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Topological Strings and (Almost) Modular Forms

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Abstract

The B-model topological string theory on a Calabi-Yau threefold X has a symmetry group Γ, generated by monodromies of the periods of X. This acts on the topological string wave function in a natural way, governed by the quantum mechanics of the phase space $H^3(X)$. We show that, depending on the choice of polarization, the genus g topological string amplitude is either a holomorphic quasi-modular form or an almost holomorphic modular form of weight 0 under Γ. Moreover, at each genus, certain combinations of genus g amplitudes are both modular and holomorphic. We illustrate this for the local Calabi-Yau manifolds giving rise to Seiberg-Witten gauge theories in four dimensions and local $\mathbb{P}_2$ and $\mathbb{P}_1 \times \mathbb{P}_1$. As a byproduct, we also obtain a simple way of relating the topological string amplitudes near different points in the moduli space, which we use to give predictions for Gromov-Witten invariants of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$.

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1. Introduction

Topological string theory has led to many insights in both physics and mathematics. Physically, it computes non-perturbative F-terms of effective supersymmetric gauge and gravity theories in string compactifications. Moreover, many dualities of superstring theory are better understood in terms of topological strings. Mathematically, the A-model explores the symplectic geometry and can be written in terms of Gromov-Witten, Donaldson-Thomas or Gopakumar-Vafa invariants, while the mirror B-model depends on the complex structure deformations and usually provides a more effective tool for calculations.

The topological string is well understood for non-compact toric Calabi-Yau manifolds. For example, the B-model on all non-compact toric Calabi-Yau manifolds was solved to all genera in [1] using the W_∞ symmetries of the theory. Geometrically, the W_∞ symmetries are the ω-preserving diffeomorphisms of the Calabi-Yau manifold, where ω is the $(3,0)$ holomorphic volume form. By contrast, for compact Calabi-Yau manifolds the genus expansion of the topological string is much harder to compute and so far only known up to genus four in certain cases, for instance for the quintic Calabi-Yau threefold. It is natural to think that understanding quantum symmetries of the theory may hold the key in the compact case as well.

In this paper, we will not deal with the full diffeomorphism group, but we will ask how does the finite subgroup Γ of large, ω-preserving diffeomorphisms, constrain the amplitudes. In other words, we ask: what can we learn from the study of the group of symmetries Γ generated by monodromies of the periods of the Calabi-Yau? For this, we need to know how Γ acts in the quantum theory. The remarkable fact about the topological string is that its partition function $Z = \exp(\sum_g g_s^{2g-2} \mathcal{F}_g)$ is a wave function in a Hilbert space obtained by quantizing $H_3(X)$, where g_s^2 plays the role of $\hbar$.[1] Classically, Γ acts on $H_3(X)$ as a discrete subgroup of the group $Sp(2n, \mathbb{Z})$ of symmetries that preserve the symplectic form, where $n = \frac{1}{2} b_3(X)$. This has a natural lift to the quantum theory.

The answer turns out to be beautiful. Namely, the $\mathcal{F}_g$'s turn out to be (almost) modular forms of Γ. By "(almost) modular form" we mean one of two things: a form which is holomorphic, but quasi-modular (i.e. it transforms with shifts), or a form which is modular, but not quite holomorphic. By studying monodromy transformations of the topological string partition function in "real polarization", where Z depends holomorphically on the moduli space, we find that it is a quasi-modular form of Γ of weight 0. The symmetry

[1] This fact was also recently explored in [18,14,31,39].

transformations under Γ imply that the genus g partition function $\mathcal{F}_g$ is fixed *recursively* in terms of *lower genus data*, up to the addition of a holomorphic modular form. Thus, modular invariance constrains the wave function, but does not determine it uniquely. The holomorphic modular form that is picked out by the topological string can be deduced (at least in principle) by its behavior at the boundaries on the moduli space. On the other hand, if we consider the topological string partition function in "holomorphic polarization", this turns out to be a modular form of weight 0, which is *not* holomorphic on the moduli space. While it fails to be holomorphic, it turns out to be "almost holomorphic" in a precise sense. Moreover, it is again determined recursively, up to the holomorphic modular form. Thus, the price to pay for insisting on holomorphicity is that the $\mathcal{F}_g$'s fail to be precisely modular, and the price of modularity is failure of holomorphicity!

The recursive relations we obtain contain exactly the same information as what was extracted in [6] from the holomorphic anomaly equation. In [6], through a beautiful study of topological sigma models coupled to gravity, the authors extracted a set of equations that the genus g partition function $\mathcal{F}_g$ satisfies, expressing an anomaly in holomorphicity of $\mathcal{F}_g$. The equations turn out to fix $\mathcal{F}_g$ in terms of lower genus data, up to an holomorphic function with a finite set of undetermined coefficients. Here, we have formulated the solutions to the holomorphic anomaly equation by exploiting the underlying symmetry of the theory. In the context of [6], solving the equations was laborious, the particularly difficult part being the construction of certain "propagators". From our perspective, the propagators are simply the "generators" of (almost) modular forms, that is the analogues of the second Eisenstein series of $SL(2,\mathbb{Z})$ and its non-holomorphic counterpart! That a reinterpretation of [6] in the language of (almost) modular forms should exist was anticipated by R. Dijkgraaf in [13]. For local Calabi-Yau manifolds, the relevant modular forms are Siegel modular forms. In the compact Calabi-Yau manifold case, our formalism seems to predict the existence of a new theory of modular forms of (subgroups of) $Sp(2n,\mathbb{Z})$, defined on spaces with Lorentzian signature (instead of the usual Siegel upper half-space).

The paper is structured as follows. In section 2, we describe the B-model topological string theory, from a wave function perspective, for both compact and non-compact target spaces. In section 3, we take a first look at how the topological string wave function behaves under the symmetry group Γ generated by the monodromies. Then, we give a more precise analysis of the resulting constraints on the wave function in section 4. We also explain the close relationship between the topological string amplitudes and (almost) modular forms in this section. In the remaining sections we give examples of our formalism:

in section 5 we study $SU(N)$ Seiberg-Witten theory, in section 6 local $\mathbb{P}^2$ — where we also use the wave function formalism to extract the Gromov-Witten invariants of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, and in section 7 local $\mathbb{P}^1\times\mathbb{P}^1$. To conclude our work, in section 8 we present some open questions, speculations and ideas for future research. Finally, Appendix A and B are devoted to a review of essential facts and conventions about modular forms, quasi-modular forms and Siegel modular forms.

2. B-model and the Quantum Geometry of $H^3(X,\mathbb{C})$

The B-model topological string on a Calabi-Yau manifold X can be obtained by a particular topological twisting of the "physical" string theory, two-dimensional $(2,2)$ supersymmetric sigma model on X coupled to gravity. The genus zero partition function of the B-model $\mathcal{F}_0$ is determined by the variations of complex structures on X. The higher genus amplitudes $\mathcal{F}_{g>0}$ can be thought of as quantizing this. When X has a mirror Y, this is dual to the A-model topological string, which is the Gromov-Witten theory of Y, obtained by an A-type twist of the physical theory on Y. As is often the case, many properties of the theory become transparent when the moduli of X and Y are allowed to vary, and the global structure of the fibration of the theory over its moduli space is considered. This is quite hard to do in the A-model directly, but the mirror B-model is ideally suited for these types of questions.

2.1. Real Polarization

Let us first recall the classical geometry of $H^3(X,\mathbb{C}) = H^3(X,\mathbb{Z})\otimes\mathbb{C}$. In the following, we will assume that X is a compact Calabi-Yau manifold, and later explain the modifications that ensue in the non-compact, local case.

Choose a complex structure on X by picking a particular 3-form ω in $H^3(X,\mathbb{C})$. Any other 3-form differing from this by a multiplication by a non-zero complex number determines the same complex structure. The set of $(3,0)$-forms is a line bundle $\mathcal{L}$ over the moduli space $\mathcal{M}$ of complex structures. Given a symplectic basis of $H_3(X,\mathbb{Z})$,

$$A^I\cap B_J = \delta^I_J,$$

where $I,J = 1,\dots n$, and $n=\frac{1}{2}b_3(X)$, we can parameterize the choices of complex structures by the periods

$$x^I = \int_{A^I}\omega, \qquad p_I = \int_{B_I}\omega.$$

The periods are not independent, but satisfy the special geometry relation:

$$p_I(x) = \frac{\partial}{\partial x^I}\mathcal{F}_0(x). \tag{2.1}$$

As is well known, $\mathcal{F}_0$ turns out to be given in terms of the classical, genus zero, free energy of the topological strings on X.

In the above, we picked a symplectic basis of H_3. Different choices of symplectic basis differ by $Sp(2n,\mathbb{Z})$ transformations:

$$\begin{aligned} \tilde{p}_I &= A_I{}^J p_J + B_{IJ} x^J \\ \tilde{x}^I &= C^{IJ} p_J + D^I{}_J x^J \end{aligned} \tag{2.2}$$

where

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{Z}).$$

For future reference, note that the period matrix τ, defined by

$$\tau_{IJ} = \frac{\partial}{\partial x^J} p_I$$

transforms as

$$\tilde{\tau} = (A\tau + B)(C\tau + D)^{-1}. \tag{2.3}$$

For a discrete subgroup $\Gamma \subset Sp(2n,\mathbb{Z})$, the changes of basis can be undone by picking a different 3-form ω. Conversely, we should identify the choices of complex structure that are related by changes of basis of $H_3(X,\mathbb{Z})$. The x's can be viewed as projective coordinates on the Teichmuller space $\mathcal{T}$ of X, on which Γ acts as the mapping class group. Consequently, the space of inequivalent complex structures is

$$\mathcal{M} = \mathcal{T}/\Gamma.$$

Generically, the moduli space $\mathcal{M}$ has singularities in complex codimension one, and Γ is generated by monodromies around the singular loci.

It is natural to think of $H^3(X,\mathbb{Z})$ as a classical phase space, with symplectic form,

$$dx^I \wedge dp_I,$$

and (2.1) as giving a lagrangian inside it. In fact, the analogy is precise. As shown in [40], in the quantum theory x^I and p_J become canonically conjugate operators

$$[\,p_I, x^J] = g_s^2\,\delta_I^J \tag{2.4}$$

where g_s^2 plays the role of $\hbar$, and the topological string partition function

$$Z(x^I) = g_s^{\frac{\chi}{24}-1} \exp\,[\sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(x^I)], \tag{2.5}$$

where $\mathcal{F}_g$ is the genus g free energy of the topological string, becomes a wave function.

More precisely, the B-model topological string theory determines a *particular* state $|Z\rangle$ in the Hilbert space obtained by quantizing $H^3(X,\mathbb{Z})$. The wave function,

$$\langle x^I | Z\rangle = Z(x^I)$$

describes the topological string partition function in one, "real" polarization[2] of $H^3(X)$. The semi-classical, genus zero approximation to the topological string wave function is determined by the classical geometry of X, and the lagrangian (2.1):

$$p_I Z(x) = g_s^2 \frac{\partial}{\partial x^I} Z(x) \sim (\frac{\partial}{\partial x^I}\mathcal{F}_0)\; Z(x).$$

The lagrangian does not determine the full quantum wave function. In general, there are normal ordering ambiguities, and to resolve them, the full topological B-model string theory is needed.[3]

The partition function Z implicitly depends on the choice of symplectic basis. Classically, changes of basis $(p,x) \to (\tilde{p},\tilde{x})$ which preserve the symplectic form are canonical transformations of the phase space. For the transformation in (2.2), the corresponding generating function $S(x,\tilde{x})$ that satisfies

$$dS = p_I dx^I - \tilde{p}_I d\tilde{x}^I \tag{2.6}$$

is given by[4]

$$S(x,\tilde{x}) = -\frac{1}{2}(C^{-1}D)_{JK} x^J x^K + (C^{-1})_{JK} x^J \tilde{x}^K - \frac{1}{2}(AC^{-1})_{JK}\tilde{x}^J\tilde{x}^K. \tag{2.7}$$

[2] For us, ω naturally lives in the complexification $H^3(X,\mathbb{C}) = \mathbb{C}\otimes H^3(X,\mathbb{R})$, so "real" polarization is a bit of a misnomer.

[3] Note that due to (2.4), g_s is a section of $\mathcal{L}$, so that $\mathcal{F}_g$ is a section of $\mathcal{L}^{2-2g}$. The full partition function is a section of $\mathcal{L}^{\frac{\chi}{24}-1}$, where χ is the Euler characteristic of the Calabi-Yau, due to the prefactor.

[4] Note that (2.6) only defines S up to an addition of a *constant* on the moduli space. This ambiguity can be absorbed in $\mathcal{F}_1$, since only derivatives of it are physical anyhow.

This has an unambiguous lift to the quantum theory, with the wave function transforming as[5]

$$\tilde{Z}(\tilde{x}) = \int dx \; e^{-S(x,\tilde{x})/g_s^2} \, Z(x). \tag{2.8}$$

We should specify the contour used to define (2.8); however, as long as we work with the perturbative g_s^2 expansion of $Z(x)$, the choice of contour does not enter. To make sense of (2.8) then, consider the saddle point expansion of the integral.

