\omega$ are deterministic parameters. It can be shown (see [29], Section 11-4) that $X$ is a strictly stationary process. $\Diamond$

**Example 4.4**: The Brownian motion, $B(t)$, is not a stationary process because its marginal distribution, $F(x; t) = \Pr(B(t) \leq x) = \Phi(x/\sqrt{t})$, changes in time. $\Diamond$

### 4.2.2 Ergodic processes

Ergodicity deals with relating statistical, or ensemble, averages of a stationary stochastic process to time averages of its individual sample functions. The interchangeability of ensemble and time averages has considerable appeal in practice because estimates of certain properties of a process can be obtained from a single "long" sample instead of from multiple independent samples. The concept of ergodicity is very important in the practical application of random vibrations and random signal analysis because analysts frequently have very little data with which to characterize the underlying stochastic process [42].

The following is referred to as the Ergodic Theorem (see [18], Section 3.5): $\mathbf{X}(t)$ is an **ergodic stochastic process** if ensemble averages of $\mathbf{X}(t)$ equal time averages of $\mathbf{X}(t)$, *i.e.*,

$$\mathrm{E}\left[h(\mathbf{X}(t))\right] = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} h(\mathbf{X}(u))\,\mathrm{d}u \tag{4.5}$$

almost surely for any real-valued measurable function $h$ such that $\mathrm{E}\left[h(\mathbf{X}(t))\right] < \infty$ (see [21], p. 120). Weaker ergodicity conditions can be defined by restricting the form of the function $h$ in Eq. (4.5). For example, we say $\mathbf{X}(t)$ is ergodic in the mean if Eq. (4.5) holds for $h(\mathbf{x}) = \mathbf{x}$.

In general, it is very difficult to verify ergodicity properties. In practice, therefore, ergodic properties are generally regarded as hypotheses. Because of their great utility, ergodic conditions are often assumed to be valid in physical situations where we expect them to be true.

### 4.2.3 Gaussian processes

$\mathbf{X}(t)$ is a **Gaussian stochastic process** if, and only if, all of its finite dimensional distributions are Gaussian. Likewise, scalar process $X(t)$ is Gaussian if, and only if, the random vector $(X(t_1), \ldots, X(t_d))^T$ has the PDF given by Eq. (2.18) for every partition $t_1, \ldots, t_d$ of $[0, \infty)$, and every integer $d \geq 1$.

**Example 4.5**: The Brownian motion, $B(t)$, is a Gaussian process since random vector $(B(t_1), \ldots, B(t_d))^T$ can be expressed as a linear transformation of $d$ mutually independent Gaussian random variables, *i.e.*, (from [32], Chapter 6)

$$\begin{pmatrix} B(t_1) \\ B(t_2) \\ \vdots \\ B(t_d) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} B(t_1) \\ B(t_2) - B(t_1) \\ \vdots \\ B(t_d) - B(t_{d-1}) \end{pmatrix}$$

and linear transformations of Gaussian random variables are Gaussian. This expression holds for any partition $t_1, \ldots, t_d$ of $[0, \infty)$, and every integer $d \geq 1$. $\Diamond$

**Example 4.6**: Let $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ be a Gaussian random vector. Then

$$X(t) = \sum_{k=1}^{n} Z_k \, w_k(t), \; t \geq 0,$$

is a Gaussian stochastic process for any collection $\{w_1(t), \ldots, w_n(t)\}$ of real-valued and continuous functions of time $t$. This follows because $X(t)$ is a linear transformation of Gaussian random vector $\mathbf{Z}$ (see Section 2.2.3). $\Diamond$

### 4.2.4 Translation processes

Let $\mathbf{X}(t)$ be a vector Gaussian stochastic process with $d \geq 1$ coordinates, and define

$$Y_i(t) = h_i \left[ X_i(t) \right], \; i = 1, \ldots, d, \tag{4.6}$$

where each $h_i$ is a memoryless, *i.e.*, time-invariant, mapping. Vector $\mathbf{Y}(t) = (Y_1(t), \ldots, Y_d(t))^T$ defined by Eq. (4.6) is referred to as a **translation process**. It can be shown that if $X_i(t)$ is a stationary process, so is $Y_i(t)$. Further, $\mathbf{Y}(t)$ is Gaussian if, and only if, each $h_i$ is linear. Translation processes are very useful for applications because the marginal CDF of a translation process can be arbitrarily specified (see Example 4.7), and their implementation is straightforward.

**Example 4.7**: Let

$$Y(t) = h[X(t)] = F^{-1} \circ \Phi[X(t)], \; t \geq 0,$$

where $X(t)$ is a Gaussian process with zero mean and unit variance, $F$ is an arbitrary CDF, and $\Phi$ is the CDF of a standard Gaussian random variable. $Y(t)$ is a translation process with marginal distribution

$$\Pr\left(Y(t) \leq y\right) = \Pr\left(F^{-1} \circ \Phi\left[X(t)\right] \leq y\right) = \Pr\left(X(t) \leq \Phi^{-1}\left[F(y)\right]\right) = F(y).$$

This important result demonstrates that the marginal CDF of a translation process can be arbitrarily specified. $\Diamond$

**Example 4.8**: Let $F_1$ denote the CDF of an exponential random variable with parameter $\lambda > 0$ (see Example 2.2), and let $F_2$ denote the CDF of a random variable uniformly distributed on interval $[\alpha, \beta]$ (see Example 2.3). By the results from Example 4.7

$$Y_1(t) = F_1^{-1} \circ \Phi[X(t)] = -\frac{1}{\lambda} \ln\left(1 - \Phi[X(t)]\right), \text{ and}$$
$$Y_2(t) = F_2^{-1} \circ \Phi[X(t)] = \alpha + (\beta - \alpha)\, \Phi[X(t)]$$

are non-Gaussian translation processes with marginal CDFs $F_1$ and $F_2$, respectively. $\Diamond$

### 4.2.5 Diffusion processes

A stochastic differential equation (SDE) is a differential equation in which one or more terms is a stochastic process, thus resulting in a solution which is itself a stochastic process. We next consider stochastic processes that can be written as the solution to an SDE of the following type

$$\mathrm{d}\mathbf{X}(t) = \mathbf{a}(\mathbf{X}(t), t)\, \mathrm{d}t + \mathbf{b}(\mathbf{X}(t), t)\, \mathrm{d}\mathbf{B}(t),\ t \geq 0, \tag{4.7}$$

where $\mathbf{a}$ and $\mathbf{b}$ are $d \times 1$ and $d \times d'$ matrices, respectively, and $\mathbf{B}(t)$ is a vector of $d'$ independent Brownian motions. The solution to Eq. (4.7), namely $\mathbf{X}(t)$, is a **diffusion process** (also called an Itô process) with drift vector $\mathbf{a}$ and diffusion matrix $\mathbf{b}\,\mathbf{b}^T$. It can be shown that $\mathbf{X}(t)$ is a Gaussian process when $\mathbf{a}$ is linear in $\mathbf{X}$ and $\mathbf{b}$ does not depend on $\mathbf{X}$. Diffusion processes are quite practical since the numerical solution to Eq. (4.7) is straightforward using finite difference schemes.

**Example 4.9**: Let

$$\ddot{X}(t) + 2\,\zeta\,\omega_0\,\dot{X}(t) + \omega_0^2\,X(t) = W(t),\ t \geq 0$$

be the equation governing the motion of a linear, single degree-of-freedom oscillator with resonant frequency $\omega_0$, damping ratio $\zeta$, initial conditions $X(0)$ and $\dot{X}(0)$, subject to white noise $W(t)$ with intensity $a$. White noise is introduced later in Example 4.17. Let $X_1(t) = X(t)$ and $X_2(t) = \dot{X}(t)$; vector $\mathbf{X}(t) = (X_1(t), X_2(t))^T$ is a diffusion process since

$$\mathrm{d}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} X_2(t) \\ -\omega_0^2 X_1(t) - 2\zeta\omega_0 X_2(t) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ \sqrt{a\,\pi} \end{pmatrix} \mathrm{d}B(t)$$

where $W(t)$ can be interpreted as the time-derivative of $B(t)$, the Brownian motion. This is not a rigorous definition, as $B$ is not mean-square differentiable, but it is a useful interpretation for our discussion and is often used. Further, it can be shown that $X(t)$ and $\dot{X}(t)$ are Gaussian processes. $\Diamond$

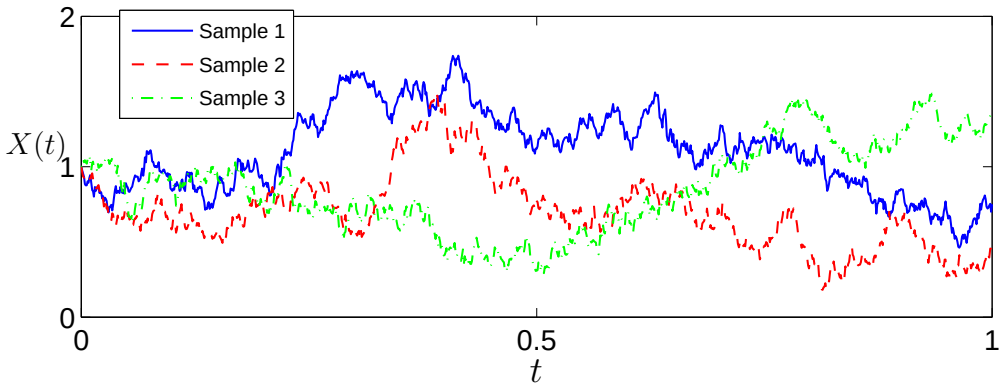


**Figure 4.4.** Three samples of the Ornstein-Uhlenbeck process with $\alpha = \beta = x_0 = 1$.

**Example 4.10**: Let

$$\mathrm{d}X(t) = -\alpha\, X(t)\,\mathrm{d}t + \beta\,\mathrm{d}B(t),\ \ X(0) = x_0,\ \ t \geq 0,$$

where $\alpha > 0$ and $\beta$ are constants; $X(t)$ is called the **Ornstein-Uhlenbeck process** [40]. A closed-form expression for $X(t)$ in terms of $B(t)$ does not exist, but the moments of any order of $X(t)$ can be calculated. Three samples of $X(t)$ are illustrated by Fig. 4.4 assuming $\alpha = \beta = x_0 = 1$. ◊

**Example 4.11**: The following stochastic differential equation

$$\mathrm{d}X(t) = \alpha\, X(t)\,\mathrm{d}t + \beta\, X(t)\,\mathrm{d}B(t),\ \ X(0) = x_0,\ \ t \geq 0$$

has an analytic solution, given by $X(t) = x_0 \exp\left[(\alpha - \beta^2/2)t + \beta B(t)\right]$. $X(t)$ is referred to as a **geometric Brownian motion** and has been used quite extensively in mathematical modeling of the financial markets. ◊

### 4.2.6 Random point processes

A random point process is a mathematical model for a physical phenomenon characterized by highly localized events distributed randomly in time and/or space. The basic building block of random point processes is the Poisson process; it is associated with counting a random numbers of points or events.

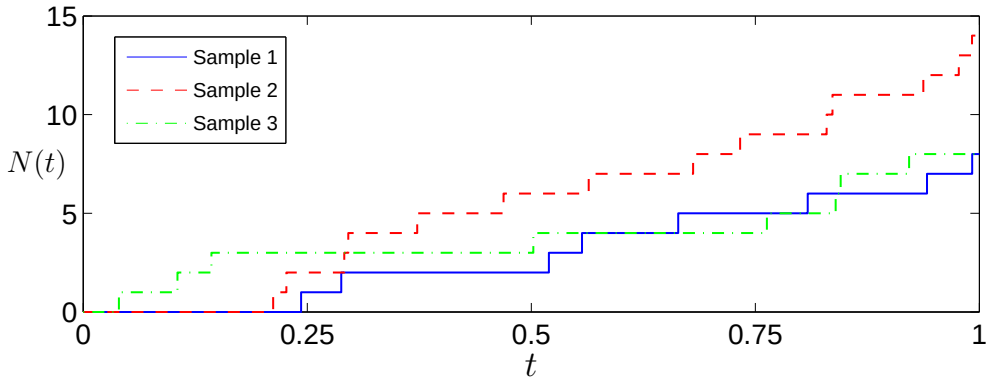


**Figure 4.5.** Three samples of the Poisson counting process, $N(t)$.

**Example 4.12**: Let $N(t)$ denote the random number of events that occur during time interval $[0, t]$ such that

$$\Pr(N(t) = n) = \frac{(\lambda\, t)^n}{n!}\, e^{-\lambda\, t}$$

is the probability of exactly $n$ events occurring in $[0, t]$, and $\lambda > 0$ is a parameter. $N(t)$ is called a homogeneous **Poisson counting process**, takes discrete values in $\{0, 1, \ldots\}$, and "jumps" from one value to the next at random times. These jump times of $N(t)$ are also of interest and are denoted by random variables $T_1, \ldots, T_{N(t)}$. We can show that:

1. $\mathrm{E}[N(t)] = \lambda\, t$ so that parameter $\lambda$ represents the average number of events occurring per unit time;

2. The increments $N(t) - N(s)$, $t > s$, over non-overlapping time intervals are stationary and independent; and

3. The inter-arrival times, defined by $X_1 = T_1$, $X_2 = T_2 - T_1$, $X_3 = T_3 - T_2$, $\ldots$ , are iid exponential random variables with parameter $\lambda$.

Three independent samples of $N(t)$, $0 \le t \le 1$, are illustrated by Fig. 4.5 for $\lambda = 10$, demonstrating that $N(1) = 8$, 14, and 8 for samples 1, 2, and 3, respectively. Further, the jump times of sample 1 are $T_1 \approx 0.24$, $T_2 \approx 0.29, \ldots, T_8 \approx 0.99$. $\Diamond$

Many random point processes are modified versions of the Poisson counting process. These include the translated, compound, thinned, and filtered Poisson processes [38]. Filtered Poisson processes are characterized by pulses of random shape and/or random magnitude arriving at random times. This type of model has been use to represent, for example, damage growth in systems, wind, earthquake, and traffic loads, and non-Gaussian white noise [26]. We have applied this model at Sandia to represent the force imparted to the nose of an aircraft due to impact with a sequence of moisture particles suspended in the atmosphere during a storm encounter [7, 34].

Let $\{N(t),\ t \geq 0\}$ be the homogeneous Poisson counting process depending on parameter $\lambda > 0$ introduced by Example 4.12, let $T_1, \ldots, T_{N(t)}$ denote the corresponding (random) jump times of $N$ during $[0, t]$, and let $Y_1, \ldots, Y_{N(t)}$ denote a sequence of random variables that define the magnitude of each pulse. The process

$$X(t) = \begin{cases} 0 & \text{if } N(t) = 0 \\ \displaystyle\sum_{k=1}^{N(t)} w(t, T_k, Y_k) & \text{if } N(t) > 0 \end{cases} \tag{4.8}$$

is a **filtered Poisson process**, where $w(t, T, Y)$ defines the shape of each pulse, occurring at time $T$ with magnitude $Y$. Further, we can show $X(t)$ is a stationary process because $\lambda$ is time-invariant. Non-stationary filtered Poisson processes depending on non-homogeneous Poisson counting processes with intensity $\lambda(t)$ are briefly considered in Example 4.23.