Given $\tilde{x}^I$, the saddle point of the integral $x^I = x^I_{cl}$ solves the classical special geometry relations that follow from (2.2) :

$$\frac{\partial S}{\partial x^I}|_{x_{cl}} = p_I(x_{cl}).$$

Expanding around the saddle point, and putting

$$x^I = x^I_{cl} + y^I,$$

we can compute the integral over y by summing Feynman diagrams where

$$\Delta_{IJ} = -(\tau + C^{-1}D)_{IJ} \tag{2.9}$$

is the inverse propagator, and derivatives of $\mathcal{F}_g$,

$$\partial_{I_1} \ldots \partial_{I_n} \mathcal{F}_g(x_{cl}), \tag{2.10}$$

the vertices. As a short hand we summarize the saddle point expansion by

$$\tilde{\mathcal{F}}_g = \mathcal{F}_g + \Gamma_g(\Delta^{IJ}, \; \partial_{I_1} \ldots \partial_{I_n} \mathcal{F}_{r<g}(x_{cl}))$$

where $\Gamma_g(\Delta^{IJ}, \; \partial_{I_1} \ldots \partial_{I_n} \mathcal{F}_{r<g}(x_{cl}))$ is a functional that is determined by the Feynman rules in terms of the lower genus vertices $\partial_{I_1} \ldots \partial_{I_n} \mathcal{F}_r(x_{cl})$ for $r < g$ and the propagator Δ^{IJ}. The latter is related to the inverse propagator Δ_{IJ} in (2.9) by $\Delta^{IJ}\Delta_{JK} = \delta^I_K$. For example, at genus 1 the functional is simply

$$\Gamma_1(\Delta^{IJ}) = \frac{1}{2} \log \det(-\Delta),$$

[5] It is important to note that this makes sense only on the *large* phase space, where the integral is over the n-dimensional space spanned by the x^I's. In particular, the choice of section of $\mathcal{L}$ does not enter.

where by Δ we mean the propagator Δ^{IJ} in matrix form. At genus two one has

$$\begin{aligned}\Gamma_2(\Delta^{IJ},\ \partial_{I_1}\ldots\partial_{I_n}\mathcal{F}_{r<2}) = \Delta^{IJ}\ &(\frac{1}{2}\partial_I\partial_J\mathcal{F}_1 + \ \frac{1}{2}\ \partial_I\mathcal{F}_1\ \partial_J\mathcal{F}_1)\\ &+ \Delta^{IJ}\Delta^{KL}(\ \frac{1}{2}\partial_I\mathcal{F}_1\ \partial_J\partial_K\partial_L\mathcal{F}_0 + \ \frac{1}{8}\ \partial_I\partial_J\partial_K\partial_L\mathcal{F}_0)\\ &+ \Delta^{IJ}\Delta^{KL}\Delta^{MN}(\frac{1}{8}\ \partial_I\partial_J\partial_K\mathcal{F}_0\ \ \partial_L\partial_M\partial_N\mathcal{F}_0\\ &\qquad + \ \frac{1}{12}\partial_I\partial_K\partial_M\mathcal{F}_0\ \ \partial_J\partial_L\partial_N\mathcal{F}_0),\end{aligned} \tag{2.11}$$

where we suppressed the argument x_{cl} for clarity.

It is easy to see from the path integral that this describes all possible degenerations of a Riemann surface of genus g to "stable" curves of lower genera, with Δ^{IJ} being the corresponding contact term, as shown in the figure below. Stable here means that the conformal Killing vectors were removed by adding punctures, so that every genus zero component has at least three punctures, and every genus one curve, one puncture.[6]

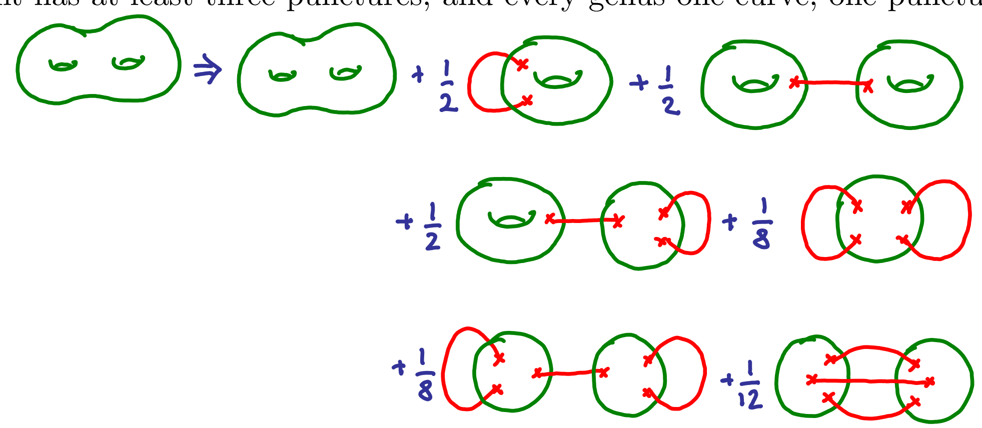

Fig. 1. Pictorial representation of the Feynman expansion at genus 2 in terms of degenerations of Riemann surfaces.

Mirror symmetry and Gromov-Witten theory picks out the real polarization which is natural at large radius where instanton corrections are suppressed, and where the classical geometry makes sense. However, also by mirror symmetry, there is a larger family of topological A-model theories which exist, though they may not have an interpretation as counting curves.

For a generic element M of $Sp(2n, \mathbb{Z})$, (2.8) simply takes one polarization into another. However, for M in the mapping class group $\Gamma \subset Sp(2n, \mathbb{Z})$, the transformation (2.8) should

[6] Note that in particular this implies that at each genus, the equations are independent of the choice of section of $\mathcal{L}$ we made, the left and the right hand side transforming in the same way.

translate into a *constraint* on $\mathcal{F}_g$, since Γ is a group of *symmetries* of the theory. We will explore the consequences of this in the rest of this paper.

2.2. Holomorphic Polarization

Instead of picking a symplectic basis of $H_3(X)$ to parameterize the variations of complex structure on X, we can choose a fixed background complex structure $\Omega \in H^3(X,\mathbb{C})$, and use it to define the Hodge decomposition of $H^3(X,\mathbb{C})$:

$$H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}.$$

Here Ω is the unique $H^{3,0}$ form and the $D_i\Omega$'s span the space of $H^{2,1}$ forms, where $D_i = \partial_i - \partial_i K$ and K is the Kähler potential $K = \log[i \int_X \Omega \wedge \bar{\Omega}]$. This implies that:

$$\omega = \varphi\,\Omega + z^i\, D_i\Omega + \bar{z}^i\, \bar{D}_i\bar{\Omega} + \bar{\varphi}\,\bar{\Omega}, \tag{2.12}$$

where (φ, z^i), and $(\bar{\varphi}, \bar{z}^i)$ become coordinates on the phase space.[7] Correspondingly we can express $|Z\rangle$ as a wave function in holomorphic polarization

$$\langle z^i, \varphi | Z\rangle = Z(z^i, \varphi).$$

The topological string partition function $Z(z^i, \varphi)$ depends on the choice of background Ω, and this dependence is not holomorphic. This is the holomorphic anomaly of [6]. One way to see this is through geometric quantization of $H^3(X)$ in this polarization [40]. We will take a different route, and exhibit this by exploring the canonical transformation from real to holomorphic polarizations. Using special geometry relations it is easy to see that

$$x^I = \int_{A^I} \omega = z^I + \ c.c$$
$$p_I = \int_{B_I} \omega = \tau_{IJ} z^J + \ c.c$$

where we defined

$$z^I = \varphi X^I + z^i D_i X^I$$

in terms of

$$X^I = \int_{A^I} \Omega, \qquad P_I = \int_{B_I} \Omega,$$

[7] Since ω for us does not live in $H^3(X,\mathbb{R})$, but rather in $H^3(X,\mathbb{C})$, $\bar{\varphi}$ and $\bar{z}^i$ are not honest complex conjugates of φ, z^i.

and where

$$\tau_{IJ} = \frac{\partial}{\partial X^I} P_J.$$

From this it easily follows that

$$dp_I \wedge dx^I = (\tau - \bar{\tau})_{IJ} dz^I \wedge d\bar{z}^J$$

and hence the canonical transformation from (x^I, p_I) to $(z^I, \bar{z}^I)$ is generated by

$$d\hat{S}(x,z) = p_I dx^I + (\tau - \bar{\tau})_{IJ} \bar{z}^I dz^J.$$

This corresponds to

$$\hat{S}(x,z) = \frac{1}{2}\bar{\tau}_{IJ} x^I x^J + x^I (\tau - \bar{\tau})_{IJ} z^J - \frac{1}{2} z^I (\tau - \bar{\tau})_{IJ} z^J + c,$$

where c is a constant, but which can now depend on the background.

In the quantum theory, this implies that the topological string partition function in the holomorphic polarization is related to that in real polarization by:

$$\hat{Z}(z; t, \bar{t}) = \int dx \; e^{-\hat{S}(x,z)/g_s^2} Z(x) \tag{2.13}$$

where t^i are local coordinates on the moduli space, parameterizing the choice of background, i.e. $X^I = X^I(t)$. Note that all the background dependence of $\hat{Z}(z)$ comes from the kernel of $\hat{S}$.[8] Let

$$c(X, \bar{X}) = -\mathcal{F}_1(X) - \frac{1}{2} \log[\det(\tau - \bar{\tau})](X, \bar{X}) - (\frac{\chi}{24} - 1) \log(g_s), \tag{2.14}$$

where χ the Euler characteristic of the Calabi-Yau.

Consider now the perturbative expansion of the integral. For simplicity, let us pick

$$\varphi = 1, \qquad z^i = 0,$$

so that $z^I = X^I$. The saddle point equation, which can be written as[9]

$$(\bar{\tau}(X) - \tau(x_{cl}))_{IJ} \, x_{cl}^J + (\tau(X) - \bar{\tau}(\bar{X}))_{IJ} \, z^J = 0,$$

[8] In what follows, we will use hats to label quantities which are not holomorphic.

[9] We used here the special geometry relation $p_I = \tau_{IJ} x^J$.

has then a simple solution,

$$x^I_{cl} = X^I.$$

Expanding around this solution,[10] we can compute the integral by summing Feynman diagrams where

$$-(\tau(X) - \bar{\tau}(\bar{X}))_{IJ} \tag{2.15}$$

is the inverse propagator, and derivatives of $\mathcal{F}_g$,

$$\partial_{I_1} \ldots \partial_{I_n} \mathcal{F}_g(X),$$

the vertices. That is, we get

$$\hat{\mathcal{F}}_g(t,\bar{t}) = \mathcal{F}_g(X) + \Gamma_g \left(- \left((\tau - \bar{\tau})^{-1} \right)^{IJ}, \ \partial_{I_1} \ldots \partial_{I_n} \mathcal{F}_{r<g}(X) \right) \tag{2.16}$$

where the properties of the functionals Γ_g obtained by the Feynman graph expansion have been discussed in the previous section.

Finally, one can show [39] that $\hat{Z}$ satisfies the holomorphic anomaly equations of [6]. Differentiating the left and the right hand side of (2.13) the with respect to $\bar{t}$ we get

$$\frac{\partial}{\partial \bar{t^{\bar{i}}}} \hat{Z} = [\frac{g_s^2}{2} \bar{C}_{\bar{i}}^{\ jk} \frac{\partial^2}{\partial z^i \partial z^j} + G_{\bar{i}j} z^j \frac{\partial}{\partial \varphi}] \ \hat{Z}$$

In the above equation, C_{ijk} is the amplitude at genus zero with three punctures, $G_{\bar{i}j}$ is the Kähler metric, and $\bar{C}_{\bar{i}}^{\ jk} = e^{2K} \bar{C}_{\bar{i}\bar{j}\bar{k}} G^{\bar{j}j} G^{\bar{k}k}$. It also satisfies the second holomorphic anomaly equation[11]

$$[\frac{\partial}{\partial t^{\bar{i}}} + \partial_i K (z^j \frac{\partial}{\partial z^j} - \varphi \frac{\partial}{\partial \varphi})] \hat{Z} = [\varphi \frac{\partial}{\partial z^i} - \partial_i \hat{\mathcal{F}}_1 - (\frac{\chi}{24} - 1) \partial_i K - \frac{1}{2 g_s^2} C_{ijk} z^j z^k] \hat{Z}.$$

The second anomaly equation implies that $\hat{Z}$ has the form

$$\hat{Z}(\varphi, z; t, \bar{t}) = \exp(\sum_{g,n} \frac{1}{n!} \ g_s^{2g-2} \hat{\mathcal{F}}^{(n)}_{g; i_1, \ldots i_n} z^{i_1} \ldots z^{i_n} \varphi^{2-2g-n} - (\frac{\chi}{24} - 1) \log \varphi)$$

[10] It should now be clear why (2.14) is natural. The above normalization of the integral ensures that $\hat{Z}$ contains no one loop term without insertions (the vanishing of genus zero terms with zero, one and two insertions is automatic in the saddle point expansion.)

[11] We used here the explicit form of $\hat{F}_1$ from [6] , from which follows that $\partial_i \hat{F}_1 + (\frac{\chi}{24} - 1) \partial_i K = \partial_i F_1 - \frac{1}{2} \partial_i \log(\tau - \bar{\tau})$.

where $\hat{\mathcal{F}}^{(n)}_{g;i_1,\ldots i_n} = D_{i_1}\ldots D_{i_n}\hat{\mathcal{F}}_g$ for $2g-2+n>0$, and zero otherwise, for some $\hat{\mathcal{F}}_g$'s, a fact that we will need later.

The holomorphic polarization, as explained in [6,40] is the natural polarization of the topological string theory, in the following sense. The topological string is obtained by twisting a physical string on the Calabi-Yau at some point in the moduli space. The physical string theory naturally depends not only on X, but also on $\bar{X}$, so the space of physical theories is labeled by $(X,\bar{X})$. After twisting, it is natural to deform by purely topological observables which are in one-to-one correspondence with the $h^{2,1}$ moduli — we have parameterized the resulting deformations by z^i above. While one would naively expect the topological theory to depend only on z, this fails and the theory depends on the background $(X,\bar{X})$ that we used to define it as well.

2.3. Local Calabi-Yau Manifolds

In the previous subsections we assumed that the Calabi-Yau X is compact. In this subsection we explain the modifications required in the local case. We can derive the results of this section by viewing the B-model on a local Calabi-Yau simply as a limit of the compact one. This is the perspective that was taken in [10,23]. Since today, there is now far more known about the topological string in the local than in the compact case, it is natural to work directly in the language of local Calabi-Yau manifolds. For a string theory on a non-compact Calabi-Yau manifold, gravity decouples. As a consequence, the moduli space is governed by *rigid* special geometry, and not *local* special geometry as in the compact Calabi-Yau case. The partition functions are no longer sections of powers of line bundle $\mathcal{L}$; the latter disappears altogether.

Consider the local Calabi-Yau manifold given by the equation

$$X\ :\ uw = H(y,z) \tag{2.17}$$

in $\mathbb{C}^4$. This has a holomorphic three-form ω given by

$$\omega = \frac{du}{u}\wedge dy\wedge dz. \tag{2.18}$$

The Calabi-Yau can be viewed as a $\mathbb{C}^*$ fibration over the $y-z$ plane where a generic fiber is given by $uw = \text{const}$. It is easy to see that the 3-cycles on X descend to 1-cycles on a Riemann surface Σ given by

$$\Sigma\ :\ 0 = H(y,z),$$

and, moreover, that the periods of the holomorphic three-form ω on X descend to the periods of a meromorphic 1-form λ on Σ

$$\int_{3-cycle} \omega = \int_{1-cycle} \lambda$$

where

$$\lambda = y dz.$$

On a genus g Riemann surface there are $2g$ compact 1-cycles that form a symplectic basis,[12] $i = 1, \ldots, g$,

$$A^i \cap B_j = \delta^i_j.$$

Let

$$x^i = \int_{A^i} \lambda, \qquad p_i = \int_{B_i} \lambda;$$

the x^i's are the *normalizable* moduli of the Calabi-Yau manifold. However, since the Calabi-Yau is non-compact, $H(y, z)$ may depend on additional parameters which are *non-normalizable* complex structure moduli s^α. Corresponding to these, there are compact 3-cycles C_α in $H_3(X)$ and 1-cycles on Σ such that

$$s^\alpha = \int_{C^\alpha} \lambda.$$

But, since the homology dual cycles to the C^α are non-compact, the metric on the moduli space along the corresponding directions will not be normalizable. As a consequence, the s^α are parameters of the model, not moduli.

This implies that the monodromy group Γ corresponds to elements of the form

$$\begin{aligned} \tilde{p}_i &= A_i{}^j p_j + B_{ij} x^j + E_{i\alpha} s^\alpha \\ \tilde{x}^i &= C^{ij} p_j + D^i{}_j x^j + F^i{}_\alpha s^\alpha \end{aligned} \tag{2.19}$$

where s^α, being parameters which do not vary, are monodromy invariant. Since Γ preserves the symplectic form

$$dx^i \wedge dp_i,$$

[12] This is a slight over-simplification. Since the Riemann surface is non-compact, it can happen that one cannot find compact representatives of the homology satisfying this, and that instead one has to work with $A^i \cap B_j = n^i_j$, with n^i_j integral. We will see examples of this in the later sections. Since it is very easy to see how this modifies the discussion of this section, we will not do this explicitly, but assume the simpler case for clarity of presentation.

we have that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

Note that, while p_i and x^j transform in a somewhat unconventional way, the period matrix

$$\tau_{ij} = \frac{\partial}{\partial x^j} p_i$$

transforms as usual:

$$\tilde{\tau} = (A\tau + B)(C\tau + D)^{-1}.$$

The corresponding generator of canonical transformations is easily found to be

$$\begin{aligned} S(x, \tilde{x}) = &- \frac{1}{2}(C^{-1}D)_{jk} x^j x^k + (C^{-1})_{jk} x^j \tilde{x}^k - \frac{1}{2}(AC^{-1})_{jk} \tilde{x}^j \tilde{x}^k \\ &+ C^{-1}_{ij} x^j F^i_\alpha s^\alpha - E_{i\alpha} \tilde{x}^i s^\alpha. \end{aligned} \tag{2.20}$$

In the quantum theory, once again x^i and p_j are promoted to operators with canonical commutation relations

$$[x^i, p_j] = g_s^2\, \delta^i_j.$$

The B-model determines a state $|Z\rangle$, and a wave function

$$Z(x^i) = \langle x^i | Z \rangle.$$

The wave function depends on the choice of real polarization, the different polarization choices being related in the usual way:

$$\tilde{Z}(\tilde{x}) = \int dx\, e^{-S(x,\tilde{x})/g_s^2}\, Z(x). \tag{2.21}$$

Computing the path integral, in the saddle point expansion around (2.19), we find that

$$\Delta_{ij} = -(\tau + C^{-1}D)_{ij} \tag{2.22}$$

is the inverse propagator, and derivatives of $\mathcal{F}_g$,

$$\partial_{i_1} \ldots \partial_{i_n} \mathcal{F}_g(x_{cl}), \tag{2.23}$$

the vertices. This implies that

$$\tilde{\mathcal{F}}_g = \mathcal{F}_g + \Gamma_g(\Delta^{ij},\ \partial_{i_1} \ldots \partial_{i_n} \mathcal{F}_{r<g}(x_{cl}))\ ,$$

where the propagator Δ^{ij} is related to (2.22) by $\Delta^{ij}\Delta_{ij} = \delta^i_k$.

Now consider the holomorphic polarization. Once again, we pick a background complex structure, this time by picking a meromorphic 1-form Λ on Σ. Since we are not allowed to vary the C^α periods, any other choice of complex structure differing from this one by *normalizable* deformations only corresponds to picking a 1-form

$$\lambda = \Lambda + z^i\,\partial_i\Lambda + \bar{z}^i\,\bar{\partial}_i\bar{\Lambda};$$

here the $\partial_i\Lambda$'s span a basis of holomorphic $(1,0)$-forms on Σ and correspond to infinitesimal deformations of complex structures. This gives us a holomorphic set of coordinates on the phase space $(z^i, \bar{z}^i)$ which are canonically conjugate, and allows us to write the wave function in the holomorphic polarization:

$$\hat{Z}(z^i) = \langle z^i | Z\rangle.$$

We also need the relation between the two polarizations. Let

$$X^i = \int_{A^i}\Lambda, \qquad P_i = \int_{B_i}\Lambda, \qquad s^\alpha = \int_{C^\alpha}\Lambda.$$

It is easy to see that

$$dx^i \wedge dp_i = (\tau_{ij} - \bar{\tau}_{ij})d\bar{Z}^i \wedge dZ^j$$

where $\tau_{ij}(X) = \partial P_i/\partial X^j$ depends on the background and we put

$$Z^i = z^j\partial_j X^i.$$

The corresponding canonical transformation is easily found:

$$\hat{S}(x,z) = \frac{1}{2}\,\bar{\tau}_{ij}\,(x-X)^i(x-X)^j + (\tau-\bar{\tau})_{ij}\,Z^i(x-X)^j - \frac{1}{2}(\tau-\bar{\tau})_{ij}\,Z^iZ^j + P_ix^i.$$

The wave functions in holomorphic and real polarizations are now simply related by

$$\hat{Z}(z) = \int dx\, e^{-\hat{S}(x,z)/g_s^2}\, Z(x) \tag{2.24}$$

The saddle point equation reads

$$\bar{\tau}(\bar{X})_{ij}(x_{cl} - X)^j + (\tau(X) - \bar{\tau}(\bar{X}))_{ij}Z^j - (p-P)_i = 0,$$

and if we put $z^i = 0$, which corresponds to Z vanishing, it has a simple solution:

$$x^i_{cl} = X^i.$$

Expanding around this, we get a Feynman graph expansion with inverse propagator

$$-(\tau(X) - \bar{\tau}(\bar{X}))_{ij}$$

and derivatives of $\mathcal{F}_g(X)$ as vertices. This gives the by now familiar expansion relating the partition functions in holomorphic and real polarizations:

$$\hat{\mathcal{F}}_g(t,\bar{t}) = \mathcal{F}_g(t) + \Gamma_g \Big(- \left((\tau - \bar{\tau})^{-1}\right)^{ij},\ \partial_{i_1} \ldots \partial_{i_n} \mathcal{F}_{r<g}(X) \Big)\ . \tag{2.25}$$

Before we go on, it is worth noting that the wave function in holomorphic polarization satisfies a set of differential equations, expressing the dependence of $\hat{Z}$ on the background — the *local holomorphic anomaly* equations. These can be derived easily by differentiating both the left and the right hand side of (2.24) with respect to $\bar{t}$ (here, t^i is the local coordinate parameterizing the choice of background, $X = X(t)$). This is straightforward, we state here only the answer:

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} \hat{Z} = \frac{1}{2}\, g_s^2\, \bar{C}_{\bar{i}}^{\,jk}\, \frac{\partial^2}{\partial z^j \partial z^k} \hat{Z} \tag{2.26}$$

where indices are raised by the inverse $g^{i\bar{j}}$ of the Kähler metric on the moduli space $g_{i\bar{j}} = \partial_i X^k (\tau - \bar{\tau})_{k\ell} \bar{\partial}_{\bar{j}} \bar{X}^\ell$.

In summary, apart from a few subtleties, the quantum mechanics of the compact and local Calabi-Yau manifolds are analogous. In the following section we will use the language of the compact theory, but everything we will say will go over, without modifications, to the non-compact case as well.

3. A First Look at the Γ Action

In this section we take a first look at how topological string amplitudes behave under monodromies. On general grounds, Γ is a group of symmetries of the physical string theory. This implies that the state $|Z\rangle$ in the Hilbert space that the topological string partition function determines should be *invariant* under monodromies. The associated wave functions, however, need not be. By definition, the wave function in real polarization requires a choice of symplectic basis of H_3 on which Γ acts nontrivially; thus, it cannot be monodromy invariant. By contrast, the wave function in the holomorphic polarization is the physical partition function. It is a well defined function[13] all over the moduli space; however, it is not holomorphic.

[13] We are assuming a definite choice of gauge, throughout. Of course, changing the gauge, the amplitudes transform as sections of the apropriate powers of $\mathcal{L}$.

3.1. The Wave Function in Real Polarization

Given a symplectic basis $\{A^I, B_I\}$, $I = 1, \ldots n$ of $H_3(X, \mathbb{Z})$, with $n = \frac{1}{2}b_3$, a pick a definite 3-form ω in $H^3(X,\mathbb{C})$. The topological string partition function determines a wave function

$$Z(x^I) = \langle x^I | Z \rangle$$

where

$$x^I = \int_{A^I} \omega,$$

and a corresponding state $|Z\rangle$ in the Hilbert space obtained by quantizing $H^3(X,\mathbb{C})$. Having picked a definite section ω of the line bundle $\mathcal{L}$, x^I's and $Z(x^I)$ are at least locally, functions on the moduli space

$$x^I = x^I(\psi).$$

where the $n-1$ variables ψ^i are some arbitrary local coordinates on $\mathcal{M}$. For definiteness, we take here the Calabi-Yau manifold to be compact, but everything carries over to the non-compact space as well, the only real modification being that there the moduli space would have dimension n, instead.

The moduli space $\mathcal{M}$ has singular loci in complex codimension 1 around which the cycles A^I, B_J undergo monodromies in Γ. As one goes around the singular locus, by sending ψ

$$\psi \rightarrow \gamma \cdot \psi,$$

for γ an element of Γ, the periods transform as

$$\begin{pmatrix} p_I \\ x^I \end{pmatrix}(\psi) \;\rightarrow\; \begin{pmatrix} p_I \\ x^I \end{pmatrix}(\gamma \cdot \psi) = \; M_\gamma \; \begin{pmatrix} p_I \\ x^I \end{pmatrix}(\psi)$$

where M_γ is a symplectic matrix corresponding to γ.

What happens in the quantum theory? The monodromy group Γ is a symmetry of the theory, so the state $|Z\rangle$ determined by the topological string partition function should be invariant under it:

$$|Z\rangle \;\rightarrow\; |Z\rangle.$$

The state $\langle x(\psi)|$, by contrast, is not invariant. There are two ways to express what happens to $\langle x|$ under monodromies. On the one hand, x^I is a function of ψ, so we get a purely classical variation of the ket vector

$$\langle x(\psi)| \;\rightarrow\; \langle x(\gamma \cdot \psi)|.$$

But on the other hand, we have seen in section 2 that *any* element $M_\gamma \in Sp(2n, \mathbb{Z})$ acting classically on the period vector has a unique lift to the quantum theory as an operator U_γ that acts on the Hilbert space. In particular,

$$\langle x(\gamma \cdot \psi)| = \langle\, x(\psi)|\, U_\gamma.$$

Putting these facts together implies that

$$\langle x(\gamma \cdot \psi)|Z\rangle = \langle x(\psi)|\, U_\gamma\, |Z\rangle,$$

or, schematically in terms of wave functions,

$$Z(x(\gamma \cdot \psi)) = \int \; e^{S_\gamma} Z\left(x(\psi)\right) \tag{3.1}$$

where $\exp(S_\gamma)$ computes the corresponding matrix element of U_γ. There is one such equation for each monodromy transformation g and its corresponding element $M_\gamma \in \Gamma$. Thus, the symmetry group Γ imposes the constraints (3.1) on Z, one for each generator.