**Example 4.13**: To illustrate the filtered Poisson process, let $\{Y_k\}$ be a sequence of independent zero-mean Gaussian random variables with a variance of 9, and let $w(t, T, Y) = Y\,(t - T)\,e^{-5(t-T)}$, $t \geq T$, denote the shape of each pulse. Samples of $X(t)$ are illustrated by Fig. 4.6 assuming $\lambda = 1/2$ and $\lambda = 5$. It can be shown that this particular $X(t)$ approaches a Gaussian process as $\lambda \to \infty$. $\Diamond$

## 4.3 Second-moment properties

In Section 2.2.2, we defined the second-moment properties of a random vector. The second-moment properties of a stochastic process are very similar except that they are, in general, functions of time. Let $t$ and $s$ denote two times; the **second-moment properties** of vector stochastic process $\mathbf{X}(t)$ are

$$\begin{aligned} \boldsymbol{\mu}(t) &= \mathrm{E}[\mathbf{X}(t)] \\ \mathbf{r}(t, s) &= \mathrm{E}[\mathbf{X}(t)\,\mathbf{X}(s)^T] \\ \mathbf{c}(t, s) &= \mathrm{E}[(\mathbf{X}(t) - \boldsymbol{\mu}(t))\,(\mathbf{X}(s) - \boldsymbol{\mu}(s))^T], \end{aligned} \tag{4.9}$$

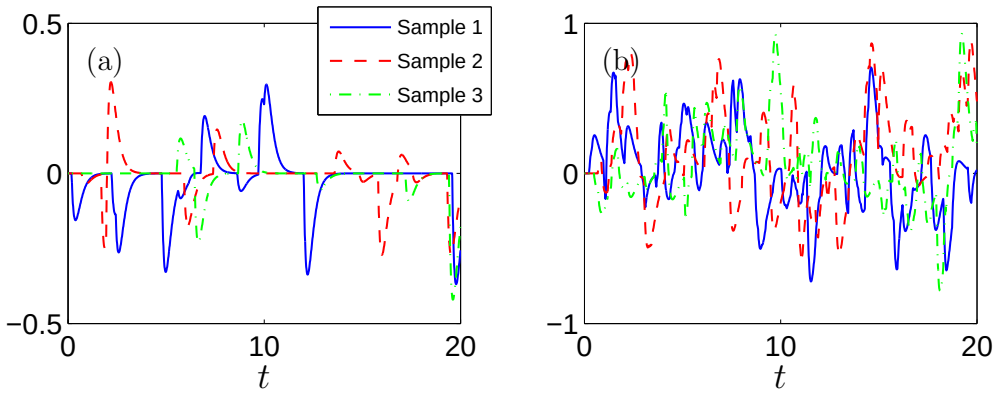


**Figure 4.6.** Three samples of a filtered Poisson process assuming: (a) $\lambda = 1/2$ and (b) $\lambda = 5$.

which correspond to the mean, correlation, and covariance functions of $\mathbf{X}$, respectively. The second-moment properties of $\mathbf{X}$ consist of the pairs $(\boldsymbol{\mu}, \mathbf{r})$ or $(\boldsymbol{\mu}, \mathbf{c})$; the triple $(\boldsymbol{\mu}, \mathbf{r}, \mathbf{c})$ is redundant since $\mathbf{c}(t,s) = \mathbf{r}(t,s) - \boldsymbol{\mu}(t)\,\boldsymbol{\mu}(s)^T$.

The **variance** and **mean-square value** of the process are special cases of the covariance and correlation functions, and they are often confused. In particular, $\mathbf{c}(t,t)$ and $\mathbf{r}(t,t)$ are the variance and mean-square value of $\mathbf{X}(t)$, respectively. By Eq. (4.9), it follows that the variance and mean-square value of $\mathbf{X}(t)$ are identical if, and only if, the mean function is zero. Further, the **standard deviation** and **root-mean-square (RMS)** values of $\mathbf{X}(t)$ are related to the square root of the variance and the square root of the mean-square values of $\mathbf{X}(t)$, respectively.

Certain properties of the correlation and covariance functions are of interest. Let $r_{i,j}(t,s)$ and $c_{i,j}(t,s)$ denote the elements of correlation and covariance matrices $\mathbf{r}(t,s)$ and $\mathbf{c}(t,s)$. We note that:

1. Elements $r_{i,i}$ and $r_{i,j}$, $i \neq j$ are referred to as the **auto- and cross-correlation functions** of $\mathbf{X}(t)$, respectively;

2. Elements $c_{i,i}$ and $c_{i,j}$, $i \neq j$ are referred to as the **auto- and cross-covariance functions** of $\mathbf{X}(t)$, respectively;

3. The cross-correlations and cross-covariances satisfy $r_{i,j}(t,s) = r_{j,i}(s,t)$ and $c_{i,j}(t,s) = c_{j,i}(s,t)$;

4. If the cross-correlations are all zero, *i.e.*, if $r_{i,j}(t,s) = 0$, $i \neq j$, $\forall t,s \geq 0$, then

coordinates $X_i(t)$ and $X_j(t)$ of $\mathbf{X}(t)$ are said to be **orthogonal**; and

5. If the cross-covariances are all zero, *i.e.*, if $c_{i,j}(t,s) = 0$, $i \neq j$, $\forall t, s \geq 0$, then coordinates $X_i(t)$ and $X_j(t)$ of $\mathbf{X}(t)$ are said to be **uncorrelated**.

The second-moment properties of a stochastic process are very useful for applications because they can be directly estimated from data. However, as is the case for random variables and vectors, knowing only the second-moment properties of a process is significantly less information than knowing the collection of CDFs or PDFs defined by Eqs. (4.2) and (4.3).

**Example 4.14**: Recall the Ornstein-Uhlenbeck process introduced by Example 4.10. The mean, correlation, and covariance functions for $X(t)$ are given by

$$\begin{aligned}
\mu(t) &= x_0\, e^{-\alpha t},\ t \geq 0 \\
r(t,s) &= \left[ x_0^2\, e^{-2\alpha \min(t,s)} + \frac{\beta^2}{2\,\alpha} \left( 1 - e^{-2\alpha \min(t,s)} \right) \right] e^{-\alpha |t-s|},\ t, s \geq 0 \\
c(t,s) &= \frac{\beta^2}{2\,\alpha} \left( 1 - e^{-2\alpha \min(t,s)} \right) e^{-\alpha |t-s|},\ t, s \geq 0,
\end{aligned}$$

and are illustrated by Fig. 4.7 assuming $x_0 = \beta = 1$ and $\alpha = 1/4$. ◊

**Example 4.15**: Recall the single degree-of-freedom oscillator driven by white noise with intensity $a > 0$ introduced by Example 4.9. For the special case of $a = 1$ and zero initial conditions, *i.e.*, $X(0) = \dot{X}(0) = 0$, then ([39], p. 176)

$$\begin{aligned}
\mu(t) &= 0,\ t \geq 0, \\
r(t,s) = c(t,s) &= \left[ \left( \cos \beta\tau + \frac{\zeta \omega_0}{\beta} \sin \beta\tau \right) \gamma(s) - \frac{\omega_0^2}{\beta} \phi(s) \sin \beta\tau \right] e^{-\zeta \omega_0 \tau},\ t \geq s,
\end{aligned}$$

where $\beta = \omega_0 \sqrt{1 - \zeta^2}$ denotes the damped natural frequency of the oscillator, $\tau = t - s \geq 0$, and

$$\begin{aligned}
\gamma(s) &= \frac{1}{4\beta^2} \left[ \frac{\beta^2 - \left( \zeta^2 \omega_0^2 + \beta^2 - \zeta^2 \omega_0^2 \cos 2\beta s + \beta \zeta \omega_0 \sin 2\beta s \right) e^{-2\zeta \omega_0 s}}{\zeta \omega_0 \left( \zeta^2 \omega_0^2 + \beta^2 \right)} \right] \\
\phi(s) &= \frac{1}{4\beta^2} \left( 1 - \cos 2\beta s \right) e^{-2\zeta \omega_0 s},\ s \geq 0.
\end{aligned}$$

◊

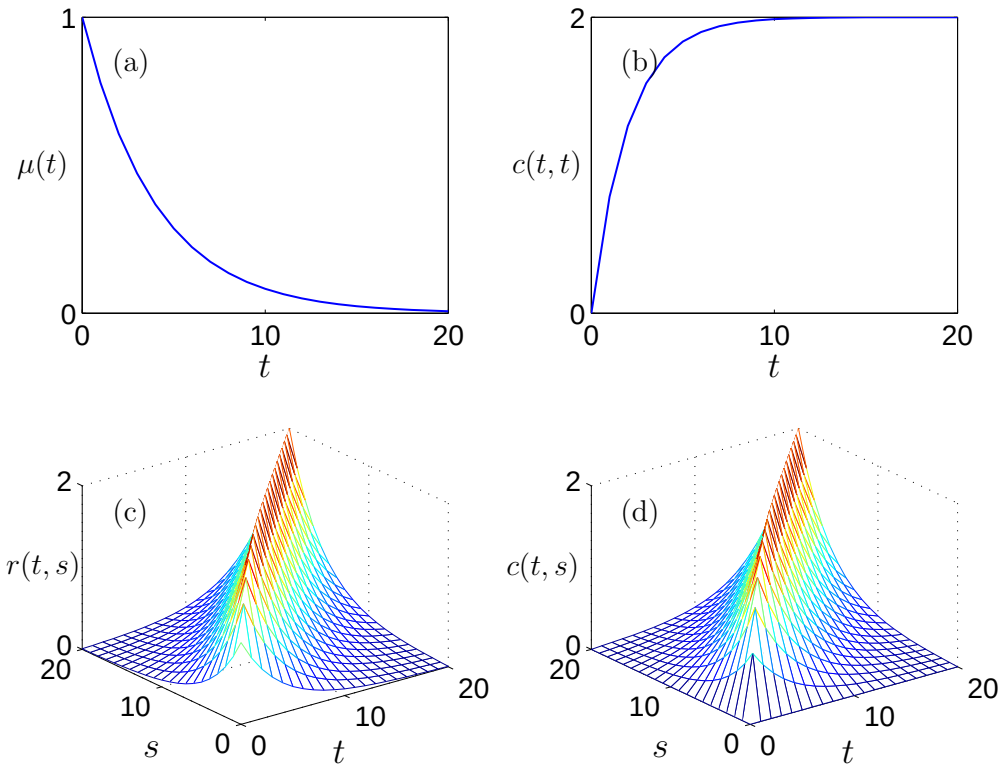


**Figure 4.7.** Second-moment properties of the Ornstein-Uhlenbeck process with $x_0 = \beta = 1$, $\alpha = 1/4$: (a) mean, $\mu(t)$, (b) variance, $c(t,t)$, (c) correlation, $r(t,s)$, and (d) covariance, $c(t,s)$.

### 4.3.1 Weakly stationary processes

Stationary stochastic processes were defined in Section 4.2.1. Weaker definitions of stationary processes can be defined that are based only on the second-moment properties. Consider process $\mathbf{X}(t)$ that satisfies the following:

1. The mean function $\boldsymbol{\mu}(t) = \boldsymbol{\mu}$ is time-invariant, and
2. The correlation and covariance functions, $\mathbf{r}(t,s) = \mathbf{r}(\tau)$ and $\mathbf{c}(t,s) = \mathbf{c}(\tau)$, depend only on the shift, $\tau = t - s$.

In this case we say that $\mathbf{X}(t)$ is **weakly stationary**. It follows from property 2 above that the variance of a weakly stationary process is given by $\mathbf{c}(0)$ and is time-invariant. Further, property 1 listed in Section 4.3 implies $r_{i,j}(\tau) = r_{j,i}(-\tau)$.

The concepts of strict stationarity and weak stationarity are very different. The former requires conditions on the finite dimensional distribution functions (see Eq. (4.4)), while the latter only requires conditions on the second-moment properties. Hence, as its name implies, strict stationarity is not implied by weak stationarity. Assuming finite variance, a process that satisfies the conditions defined by Eq. (4.4) is also stationary in the weak sense; the converse is not true in general. A notable exception is the Gaussian process, which is stationary if, and only if, it is weakly stationary.

**Example 4.16**: The Ornstein-Uhlenbeck process and the response of a linear oscillator driven by white noise are non-stationary processes because, as demonstrated by Examples 4.14 and 4.15, the covariance functions of both processes depend on two time arguments. However, both processes become weakly stationary as $t \to \infty$. This follows since, by Example 4.14

$$\lim_{t\to\infty} \mu(t) = 0 \quad \text{and} \quad \lim_{\substack{t\to\infty \\ s\to\infty}} r(t,s) = \lim_{\substack{t\to\infty \\ s\to\infty}} c(t,s) = \frac{\beta^2}{2\,\alpha}\, e^{-\alpha|t-s|}$$

demonstrating that as $t \to \infty$, the Ornstein-Uhlenbeck process becomes a weakly stationary process because the mean is time-invariant, and the correlation / covariance functions depend only on time shift $t - s$. The asymptotic behavior of the second-moment properties of this process can be observed in Fig. 4.7. For example, the mean and variance approach 0 and $\beta^2/2\alpha = 2$, respectively. Further, the correlation / covariance function of the response of the linear oscillator defined in Example 4.15 has the following asymptotic properties,

$$\lim_{\substack{t\to\infty \\ s\to\infty}} r(t,s) = \lim_{\substack{t\to\infty \\ s\to\infty}} c(t,s) = \left(\cos\beta\tau + \frac{1}{4\,\beta\,(\zeta^2\omega_0^2 + \beta^2)}\sin\beta\tau\right) e^{-\zeta\omega_0|\tau|},$$

which is a function of a single time argument, $\tau = t - s$. $\Diamond$

### 4.3.2 Spectral density

The spectral density provides an alternative representation for the second-moment properties of a weakly stationary process, and it is perhaps the most common way to define the second-moment properties of a stochastic process for engineering applications. It is also commonly called a power spectral density and abbreviated as PSD.

Consider a weakly stationary real-valued stochastic process $X(t)$, $t \geq 0$ with correlation function $r(\tau) = \mathrm{E}[X(t)\,X(t+\tau)]$. The **two-sided spectral density** function is the Fourier transform of the correlation function, *i.e.*,

$$s(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau}\, r(\tau)\, \mathrm{d}\tau,\ -\infty < \omega < \infty, \tag{4.10}$$

where $i = \sqrt{-1}$. It is common to instead use the **one-sided spectral density** function defined for non-negative frequencies,

$$g(\omega) = \frac{1}{\pi} \int_{0}^{\infty} r(\tau)\, \cos(\omega\tau)\, \mathrm{d}\tau,\ \omega \geq 0, \tag{4.11}$$

where $g(\omega) = 2\,s(\omega)$, $\omega \geq 0$. We note that: (i) $s(\omega), g(\omega) \geq 0$, (ii) the area under each is equal to $r(0)$, and (iii) the spectral density contains the same information as the correlation function, so the second moment properties of $X(t)$ may be completely specified by the mean and spectral density. Some common spectral density / correlation function pairs are listed in Table 4.1.