Using the results of section 2, equation (3.1) implies constraints on the free energy, genus by genus. For example, (3.1) implies that the free energies satisfy[14]

$$\mathcal{F}_g(x(\gamma \cdot \psi)) = \mathcal{F}_g(x(\psi)) + \Gamma_g\left(\Delta^{IJ}_{M_\gamma},\ \partial_{I_1}\ldots\partial_{I_N}\mathcal{F}_{r<g}\right) \tag{3.2}$$

with Δ_{M_γ} given by

$$(\Delta_{M_\gamma})^{IJ} = -\left((\tau + C^{-1}D)^{-1}\right)^{IJ}, \tag{3.3}$$

where

$$M_\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3.4}$$

To summarize, non-trivial monodromy (with $\det(C) \neq 0$) around a point in the moduli space corresponds to choosing A-cycles which are *not well* defined there, but instead transform by

$$x^I \to C^{IJ}p_J + D^I{}_J x^J.$$

This leads to an obstruction to analytic continuation of the amplitudes all over the moduli space. It also lead us to the notion of "good variables" in the moduli space, which are implicit in Gromov-Witten computations: near a point in the moduli space, the "good" variables are those with no non-trivial monodromy, meaning that $C^{IJ} = 0$.

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[14] It is important to emphasize that this does not depend on the choice of section either. We could have written here simply $x^I(\psi) = x^I$ and $x^I(\gamma \cdot \psi) = C^{IJ}p_J(x) + D^I{}_J x^J$.

3.2. Another Perspective

Consider instead the wave function in holomorphic polarization. Pick a background complex structure Ω, and write ω as in (2.12)

$$\omega = \varphi\,\Omega + z^i\,D_i\Omega + \bar{z}^i\,\bar{D}_i\bar{\Omega} + \bar{\varphi}\,\bar{\Omega}.$$

Using φ and z^i as coordinates, we can write $|Z\rangle$ as a wave function in holomorphic polarization

$$\hat{Z}(\varphi, z^i) = \langle \varphi, z^i | Z \rangle$$

Note that z^i are coordinates on $\mathcal{M}$, centered at Ω.

How does $Z(\varphi, z^i)$ transform under Γ? In real polarization, the non-trivial transformation law of the wave function came about from having to pick a basis of periods $\langle x^I|$, which were not invariant under Γ. In writing down the wave function in holomorphic polarization, that is in defining $\langle\varphi, z^i|$, we made no reference to the periods, so $\hat{Z}(\varphi, z^i)$ has to be invariant. There is another, independent reason why this has to be so. Namely, $Z(\varphi, z^i)$ is the physical wave function everywhere on $\mathcal{M}$ and as such, it better be well defined everywhere!

We have seen above that the wave function in real polarization has rather complicated monodromy transformations under Γ, while the wave function in holomorphic polarization is invariant. Since the two polarizations are related in a simple way, we could have derived the transformation properties of one from that of the other. Consider for example the genus two amplitudes in (2.16) for a compact Calabi-Yau, and in (2.25) for a non-compact one. While on the left hand side $\hat{\mathcal{F}}_2$ is manifestly invariant under Γ, on the right hand side all the ingredients have non-trivial monodromy transformations. In fact, we have

$$\left((\tau-\bar{\tau})^{-1}\right)^{IJ} \rightarrow (C\tau+D)^I{}_K\,(C\tau+D)^J{}_L\left((\tau-\bar{\tau})^{-1}\right)^{KL} - C^{IL}(C\tau+D)^J{}_L, \qquad (3.5)$$

where C, D enter M_Γ as in (3.4) , and analogously in the local case. These quasi-modular transformations of $(\tau-\bar{\tau})^{-1}$ must precisely cancel the transformations of the genus zero, one and two amplitudes in real polarization. We will come back to this in the next section.

4. Topological Strings and Modular Forms

In the previous section we took a first look at how the topological string partition functions transform under Γ. In this section we give a simple and precise description of how, and to which extent, can the discrete symmetry group Γ constrain the topological string amplitudes. Along the way, we will discover a close relationship of topological string partition functions and modular forms.

On the one hand, we have seen in the previous sections that the partition function in *holomorphic* polarization satisfies

i. $\hat{\mathcal{F}}_g(x,\bar{x})$ is invariant under Γ — that is, it is a modular form of Γ of weight zero.

ii. $\hat{\mathcal{F}}_g(x,\bar{x})$ is "almost" holomorphic — its anti-holomorphic dependence can be summarized in a finite power series in $(\tau-\bar{\tau})^{-1}$.

On the other hand, the topological string partition function in *real* polarization satisfies

iii. $\mathcal{F}_g(x)$ is holomorphic, but not modular in the usual sense.

iv. $\mathcal{F}_g(x)$ is the constant part of the series expansion of $\hat{\mathcal{F}}_g(x,\bar{x})$ in $(\tau-\bar{\tau})^{-1}$.

Forms of this type were considered by Kaneko and Zagier [23].[15] In [23] forms satisfying *i.* and *ii.* (with arbitrary weight) are called *almost holomorphic modular forms* of Γ. Moreover, for every almost holomorphic modular form, [23] defines the associated *quasi-modular form* as that satisfying *iii.* and *iv.* These are holomorphic forms which are not modular in the usual sense. This suggests that the genus g amplitudes are in fact naturally (almost) modular functions of τ (and $\bar{\tau}$ in holomorphic polarization), which can be extended from functions on the moduli space $\mathcal{M}$ of complex structures to the space $\mathcal{H}_X$ parameterized by the period matrix τ_{IJ} on X modulo Γ. In the following, we will mainly study this in the local Calabi-Yau examples, and show that this indeed is the case, leaving compact Calabi-Yau manifolds for future work.

Now, take a holomorphic, quasi-modular form $E_{IJ}(\tau)$ of Γ, such that

$$\hat{E}^{IJ}(\tau,\bar{\tau}) = E^{IJ}(\tau) + \left((\tau-\bar{\tau})^{-1}\right)^{IJ} \tag{4.1}$$

[15] To be precise, [23] considers only modular forms of $SL(2,\mathbb{Z})$. However, this has an obvious generalization, at least in principle, to (subgroups of) $Sp(2n,\mathbb{Z})$.

is a modular form, albeit an almost holomorphic one. Since $(\tau-\bar{\tau})^{-1}$ transforms under Γ as in (3.5), for $\hat{E}^{IJ}$ to be modular, E^{IJ} must transform as

$$E^{IJ}(\tau) \to (C\tau+D)^I{}_K\,(C\tau+D)^J{}_L\,E^{KL}(\tau) + C^{IL}(C\tau+D)^J{}_L. \tag{4.2}$$

Then $\hat{E}$ transforms simply as

$$\hat{E}^{IJ}(\tau,\bar{\tau}) \to (C\tau+D)^I{}_K\,(C\tau+D)^J{}_L\,\hat{E}^{KL}(\tau,\bar{\tau}) \tag{4.3}$$

Of course, E^{IJ} and $\hat{E}^{IJ}$ are just $\Gamma \subset Sp(2n,\mathbb{Z})$ analogues (up to normalization) of the second Eisenstein series $E_2(\tau)$ of $SL(2,\mathbb{Z})$, and its modular but non-holomorphic counterpart $E_2^*(\tau,\bar{\tau})$ — see Appendix A. It is important to note that the transformation properties given above *do not* define E and $\hat{E}$ uniquely: shifting E^{IJ} by any holomorphic modular form e^{IJ} of Γ,

$$E^{IJ}(\tau) \to E^{IJ}(\tau) + e^{IJ}(\tau)$$

with $e^{IJ}(\tau)$ transforming as

$$e^{IJ}(\tau) \to (C\tau+D)^I{}_K\,(C\tau+D)^J{}_L\,e^{KL}(\tau),$$

we still get a solution of (4.2).

With this in hand, one can reorganize each $\mathcal{F}_g$ as a finite power series in E with coefficients that are strictly holomorphic modular forms [23]. In particular, the free energy at genus g in holomorphic polarization can be written as

$$\hat{\mathcal{F}}_g(\tau,\bar{\tau}) = h_g^{(0)}(\tau) + (h_g^{(1)})_{IJ}\,\hat{E}^{IJ}(\tau,\bar{\tau}) + \ldots + (h_g^{(3g-3)})_{I_1\ldots I_{6g-6}}\,\hat{E}^{I_1I_2}(\tau,\bar{\tau})\ldots\hat{E}^{I_{6g-7}I_{6g-6}}(\tau,\bar{\tau}), \tag{4.4}$$

where $h_g^{(k)}(\tau)$ are holomorphic modular forms of Γ in the usual sense. Moreover, taking $\hat{\mathcal{F}}_g(\tau,\bar{\tau})$ and sending $\bar{\tau}$ to infinity,[16]

$$\mathcal{F}_g(\tau) = \lim_{\bar{\tau}\to\infty} \hat{\mathcal{F}}_g(\tau,\bar{\tau})$$

[16] By sending $\bar{\tau}$ to infinity what we really mean is keeping the constant term in the finite power series in $(\tau-\bar{\tau})^{-1}$. For $SL(2,\mathbb{Z})$, this is simply the isomorphism between the rings of almost holomorphic modular forms and quasi-modular forms described in [23], which can be easily generalized to $Sp(2n,\mathbb{Z})$.

we recover the modular expansion of the partition function in real polarization:

$$\mathcal{F}_g(\tau) = h_g^{(0)}(\tau) + (h_g^{(1)})_{IJ}\, E^{IJ}(\tau) + \ldots + (h_g^{(3g-3)})_{I_1\ldots I_{6g-6}}\, E^{I_1 I_2}(\tau)\ldots E^{I_{6g-7}I_{6g-6}}(\tau).$$

This gives us a way to construct modular invariant quantities out of the free energy and correlation functions. For example, it is easy to see that the highest order term in the $(\tau-\bar{\tau})^{-1}$ expansion of $\hat{\mathcal{F}}_g$ is always modular. It is constructed solely out of genus zero amplitudes, as it corresponds to the most degenerate genus g Riemann surface that breaks up into $(2g-2)$ genus zero components with three punctures each. Moreover, it follows that $\partial_I\partial_J\partial_K\mathcal{F}_0$ is itself modular and corresponds to an irreducible representation — a third rank symmetric tensor:

$$\partial_I\partial_J\partial_K\mathcal{F}_0 \to \left((C\tau+D)^{-1}\right)^{I'}{}_{I}\ \left((C\tau+D)^{-1}\right)^{J'}{}_{J}\ \left((C\tau+D)^{-1}\right)^{K'}{}_{K}\ \partial_{I'}\partial_{J'}\partial_{K'}\mathcal{F}_0, \tag{4.5}$$

which can be verified directly as well.

From $h_g^{(0)}$, we get a modular forms of weight zero, constructed out of $\mathcal{F}_g$ and lower genus amplitudes via

$$(h_g^{(0)})(\tau) = \mathcal{F}_g(\tau) + \Gamma_g(E^{IJ}(\tau),\ \partial_{I_1}\ldots\partial_{I_N}\mathcal{F}_{r<g}), \tag{4.6}$$

where Γ_g is the functional introduced in the previous sections. While none of the terms on the right hand side is modular on its own, added together we get a modular invariant of Γ. We can turn this around and read this equation as follows: given the genus $r<g$ amplitudes and the propagator E^{IJ}, the free energy $\mathcal{F}_g(\tau)$ is fixed, up to the addition of a precisely modular holomorphic form $h_g^{(0)}$! In practice, this means that $h_g^{(0)}$ is a meromorphic function on the moduli space.[17]

We can write this compactly as follows. Let

$$\mathcal{H}(\tau) = \sum_{g=1}^{\infty} h_g^{(0)}(\tau)\ g_s^{2g-2}$$

[17] As stated in section 2, throughout we assumed a definite choice of a gauge, and picked a 3-form ω as a definite section of $\mathcal{L}$. Like $\mathcal{F}_g$'s, $h_g^{(0)}$ depend on this choice – they are sections of $\mathcal{L}^{2-2g}$, so $h_g^{(0)}$ is more precisely a meromorphic section of $\mathcal{L}^{2-2g}$. On a non-compact Calabi-Yau, however, it is simply a meromorphic function.

be the generating functional of weight zero modular forms, and define the generating function of correlation functions

$$\mathcal{W}(y,x) = \sum_{g,n} \frac{1}{n!}\, \partial_{I_1} \ldots \partial_{I_n} \mathcal{F}_g(x)\, y^{I_1} \ldots y^{I_n}\, g_s^{2g-2}$$

where the sum over n runs from zero to infinity, except at genus zero and one, where it starts at $n=3$ and $n=1$, respectively. Then, the above can be summarized by writing

$$\exp(\,\mathcal{H}(x)) = \int dy\, \exp(-\frac{1}{2g_s^2}\, E_{IJ}\, y^I y^J)\, \exp(\,\mathcal{W}(y,x)\,)$$

where E_{IJ} is the inverse of E^{IJ},

$$E^{IK} E_{KJ} = \delta^I_J.$$

This follows directly from the path integral of section 2 relating the wave functions in the real and holomorphic polarizations, which we can be written as

$$\hat{Z}(x,\bar{x}) = \int dy(-\frac{1}{2g_s^2}\,(E-\hat{E})_{IJ}\, y^I y^J)\, \exp(\,\mathcal{W}(y,x)\,)$$

where one views $\hat{E}$ as a perturbation.

Furthermore, one can show that similar equations hold when $\mathcal{F}$ and E are replaced by their non-holomorphic counterparts. To see this, note that the inverse of (2.13) is

$$Z(x) = \int dz\, e^{\hat{S}(x,z)/g_s^2}\, \hat{Z}(z; X, \bar{X}), \tag{4.7}$$

with all the quantities as defined in section 2. If we choose the background $X^I = x^I$, this has a saddle point at $z^I = x^I$. Expanding around it, by putting $z^I = x^I + y^I$ where $y^I = -\varphi x^I + z^i D_i x^I$, and integrating over y, we get

$$Z(x) = \int dy\, \exp(-\frac{1}{g_s^2}\,(\tau-\bar{\tau})_{IJ} y^I y^I)\exp(\,\hat{\mathcal{W}}(y;x,\bar{x})\,),$$

where

$$\begin{aligned}\hat{\mathcal{W}}(y;x,\bar{x}) &= \sum_g\, g_s^{2g-2}\, \hat{\mathcal{F}}_g((1-\varphi)x + z^i D_i x, \bar{x}) \\ &= \sum_{n,g} \frac{1}{n!}\, g_s^{2g-2}\,(1-\varphi)^{2-2g-n}\, z^{i_1} \ldots z^{i_n} D_{i_1} \ldots D_{i_n} \hat{\mathcal{F}}_g(x,\bar{x}) - (\frac{\chi}{24} - 1)\log(1-\varphi).\end{aligned}$$

From this, and thinking about $Z(x)$ in terms of a power series in E, it follows immediately that

$$\exp(\mathcal{H}(x)) = \int dy\ \exp(-\frac{1}{2g_s^2}\,\hat{E}_{IJ}(x,\bar{x})\,y^I y^J)\ \exp(\,\hat{\mathcal{W}}(y,x,\bar{x})\,). \tag{4.8}$$

The equation (4.8) has appeared before. In the seminal paper [6] the authors derived a set of equations that the physical free energies $\hat{\mathcal{F}}_g$ must satisfy, through analysis of the worldsheet theory. These equations were interpreted in [40] as saying that the topological string partition function is a wave function in the Hilbert space obtained from the geometric quantization of $H^3(X,\mathbb{C})$, the fact that we used repeatedly here. Holomorphic anomaly equations (and modular invariance) constrain what the topological string amplitudes can be. Here we described the solutions to the equations using symmetry alone. The construction of the propagators $\hat{E}$, which was the guts of the method of [6] for solving the equations, was quite complicated. The answers were messy, with ambiguities that had no clear interpretation. Now, the meaning of the propagators $\hat{E}^{IJ}$ and E^{IJ} is simple and beautiful — they are simply generators of (almost) modular forms of the symmetry group Γ!

The only remaining thing to show is that the propagators of our expansion and of [6] agree. In [6] the authors gave a set of relations that the inverse propagators satisfy (p. 103 of [6]). It is easily shown that our propagators (4.1) satisfy these relations (for any holomorphic form E_{IJ}). Let

$$\hat{E}_{\varphi\varphi} = \hat{E}_{IJ}\,x^I\,x^J, \qquad \hat{E}_{\varphi i} = \hat{E}_{IJ}\,x^I\,D_i x^J, \qquad \hat{E}_{ij} = \hat{E}_{IJ}\,D_i x^I\,D_j x^J,$$

where D_i is the Kähler covariant derivative $D_i = \partial_i - \partial_i K$ and K is the Kähler form of the special geometry of X. Then, with a bit of algebra it follows that these satisfy

$$\begin{aligned}\bar{\partial}_{\bar{i}}\hat{E}_{jk} &= \bar{C}_{\bar{i}}^{mn}\hat{E}_{mj}\hat{E}_{nk} + G_{\bar{i}j}\hat{E}_{\varphi k} + G_{\bar{i}k}\hat{E}_{\varphi j}\\ \bar{\partial}_{\bar{i}}\hat{E}_{j\varphi} &= \bar{C}_{\bar{i}}^{mn}\hat{E}_{mj}\hat{E}_{n\varphi} + G_{\bar{i}j}\hat{E}_{\varphi\varphi}\\ \bar{\partial}_{\bar{i}}\hat{E}_{\varphi\varphi} &= \bar{C}_{\bar{i}}^{mn}\hat{E}_{m\varphi}\hat{E}_{n\varphi}\end{aligned} \tag{4.9}$$

where

$$G_{\bar{i}j} = \bar{\partial}_{\bar{i}}\partial_j K,\ \ \bar{C}_{\bar{i}}^{mn} = e^{-2K}G^{m\bar{m}}G^{n\bar{n}}\bar{C}_{\bar{i}\bar{m}\bar{n}},\ \ \bar{C}_{\bar{i}\bar{m}\bar{n}} = \bar{C}_{IJK}\bar{D}_{\bar{i}}\bar{x}^I\bar{D}_{\bar{j}}\bar{x}^J\bar{D}_{\bar{k}}\bar{x}^K.$$

The equations (4.9) are exactly the equations of [6] with obvious substitutions.

4.1. A Mathematical Subtlety

As we have shown in the previous sections, our results are completely general and apply to both non-compact and compact Calabi-Yau threefolds. However, to make contact with the theory of modular forms in mathematics there is an important subtlety that we have not mentioned yet.

In the theory of modular forms, the period matrix τ_{IJ} acquires a crucial role. A modular form is defined to be a holomorphic function $f : \mathcal{H}_k \to \mathbb{C}$ satisfying certain transformation properties, where $\mathcal{H}_k$ is the Siegel upper half-space:

$$\mathcal{H}_k = \{\tau \in \mathrm{Mat}_{k\times k}(\mathbb{C}) | \, \tau^T = \tau, \tau - \bar{\tau} > 0\},$$

which is the space of $k \times k$ symmetric matrices with positive definite imaginary part. The period matrix is the τ in the definition of the Siegel upper half-space. Note that strictly speaking, this defines Siegel modular forms; proper modular forms are obtained for $k = 1$.[18]

For the non-compact case, the mirror symmetric geometry reduces to a family of Riemann surfaces of a certain genus. Thus, it is clear that the period matrix τ_{IJ} has positive definite imaginary part. Therefore, in this case our results should be interpreted mathematically as Siegel modular forms, where k depends on the genus g of the Riemann surface. In particular, if the mirror geometry is a family of elliptic curves, $k = 1$, and we recover proper modular forms.

However, in the compact case the situation changes slightly. The period matrix τ_{IJ} does not have positive definite imaginary part anymore; it has signature $(h^{2,1}, 1)$, as explained for instance in [15]. Thus, in this case the Siegel upper half-space is not the relevant object anymore, and we cannot make contact directly with Siegel modular forms. This seems to call for a new theory of modular forms defined on spaces with indefinite signature. It would be very interesting to develop this mathematically.

Another possibility, in order to make contact with already known mathematical concepts in the compact case, is to replace the period matrix τ_{IJ} by a different but related matrix $\mathcal{N}_{IJ}$ — see for instance [15] for a definition — which has positive definite imaginary part, but is not holomorphic. This is usually done in the context of supergravity. Roughly speaking, it amounts to replacing the intersection pairing by the Hodge star pairing. In that way perhaps we can come back into the realm of Siegel modular forms, perhaps along the lines of what was done in [15] in a related context.

In the following sections we will give applications of the modular approach we have developed so far, for local Calabi-Yau threefolds.

[18] See Appendix A and B for definitions and conventions.

5. Seiberg-Witten Theory

As is well known, type II string theory compactified on local Calabi-Yau manifolds gives rise to $\mathcal{N}=2$ gauge theories in four dimensions. The topological string theory on these manifolds computes topological terms in the effective action of $N=2$ Seiberg-Witten theory with gauge group G [6,24]. These terms are summarized in a partition function

$$Z_{SW} = \exp(\lambda^{2g-2}\mathcal{F}_g(a)) \ , \tag{5.1}$$

where $\mathcal{F}_g$ coincides with the genus g topological string free energy, and the a's are local parameters in the vacuum manifold of the gauge theory. Each term in (5.1) has a physical meaning in the effective action of the $\mathcal{N}=2$ gauge theory. The genus zero topological string amplitude yields the exact gauge coupling

$$\tau_{ij} = \frac{\partial^2\mathcal{F}_0}{\partial a_i \partial a_j} \ , \tag{5.2}$$

with $i,j=1,\dots r$, where $r=\mathrm{rank}(G)$, while the higher genus topological string amplitudes yield the gravitational coupling of the self-dual part of the curvature R_+ to the self-dual part of the graviphoton field strength $\int \mathrm{d}x^4 \mathcal{F}_g R_+^2 F_+^{2g-2}$. The $\mathcal{F}_g(a)$'s for $g>1$ were in fact extensively studied in the weak electric coupling limit [32].

The corresponding Calabi-Yau manifold is given by an equation of the form (2.17) with an appropriate $H(y,z)$ depending on the theory. For example, for $G=SU(n)$ without matter,

$$H(y,z) = y^2 - (P_n(z))^2 + 1 \tag{5.3}$$

where $P_n(z) = z^n + u_2 z^{n-2} + \dots u_n$, and the holomorphic 3-form is given by (2.18). The parameters u_i are complex coordinates on the moduli space of the Calabi-Yau. In the gauge theory, they correspond to the expectation values of the gauge invariant observables

$$u_k = \frac{1}{k}\mathrm{Tr}\langle\phi^k\rangle + \text{products of lower order Casimirs}, \tag{5.4}$$

where ϕ is the adjoint valued Higgs field.

The family of Riemann surfaces obtained by setting

$$\Sigma_g : \qquad H(y,z) = 0$$

is the Seiberg-Witten curve of the gauge theory. The genus g of the Riemann surface is the rank of the gauge group r. The gauge coupling constant $\mathrm{Im}(\tau_{ij})$ is the period matrix of the Riemann surface. Alternatively, τ_{ij} is the complex structure of the Jacobian of the Riemann surface Σ_g, which is an abelian variety. The abelian variety is spanned by the periods

$$\begin{pmatrix} a_{D_i} \\ a_i \end{pmatrix} = \begin{pmatrix} p_i \\ x^i \end{pmatrix} = \begin{pmatrix} \int_{B_i} \lambda \\ \int_{A^i} \lambda \end{pmatrix} , \tag{5.5}$$

with $i = 1, \ldots r$, and where the A- and B-cycles generate the symplectic integer basis of $H_1(\Sigma_g, \mathbb{Z})$. Here λ is a meromorphic differential, which is part of the data of the theory. As explained in section 2, in the string theory context, λ comes from the reduction of the holomorphic 3-form of the parent Calabi-Yau threefold to a one-form on Σ_g. For theories with matter, there can be additional periods on Σ_g — λ then has poles whose residues correspond to the mass parameters.

The monodromy group Γ of the curve Σ_g, which is naturally a subgroup of $\mathrm{Sp}(2r, \mathbb{Z})$, played the central role in [35]. It is generated by the BPS particles going massless at a codimension one loci in the moduli space and captures the non-perturbative duality symmetries of the $\mathcal{N} = 2$ gauge theory, since it acts non-trivially on the coupling constant τ_{ij}. From the monodromies of the periods around the perturbative limits in the moduli space, [35] showed that one can deduce the periods themselves everywhere in the moduli space — this is the Riemann-Hilbert problem — and hence also τ_{ij} and $\mathcal{F}_0$. It is then very natural to ask what does the group Γ of symmetries imply about the full partition function Z_{SW}. In fact, this question, and the close relation of Seiberg-Witten theory and topological strings in general, is what motivated this paper.

The topological string partition function is a wave function for both compact Calabi-Yau threefolds, studied in [6], and non-compact Calabi-Yau threefolds, as we have seen in section 2. This implies that the Seiberg-Witten partition function [40] Z_{SW} is a wave function, arising by geometric quantization of $H_1(\Sigma_g)$ — see [21]. In particular, in holomorphic polarization, it satisfies the local holomorphic anomaly equation (2.26). In fact, it would be very interesting to derive this directly from the $\mathcal{N} = 2$ gauge theory.

Since the partition function Z_{SW} is known, this gives us a testing ground for exploring the restrictions that follow from the duality symmetries generated by Γ, but now acting on the full quantum wave function Z_{SW}.[19]

[19] The observation that duality transformations imply quasi-modular properties of the $\mathcal{F}_g$'s has been made earlier in [12]. However, their results are different from ours in that their partition function $Z = \exp \mathcal{F}$ does not transform like a wave function; rather, it transforms by Legendre transformations of $\mathcal{F}$.