**Example 4.17**: Let $W(t)$, $t \geq 0$, be a stochastic process with one-sided PSD $g(\omega) = a$, $0 \leq \omega < \infty$, and correlation function $r(\tau) = \pi\, a\, \delta(\tau)$, where $\delta(\tau)$ is the Dirac delta function, and parameter $a > 0$ is referred to as the intensity of $W(t)$. By this construction, for any two times $t_1 \neq t_2$, $W(t_1)$ and $W(t_2)$ are uncorrelated. Further, the variance of $W(t)$ is unbounded for any $t$. Process $W$ is commonly referred to as **white noise** because of its similarity to "white light", which has the property that its spectral density is flat over the visible portion of the electromagnetic spectrum. White noise is used extensively in applied science and engineering to approximate a great number of physical phenomena. However, it should be obvious that such a process is an abstraction or a limiting process; it is not physically realizable. ♢

**Example 4.18**: Recall again the linear, single degree-of-freedom oscillator introduced in Example 4.9. In this example, we replace white noise $W(t)$ with a zero-mean, stationary process with one-sided PSD $g(\omega)$ that is not necessarily white. Due to the presence of

**Table 4.1.** Some common correlation and spectral density functions (taken from [39], Table 2.1).

| Name | Correlation, $r(\tau)$ | One-sided PSD, $g(\omega)$ |
|---|---|---|
| White noise | $\pi\, a\, \delta(\tau)$ | $a$ |
| Band-limited white noise | $\dfrac{a}{\tau}\, \sin(\bar{\omega}\, \tau)$ | $\begin{cases} a & 0 \leq \omega \leq \bar{\omega} \\ 0 & \omega > \bar{\omega} \end{cases}$ |
| Rectangular pulse | $\dfrac{2\, a}{\tau}\, \sin\left(\dfrac{(\omega_2 - \omega_1)\, \tau}{2}\right) \cos\left(\dfrac{(\omega_2 - \omega_1)\, \tau}{2}\right)$ | $\begin{cases} a & \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{else} \end{cases}$ |
| First-order Markov | $\sigma^2\, \exp\left(-\lambda\, \lvert\tau\rvert\right)$ | $\dfrac{2\, \sigma^2\, \lambda}{\pi\, (\omega^2 + \lambda^2)}$ |
| Second-order Markov | $\sigma^2\, \exp\left(-\lambda\, \lvert\tau\rvert\right)(1 + \lambda\, \lvert\tau\rvert)$ | $\dfrac{4\, \sigma^2\, \lambda^3}{\pi\, (\omega^2 + \lambda^2)^2}$ |

damping ($\zeta > 0$), response $X(t)$ will approach a stationary process as $t \to \infty$ with one-sided spectral density (see [39], p. 196)

$$g_X(\omega) = \frac{g(\omega)}{(\omega^2 - \omega_0^2)^2 + (2\,\zeta\,\omega\,\omega_0)^2},\ \omega \geq 0.$$

$\Diamond$

**Example 4.19**: Let $X(t)$ be a zero-mean, unit-variance, band-limited Gaussian process with parameters $a = 1/\bar{\omega}$ and $\bar{\omega} = 5{,}000$ rad/sec (see Table 4.1). Let $Y(t)$ be a non-Gaussian stochastic process with identical second-moment properties, but marginal distribution of a student-t random variable (see [43], Section A.2). One sample of band-limited white noise processes $X(t)$ and $Y(t)$ are illustrated by Fig. 4.8(a) and (b), respectively. The time history plots of each sample differ considerably, but $X(t)$ and $Y(t)$ have identical one-sided PSD. In fact, there are an infinite number of non-Gaussian stochastic processes, each with the same PSD as the Gaussian process $X(t)$, but with very different sample functions; the process $Y(t)$ shown in Fig. 4.8(b) is simply one example. This example illustrates yet again that second-moment properties do not provide enough information to uniquely define a stochastic process. $\Diamond$

The concept of spectral density can be extended to the case of vector stochastic processes. If $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))^T$ is a weakly stationary vector stochastic process with correlation functions $r_{k,l}(\tau) = \mathrm{E}[X_k(t)\,X_l(t+\tau)]$, $k, l = 1, \ldots, d$, then the two-sided spectral density of $\mathbf{X}(t)$ is a matrix with coordinates

$$s_{k,l}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r_{k,l}(\tau)\, e^{-i\omega\tau}\, \mathrm{d}\tau,\ k, l = 1, \ldots, d. \tag{4.12}$$

The diagonal and off-diagonal elements of this matrix are referred to as **auto- and cross-spectral densities**, respectively. Alternate definitions for the spectral density of $\mathbf{X}(t)$ are given by

$$\begin{aligned} \mathbf{g}(\omega) &= \quad \mathbf{s}(\omega) + \mathbf{s}(-\omega), \quad \text{and} \\ \mathbf{h}(\omega) &= -i\,(\mathbf{s}(\omega) - \mathbf{s}(-\omega))\,. \end{aligned} \tag{4.13}$$

As discussed in Section 4.5, it is possible to extend the concept of spectral density to special types of non-stationary processes. One such extension, referred to as an **evolutionary spectral density**, is a function of time and frequency and captures the frequency content of a non-stationary process in a small vicinity of each time [31].

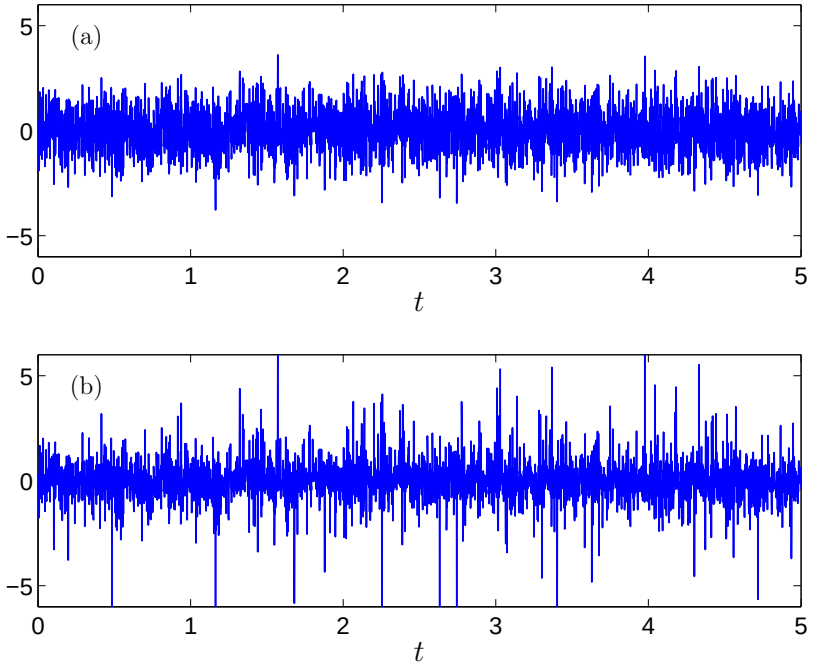


**Figure 4.8.** One sample of two stochastic processes with identical PSD: (a) $X(t)$, and (b) $Y(t)$.

## 4.4 Spectral representation of stochastic processes

For applications, it is often useful to use a spectral representation for a stochastic process. We consider two types. First, we use the spectral representation theorem to express any weakly stationary process by a superposition of harmonics with random amplitude and/or phase. Second, we apply the Karhunen-Loéve representation to express any process with finite variance as a linear combination of a countable number of deterministic functions with random coefficients. Methods to create samples of stochastic processes from either of these representations will be discussed in Chapter 5.

### 4.4.1 Spectral representation theorem

If $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))^T$ is a weakly stationary vector stochastic process with spectral densities given by Eq. (4.12) then, by the **spectral representation theorem**, $\mathbf{X}(t)$ can be expressed via the following mean-square integral

$$\mathbf{X}(t) = \int_0^\infty \left[\cos(\omega t)\, \mathrm{d}\mathbf{U}(\omega) + \sin(\omega t)\, \mathrm{d}\mathbf{V}(\omega)\right], \tag{4.14}$$

where $\mathbf{U}$ and $\mathbf{V}$ are zero-mean stationary vector processes with orthogonal increments, *i.e.*,

$$\begin{aligned} \mathrm{E}\left[\mathrm{d}U_k(\omega)\, \mathrm{d}U_l(\nu)\right] = \quad & \mathrm{E}\left[\mathrm{d}V_k(\omega)\, \mathrm{d}V_l(\nu)\right] = \delta(\omega - \nu)\, g_{k,l}(\omega)\, \mathrm{d}(\omega), \\ \mathrm{E}\left[\mathrm{d}U_k(\omega)\, \mathrm{d}V_l(\nu)\right] = -& \mathrm{E}\left[\mathrm{d}V_k(\omega)\, \mathrm{d}U_l(\nu)\right] = \delta(\omega - \nu)\, h_{k,l}(\omega)\, \mathrm{d}(\omega), \end{aligned} \tag{4.15}$$

for $k, l = 1, \ldots, d$, where $\delta(\omega) = 1$ for $\omega = 0$ and zero otherwise. The discussion of mean-square integration and second-moment calculus in general is beyond the scope of this report; see [21], Section 3.9, for more information on these topics. By Eq. (4.14), any weakly stationary stochastic process can be represented by a superposition of harmonics with random amplitude and phase.

### 4.4.2 Karhunen-Loéve representation

An alternative to the spectral representation theorem is as follows. Let $X(t)$, $a \leq t \leq b$, be a stochastic process with zero mean and covariance function $c(t, s) = \mathrm{E}[X(t)\, X(s)]$; the derivation that follows can be generalized for the case of vector processes. The **Karhunen-Loéve representation** (K-L) of $X(t)$ is given by

$$X(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \phi_k(t)\, U_k, \tag{4.16}$$

and can be used to obtain a parametric representation for $X(t)$ [17]. In this representation, $\{\lambda_k, \phi_k(t),\ k \geq 1\}$ are the eigenvalues and eigenfunctions, respectively, of $c(t,s)$, and satisfy the integral equation

$$\int_a^b c(t,s)\,\phi_k(s)\,\mathrm{d}s = \lambda_k\,\phi_k(t), \tag{4.17}$$

and each $U_k$ is a random variable such that

$$U_k = \frac{1}{\sqrt{\lambda_k}} \int_a^b X(t)\,\phi_k(t)\,\mathrm{d}t. \tag{4.18}$$

It can be shown that the $U_k$ are iid $N(0,1)$ random variables if $X(t)$ is a Gaussian process. The K-L representation can be useful in applications since it provides alternative definitions for stochastic processes as functions of a countable number of random variables. However, the K-L representation can be difficult to obtain.

**Example 4.20**: Consider the Brownian motion $B(t)$ introduced in Example 4.2. The K-L representation of $B(t)$ defined on interval $0 \leq t \leq a$ is given by Eq. (4.16) with (see [29], p. 415)

$$\lambda_k = \left(\frac{2\,a}{\pi(2k+1)}\right)^2 \quad \phi_k(t) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{\pi(2k+1)t}{2a}\right)$$

and $\{U_k\}$ a sequence of independent $N(0,1)$ random variables. $\Diamond$

**Example 4.21**: Let $X(t)$, $-a \leq t \leq a$, be a stationary Gaussian process with zero mean and correlation $r(\tau) = \mathrm{E}[X(t)\,X(t+\tau)] = e^{-|\tau|}$. The K-L representation of $X(t)$ is given by Eq. (4.16) with (taken from [41]),

$$\lambda_k = \frac{2}{\theta_k^2 + 1},$$
$$\phi_{2k}(t) = \frac{\cos(\theta_{2k}t)}{\sqrt{a + \frac{\sin(2a\theta_{2k})}{2\theta_{2k}}}}, \quad \text{and} \quad \phi_{2k-1}(t) = \frac{\sin(\theta_{2k-1}t)}{\sqrt{a - \frac{\sin(2a\theta_{2k-1})}{2\theta_{2k-1}}}},$$

where the $\theta_k$ come from the solution to the following characteristic equation,

$$1 - \theta_{2k}\tan(a\,\theta_{2k}) = 0, \quad \text{and}$$
$$\theta_{2k-1} + \tan(a\,\theta_{2k-1}) = 0.$$

and $\{U_k\}$ form a sequence of iid $N(0,1)$ random variables.

This example illustrates one unfavorable feature of the K-L representation when used for applications. Suppose we truncate the K-L representation for $X(t)$ at $2n$ terms, *i.e.*,

$$X^{(2n)}(t) = \sum_{k=1}^{2n} \sqrt{\lambda_k}\, \phi_k(t)\, U_k,$$

Process $X(t)$ is known to be stationary, but

$$\begin{aligned} \mathrm{E}[X^{(2n)}(t)\, X^{(2n)}(s)] &= \sum_{k=1}^{2n} \lambda_k\, \phi_k(t)\, \phi_k(s) \\ &= \sum_{k=1}^{n} \frac{2}{\theta_{2k}^2 + 1}\, \phi_{2k}(t)\, \phi_{2k}(s) + \sum_{k=1}^{n} \frac{2}{\theta_{2k-1}^2 + 1}\, \phi_{2k-1}(t)\, \phi_{2k-1}(s) \end{aligned}$$

cannot be written as a function of time lag $\tau = t - s$, meaning that the K-L representation for $X(t)$ based on $2n$ random variables is not stationary (refer to Section 4.3.1), and this is true no matter how may terms $n$ are retained in the sum. Hence, truncated K-L representations for stationary processes are, in general, non-stationary. The repercussions of this result must be assessed on a case by case basis; refer to [11] for a detailed discussion on a specific example. ◊

## 4.5 Special types of non-stationary processes

Let $X_{\mathrm{s}}(t)$ be a stationary stochastic process with zero mean, unit variance, correlation function $r(\tau) = \mathrm{E}[X_{\mathrm{s}}(t)\, X_{\mathrm{s}}(t+\tau)]$, and one-sided PSD $g(\omega)$. We consider two special types of non-stationary processes that can be viewed as manipulations of $X_{\mathrm{s}}(t)$, referred to as: (1) transformations of stationary processes, and (2) oscillatory processes. As illustrated by Fig. 4.9, there is some overlap between types (1) and (2), and the types considered do not completely cover the space of non-stationary processes, *i.e.*, there exists a collection of non-stationary processes that cannot be represented by either type. A more detailed description of these special types of non-stationary process is given in [22].

### 4.5.1 Transformations of stationary processes

We first consider the collection of non-stationary processes that can be expressed as direct transformations of stationary process $X_{\mathrm{s}}(t)$ defined above. There are two special cases, referred to as uniformly modulated stationary processes, and stationary processes under time shift. The former has been used to represent, for example, turbulent flow on a re-entry vehicle

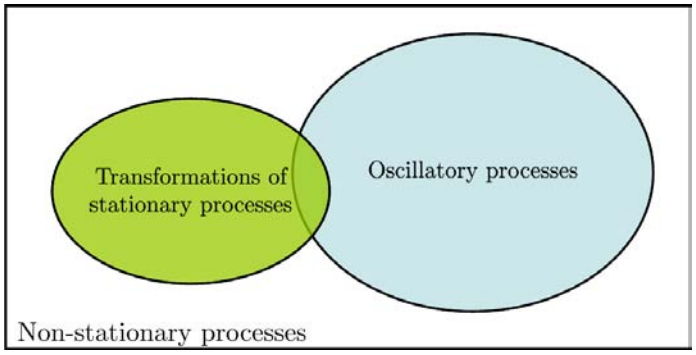


**Figure 4.9.** The class of non-stationary processes with finite variance.

[9], where the modulation function represents the effects of increasing dynamic pressure. The latter case has been used to represent moisture particle impacts on a decelerating re-entry vehicle [7], where the concept of a time shift is used to model the effects of vehicle deceleration on particle arrival times.

Let

$$X(t) = \alpha(t)\, X_{\mathrm{s}}(t),\ t \geq 0, \tag{4.19}$$

where $\alpha(t) \geq 0$ is a deterministic function. Process $X(t)$ is a type of non-stationary process called a **uniformly modulated stationary process** with zero mean and correlation function

$$\mathrm{E}[X(t)\, X(s)] = \alpha(t)\, \alpha(s)\, r(t-s), \tag{4.20}$$

where $r(\tau) = \mathrm{E}[X_{\mathrm{s}}(t)\, X_{\mathrm{s}}(t+\tau)]$. Because $\mathrm{E}[X(t)\, X(s)]$ cannot be written in terms of $t-s$, $X(t)$ is indeed a non-stationary process. Further, we can show that $X(t)$ has one-sided evolutionary spectral density

$$\psi(t,\omega) = \alpha^2(t)\, g(\omega), \tag{4.21}$$

where $g(\omega)$ denotes the one-sided (time-invariant) spectral density of stationary process $X_{\mathrm{s}}$.

Next let

$$X(t) = X_{\mathrm{s}}\left[h(t)\right],\ t \geq 0, \tag{4.22}$$

where $h(t) \geq 0$ is a deterministic function such that $h(0) = 0$ and $h'(t) = \mathrm{d}h(t)/\mathrm{d}t > 0$. Process $X(t)$ is a **stationary process under time shift** $h$ with zero mean, correlation

function

$$\mathrm{E}[X(t)\,X(s)] = \mathrm{E}\left[X_\mathrm{s}[h(t)]\,X_\mathrm{s}[h(s)]\right] = r\left[h(t) - h(s)\right], \tag{4.23}$$

and one-sided evolutionary spectral density

$$\psi(t,\omega) = g\left(\frac{\omega\, t}{h(t)}\right). \tag{4.24}$$

We provide detailed examples of the uniformly modulated stationary process and stationary process under time shift in Section 5.2.1.

### 4.5.2 Oscillatory processes

Recall from Section 4.4.1 that real-valued stationary process $X_\mathrm{s}(t)$ with one-sided PSD $g(\omega)$ admits the following spectral representation

$$X_\mathrm{s}(t) = \int_0^\infty \left[\cos(\omega t)\,\mathrm{d}U(\omega) + \sin(\omega t)\,\mathrm{d}V(\omega)\right], \tag{4.25}$$

where $U$ and $V$ are zero-mean stationary processes with increments that satisfy Eq. (4.15). Process

$$X(t) = \int_0^\infty a(t,\omega)\left[\cos(\omega t)\,\mathrm{d}U(\omega) + \sin(\omega t)\,\mathrm{d}V(\omega)\right] \tag{4.26}$$

created from the spectral representation of $X_\mathrm{s}(t)$, where $a(t,\omega)$ is a slowly-varying function of time $t$ for all frequencies $\omega$, is a type of non-stationary stochastic process called an **oscillatory process** (or Priestley process [31]). It can be shown that $X(t)$ has zero mean and one-sided evolutionary spectral density

$$\psi(t,\omega) = a(t,\omega)^2\, g(\omega) \tag{4.27}$$

Note that if we choose $X_\mathrm{s}(t)$ defined by Eq. (4.25) to be white noise with $g(\omega) = 1$, $X(t)$ has one-sided evolutionary spectral density $a(t,\omega)^2$.

**Example 4.22**: The uniformly modulated process defined by Eq. (4.19) is an oscillatory process with $a(t,\omega) = \alpha(t)$. The process defined by Eq. (4.22) may or may not be an oscillatory process depending on the rate of change of function $h(t)$. ◊

**Example 4.23**: Recall the filtered Poisson process considered in Example 4.13

$$X(t) = \begin{cases} 0 & \text{if } N(t) = 0 \\ \displaystyle\sum_{k=1}^{N(t;\lambda(t))} Y_k \, (t - T_k) \, e^{-5(t-T_k)} & \text{if } N(t) > 0 \end{cases}$$

where $\{Y_k\}$ was a sequence of independent zero-mean Gaussian random variables with variance $\sigma^2$, and $\lambda(t) = \lambda > 0$ was assumed time-invariant. We consider here instead the case where $\lambda(t)$ is time-varying so that $X(t)$ is a non-stationary process. It can be shown that, under this assumption, $X(t)$ can be expressed as an oscillatory process with (see [26], Section 3.2)

$$a(t,\omega) = \int_{-\infty}^{\infty} \sqrt{\mathrm{E}\left[\lambda(t-u)\right]} \, u \, e^{-5u} \, e^{-i\omega u} \, \mathrm{d}u \quad \text{and} \quad g(\omega) = \frac{\sigma^2}{2\pi}$$

$\Diamond$

**Example 4.24**: Suppose that a collection of $m \geq 1$ one-sided time-invariant PSDs are provided, denoted by $g_1(\omega), \ldots, g_m(\omega)$, that define the spectral content of $X(t)$ at known fixed distinct times $0 \leq t_1 < \cdots < t_m$. We can construct an oscillatory process as defined by Eq. (4.26) to be consistent with this information. This approach was used to represent turbulent pressure fluctuations in an attached boundary layer [8]. Let $X_{\mathrm{s}}(t)$ be a stationary white noise with $g(\omega) = 1$ and let

$$\psi(t,\omega) = a(t,\omega)^2 = \sum_{j=1}^{m} g_j(\omega) \, \phi(t - t_j),$$

where

$$\phi(t - t_j) = \begin{cases} 1 + \dfrac{1}{\Delta t}(t - t_j), & \text{if } t_j - \Delta t \leq t < t_j \\ 1 - \dfrac{1}{\Delta t}(t - t_j), & \text{if } t_j \leq t < t_j + \Delta t \\ 0, & \text{else} \end{cases}$$

is an interpolating function with boundary conditions $\phi = 1$ for $t < t_1$ and $t > t_m$, and $\Delta t = t_{j+1} - t_j$, $j = 1, \ldots, m-1$, denotes the time elapsed between successive $t_j$, assumed constant and sufficiently large so that $a(t,\omega)$ is slowly-varying in time for all $\omega$. By this construction, the evolutionary spectral density of $X(t)$ in the vicinity of time $t_i$ is $g_i(\omega)$,

$i = 1, \ldots, m$; for other times $t \in (t_i, t_{i+1})$ it is a linear combination of $g_i(\omega)$ and $g_{i+1}(\omega)$, $i = 1, \ldots, m-1$. Further, we note that

$$\mathrm{Var}[X(t)] = \sigma^2(t) = \sum_{j=1}^{m} \sigma_j^2 \, \phi(t - t_j) \quad \text{where} \quad \sigma_j^2 = \int_0^\infty g_j(\omega) \, \mathrm{d}\omega.$$

$\Diamond$

# Chapter 5

# Simulation of stochastic processes

Let $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))^T$ be a vector stochastic process; definitions and properties of $\mathbf{X}(t)$ were discussed in Chapter 4. Our objective in this Chapter is to generate independent samples of $\mathbf{X}(t)$. In Section 5.1, we consider the simplest case where $\mathbf{X}(t)$ is stationary and Gaussian. We then consider the simulation of special types of non-stationary Gaussian and non-Gaussian processes in Sections 5.2 and 5.3, respectively. MATLAB implementations of the algorithms developed are presented in Appendix A. In all cases, we assume $\mathbf{X}(t)$ to have zero mean; we note that if the mean of $\mathbf{X}(t)$ is not zero, we generate samples of $\mathbf{X}(t) - \mathrm{E}[\mathbf{X}(t)]$ and add the mean function $\mathrm{E}[\mathbf{X}(t)]$ to each sample. The discussion is concluded by a brief summary of possible verification checks in Section 5.4.

## 5.1 Stationary Gaussian processes

Let $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))^T$ be a stationary Gaussian vector process with zero mean, covariance function $\mathbf{c}(\tau) = \mathrm{E}[\mathbf{X}(t)\,\mathbf{X}(t+\tau)^T]$, and two-sided spectral density $\mathbf{s}(\omega) = \{s_{k,l}(\omega)\}$, $k, l = 1, \ldots, d$. Our main tool to generate samples of $\mathbf{X}(t)$ is the spectral representation theorem discussed in Section 4.4.1; alternative methods are discussed in [21], Section 5.3.1. We do not include the Karhunen-Loève representation here because, as mentioned in Example 4.21, the K-L representation of a stationary process is, in general, not stationary. The use of the spectral representation requires two steps. First, we define stochastic process $\mathbf{X}^{(n)}(t)$ that is an approximation for $\mathbf{X}(t)$ depending on $n$ random variables that converges in some sense to $\mathbf{X}(t)$. Second, we develop efficient computer algorithms to generate independent samples of $\mathbf{X}^{(n)}(t)$.

### 5.1.1 Parametric model with fixed frequencies

Recall that, by the spectral representation theorem, $\mathbf{X}(t)$ can be expressed as a mean-square integral as defined by Eq. (4.14). A finite-dimensional approximation for this integral

provides a model to be used for Monte Carlo simulation. Define, for each $k = 1, \ldots, d$, a cut-off frequency $\omega_k^\star$, such that

$$\int_{-\omega_k^\star}^{\omega_k^\star} s_{kk}(\omega)\,\mathrm{d}\omega \approx \int_{-\infty}^{\infty} s_{kk}(\omega)\,\mathrm{d}\omega, \tag{5.1}$$

and let $\omega^\star = \max_{1\le k\le d}\omega_k^\star$. Let $(u_{j-1}, u_j)$, $j = 1, \ldots, n$, with $u_0 = 0$ and $u_n = \omega^\star$, be a partition of frequency range $(0, \omega^\star)$ in $n$ non-overlapping intervals of length $\Delta\omega_j = u_j - u_{j-1}$. Denote by $\omega_j$ the midpoint of $(u_{j-1}, u_j)$, $j = 1, \ldots, n$. The following is an approximation for $\mathbf{X}(t)$ of order $n$ (see [20], p. 174):

$$\mathbf{X}^{(n)}(t) = \sum_{j=1}^{n} \left[\mathbf{A}_j\, \cos(\omega_j t) + \mathbf{B}_j\, \sin(\omega_j t)\right], \tag{5.2}$$

where $\mathbf{A}_j$ and $\mathbf{B}_j$ are $\mathbb{R}^d$–valued zero-mean Gaussian random vectors with the following second-moment properties for $i, j = 1, \ldots, n$ and $k, l = 1, \ldots, d$:

$$\begin{aligned}
\mathrm{E}\left[A_{i,k}\, A_{j,l}\right] = \quad \mathrm{E}\left[B_{i,k}\, B_{j,l}\right] &= \delta_{i,j} \int_{u_{i-1}}^{u_i} g_{k,l}(\omega)\,\mathrm{d}\omega \approx \delta_{i,j}\, g_{k,l}(\omega_i)\,\Delta\omega_i, \\
\mathrm{E}\left[A_{i,k}\, B_{j,l}\right] = -\mathrm{E}\left[B_{i,k}\, A_{j,l}\right] &= \delta_{i,j} \int_{u_{i-1}}^{u_i} h_{k,l}(\omega)\,\mathrm{d}\omega \approx \delta_{i,j}\, h_{k,l}(\omega_i)\,\Delta\omega_i,
\end{aligned} \tag{5.3}$$

where $\delta_{i,j} = 1$ if $i = j$ and zero otherwise, and $g_{k,l}(\omega)$ and $h_{k,l}(\omega)$ are defined by Eq. (4.13).

The model defined by Eq. (5.2) has some desirable properties. First, $\mathbf{X}^{(n)}(t)$ is a zero-mean, stationary Gaussian process for any $n \ge 1$. This follows since $\mathbf{X}^{(n)}(t)$ is a linear combination of Gaussian random variables, $\mathrm{E}\left[\mathbf{X}^{(n)}(t)\right] = \mathbf{0}$, and

$$\begin{aligned}
\mathrm{E}\left[\mathbf{X}^{(n)}(t)\,\mathbf{X}^{(n)}(s)^T\right] &= \sum_{k=1}^{n} \mathrm{E}[\mathbf{A}_k\,\mathbf{A}_k^T]\cos(\omega_k(t-s)) \\
&= \sum_{k=1}^{n} \mathrm{E}[\mathbf{A}_k\,\mathbf{A}_k^T]\cos(\omega_k\,\tau),\ t \ge s,
\end{aligned} \tag{5.4}$$

where $\tau = t - s$. Second, it can be shown that, as $n \to \infty$: (i) $\mathbf{X}^{(n)}$ approaches $\mathbf{X}$ in the mean-square sense, (ii) the covariance functions of $\mathbf{X}^{(n)}$ approach the covariance functions of $\mathbf{X}$, and (iii) $\mathbf{X}^{(n)}$ becomes a version (*i.e.*, has an identical finite dimensional distribution) of $\mathbf{X}$. Finally, the Fast Fourier Transform (FFT) provides a very efficient algorithm for sample generation.

It should be noted that samples of $\mathbf{X}^{(n)}$ are periodic with a period defined by the smallest frequency line in the partition of $(0, \omega^\star)$; for example, the period is $2\pi/\omega_1$ assuming $\Delta\omega_j =$

$\omega^\star/n$ and $\omega_j = (j - 1/2)\,\Delta\omega_j$. Samples of $\mathbf{X}^{(n)}(t)$ for times longer than $2\pi/\omega_1$ therefore provide the same information as samples of length $2\pi/\omega_1$. A procedure to generate samples of arbitrary length by applying smoothing windows to a collection of overlapped samples of $\mathbf{X}^{(n)}$ has been developed [37].

**Example 5.1**: Consider the case of a zero-mean scalar (*i.e.*, $d = 1$) stationary Gaussian process, denoted by $X(t)$, with covariance function $c(\tau) = \mathrm{E}[X(t)\,X(t+\tau)]$ and one-sided PSD $g(\omega)$. By Eqs. (5.2) and (5.3),

$$X^{(n)}(t) = \sum_{j=1}^{n} \left[A_j\,\cos(\omega_j t) + B_j\,\sin(\omega_j t)\right],$$

is an approximation for $X(t)$, and $A_j, B_j$ are zero-mean Gaussian random variables such that, for $i, j = 1, \ldots, n$,

$$\begin{aligned}
&\mathrm{E}[A_i\,B_j] = 0,\\
&\mathrm{E}[A_i\,A_j] = \mathrm{E}[B_i\,B_j] = \delta_{ij}\int_{u_{i-1}}^{u_i} g(\omega)\,\mathrm{d}\omega \approx \delta_{ij}\,g(\omega_i)\,\Delta\omega_i,
\end{aligned}$$

where, by Eq. (4.13), $g(\omega) = s_{11}(\omega) + s_{11}(-\omega)$ is the one-sided PSD of $X(t)$. We note that $X^{(n)}(t)$ can also be written as

$$X^{(n)}(t) = \sum_{j=1}^{n} \sigma_j\,\left[\bar{A}_j\,\cos\left(\omega_j t\right) + \bar{B}_j\,\sin\left(\omega_j t\right)\right]$$

where $\bar{A}_j$ and $\bar{B}_j$ are independent, standard Gaussian random variables, and $\sigma_j^2 = g(\omega_j)\,\Delta\omega_j$. Figure 5.1 shows the approximation for $g(\omega)$ used for calculations. The use of the FFT algorithm follows directly since [36]

$$\begin{aligned}
X^{(n)}(t) &= \sum_{k=1}^{n} \sigma_k\,\left[\bar{A}_k\,\cos\left(\omega_k t\right) + \bar{B}_k\,\sin\left(\omega_k t\right)\right]\\
&= \mathrm{Real}\left[\sum_{k=1}^{n} D_k\,e^{i\,\omega_k\,t}\right] = n\cdot\mathrm{Real}\left[\texttt{ifft}(\{D_k\})\right],
\end{aligned}$$

where

$$\begin{aligned}
D_k &= \sigma_k\,\left(\bar{A}_k^2 + \bar{B}_k^2\right)^{1/2}\,e^{i\,\Psi_k}\\
\Psi_k &= -\arctan\left(\frac{-\bar{B}_k}{\bar{A}_k}\right)
\end{aligned}$$

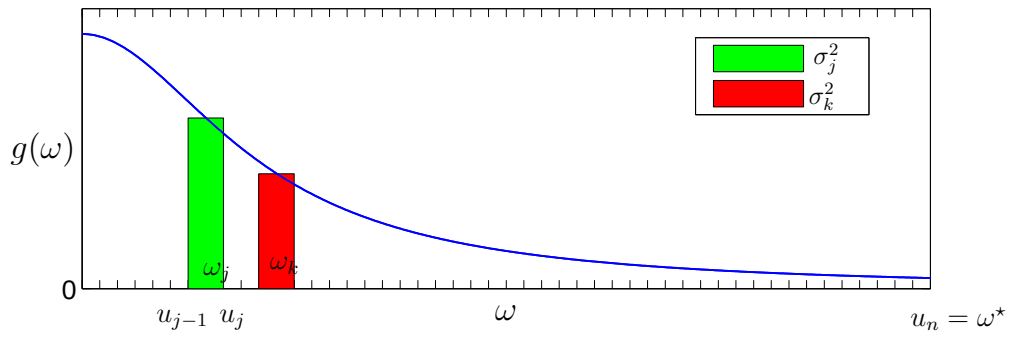


**Figure 5.1.** Discrete approximation for one-sided PSD.

and $i = \sqrt{-1}$.