5.1. Seiberg-Witten Theory and Modular Forms

One crucial property of the abelian variety is that $\mathrm{Im}(\tau_{ij}) > 0$, which ensures positivity of the kinetic terms of the vector multiplet. Thus, in this case the period matrix τ_{ij} can be used to define the Siegel upper half space $\mathcal{H}_r$ as

$$\mathcal{H}_r = \{\tau \in \mathrm{Mat}_{r\times r}(\mathbb{C}) | \tau^T = \tau, \mathrm{Im}(\tau) > 0\}. \tag{5.6}$$

The monodromy group $\Gamma \subset Sp(2r, \mathbb{Z})$ of the family of Riemann surfaces Σ_g acts on τ_{ij} as

$$\tau \to (A\tau + B)(C\tau + D)^{-1} \quad \text{for} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

Thus, in principle, we should be able to give explicit expressions for the Seiberg-Witten higher genus amplitudes in terms of Siegel modular forms under the corresponding subgroup $\Gamma \subset Sp(2r, \mathbb{Z})$ (see appendix B for a brief review of Siegel modular forms). To start with, however, let us consider $SU(2)$ gauge theory, where the modular group $\Gamma \subset SL(2, \mathbb{Z})$, and correspondingly standard modular forms suffice.

i. $SU(2)$ Seiberg-Witten theory

The curve of the $SU(2)$ gauge theory can be written as[20]

$$y^2 = (x^2 - 1)(x - u). \tag{5.7}$$

There are three singular points in the moduli space, corresponding to $u = \pm 1, \infty$ with monodromies

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{5.8}$$

acting on

$$\Pi = \begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} \int_B \lambda \\ \int_A \lambda \end{pmatrix}$$

[20] As explained in [36] there are two curves corresponding to this gauge theory, differing by a factor of 2 in the normalization of the A-period and electric charge. The curve at hand has $\#(A \cap B) = 2$ between the generators of $H_1(\Sigma, \mathbb{Z})$. The curve which is the $n = 2$ specialization of (5.3) has the A-period $A' = A/2$. Correspondingly, the modular groups will differ: in the second case we would get the $\Gamma_0(4)$ subgroup of $SL(2, \mathbb{Z})$ instead of $\Gamma(2)$.

where

$$\#(A\cap B) = 2. \tag{5.9}$$

The monodromies (5.8) generate the $\Gamma(2)$ subgroup of $SL(2,\mathbb{Z})$; that is, the subgroup of 2×2 matrices congruent to the identity matrix, modulo 2. The $x=a$, $p=a_D$ are by now canonical variables of Seiberg-Witten theory [35], so we will mainly use that notation.

The periods a, a_D solve the Picard-Fuchs equation

$$\mathcal{L}\Pi = 0,$$

where $\mathcal{L} = \theta(\theta-1) - u^2(\theta-\frac{1}{2})^2$ and $\theta = u\frac{\partial}{\partial u}$. From the previous sections, we can predict that the genus g amplitudes $\mathcal{F}_g$ of this theory are (almost) modular forms of $\Gamma(2)$, with definite transformation properties. Since the higher genus amplitudes are known from [35,31], they will provide a direct check of our predictions.

The parameter τ of the modular curve is defined by $\tau = \frac{\partial p}{\partial x}$, or in usual Seiberg-Witten notation

$$\tau = \frac{\partial a_D}{\partial a} = 2\frac{\partial^2}{\partial a^2}\mathcal{F}_0(a). \tag{5.10}$$

Solving the Picard-Fuchs equation for the periods, we can obtain τ as a function of u. Alternatively, we can proceed as follows. Recall that the j-function of the elliptic curve, which has the q-expansion

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \ldots$$

where $q = e^{2\pi i\tau}$, provides a coordinate independent way of characterizing the curve. Roughly speaking, elliptic curves are the same if their j-functions are equal. Bringing the equation (5.7) of the family of elliptic curves in Weierstrass form

$$y^2 = 4x^3 - g_2 x - g_3 \tag{5.11}$$

the j function can be computed as

$$j = 1728\frac{g_2^3}{g_2^3 - 27g_3^2}. \tag{5.12}$$

For the family of elliptic curves (5.7), this gives

$$j(\tau) = \frac{64(3+u^2)^3}{(u^2-1)^2}. \tag{5.13}$$

Then, using the q-expansion of the j-function, we get a q-expansion for u, in the large u limit:

$$u = \frac{1}{8}q^{-1/2} + \frac{5}{2}q^{1/2} - \frac{31}{4}q^{3/2} + 27q^{5/2} + \mathcal{O}(q^{7/2}).$$

However, what we want is an expression of u in terms of τ which is valid everywhere in the moduli space, not just a q-expansion when u is large; in other words, we want to find the modular form of $\Gamma(2)$ which has the above q-expansion. Since u is a good coordinate on the moduli space, which is the quotient of the Teichmuller space by $\Gamma(2)$, it has to be invariant under $\Gamma(2)$; i.e., it must be a modular form of weight zero. For a brief review of modular forms of $\Gamma(2)$, see Appendix A.

The modular forms of $\Gamma(2)$ are generated by the following θ-constants:

$$b(\tau) := \theta_2^4(\tau), \qquad c(\tau) := \theta_3^4(\tau), \qquad d(\tau) := \theta_4^4(\tau)$$

which all have weight 2. These are not independent, but satisfy the relation

$$c = b + d.$$

It is easy to show that [21]

$$u(\tau) = \frac{c+d}{b}(\tau), \tag{5.14}$$

which is modular invariant, as claimed.

The genus one amplitude [30]

$$\mathcal{F}_1 = -\frac{1}{2}\log\left(\det\left(\frac{\partial a}{\partial u}\right)\right) - \frac{1}{12}\log(u^2-1) \tag{5.15}$$

can be rewritten, using the results we have obtained so far, as [26]

$$\mathcal{F}_1(\tau) = -\log\eta(\tau) \tag{5.16}$$

where $\eta(\tau)$ is the Dedekind η-function. Note that this transforms under modular transformation in $\Gamma(2)$ exactly as predicted in section 2, namely

$$\mathcal{F}_1\left(\frac{A\tau+B}{C\tau+D}\right) = \mathcal{F}_1(\tau) + \frac{1}{2}\log\frac{1}{\tau + C^{-1}D}$$

(up to a constant that is irrelevant, as only $\partial\mathcal{F}_1$ is well defined).[21]

[21] In this case, $\mathcal{F}_1$ transforms in this way under the whole $SL(2,\mathbb{Z})$, but this is an accident of the model. In particular, had we worked with $\Gamma_0(4)$ (and hence with $\tau' = \tau/2$), $\mathcal{F}_1$ would transform like this under $\Gamma_0(4)$, but not under the full $SL(2,\mathbb{Z})$.

Next, from section 4, we expect that $\frac{\partial^3 \mathcal{F}_0}{\partial a^3} = \frac{1}{2}\frac{\partial \tau}{\partial a}$ is a modular form of weight -3. Using the fact that $\frac{\partial}{\partial a} = \frac{\partial u}{\partial a}\frac{\partial}{\partial u}$, the modular expression for u (5.14) and the modular expression for $\frac{\partial u}{\partial a}$ obtained by combining (5.15) and (5.16), we get

$$\frac{\partial^3}{\partial a^3}\mathcal{F}_0(a) = -\frac{\sqrt{b}}{c\, d} \tag{5.17}$$

which indeed transforms as expected.

Now, consider the genus two amplitude. In [21] it was shown that this can be written as

$$\mathcal{F}_2(\tau) = h_2^{(0)}(\tau) + h_2^{(1)}(\tau)\; E(\tau) + h_2^{(2)}(\tau)\; (E(\tau))^2 + h_2^{(3)}(\tau)\; (E(\tau))^3 \tag{5.18}$$

where the propagator $E(\tau)$ is given in terms of the second Eisenstein series

$$E(\tau) = \frac{2\pi i}{6} E_2(\tau),$$

and the modular coefficients are

$$\begin{aligned} h_2^{(0)} &= \frac{1}{30}(c+d)(16b^2+19cd)\, X \\ h_2^{(1)} &= -2\left(\frac{6}{2\pi i}\right)(b^2+cd)\, X \\ h_2^{(2)} &= 3\left(\frac{6}{2\pi i}\right)^2 (c+d)\, X \\ h_2^{(3)} &= -\frac{5}{3}\left(\frac{6}{2\pi i}\right)^3 X \end{aligned} \tag{5.19}$$

where we defined

$$X = \frac{1}{1728}\frac{b}{c^2 d^2}.$$

We will now see that this is exactly as predicted in section 4!

First, consider how this transforms under modular transformations in Γ. Note that the coefficients $h_2^{(k)}$ are modular forms of Γ of weight $(-3k)$:

$$h_2^{(k)}(\,(A\tau+B)/(C\tau+D)\,) = (C\tau+D)^{-3k}\, h_2^{(k)}(\tau)$$

Moreover, k ranges from zero to $3g-3$, where $g=2$ in this case.

On the other hand $E(\tau)$ transforms as a quasi-modular form:

$$E((A\tau+B)/(C\tau+D)) = (C\tau+D)^2 E(\tau) + 2\, C(C\tau+D); \tag{5.20}$$

in other words it is a holomorphic form, modular up to shifts (cf. (4.2)). The fact that $\mathcal{F}_2$ is a finite power series in $E(\tau)$, with coefficients that are strictly modular forms of $\Gamma(2)$ means that $\mathcal{F}_2$ is itself a quasi-modular form of $\Gamma(2)$, per definition. Note that the propagator in (5.20) transforms by a factor of 2 relative to (4.2). This factor of two is a consequence of the fact that the intersection number of the A and the B periods of the curve is twice bigger than the conventional one (5.9). It is very easy to derive this from section 2 and 3 (see footnote 6).

Moreover, it is easy to check, starting from (5.16), (5.17) and (5.18) (with the help of some standard modular formulae given in appendix A), that $\mathcal{F}_2$ transforms under modular transformations exactly as predicted in section 3. To do so, note that, looping around $u = 1$ for example, simply acts on τ by the $\Gamma(2) \subset SL(2, \mathbb{Z})$ transformation M_1 given in (5.8). Using the usual transformation properties of modular forms and the expression (5.18) for $\mathcal{F}_2$ in terms of modular forms of $\Gamma(2)$, it is then easy to work out the transformation property of $\mathcal{F}_2$ under M_1.

Furthermore, while the $\mathcal{F}_g$ and the vertices $\partial_{i_1}, \ldots, \partial_{in}\mathcal{F}_g$ are not quite modular, the combinations

$$\mathcal{F}_g(\tau) + \Gamma_g(E(\tau),\ \partial_{i_1} \ldots \partial_{i_n}\mathcal{F}_{r<g}) \ = \ h_g^{(0)}(\tau) \tag{5.21}$$

are exactly invariant under modular transformations and agree with $h_g^{(0)}(\tau)$, as expected from section 4.

We can trade quasi-modular forms for almost holomorphic forms by replacing $E(\tau)$ in all formulae by its modular, but not holomorphic counterpart

$$\hat{E}(\tau, \bar{\tau}) = E(\tau) + \frac{2}{\tau - \bar{\tau}}$$

which transforms as

$$\hat{E}(\ (A\tau + B)/(C\tau + D),\ (A\bar{\tau} + B)/(C\bar{\tau} + D)\ ) = (C\tau + D)^2 \hat{E}(\tau, \bar{\tau}).$$

Also, note that $\mathcal{F}_1$ can be made exactly modular by writing

$$\hat{\mathcal{F}}_1(\tau, \bar{\tau}) = -\log((\tau - \bar{\tau})^{\frac{1}{2}} |\eta(\tau))|^2).$$

This is exactly the one-loop amplitude of the local Calabi-Yau in holomorphic polarization. More precisely, it is only the holomorphic derivatives $\frac{\partial}{\partial a}\mathcal{F}_1$, and $\frac{\partial}{\partial a}\hat{\mathcal{F}}_1$ that are physical, but this is the natural way to write it.

Finally, $\hat{E}(\tau, \bar{\tau})$ is exactly the propagator of [6]! One has that

$$\hat{\mathcal{F}}_g(\tau, \bar{\tau}) + \Gamma_g(\hat{E}(\tau, \bar{\tau}),\ \partial_{i_1}, \ldots, \partial_{i_n}\mathcal{F}_{r<g}(\tau, \bar{\tau})) = h_g^{(0)}(\tau) \tag{5.22}$$

is strictly holomorphic, with the same modular form $h_g^{(0)}(\tau)$ as in (5.21).

In the next subsection, we consider gauge groups of higher rank, corresponding to Riemann surfaces of genus higher than one.

5.2. The $SU(n)$, $n > 2$ Seiberg-Witten Theory

As mentioned earlier, the Riemann surface corresponding to $SU(n)$ Seiberg Witten theory is a genus $g = n - 1$ curve

$$y^2 - (P_n(z))^2 + \Lambda^{2n} = 0 \tag{5.23}$$

where

$$P_n(z) = z^n + u_2 z^{n-2} + \dots u_{n+1}.$$

The singular loci in the moduli space correspond to the zeroes of the discriminant

$$\Delta = \prod_{i<j} (e_i(u) - e_j(u))^2 \tag{5.24}$$

where $e_i(u)$ are roots of $P_n(z,u)^2 - \Lambda^{2n}$. That is, at the values of the moduli u for which any pair of roots come together $e_i(u) \to e_j(u)$, the curve becomes singular. There is a natural basis of $(n-1)$ A-cycles corresponding to pairs of branch points that pair up as Λ goes to zero. This corresponds to points where the non-abelian gauge bosons become massless in the classical theory. The monodromy group $\Gamma \subset Sp(2g, \mathbb{Z})$ of the quantum theory can be determined [25], by following the exchange paths of the branch points.

We will leave the detailed analysis of this and the corresponding implications for the structure of the topological string amplitudes as an interesting exercise, and only consider briefly the one-loop amplitude.