MATLAB code `gsvpfft.m` and `gsvpfft_AB.m` listed in Appendix A.1 can be used to generate samples of $X^{(n)}(t)$. For illustration, let

$$c(\tau) = \frac{\sin(\bar{\omega}\,\tau)}{\bar{\omega}\,\tau} \quad \text{and} \quad g(\omega) = \begin{cases} \dfrac{1}{\bar{\omega}} & \text{if } 0 \le \omega \le \bar{\omega} \\ 0 & \text{else} \end{cases}$$

denote the covariance and one-sided PSD functions, respectively, that define the second-moment properties of $X(t)$, where parameter $\bar{\omega} = 200$ rad/sec. Shown in Fig. 5.2 is one sample of $X^{(n)}(t)$ for the case of $n = 200$. Also shown are estimates of $c_n(\tau) = \mathrm{E}[X^{(n)}(t)\,X^{(n)}(t+\tau)]$, for the case of $n = 50$, $n = 100$, and $n = 200$ using 1,000 Monte Carlo samples. By Fig. 5.2, we note that as $n$ increases, $c_n(\tau)$ approaches $c(\tau)$ for all $\tau$. $\Diamond$

**Example 5.2**: We next consider the case of an $\mathbb{R}^d$–valued stationary Gaussian process with $d = 11$ coordinates, *i.e.*, $\mathbf{X}(t) = (X_1(t), \ldots, X_{11}(t))^T$. This can be interpreted as a scalar random function of time and space applied at $d = 11$ distinct spatial locations; for simplicity, the spatial locations are assumed separated by uniform spacing $\Delta x > 0$. Let

$$c_{k,l}(\tau) = \mathrm{E}[X_k(t)\,X_l(t+\tau)] = \frac{\sin(\bar{\omega}\,\tau)}{\bar{\omega}\,\tau}\,\exp\left(-\theta\,\Delta x\,|k-l|\right),\ k,l = 1,\ldots,11,$$

define the covariance function of $\mathbf{X}(t)$, where $\bar{\omega}$=2,000 rad/sec and $\Delta x = 1/10$. Parameter $\theta > 0$ is related to the spatial correlation length of the process. For large $\theta$, processes $X_k(t)$ and $X_l(t)$ are nearly uncorrelated; for small $\theta$, $X_k(t)$ and $X_l(t)$ are nearly perfectly correlated, meaning they are nearly identical for all $t$. For calculations, we use $\theta = 1$.

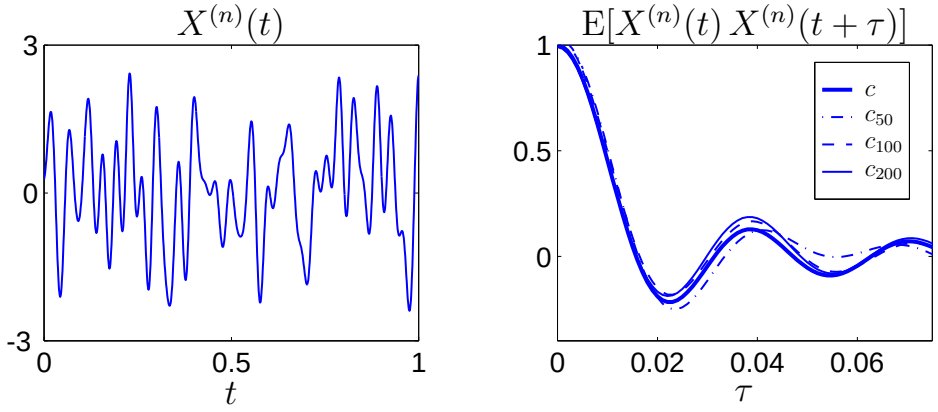


**Figure 5.2.** Samples and estimates of second-moment properties of a stationary Gaussian process. Shown are one sample of $X^{(n)}(t)$ for $n = 200$ and estimates of $c_n(\tau)$ for $n = 50$, 100, and 200. The exact covariance function, $c(\tau)$ is also plotted.

We generated 100 independent samples of $\mathbf{X}^{(n)}$ using MATLAB code `gsvpfft.m` listed in Appendix A.1 with $n = 1000$; results are illustrated by Fig. 5.3. One sample of coordinates $X_1(t)$, $X_2(t)$, and $X_{11}(t)$ are shown in Fig. 5.3(a), where the effect of the spatial correlation of the process is clearly evident. Values for $X_1(t)$ and $X_2(t)$, which correspond to two spatial points separated by $\Delta x$, are quite similar, while values for $X_1(t)$ and $X_{11}(t)$, which correspond to two spatial points separated by $10\Delta x$, differ significantly. Statistical estimates of the temporal and spatial correlation functions are illustrated by Figs. 5.3(b)–(c), where $\eta = \Delta x|k-l|$ denotes the spatial distance between coordinates of $\mathbf{X}$(t); the exact correlation functions as specified above are also shown. ◊

### 5.1.2 Parametric model with random frequencies

Let $X(t)$ be a stationary Gaussian process with zero mean, variance $\sigma^2$, and one-sided spectral density $g(\omega)$. We provide an approximation for $X(t)$ that is an alternative to $X^{(n)}(t)$ defined by Eq. (5.2). We limit the discussion to scalar Gaussian processes for simplicity; the derivations can be extended to represent vector processes if needed.

Let $N(\omega)$, $\omega > 0$, denote a homogeneous Poisson counting process (see Example 4.12) with intensity $\lambda > 0$ that depends on temporal frequency $\omega$ rather than time. Let random variables $N_1$ and $N_2$ denote two samples of $N$; the corresponding "jump frequencies" are de-

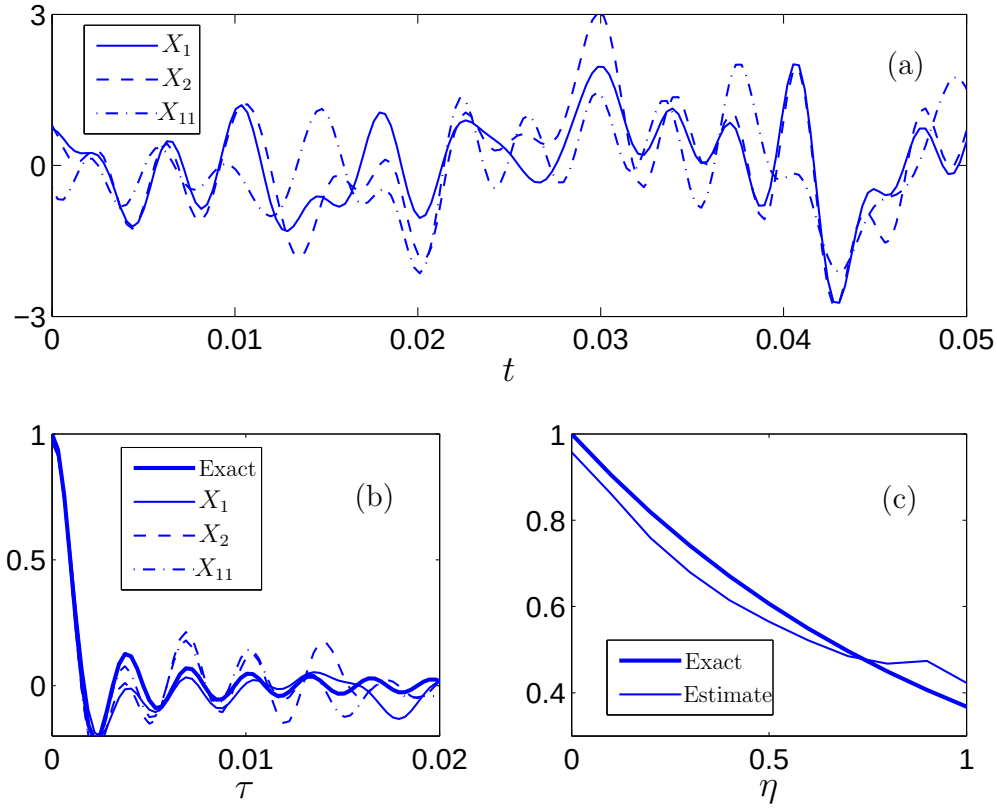


**Figure 5.3.** Samples and estimates of second-moment properties of a stationary vector Gaussian process: (a) one sample of coordinates $X_1^{(n)}$, $X_2^{(n)}$, and $X_{11}^{(n)}$, (b) temporal correlation function of $\mathbf{X}^{(n)}$, and (c) spatial correlation function of $\mathbf{X}^{(n)}$.

noted by $(U_1, \ldots, U_{N_1})^T$ and $(V_1, \ldots, V_{N_2})^T$, respectively. The following is an approximation for $X(t)$

$$X^{(N,\lambda)}(t) = \sum_{j=1}^{N_1} A_j \, h(U_j) \, \cos(U_j \, t) + \sum_{k=1}^{N_2} B_k \, h(V_k) \, \sin(V_k \, t) \tag{5.5}$$

where

$$h^2(\omega) = \frac{1}{\lambda \, \sigma^2} \, g(\omega), \; \omega \geq 0, \tag{5.6}$$

and $\{A_j\}$ and $\{B_k\}$ are iid Gaussian random variables with zero mean and variance $\sigma^2$. By Eq. (5.5), $X^{(N,\lambda)}(t)$ is a superposition of harmonics with both random amplitudes and frequencies, in contrast to the model defined by Eq. (5.2) that has random amplitudes but fixed frequencies.

The model defined by Eq. (5.5) has some desirable properties. First, $X^{(N,\lambda)}(t)$ is a weakly stationary process for any $\lambda > 0$. Second, the second-moment properties of $X^{(N,\lambda)}(t)$ approach the second-moment properties of $X(t)$ as $\omega^\star \to \infty$ for any $\lambda > 0$. These statements follow since

$$\begin{aligned} \mathrm{E}[X^{(N,\lambda)}(t)] &= 0, \quad \text{and} \\ \mathrm{E}[X^{(N,\lambda)}(t) \, X^{(N,\lambda)}(s)] &= \int_0^{\omega^\star} g(\omega) \, \cos(\omega(t-s)) \, \mathrm{d}\omega. \end{aligned} \tag{5.7}$$

Third, $X^{(N,\lambda)}(t)$ converges to a Gaussian process as $\lambda \to \infty$ for any $\omega^\star > 0$ [20]. Fourth, samples of $X^{(N,\lambda)}(t)$ are not periodic. However, $X^{(N,\lambda)}(t)$ is a non-Gaussian process for any finite intensity $\lambda$ and, unlike the approach described in Section 5.1.1, the FFT algorithm cannot be used for efficient sample generation.

In summary, we conclude that approximation $X^{(n)}$ discussed in Section 5.1.1 using fixed frequencies is preferred over approximation $X^{(N,\lambda)}$ using random frequencies defined in this section except for the case where very long samples of $X$ are needed. This is because long samples of $X$ require that a large number of frequency lines be considered so as to avoid samples of $X^{(n)}$ that are $2\pi/\omega_1$-periodic, where $\omega_1$ is the smallest frequency considered (see Fig. 5.1).

## 5.2 Non-stationary Gaussian processes

We next consider non-stationary Gaussian processes; we will limit the discussion to scalar processes for clarity. Let $X(t)$ denote a non-stationary scalar Gaussian process with zero

mean and covariance function $c(t,s) = \mathrm{E}[X(t)\,X(s)]$. In this section, we present methods to generate independent samples of $X(t)$. Three approaches are considered, corresponding to the three classes of non-stationary Gaussian processes illustrated by Fig. 4.9: (1) transformations of stationary Gaussian processes, (2) oscillatory Gaussian processes, and (3) general Gaussian processes. In general, the approaches are presented in order of increasing generality and complexity.

### 5.2.1 Transformations of stationary Gaussian processes

Let $X_{\mathrm{s}}(t)$ be a zero-mean stationary Gaussian stochastic process. We first consider the class of non-stationary, Gaussian processes that can be expressed as

$$X(t) = \alpha(t)\,X_{\mathrm{s}}\left[h(t)\right],\ t \geq 0, \tag{5.8}$$

where $\alpha(t) > 0$ and $h(t)$ are deterministic functions of time, $t$, and $h$ satisfies $h(0) = 0$ and $h'(t) = \mathrm{d}h(t)/\mathrm{d}t > 0$. The class of processes defined by Eq. (5.8) includes uniformly modulated stationary processes and stationary processes under time shift, as defined in Section 4.5.1.

The procedure to generate `ns` independent samples of $X(t)$ is straightforward and involves three steps:

1. Generate `ns` samples of stationary Gaussian process $X_{\mathrm{s}}(t)$ using any of the methods of Section 5.1;
2. Apply time shift $h(t)$ to each sample of $X_{\mathrm{s}}(t)$, creating `ns` samples of $X_{\mathrm{s}}\left[h(t)\right]$; and
3. Multiply each sample of $X_{\mathrm{s}}\left[h(t)\right]$ by modulation function $\alpha(t)$ to achieve `ns` samples of $X(t)$.

**Example 5.3**: Let $X_{\mathrm{s}}(t)$ be a stationary Gaussian process with zero mean, unit variance, and correlation $\mathrm{E}[X_{\mathrm{s}}(t)\,X_{\mathrm{s}}(t+\tau)] = e^{-200|\tau|}$. The corresponding one-sided PSD of $X_{\mathrm{s}}(t)$ (see Table 4.1) is $g(\omega) = 200/\pi(\omega^2 + 100^2)$. One sample of $X_{\mathrm{s}}(t)$, $0 \leq t \leq 2$, is illustrated by Fig. 5.4(a). The FFT algorithm discussed in Example 5.1 was used to create these samples of $X_{\mathrm{s}}(t)$. The corresponding MATLAB code, `gsvpfft.m`, is listed in Appendix A.1.

Define $X_1(t) = e^{-t}\,X_{\mathrm{s}}(t)$ and $X_2(t) = X_{\mathrm{s}}[h(t)]$ where $h(t) = \rho\,e^t - \rho$ and $\rho = 2/(e^2 - 1)$. By Eq. (4.19), $X_1(t)$ is a uniformly modulated stationary process with modulation function $\alpha(t) = e^{-t}$ and, by Eq. (4.22), $X_2(t)$ is a stationary process under time shift $h(t)$. Samples of $X_1(t)$ and $X_2(t)$ are illustrated by Figs. 5.4(b) and (c), respectively; both are non-stationary, Gaussian processes.

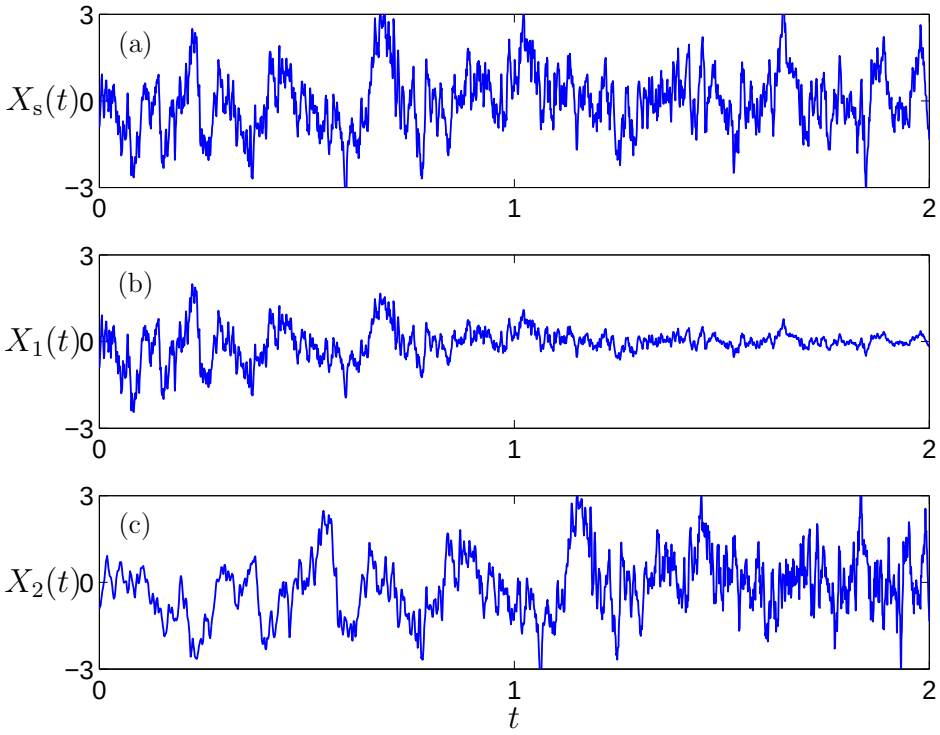


**Figure 5.4.** One sample of three zero-mean Gaussian processes: (a) stationary process $X_s(t)$, (b) uniformly modulated process $X_1(t)$, and (c) time shifted process $X_2(t)$.

Time shift $h(t)$ and its first derivative, $h'(t) = \mathrm{d}h(t)/\mathrm{d}t$, are illustrated by Fig. 5.5 for $0 \leq t \leq 2$. For early times, $h(t) < t$ and $h'(t) < 1$, meaning that time is effectively "slowed down" for process $X_2(t)$; the time axis in Fig. 5.4(c) therefore appears "stretched out" when compared to the time axis in Fig. 5.4(a). For $t > t_1 = \ln(1/\rho) \approx 1.16$, $h'(t) > 1$ meaning that time "speeds up". Accordingly, the time axis in Fig. 5.4(c) appears compressed near $t = 2$ when compared to the time axis in Fig. 5.4(a). $\Diamond$

### 5.2.2 Methods based on evolutionary spectral density

We next consider the case where $X(t)$ is a zero-mean, non-stationary Gaussian process with prescribed one-sided evolutionary spectral density $\psi(t, \omega)$. Assuming $\psi(t, \omega)$ is slowly-varying in time for all frequencies $\omega$, $X(t)$ can be expressed as an oscillatory process with spectral representation given by Eq. (4.26) with $a(t, \omega)^2 = \psi(t, \omega)$. Let $\omega^\star$ denote a cut-off

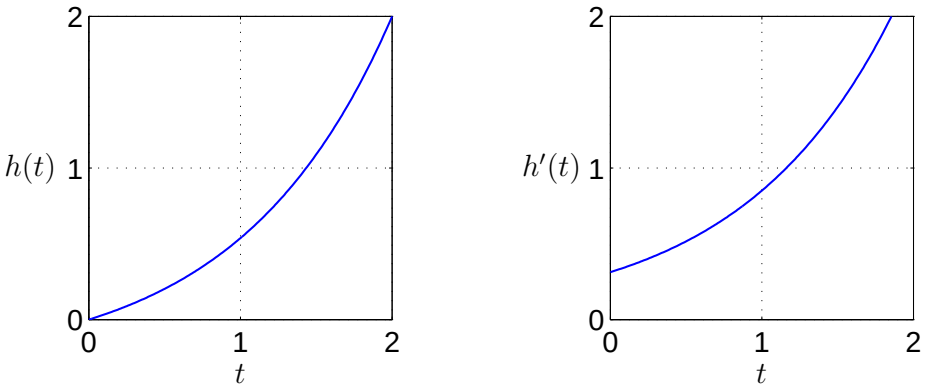


**Figure 5.5.** Time shift function $h(t)$ and $h'(t) = \mathrm{d}h(t)/\mathrm{d}t$ used in Example 5.3.

frequency, *i.e.*, $\psi(t,\omega) \approx 0$ for $\omega > \omega^\star$. The following is an approximation for $X(t)$ depending on $2n$ random variables

$$X^{(n)}(t) = \sum_{j=1}^{n} \sigma_j(t) \left[\bar{A}_j \cos(\omega_j t) + \bar{B}_j \sin(\omega_j t)\right], \tag{5.9}$$

where $\omega_1 < \cdots < \omega_n$ form a partition of $[0, \omega^\star]$, $\bar{A}_j, \bar{B}_j$ are standard uncorrelated Gaussian random variables, and $\sigma_j^2(t) = \psi(t,\omega_j)\Delta\omega_j$. The model defined by Eq. (5.9) is identical to the model used for the stationary case (see Example 5.1), with the exception that $\sigma_j(t)$ is now time-varying.

**Example 5.4**: Recall Example 4.24, where we used the oscillatory process to represent non-stationary process $X(t)$, assumed stationary in the vicinity of a collection of fixed distinct times $0 \le t_1 < \cdots < t_m$, with corresponding one-sided PSDs $g_1(\omega), \ldots, g_m(\omega)$. Here, we consider the following $m = 2$ time-invariant PSDs

$$g_1(\omega) = \frac{2 \cdot 100\, \sigma_1^2}{\pi(\omega^2 + 100^2)} \quad \text{and} \quad g_2(\omega) = \frac{4 \cdot 100^3\, \sigma_2^2}{\pi(\omega^2 + 100^2)^2} \tag{5.10}$$

valid in the vicinity of times $t_1 = 1$ and $t_2 = 2$, where $\sigma_1^2 = 1$ and $\sigma_2^2 = 2$. One sample of $X(t)$ and the variance of $X(t)$ are illustrated by Figs. 5.6(a) and (b), respectively, where it is evident that the frequency content of $X(t)$ decreases and the variance of $X(t)$ increases with increasing time; these features are consistent with $g_1(\omega)$ and $g_2(\omega)$. The one-sided evolutionary PSD of non-stationary process $X(t)$ is illustrated by Fig. 5.6(c); the two time-invariant PSDs $g_1$ and $g_2$, indicated by dark lines, are also shown at times $t_1 = 1$ and $t_2 = 2$, respectively. ◊

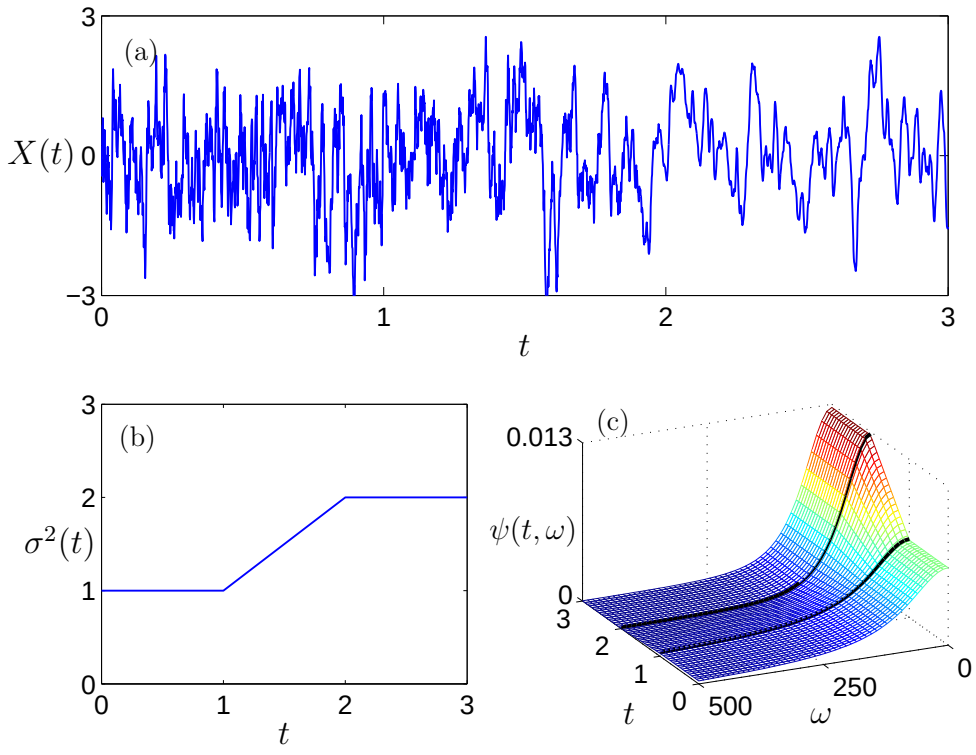


**Figure 5.6.** Oscillatory Gaussian process: (a) one sample of non-stationary process $X(t)$, (b) variance of $X(t)$, and (c) one-sided evolutionary PSD of $X(t)$.

### 5.2.3 Methods based on covariance function

The approaches considered in Sections 5.2.1 and 5.2.2 can only represent a subset of the class of non-stationary Gaussian processes (refer to Fig. 4.9). The approach considered in this section is the most general as it can be used to generate samples of arbitrary non-stationary Gaussian processes. However, the method requires full knowledge of the covariance function, which often may not be available for practical problems. It is shown that a special case of this approach is equivalent to the Karhunen-Loève representation introduced in Section 4.4.2.

Let $X(t)$ be a non-stationary Gaussian process with zero mean and covariance function $c(t,s) = \mathrm{E}\left[X(t)\,X(s)\right]$, and let $X^{(n)}(t)$ denote an approximation for $X(t)$ depending on $n$ random variables. To generate samples of $X^{(n)}(t)$ on $0 \le t \le T$, we partition $[0, \bar{t}\,]$ into $n-1$ non-overlapping intervals of width $\Delta t > 0$, and let

$$\mathbf{c} = \begin{bmatrix} c(0,0) & c(0,\Delta t) & \cdots & c(0,(n-1)\Delta t) \\ c(\Delta t,0) & c(\Delta t,\Delta t) & \cdots & c(\Delta t,(n-1)\Delta t) \\ \vdots & \vdots & \ddots & \vdots \\ c((n-1)\Delta t,0) & c((n-1)\Delta t,\Delta t) & \cdots & c((n-1)\Delta t,(n-1)\Delta t) \end{bmatrix} \tag{5.11}$$

denote an $n \times n$ matrix approximating $c(t,s)$ on $[0,\bar{t}\,] \times [0,\bar{t}\,]$. One sample of $X^{(n)}(t)$ can be obtained by the following matrix-vector multiplication [14]

$$\mathbf{X}^{(n)} = \mathbf{b}\,\mathbf{W} \tag{5.12}$$

where $\mathbf{X}^{(n)} = (X^{(n)}(0), X^{(n)}(\Delta t), \ldots X^{(n)}((n-1)\Delta t))^T$, $\mathbf{W} = (W_1, \ldots, W_n)^T$ is an $n \times 1$ vector of iid $N(0,1)$ random variables, and $\mathbf{b}$ is an $n \times n$ lower-triangular matrix such that $\mathbf{b}\mathbf{b}^T = \mathbf{c}$, *i.e.*, $\mathbf{b}$ is the Cholesky factorization of $\mathbf{c}$ (see [19], Section 4.2). MATLAB functions `gnsspchol.m` and `cholcov.m` listed in Appendix A.2 can be used to implement this approach; the latter is used to provide approximations for the Cholesky factorization for large matrices.

An alternative approach is to instead replace matrix $\mathbf{b}$ defined by Eq. (5.12) with

$$\mathbf{b} = \begin{bmatrix} \sqrt{\lambda_1}\,\boldsymbol{\phi}_1 & \sqrt{\lambda_2}\,\boldsymbol{\phi}_2 & \cdots & \sqrt{\lambda_n}\,\boldsymbol{\phi}_n \end{bmatrix} \tag{5.13}$$

where $\{\lambda_j\}$ and $\{\boldsymbol{\phi}_j\}$ denote the collection of $n$ eigenvalues and $n \times 1$ eigenvectors, respectively, of covariance matrix $\mathbf{c}$ defined by Eq. (5.11), *i.e.*,

$$\mathbf{c}\,\boldsymbol{\phi}_j = \lambda_j\,\boldsymbol{\phi}_j, j = 1, \ldots, n. \tag{5.14}$$

By using the $\mathbf{b}$ matrix defined by Eq. (5.13), we can show that

$$X^{(n)}((k-1)\Delta t) = \sum_{j=1}^{n} \sqrt{\lambda_j}\,\phi_j((k-1)\Delta t)\,W_j,\ k = 1, \ldots, n, \tag{5.15}$$

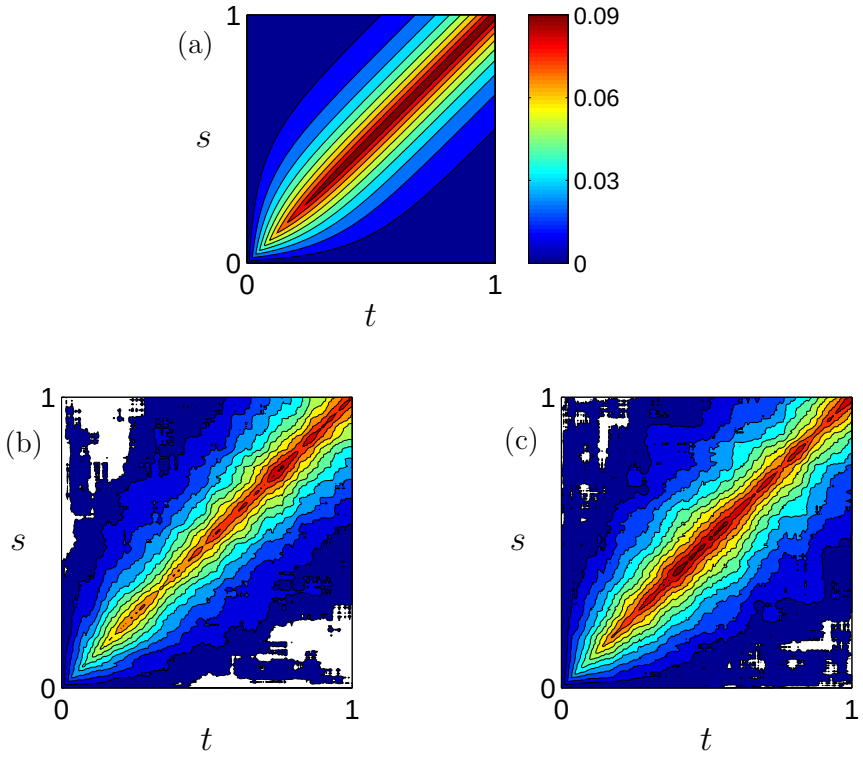


**Figure 5.7.** Contours of the covariance function, $c(t, s)$, of the Ornstein-Uhlenbeck process: (a) exact, and estimates from 500 Monte Carlo samples generated by (b) `gnsspchol.m` and (c) `gnsspKL.m` (see Appendix A.2).

so that approximation $X^{(n)}(t)$ is a Karhunen-Loève representation for $X(t)$, truncated at $n$ terms (see Eq. (4.16)). MATLAB code `gnsspKL.m` listed in Appendix A.2 can be used to implement this approach.