On general grounds [5,6], the one-loop amplitude in the topological string theory has the universal form

$$\mathcal{F}_1(\tau) = -\frac{1}{2}\log(\det(D_i X)) - \frac{1}{12}\log(\Delta). \tag{5.25}$$

This result was also derived in a purely gauge theory context in [30,29]. There, the authors computed the one-loop amplitude of the (twisted) $\mathcal{N} = 2$ gauge theory on a curved four-manifold, namely the coefficients of the $\int R^2$ term in the effective action. Restricting the curvature to be anti-self dual, $R_- = 0$, this is precisely the term that the topological string computes.[22] This gives

$$\mathcal{F}_1(\tau) = -\frac{1}{2}\log\left(\det\left(\frac{\partial a_i}{\partial u_k}\right)\right) - \frac{1}{12}\log(\Delta), \tag{5.26}$$

[22] Practically, in terms of [30,29] this corresponds to setting the signature σ of the four-manifold equal to $\sigma = -\frac{2}{3}\chi$ where χ is its Euler character. One way to see this is that it holds exactly for the $K3$, for example, where the curvature is anti-self dual.

where Δ is the discriminant of the Seiberg-Witten curve. For example, for $G = SU(n)$ with curve given by (5.23), Δ is (5.24).

Note that the u's are necessarily modular invariants of Γ, as they are just parameters entering into the algebraic definition of the curve, and hence they do not 'talk' to its periods. On the other hand, Δ is simply a rational function of u, so also necessarily a Siegel modular form of Γ of weight zero.

To write the full amplitude in terms of modular forms, note that from [30,29] we have

$$\left(\det\left(\frac{\partial a_j}{\partial u_i}\right)\right)^{\frac{1}{2}} \Delta^{\frac{1}{8}} = \theta \begin{bmatrix} 0 \\ \vec{\delta} \end{bmatrix} (0,\tau) \tag{5.27}$$

where $\vec{\delta} = \left[\frac{1}{2}, \ldots, \frac{1}{2}\right]$ and we defined the 'generalized' θ-functions with characteristic in appendix B. As a consequence we can write

$$\mathcal{F}_1(\tau) = -\log\left(\theta \begin{bmatrix} 0 \\ \vec{\delta} \end{bmatrix} (0,\tau)\right) + \frac{1}{24}\log(\Delta) \ .$$

This is consistent with the transformation properties of $\mathcal{F}_1$, since $\theta\left[\begin{smallmatrix} 0 \\ \vec{\delta} \end{smallmatrix}\right]$ is a scalar Siegel modular form of weight 1/2.

6. Local $\mathbb{P}^2$

We now study the local $\mathbb{P}^2$, from the mirror B-model point of view. In this case the mirror is a family of elliptic curves Σ with monodromy group $\Gamma(3)$. The Gromov-Witten theory of the local $\mathbb{P}^2$ at large radius was solved in [3,4] . Using those results, we can show explicitly that the predictions for modular properties of the topological string amplitudes are satisfied.

Another interesting point in the moduli space of the local $\mathbb{P}^2$ is the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold point. One can in principle formulate the Gromov-Witten theory of the orbifold point as well, however the amplitudes are not yet available [34,7]. We now have a simple prescription to carry over the large radius results to other points in the moduli space, the orbifold point in particular, so we can make new predictions there.

6.1. Mirror Family of Elliptic Curves

The mirror data is a family of elliptic curves Σ, given by the equation

$$\sum_{i=1}^{3} x_i^3 - 3\psi \prod_{i=1}^{3} x_i = 0 \tag{6.1}$$

in $\mathbb{P}^3$, and a meromorphic 1-form $\lambda = \log(x_2/x_3)dx_1/x_1$. This has an obvious $\mathbb{Z}_3$ symmetry

$$\psi \to \alpha\psi, \qquad \alpha = e^{2\pi i/3},$$

since it can be undone by a coordinate transformation $x_1 \to \alpha^{-1}x_1$ that affects neither Σ nor λ. The discriminant Δ of the curve is

$$\Delta = (1 - \psi^3).$$

This vanishes at the three singular points $\psi^3 = 1$, corresponding to conifold singularities.

To make contact with standard elliptic functions and their modular properties we make a $PGL(3,\mathbb{C})$ transform to bring the equation of the curve to its Weierstrass form

$$y^2 = 4x^3 - g_2 x - g_3$$

with

$$g_2 = \frac{\alpha(8+\psi^3)}{2^{(2/3)}24\psi^3}, \qquad g_3 = \frac{8 + 20\psi^3 - \psi^6}{864\psi^6},$$

so that its j-function is given by

$$j(\tau) = -\frac{27\psi^3(8+\psi^3)^3}{(1-\psi^3)^3}\ . \tag{6.2}$$

As usual,

$$\tau = \frac{\partial p}{\partial x} \tag{6.3}$$

is the standard complex structure modulus of the family of elliptic curves, where we view Σ as a quotient of a complex plane by a lattice generated by 1 and τ. Here[23]

$$p = \int_B \lambda(\psi), \qquad x = \int_A \lambda(\psi)$$

[23] We use x to denote both the coordinate on the Riemann surface and the period of λ. It should be clear from the context which meaning we assign to x.

where $\lambda(\psi) = \log(x) dy/y$. Our j-function is normalized to

$$j = \frac{1}{q} + 744 + 196884q + \mathcal{O}(q^2), \tag{6.4}$$

where $q = \exp(2\pi i \tau)$. Combining the two expressions for the j-function, we find a series expansion for $\psi(q)$ in the large ψ limit:

$$3\psi = \frac{1}{q^{\frac{1}{3}}} + 5q^{\frac{2}{3}} - 7q^{\frac{5}{3}} + \mathcal{O}(q^{\frac{8}{3}}) \ . \tag{6.5}$$

Alternatively, we can obtain the same expansion by first using the Picard-Fuchs equations to find the periods $x(\psi)$, $p(\psi)$, and then computing $\tau(\psi)$ directly using the definition (6.3). We will study in more details the Picard-Fuchs equation and its solutions in the next subsection. For now, we only note one interesting aspect to this. Namely, as discussed in section 2.3, due to the non-compactness of the Calabi-Yau, it may not be possible to find a basis of periods that are normalized canonically. This occurs in the present example: the compact B period satisfies

$$\#(A \cap B) = -3. \tag{6.6}$$

One way to see this is in the mirror A-model: the compact parts of H_4 and H_2 of the manifold are generated by the $\mathbb{P}^2$, which we take to be mirror to the B-period, and the $\mathbb{P}^1$ line inside it, mirror to the A period. In the Calabi-Yau, these do intersect, but the intersection number is -3. Correspondingly, if we put $x = t$,

$$p = -3 \frac{\partial}{dt} \mathcal{F}_0(t),$$

and therefore $\tau = -3 \frac{\partial^2}{dt^2} \mathcal{F}_0(t)$.

The above expression for $\psi(\tau)$ is valid for $\mathrm{Im}(\tau) \to \infty$. In the next subsection, we will show that the local $\mathbb{P}^2$ is governed by a $\Gamma(3)$ subgroup of $SL(2, Z)$. This will allow us to give a globally well defined expression for ψ in terms of modular forms under $\Gamma(3)$.

6.2. The Monodromy Group

The meromorphic 1-form λ turns out to have a non-vanishing residue: in addition to the usual A and B periods — by this we mean the periods associated to the A and B cycles — of the genus one Riemann surface, it has an additional period, which we will call C. As discussed in section 2.3, the extra period does not correspond to a modulus of the Riemann surface, but to an auxiliary parameter. While the monodromies mix up all the periods, the monodromy action on the extra period C should be highly constrained. To derive the monodromy action on the full period vector

$$\Pi = \begin{pmatrix} \int_B \lambda \\ \int_A \lambda \\ \int_C \lambda \end{pmatrix}$$

we will solve the Picard-Fuchs (PF) differential equations that Π satisfies

$$\mathcal{L}\Pi = 0, \tag{6.7}$$

everywhere in the moduli space. A certain linear combination of the solutions to equation (6.7) will have the property that its monodromies are integral, and that gives Π.

Before doing that, note that, since the additional period C is just an auxiliary parameter, the modular properties of the topological string amplitudes should be governed by the monodromy group of the family of elliptic curves Σ. It is well known that this is a $\Gamma(3)$ subgroup of $SL(2,\mathbb{Z})$, when viewed as a fibration over the punctured ψ plane. We will see below that this is indeed the case.

Now let us come back to the study of the full Picard-Fuchs equation. It is convenient to work in the coordinate z, centered at large radius:

$$z = -\frac{1}{(3\psi)^3}. \tag{6.8}$$

There are three special points in the z plane. In addition to the large radius point at $z = 0$, there is also the conifold point, coming from $\psi^3 = 1$, and the orbifold point $z = \infty$, with $\mathbb{Z}_3$ monodromy around it. In this coordinate the Picard-Fuchs differential operator $\mathcal{L}$ has the form

$$\mathcal{L} = \theta_z^3 + 3z(3\theta_z + 2)(3\theta_z + 1)\theta \ .$$

This has three independent solutions, one of which is a constant, corresponding to the period of λ around the $C-$cycle. The corresponding new cycle C encircles the residue of $\lambda(\psi)$.

The solutions near large radius ($z = 0$) can be found by the Frobenius method from the generating function

$$\omega(z,s) := \sum_{n=1}^{\infty} \frac{z^{s+n}}{\Gamma(-3(n+s)+1)\Gamma^3(n+s+1)},$$

with $\mathcal{L}\omega(z,s)=0$. This gives 3 independent solutions,

$$\omega_i = \frac{1}{(2\pi i)^i}\,\frac{\mathrm{d}^i}{\mathrm{d}^i s}\omega(z,s)\bigg|_{s=0}\,,$$

i.e. $\omega_0 = 1$, $\omega_1 = \frac{1}{2\pi i}(\log(z)+\sigma_1(z))$ and $\omega_2 = \frac{1}{(2\pi i)^2}(\log(z)^2 + 2\sigma_1\log(z)+\sigma_2(z))$, where the first orders are $\sigma_1 = -6\,z + 45\,z^2 + \mathcal{O}(z^3)$ and $\sigma_2 = -18\,z + \frac{423\,z^2}{2} + \mathcal{O}(z^3)$.

Linear combinations of these solutions will give the periods over cycles in integer cohomology. This requires analytic continuation to all singular points. The result is

$$\Pi = \begin{pmatrix} -3\partial_t\mathcal{F}_0 \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\omega_2 - \frac{1}{2}\omega_1 - \frac{1}{4} \\ \omega_1 \\ 1 \end{pmatrix}. \tag{6.9}$$

The factor of -3 in the above equation comes from (6.6) as we explained earlier. From above, we can read off the mirror map, giving the A-period in terms of the coordinates on the moduli space, and its inverse:

$$z(Q) = Q + 6\,Q^2 + 9\,Q^3 + 56\,Q^4 + \mathcal{O}(Q^5)\ . \tag{6.10}$$

where $Q = e^{2\pi i t}$, and z is defined in (6.8).[24]

From this, we can also read off the monodromy of the periods Π around large radius, i.e. around $z=0$ (or $\psi=\infty$). From (6.10) it follows that this is equivalent to shifting t by one, and, since $-3\partial\mathcal{F}_0 = \frac{1}{2}t^2 - \frac{t}{2} - \frac{1}{4} + O(e^{\pi i t})$, this gives

$$M_\infty = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.11}$$

Expanding the periods at the conifold point $\psi^3=1$, one finds the monodromy

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.12}$$

[24] For the genus zero partition function this gives

$$\partial_t\mathcal{F}_0 = -\frac{t^2}{6} + \frac{t}{6} + \frac{1}{12} + 3\,Q - \frac{45\,Q^2}{4} + \frac{244\,Q^3}{3} - \frac{12333\,Q^4}{16} + \mathcal{O}(Q^5),$$

which agrees with the Gromow-Witten large radius expansion. Using this, and the definition of τ we can explicitly check (6.5).

This is the Picard-Lefshetz monodromy around the shrinking B-cycle with intersection form (6.6). The C-period corresponds to an auxiliary parameter, and correspondingly the C-cycle does not intersect the A and B cycles.

From M_∞ and M_1, we can recover the monodromy around the orbifold point M_0, as holomorphy requires

$$M_0 M_1 M_\infty = 1,$$
$$M_0 = \begin{pmatrix} -2 & -1 & 1 \\ 3 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.13}$$

This satisfies $(M_0)^3 = 1$, as it should, since the monodromy is of third order. Note that in all three cases, the monodromies act trivially on the C-period, which is consistent with the fact that this corresponds simply to a parameter. Moreover, the monodromy action on the A and the B periods generates the $\Gamma_0(3)$ subgroup of $SL(2,\mathbb{Z})$.

If instead of the z-plane, we choose to work with the ψ-plane, then $\psi = 0$ is a regular point, with trivial monodromy around it, but instead we have three conifold singularities, at $\psi = 1, \alpha, \alpha^2$, with $\alpha = e^{\frac{2\pi i}{3}}$. The monodromies $\tilde{M}$ in the ψ-plane can be derived from the expressions above. For example,

$$\tilde{M}_1 = M_1, \qquad \tilde{M}_\alpha = M_0 M_1 M_0^{-1}, \qquad \tilde{M}_{\alpha^2} = M_0^2 M_1 M_0^{-2}$$

with monodromy at infinity given by $\tilde{M}_\infty = \tilde{M}_1 \tilde{M}_\alpha \tilde{M}_{\alpha^2}$. These turn out to generate the $\Gamma(3)$ subgroup of $SL(2,\mathbb{Z})$. Below, we will choose to work with modular forms of $\Gamma(3)$, in terms of which both ψ and z will be given by exactly modular forms.