**Example 5.5**: Let $X(t)$ denote the Ornstein-Uhlenbeck process considered previously with parameters $\alpha = 5$, $\beta = 1$, and $x_0 = 1$. We generate 500 independent samples of $X(t)$ on $[0, \bar{t}\,] = [0, 1]$ with a time step of $\Delta t = 0.004$ using both `gnsspchol.m` and `gnsspKL.m`. As a check, we can estimate the covariance function from samples of $X(t)$ and compare with the known result listed in Example 4.14. Contours of these estimates are compared with contours of the exact covariance function in Fig. 5.7; good agreement with the exact solution can be observed using both methods. ◊

## 5.3 Non-Gaussian processes

We next consider the simulation of certain types of non-Gaussian processes. In particular, we will study non-Gaussian processes defined by transformations with and without memory of Gaussian processes, as well as random point processes; these three types of non-Gaussian processes are discussed in Sections 5.3.1, 5.3.2, and 5.3.3, respectively. There are many other types of non-Gaussian processes which are beyond the scope of this report, but we believe the types presented are applicable to a wide variety of problems in applied science and engineering. As in Section 5.2, we will limit the discussion to scalar processes for clarity.

### 5.3.1 Memoryless transformations of Gaussian processes

Let $X(t)$, $t \geq 0$, be a stationary Gaussian process with zero mean, unit variance, and covariance function $\rho(\tau) = \mathrm{E}\left[X(t)\, X(t+\tau)\right]$. In this section, we consider non-Gaussian processes that can be expressed via the following memoryless transformation of $X(t)$

$$Y(t) = F^{-1} \circ \Phi\left[X(t)\right] = h\left[X(t)\right], \tag{5.16}$$

where $\Phi$ is the CDF of a $N(0,1)$ random variable and $F$ is an arbitrary CDF. It was shown in Section 4.2.4 that $Y(t)$ is a stationary process with marginal CDF $F$. The covariance structure of $X$ is, in general, not preserved by the transformation defined by Eq. (5.16) meaning that the covariance of $Y(t)$ is not, in general, equal to $\rho(\tau)$. There are many cases where the change to the covariance function is not significant and can be ignored; one example is when $F$ is the CDF of a symmetric beta random variable (see [10]). Sometimes, however, the covariance functions of $X$ and $Y$ are very different, and this difference cannot be neglected. This is an area of active research; see, for example, [4].

The procedure to generate `ns` independent samples of $Y(t)$ is straightforward and involves two steps:

1. Generate `ns` samples of stationary Gaussian process $X(t)$ using any of the methods of Section 5.1; and
2. Apply translation $h$ to each sample of $X(t)$, creating `ns` samples of $Y(t)$.

**Example 5.6**: Suppose $Y(t)$ is a non-Gaussian stationary stochastic process with zero mean, covariance function

$$\mathrm{E}\left[Y(t)Y(t+\tau)\right] = \left(\frac{3 + 2e^{-2\alpha|\tau|}}{5}\right) e^{-\alpha|\tau|}$$

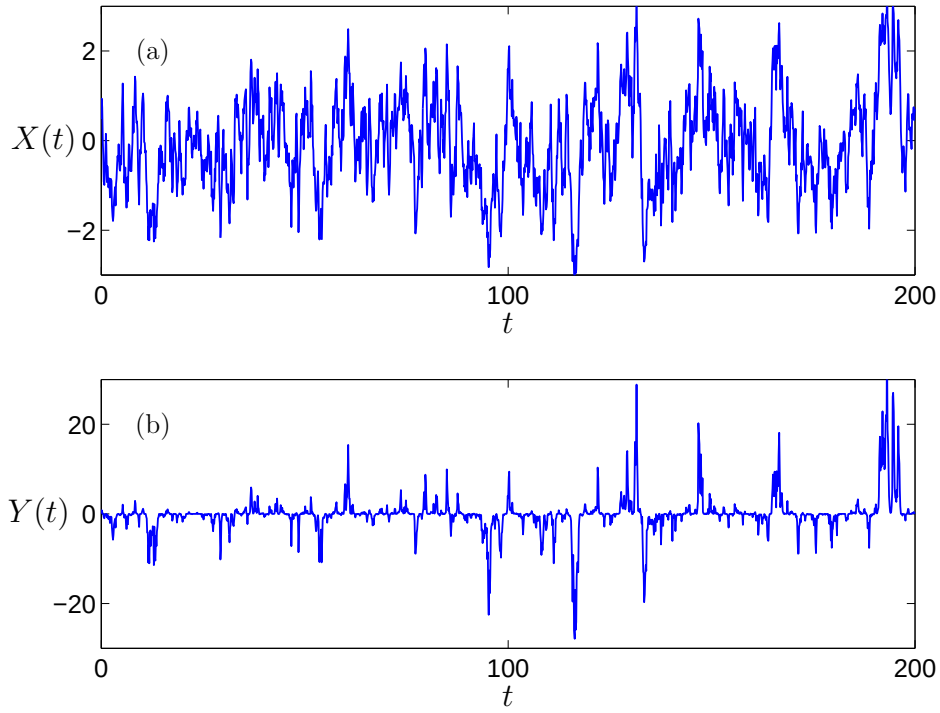


**Figure 5.8.** One sample of: (a) Gaussian process $X(t)$ with covariance function $\mathrm{E}\left[X(t)X(t+\tau)\right] = e^{-\alpha|\tau|}$ and (b) non-Gaussian translation process $Y(t) = X(t)^3$.

where $\alpha > 0$ is a constant, and marginal distribution

$$F(x) = \Phi\left(|x|^{1/3}\mathrm{sign}(x)\right),$$

where $\Phi$ denotes the CDF of a $N(0,1)$ random variable. This process can be expressed as $Y(t) = X(t)^3$, where $X(t)$ is a stationary Gaussian process with zero mean and covariance function $\mathrm{E}\left[X(t)X(t+\tau)\right] = e^{-\alpha|\tau|}$ (see [20], Section 3.1.1). One sample of processes $X(t)$ and $Y(t)$ are illustrated by Fig. 5.8 assuming $\alpha = 1$. The corresponding covariance functions of $X$ and $Y$ are illustrated by Fig. 5.9, demonstrating that $c \approx \rho$ in this case. $\Diamond$

### 5.3.2 Transformations with memory of Gaussian processes

If the current value of stochastic process $Y(t)$ is obtained from the past history of another process $X(t)$, the mapping from $X$ to $Y$ is said to have memory. Transformations with memory can be defined by differential equations with input $X$ and output $Y$. Our objective

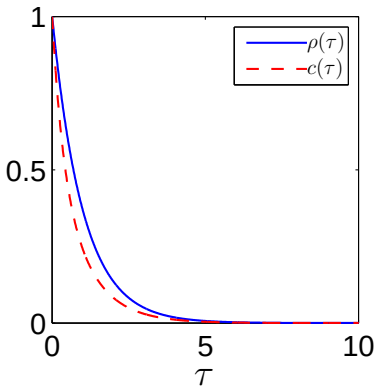


**Figure 5.9.** Covariance functions $\rho(\tau) = \mathrm{E}\,[X(t)X(t+\tau)]$ and $c(\tau) = \mathrm{E}\,[Y(t)Y(t+\tau)]$.

in this section is to generate samples of $Y$ by numerical solution of a special class of stochastic differential equations (SDEs). It is assumed that: (i) $Y$ is a scalar stochastic process defined by a differential equation of special form with random input $X$; (ii) the defining differential equation for $X$ is known and of a special form; and (iii) $X$ is Gaussian noise. Processes $Y$ that can be defined in this way are referred to as diffusion processes. More general types of transformations with memory can be considered (see, for example, [3] or [21], Section 5.3.3).

We first assume input $X(t)$ is a Gaussian white noise process. Output $Y(t)$ in this case is the solution to the following SDE

$$\mathrm{d}Y(t) = a\,(Y(t),t)\,\mathrm{d}t + b\,(Y(t),t)\,\mathrm{d}B(t),\ t \geq 0, \tag{5.17}$$

where $a$ and $b$ are, in general, time-varying functions of the state, and $B(t)$ denotes the Brownian motion; $a$ is commonly referred to as the drift term, and $b^2$ is the diffusion term. The Ornstein-Uhlenbeck process (see Example 4.10) is a special case of Eq. (5.17) with linear, time-invariant drift and constant diffusion.

The numerical solution of the SDE defined by Eq. (5.17) is possible by standard finite difference schemes. Euler's method (see [15], Section 5.2) is perhaps the simplest and is used in the following example; more accurate and efficient finite difference schemes can also be applied [25].

**Example 5.7**: Consider the following stochastic differential equation (SDE)

$$\mathrm{d}Y(t) = \sigma^2\,\mathrm{d}t + 2\,\sigma\,\sqrt{Y(t)}\,\mathrm{d}B(t),\ t \geq 0$$

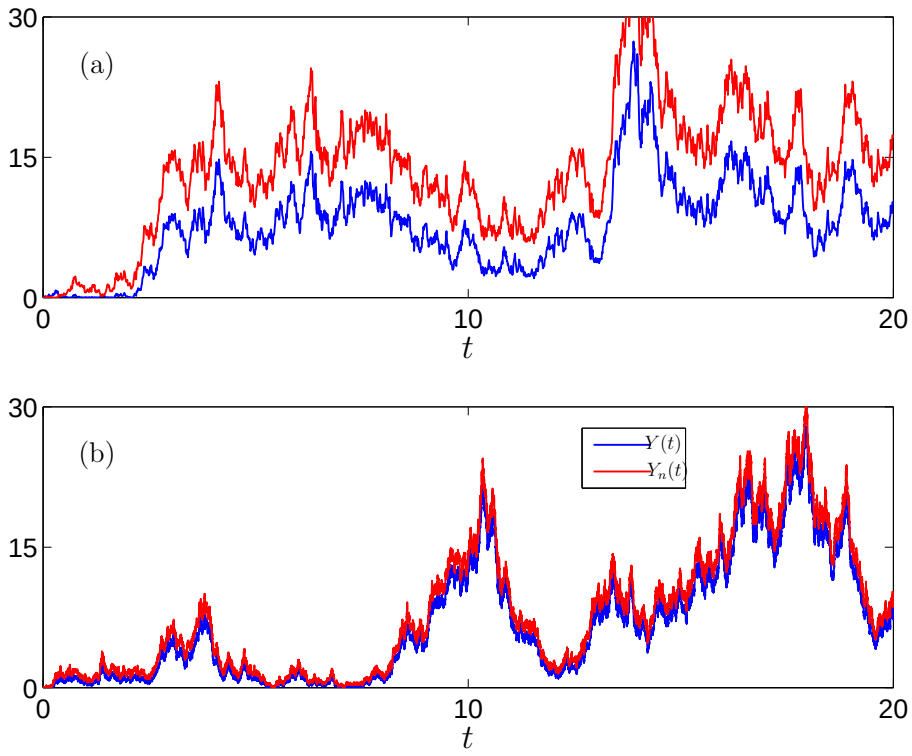


**Figure 5.10.** One sample of exact solution, $Y(t)$, and numerical solution, $Y_n(t)$, using Euler's method with: (a) $\Delta t = 1/100$, and (b) $\Delta t = 1/1000$.

with exact solution (see [20], p. 199)

$$Y(t) = \left(\sqrt{Y(0)} + \sigma\, B(t)\right)^2, \; t \geq 0.$$

Figure 5.10 shows realizations of the solution of the SDE assuming $\sigma = 1$ and $X(0) = 0$, calculated along the same path of the Brownian motion process $B(t)$. The solutions are plotted with time steps $\Delta t = 1/100$ and $\Delta t = 1/1000$ in Fig. 5.10(a) and (b), respectively. The exact solution, $Y(t)$, is shown in blue, while the Euler approximation of the solution, $Y_n(t)$, is shown in red. We note that $Y_n(t)$ can differ significantly from $Y(t)$ when the time step is large; this can also occur when the magnitude of the driving noise, $\sigma$, is large. The Euler approximation approaches the exact solution as $\Delta t \to 0$. The sample of process $Y_n(t)$ was generated by MATLAB code `ngvpITO.m` listed in Appendix A.3. ♢

If, instead, input process $X(t)$ is not a white noise but can itself be expressed as the

following SDE

$$\mathrm{d}X(t) = \alpha\left(X(t), t\right)\mathrm{d}t + \beta\left(X(t), t\right)\mathrm{d}B(t),\ t \geq 0, \tag{5.18}$$

then

$$\mathrm{d}\begin{pmatrix} Y(t) \\ X(t) \end{pmatrix} = \begin{pmatrix} a\left(Y(t), t\right) + b\left(Y(t), t\right)\ X(t) \\ \alpha\left(X(t), t\right) \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ \beta\left(X(t), t\right) \end{pmatrix}\mathrm{d}B(t),\ t \geq 0, \tag{5.19}$$

governs the evolution in time of vector process $(Y(t), X(t))^T$. Finite difference schemes such as Euler's method can be applied to provide approximations for $(Y(t), X(t))^T$.

### 5.3.3 Random point processes

We have introduced two types of random point processes, namely the Poisson counting process and filtered Poisson process, in Section 4.2.6. This section provides algorithms for generating samples of these processes. A detailed discussion on more general types of random point processes is given by [38].

The generation of one sample of the homogeneous Poisson counting process, $N(t)$, $0 \leq t \leq \bar{t}$, involves three steps:

1. Generate iid samples of exponential random variable $X$ with parameter $\lambda > 0$ until the cumulative sum of each sample exceeds time $\bar{t}$, *i.e.*, find $n$ such that $X_1 + \cdots X_n < \bar{t} \leq X_1 + \cdots X_{n+1}$;

2. Discard sample $X_{n+1}$ and map the remaining samples of $X$ to the arrival or jump times of $N(t)$ on $[0, \bar{t}\,]$, *i.e.*,

$$\begin{aligned} T_1 &= X_1 \\ T_2 &= X_1 + X_2 \\ &\vdots \\ T_n &= X_1 + X_2 + \cdots + X_n; \text{ and} \end{aligned}$$

3. Define $N(t)$, $0 \leq t \leq \bar{t}$, as

$$N(t) = \begin{cases} 0 & 0 \leq t < T_1 \\ 1 & T_1 \leq t < T_2 \\ \vdots & \vdots \\ n & T_n \leq t \leq \bar{t} \end{cases}$$

This three-step algorithm is implemented in MATLAB code `pcp.m` listed in Appendix A.3. Three samples of $N(t)$ are illustrated by Fig. 4.5 assuming $\lambda = 10$ and $[0, \bar{t}\,] = [0, 1]$.

To generate a single sample of the filtered Poisson process, $X(t)$ as defined by Eq. (4.8), on time interval $[0, \bar{t}\,]$, we do the following:

1. Generate one sample of homogeneous Poisson counting process $N(t)$ on $[0, \bar{t}\,]$ with jump times $T_1, \ldots, T_n$ as outlined above;
2. If $n > 0$, generate $n$ independent samples of random variable $Y$ that defines the random magnitude of each pulse;
3. Evaluate $w(t, T_k, Y_k)$, $k = 1, \ldots, n$, the shape of each pulse; and
4. Perform the summation in Eq. (4.8).

This four step algorithm is implemented in MATLAB code `fpp.m` listed in Appendix A.3; samples of a filtered Poisson process using this code are illustrated by Fig.4.6.

## 5.4 Verification

Various checks are available to verify that the samples generated using the methods of Sections 5.1, 5.2, and 5.3 satisfy the desired properties. In general, it is useful to verify the mean function, the covariance function, and the marginal distribution. For the case of stationary processes, the spectral density should also be verified. Entire textbooks are written on efficient methods for verification checking of stochastic processes; see, for example, [2] and [27]. We present only the minimum necessary for completeness.

Estimates of the mean and marginal distribution functions can be obtained by standard MATLAB functions `mean` and `hist` as discussed in Section 3.3. Methods to estimate the second-moment properties of a process depend on whether or not the process is ergodic, meaning that ensemble averages can be replaced by time averages (see Section 4.2.2).

Suppose first $X(t)$ is a stationary and ergodic process. Estimates of the covariance and power spectral density functions can be obtained from a single sample of $X(t)$ by using MATLAB codes `xcov2.m` and `xsd1.m`, respectively, listed in Appendix A.4. For example, the covariance function estimates illustrated by Fig. 5.2 and Fig. 5.3(b) were computed using `xcov2.m`. Next suppose $X(t)$ is a non-ergodic process. Estimates of the second-moment properties of $X(t)$ can be obtained from multiple independent samples of $X(t)$ using MATLAB code `smpMS.m` listed in Appendix A.4. The estimates illustrated by Fig. 5.7(b) and (c) were calculated using this function.

# Chapter 6

# Conclusions

There has been considerable work done at Sandia National Laboratories using random variables and vectors to represent parametric uncertainty within our current modeling and simulation environment. However, many phenomena of interest can be viewed as random in space and/or time, and the use of more general stochastic models to represent such phenomena have been rather limited. Examples include turbulent flow over aerospace vehicles and material properties within epoxy foams. The purpose of this report was to provide some theoretical background on stochastic processes and random fields that can be used to model phenomena that are random in space and/or time, and to provide a collection of simple algorithms that can be used to generate sample functions of these processes or fields.

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# Appendix A

# MATLAB code

## A.1 Stationary Gaussian processes

Two MATLAB functions are provided to generate samples of stationary Gaussian processes:

1. `gsvpfft.m`

   Generate samples of a zero-mean stationary Gaussian vector process on $[0, T]$ using the FFT algorithm.

2. `gsvpfft_AB.m`

   Generate samples of Gaussian vectors $A$ and $B$ that satisfy Eq. (5.3).

---

```
function [t,X]=gsvpfft(psdparams,psdfun,om_star,T,nseed,d,ns)
%GSVPFFT
% Function to generate ns independent samples of a zero-mean,
% stationary, Rd-valued Gaussian process on [0,T] using the FFT
% algorithm.  The two-sided PSD is defined by m-file psdfun with
% parameters psdparams, and cut-off frequency om_star (rad/s).

% om_star and T specify time step and number of time points
dt = (2*pi)/om_star;
m = floor( (om_star*T)/(2*pi) );

% time vector
t=0:dt:(m-1)*dt;t=t(:);

% discretize frequency vector (rad/s)
om=linspace(0,om_star,m);del_om=om(2)-om(1);
```

```
% initialize
randn('seed',nseed);
X=zeros(m,d,ns);
IMAG=sqrt(-1);

for j=1:ns,

    % get m samples of d x 1 vectors A and B
    [A,B]=gsvpfft_AB(psdfun,psdparams,om,del_om,d);

    % map to samples of C and Psi
    C = sqrt( (A.^2 + B.^2) );
    Psi = -atan2(B,A);

    % map to samples of D
    D = C .* exp(IMAG*Psi);

    % use ifft to get sample of X
    X(:,:,j)=m*real(ifft(transpose(D)));

end
```

---

```
function [A,B]=gsvpfft_AB(psdfun,psdparams,om,del_om,d)
%GSVPFFT_AB
% Function to compute m independent samples of d x 1 Gaussian
% vectors A and B that satisfy Eq.  (5.3).

% setup
IMAG=sqrt(-1);
m=length(om);
A=zeros(d,m);B=A;

% number of frequency / time points
for j=1:m,

    % evaluate g(om_j) and h(om_j) (from Eq.  (4.13))
    om_j=om(j);
    g=feval(psdfun,om_j,psdparams) + feval(psdfun,-om_j,psdparams);
```

```
    h=-IMAG*(feval(psdfun,om_j,psdparams) - ...
       feval(psdfun,-om_j,psdparams));

    % covariance matrix of 2d x 1 vector C = A;B
    covC=zeros(2*d,2*d);
    covC(1:d,1:d) = del_om*g;
    covC(d+1:2*d,d+1:2*d) = del_om*g;
    covC(1:d,d+1:2*d) = del_om*h;
    covC(d+1:2*d,1:d) = -del_om*h;

    % one sample of C=A;B
    b=chol(covC)';
    C=b*randn(2*d,1);

    % recover samples of A and B
    A(:,j)=C(1:d);B(:,j)=C(d+1:2*d);

end
```

---

## A.2 Non-Stationary Gaussian processes

Three MATLAB functions are provided to generate samples of non-stationary Gaussian processes:

1. `gnsspchol.m`

   Generate samples of a zero-mean non-stationary Gaussian process on $[0, T]$ using the Cholesky decomposition of the covariance matrix.

2. `cholcov.m`

   Compute the Cholesky decomposition of a covariance matrix.

3. `gnsspKL.m`

   Generate samples of a zero-mean non-stationary Gaussian process on $[0, T]$ using the Karhunen-Loéve representation of the covariance matrix.

---

```
function [t,X]=gnsspchol(p,covfun,dt,T,nseed,ns)
%GSSPCHOL
% Function to generate ns independent samples of a zero-mean,
% non-stationary, real-valued Gaussian process on [0,T] using
% the Cholesky decomposition.  The covariance function of X is
% specified by m-file covfun with parameter vector p.

% initialize randn
randn('seed',nseed)

% time vector
t=0:dt:T;t=t(:);
n=length(t);

% covariance matrix
cflag=0;
c=feval(covfun,p,t,t);

% if Var[X(0)]=0, then X(0)=0 a.s.
if c(1,1)==0,
    cflag=1;
    warning('gnsspchol():  variance is zero at t=0')
```

```
    c=c(2:n,2:n);t=t(2:n);n=n-1;
end

% Cholesky decomposition of c
b=cholcov(c,1/100);

% main loop
X=zeros(n,ns);
for k=1:ns,
    X(:,k) = b*randn(n,1); % Eq. (5.12)
end

if cflag,
    t=[0;t];
    X=[zeros(1,ns);X];
end
```

---

```
function b=cholcov(c,tol)
%CHOLCOV
% Function to compute Cholesky decomposition of covariance matrix
% c.  The sparsity of c is exploited to do the decomposition more
% efficiently, and any entries of c less than tol are set to zero.

% construct sparse matrix that only includes values of magnitude at
% least tol
[rows, cols] = find(abs(c) >= tol);
ii = sub2ind(size(c), rows, cols);
c_sparse = sparse(rows, cols, c(ii));

% find a good ordering of equations for the Cholesky factorization
P = symamd(c_sparse);
R = chol(c_sparse(P,P));

% output
b=transpose(R);
```

---

```
function [t,X]=gnsspKL(p,covfun,dt,T,nseed,ns)
%GSSPKL
```

```
% Function to generate ns independent samples of a zero-mean,
% non-stationary, real-valued Gaussian process on [0,T] using
% the Karhunen-Loeve representation.  The covariance function
% of X is specified by m-file covfun with parameter vector p.

% initialize randn
randn('seed',nseed)

% time vector
t=0:dt:T;t=t(:);
n=length(t);

% covariance matrix
c=feval(covfun,p,t,t);

% eigensolution of c
[PHI,LAM]=eig(c);

% construct b matrix, Eq. (5.13)
for j=1:n,
    b(:,j) = sqrt(LAM(j,j)) * PHI(:,j);
end

% main loop
X=zeros(n,ns);
for k=1:ns,
    X(:,k) = b*randn(n,1); % Eq. (5.12)
end
```

---

## A.3 Non-Gaussian processes

Three MATLAB functions are provided to generate samples of non-Gaussian processes:

1. `ngvpITO.m`

   Generate samples of non-Gaussian, non-stationary vector diffusion process defined by Eq. (4.7) using Euler's method.

2. `fpp.m`

   Generate samples of stationary filtered Poisson process defined by Eq. (4.8). The pulses have Gaussian magnitudes, which can be modified in the code.

3. `pcp.m`

   Generate samples of homogeneous Poisson counting process discussed in Example 4.12.

---

```
function [t,Xn]=ngvpITO(afun,bfun,aparams,bparams,x0,d,dp,n,T,ns,nseed)
%NGVPITO
% Function to compute ns independent samples of the Rd-valued diffusion
% process defined by the following Ito differential equation:
%
% dX(t) = a(X(t)) dt + b(X(t)) dB(t), 0 <= t <= T, X(0) = x0,
%
% where X is a Rd-valued non-Gaussian process, a is an d x 1 drift
% term, b is an d x dp diffusion term, and B is a dp x 1 Brownian
% motion.  The solution is approximated using Euler's method.  The
% drift and diffusion are defined by external m-files.

% time vector
t=linspace(0,T,n);dt=t(2)-t(1);

% setup
randn('seed',nseed);
Xn=zeros(d,n,ns);
for i=1:ns,
    Xn(:,1,i) = x0; % set initial conditions
end

% loop over number of samples
```

```
for i=1:ns,
    % Euler method - see [21], p.  277
    for k=2:n,
       % increment of Brownian motion
       dB=sqrt(dt)*randn(dp,1);
       % drift term
       a=feval(afun,aparams,Xn(:,k-1,i));
       % diffusion term
       b=feval(bfun,bparams,Xn(:,k-1,i));
       % update state vector
       Xn(:,k,i) = a*dt + b*dB + Xn(:,k-1,i);
    end
end
```

---

```
function [t,X]=fpp(lam,s2,wfun,wparams,T,n,ns,nseed)
%FPP
% Function to compute ns independent samples of a filtered Poisson
% process on [0,T]. The process depends on (1) a homogeneous
% Poisson counting process of intensity lam>0, (2) independent
% Gaussian pulses with mean zero and variance s2>0, and (3) a
% shape function w(t,tau,y), defined by external m-file.  Each
% column of X is a sample.  See Example (4.13)

% time vector
dt=T/n;t=0:dt:T;

% setup
randn('seed',nseed);rand('seed',nseed);
X=zeros(n+1,ns);

% ns realizations of Poisson counting process in [0,T]
[N,Tau]=pcp(lam,T,ns);

% loop over samples
for i=1:ns,
    % number of points and their arrival times
    Ni=N(i);Taui=Tau(1:Ni,i);
    if Ni>0,
       % Y is N(0,s2)
```

```
      y=sqrt(s2)*randn(Ni,1);
      % loop over time points
      for j=1:n+1,
         % evaluate weight function
         wj=feval(wfun,wparams,t(j),Taui,y);
         % Eq. (4.8)
         X(j,i)=sum(wj);
      end
   end
end
```

---

```
function [N,Tau]=pcp(lam,T,ns)
%PCP
% Function to generate ns samples of the homogeneous Poisson
% counting process of intensity lam > 0 in the time interval
% [0,T]. Also computed are the random times, Tau, at which
% the events occur.  See Example 4.12.

% setup
Nguess = floor(1.5*lam*T);     % Initial guess for N: 1.5*E[N]=1.5*lam*T
T=zeros(Nguess,ns);

% loop over samples
for i=1:ns,
    Ei=exprnd(1/lam,Nguess,1); % interarrival times are Exp RVs
    Ti=cumsum(Ei);T(:,i)=Ti;   % jump times of N
    ndx=find(Ti>T);            % keep times that do not exceed T
    if isempty(ndx),
       error('pcp():  Nguess not large enough');
    else
       N(i)=ndx(1)-1;
    end
end
Nmax=max(N);

% all jumps that exceed T will have infinite arrival time
Tau=Inf*ones(Nmax,ns);
for i=1:ns,
    Tau(1:N(i),i)=T(1:N(i),i);
```

```
end
```

---

## A.4 Verification

Three MATLAB functions are provided to perform statistical checks on a collection of samples generated by any of the methods listed in Sections A.1-A.3:

1. `smpMS.m`

   Estimate (time-varying) second-moment properties of non-stationary process from multiple samples.

2. `xcov2.m`

   Estimate the auto- or cross-covariance function of a stationary, ergodic process.

3. `xsd1.m`

   Estimate the one-sided auto- or cross-spectral density of a stationary, ergodic process.

---

```
function [mn,c,r]=smpMS(xx)
%SMPMS
% Function to estimate second-moment properties of a non-ergodic
% stochastic process from input xx, an n x ns array of sample functions.
% See Chapter 12 from [42].
%
% Note:  diag(c) is the variance estimate

% setup
[n,ns]=size(xx);

% mean estimate
mn = mean(transpose(xx));

for i=1:n,
    xx_i = xx(i,:);
    for j=1:i,
       xx_j = xx(j,:);
       % lower triangle of correlation matrix
       r(i,j) = mean(xx_i .* xx_j);
    end
end
% upper triangle of correlation matrix
```

```
r = r + transpose(r) - diag(diag(r));

% covariance estimate
c = r - transpose(mn)*mn;
```

---

```
function [tau,c]=xcov2(x,y,dt,ndt)
%XCOV2
% Function to provide an estimate of the cross covariance function
% of stationary processes X and Y, given one sample of each.
%
% Note:  [tau,c]=xcov2(x,x,dt,ndt) estimates the auto-covariance
% of x.

% use Matlab's xcov.m
[c,lags]=xcov(x,y,ndt);

% scale output
n=length(x);
tau= dt * lags(ndt+1:end);tau=tau(:);
c = 1/n * c(ndt+1:end);c=c(:);
```

---

```
function [g,f]=xsd1(x,y,dt,nd)
%XSD1
% Function to estimate the one-sided cross-spectral density function
% of stationary processes X,Y given one sample of each.  The method is
% based on Sections 11.5 and 11.6 from Ref. [2].
%
% Note:  [g,f]=xsd1(x,x,dt,nd) estimates the one-sided auto-spectral
% density of x.

% setup
m=length(x);N=floor(m/nd);

% remove mean functions
x=x-mean(x);y=y-mean(y);

% partition data
```

```
k=0;
for i=1:nd,
    for j=1:N,
       k=k+1;
       xx(i,j)=x(k);
       yy(i,j)=y(k);
       f(j)=(j-1)/(N*dt);
    end
end

% Fourier coefficients
for i=1:nd,
    X(i,:)=dt*(fft(xx(i,:)));
    Y(i,:)=dt*(fft(yy(i,:)));
end

% One-sided cross-spectral density estimate
for k=0:N/2,
    temp=0;
    for i=1:nd,
       temp = temp + ( X(i,k+1) * conj( Y(i,k+1) ) );
    end
    g(k+1)=2/(nd*N*dt)*temp;
end
f=f(1:floor(N/2)+1);
f=f(:);g=g(:);
```

---

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