6.3. Topological Strings on Local $\mathbb{P}^2$ and Modular Forms

To get modular expressions for the topological string amplitudes we need to know a bit about modular forms of the subgroup $\Gamma(3)$ of $SL(2,\mathbb{Z})$. Essential facts about them are reviewed in Appendix A; for a detailed study of modular forms of $\Gamma(3)$, see [17].

The set of θ-constants that generate modular forms of $\Gamma(3)$ is:

$$a := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}, \quad b := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}, \quad c := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}, \quad d := \theta^3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \end{bmatrix},$$

which all have weight 3/2. They satisfy the relations [17]

$$c = b - a, \qquad d = a + \alpha b \ ,$$

and the Dedekind η-function is given by $\eta^{12} = \frac{i}{3^{3/2}} abcd$. To begin with, note that since ψ is a coordinate on the moduli space, it has to be a weight zero modular form of $\Gamma(3)$. Indeed, we find that

$$\psi(\tau) = \frac{a-c-d}{d} \ . \tag{6.14}$$

From [5] we know that the genus one free energy is given by

$$\mathcal{F}_1 = -\frac{1}{2}\log\left(\frac{\partial t}{\partial \psi}\right) - \frac{1}{12}\log(1-\psi^3) \ .$$

It is easy to show, using the Q-expansion of z around $z=0$, that

$$\frac{\partial t}{\partial \psi} = -\sqrt{3}\frac{d}{\eta}, \tag{6.15}$$

and that, on the other hand,

$$\Delta = 1-\psi^3 = -3^3\frac{\eta^{12}}{d^4}.$$

Combining these three expressions, we get

$$\mathcal{F}_1(\tau) = -\log(\eta(\tau)) + \frac{1}{24}\log(\Delta) \ = -\frac{1}{6}\log(d\eta^3),$$

up to an irrelevant constant term. This transforms under Γ as $-\log(\eta)$ does, since the discriminant Δ is invariant, which is exactly what we predicted. As a consistency check, if we use the Q-expansion of q and the modular expression for $\mathcal{F}_1$ we get the expansion

$$\mathcal{F}_1 = -\frac{1}{12}\log Q + \frac{Q}{4} - \frac{3Q^2}{8} - \frac{23Q^3}{3} + \mathcal{O}(Q^4),$$

which is precisely the genus 1 amplitude of local $\mathbb{P}^2$.

6.4. Higher Genus Amplitudes

To find the higher genus amplitudes, we need the modular expression for the Yukawa coupling $C_{ttt} = \frac{\partial^3}{\partial t^3}\mathcal{F}_0$. We know that

$$C_{ttt} = -\frac{1}{3}\frac{\partial \tau}{\partial t} = -\frac{1}{3}\frac{\partial \psi}{\partial t}\frac{\partial \tau}{\partial \psi}.$$

Using the modular expressions for ψ (6.14), for $\frac{\partial t}{\partial \psi}$ (6.15), and the formulae for logarithmic derivatives derived in Appendix A, we get

$$C_{ttt} = -\frac{1}{3^{5/2}}\frac{d}{\eta^9}. \tag{6.16}$$

Another useful object is the $\Gamma(3)$-invariant Yukawa coupling, expressed in terms of the globally defined variable ψ. We obtain

$$C_{\psi\psi\psi} = \left(\frac{\partial t}{\partial \psi}\right)^3 C_{ttt} = -\frac{9}{\Delta}. \tag{6.17}$$

Using the results of the previous subsection, we can now find a modular expression for higher genus amplitudes, through their Feynman expansions. The propagator $E(\tau)$ must transform under modular transformations as in (4.2)

$$E((A\tau + B)/(C\tau + D)) = (C\tau + D)^2 E(\tau) - 3\, C\, (C\tau + D);$$

the factor of -3 comes from the intersection numbers (6.6). For example, we can take

$$E = -\frac{2\pi i}{4} E_2(\tau).$$

We could have worked with the full $E' = 6\frac{\partial}{\partial \tau}\mathcal{F}_1$ as well, since the propagator is defined up to a modular invariant piece; it would have only changed the modular invariant $h_2^{(0)}$.

We obtain that the general form of the higher genus amplitudes reads

$$\mathcal{F}_g = X^{g-1} \sum_{k=0}^{3(g-1)} E_2^k h_g^{(3g-3-k)}(K_2, K_4, K_6) \tag{6.18}$$

where we defined the weight -6 object

$$X = \frac{d^2}{2^9 3^4 \eta^{18}} = \frac{1}{1536} C_{ttt}^2$$

and the ring of modular forms of $\Gamma(3)$ generating the weight $2d$ forms $h_g^{(d)}$ is given by

$$K_2 = -\alpha^2 \frac{(a - \alpha c)^2}{\eta^2}, \qquad K_4 = \frac{1}{\alpha^2 - 1} \frac{ac(a+c)(\alpha^2 a - c)}{\eta^4}, \qquad K_6 = \frac{(ac)^2(a+c)^2}{\eta^6}.$$

The coefficients of E_2 are fixed by the Feynman graph expansion and we obtain for example

$$\begin{aligned} h_2^{(0)} &= \mathcal{F}_2 - X\left(5E_2^3 + E_2^2 K_2 + \frac{1}{3} E_2 K_2^2\right), \\ h_3^{(0)} &= \mathcal{F}_3 - X^2 (180 E_2^6 + 240 E_2^5 K_2 + 4 E_2^4 (145 K_2^2 - 1008 K_4) \\ &\quad + \frac{32}{9} E_2^3 (199 K_2^3 - 1908 K_2 K_4 + 648 K_6) + \frac{4}{5} E_2^2 (563 K_2^4 - 7936 K_2^2 K_4 + 26496 K_4^2) \\ &\quad + \frac{16}{15} E_2 (149 K_2^5 - 2536 K_2^3 K_4 + 11952 K_2 K_4^2 - 3456 K_4 K_6)). \end{aligned} \tag{6.19}$$

Now, using known results for $\mathcal{F}_g$ in the large radius limit (obtained for instance through the topological vertex formalism), we can find the $h_g^{(0)}$'s explicitly — this corresponds to fixing the holomorphic ambiguity in the BCOV formalism. For instance, we obtain

$$\begin{aligned} h_2^{(0)} &= \frac{11}{69120} + \frac{1}{34560\Delta} - \frac{1}{7680\Delta^2}, \\ h_3^{(0)} &= \frac{17}{6289280} + \frac{269}{46448640\Delta} - \frac{19393}{278691840\Delta^2} + \frac{337}{2211840\Delta^3} - \frac{373}{4128768\Delta^4}. \end{aligned} \tag{6.20}$$

6.5. The $\mathbb{C}^3/\mathbb{Z}_3$ Orbifold Point

In this section we explain how to extract the Gromov-Witten generating functions of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ from the large radius amplitudes, through the wave function formalism.

Let us first discuss this theory from the mirror A-model point of view. The target space X is an $X = \mathbb{C}^3/\mathbb{Z}_3$ orbifold, with $\mathbb{Z}_3$ acting on the three coordinates z_i, $i = 1, 2, 3$ by

$$z_i \to \alpha z_i, \qquad \alpha = e^{\frac{2\pi i}{3}}.$$

In quantizing string theory on X, the Hilbert space splits into 3 twisted sectors, corresponding to strings closed up to α^k, $k = 0, 1, 2$ (and projecting onto $\mathbb{Z}_3$ invariant states). The supersymmetric ground states in the k-th sector correspond to the cohomology of the fixed point set of α^k. This has an interpretation in terms of the cohomology of X as well. In the case at hand, the ground states in the sector twisted by α^k correspond to the generators of $H^{k,k}(X)$. Namely, the contribution to the cohomology of X is determined by the $U(1)_L \times U(1)_R$ charges of the states, where the charge (p_i, q_i) corresponds to H^{p_i,q_i}. In the twisted sectors, however, these receive a zero-point shift: in the sector twisted by $z_i \to e^{2\pi i k_i} z_i$ with $0 \leq k_i < 1$ the shift is $(\sum_i k_i, \sum_i k_i)$. As there is precisely one such state for each k, the stringy cohomology of the orbifold agrees with the cohomology of the smooth resolution of X, i.e. the $O(-3) \to \mathbb{P}^2$, as is generally true (see however [38]).

As explained in [37], the orbifold theories have discrete *quantum* symmetries. In the present case, this is the $\mathbb{Z}_3$ symmetry which sends a state in the k'th twisted sector to itself times α^k. This is respected by interactions, so it is a well defined symmetry of the quantum theory. This implies that the only non-vanishing correlation functions are those that have net charge zero (mod 3). In particular, if we consider correlation functions of n insertions of topological observables $\mathcal{O}_\sigma$ corresponding to the generator of $H^{1,1}(X)$,

$$\langle \underbrace{\mathcal{O}_\sigma \, \mathcal{O}_\sigma \ldots \mathcal{O}_\sigma}_{n} \rangle_g$$

at any genus g, this does not vanish only if $n = 0$ (mod 3) We will describe in this section how to compute the generating functions of correlation functions at genus g

$$\mathcal{F}_g^{\rm orb}(\sigma) = \sum_n \frac{1}{n!} \langle (\mathcal{O}_\sigma)^n \rangle \sigma^n$$

and show that this is indeed the case. By $\mathcal{F}_g^{\rm orb}$ here, we mean the generating function at the orbifold point — in this section, we will denote the generating function in the large radius limit by $\mathcal{F}_g^\infty$ to avoid confusion.

From what we explained in section 3, the expectation is the following. The good coordinate in one region of the moduli space generally fails to be good at other regions of the moduli space. The good variable at large radius is t, as the corresponding monodromy is trivial (6.11), according to our criterion in section 3. However, the monodromy of the period t is not trivial around the orbifold point, being given by (6.13), as $3 \neq 0$. Correspondingly, even though we know the topological string amplitudes near the large radius point, we cannot simply analytically continue them to the orbifold point — the resulting objects would have bad singularities. Changing to good variables at the orbifold point involves a wave function transform that mixes up the genera.

What is the good variable at the orbifold point? Clearly, it is the mirror B-model realization of the parameter σ that enters the orbifold Gromov-Witten partition functions in the A-model language and corresponds to $H^{1,1}(X)$. The dual variable σ_D

$$\sigma_D = -3 \frac{\partial}{\partial \sigma} \mathcal{F}_0^{\rm orb}$$

corresponds to $H^{2,2}(X)$. To identify them in the B-model, note that, on the one hand, under the quantum symmetry $\mathbb{Z}_3$ symmetry σ and σ_D transform as

$$(1, \sigma, \sigma_D) \rightarrow (1, \alpha\, \sigma, \alpha^2\, \sigma_D).$$

On the other hand, the symmetry acts in the mirror theory by [37]

$$\psi \rightarrow \alpha\, \psi.$$

The fixed point of this, $\psi = 0$, corresponds to the elliptic curve with $\mathbb{Z}_3$ symmetry, which is mirror to the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. We can easily find the solutions to the Picard-Fuchs equations with these properties.

A basis of solutions is given by the hypergeometric system ${}_3F_2$

$$B_k(\psi) = \frac{(-1)^{\frac{k}{3}}}{k}(3\psi)^k \sum_{n=0}^{\infty} \frac{\left(\left[\frac{k}{3}\right]_n\right)^3}{\prod_{i=1}^3 \left[\frac{k+i}{3}\right]_n} \psi^{3n} \ , \tag{6.21}$$

for $k = 1, 2$, where we defined the Pochhammer symbols $[a]_n := \frac{\Gamma(a+n)}{\Gamma(a)}$. We also set $B_0(\psi) = 1$. The B's diagonalize the monodromy around the orbifold point, namely $\psi \to \alpha\psi$ takes

$$(B_0, B_1, B_2) \to (B_0, \alpha B_1, \alpha^2 B_2).$$

Consequently, we can identify

$$(1, \sigma, \sigma_D) = (B_0, B_1, B_2).$$

The relative normalization of σ and σ_D can be fixed using $\sigma_D = -3\frac{\partial \mathcal{F}_0^{\rm orb}}{\partial \sigma}$ and hence $\frac{\partial \tilde{\tau}}{\partial \sigma} = \frac{\partial^2 \sigma_D}{\partial \sigma^2} = -3C_{\psi\psi\psi}\left(\frac{\partial \psi}{\partial \sigma}\right)^3$, since ψ is globally defined.

We can already make a prediction for the genus zero free energy at the orbifold point, up to an overall constant. By integrating $\sigma_D = 3\frac{\partial \mathcal{F}_0^{\rm orb}}{\partial \sigma}$, we get

$$\mathcal{F}_0^{\rm orb}(\sigma) = \sum_{n=1}^{\infty} \frac{N_{g=0,n}^{\rm orb}}{(3n)!} \sigma^{3n}$$

where, for example

$$N_{0,1}^{\rm orb} = \frac{1}{3}, \quad N_{0,2}^{\rm orb} = -\frac{1}{3^3}, \quad N_{0,3}^{\rm orb} = \frac{1}{3^2}, \quad N_{0,4}^{\rm orb} = -\frac{1093}{3^6},$$
$$N_{0,5}^{\rm orb} = \frac{119401}{3^7}, \quad N_{0,6}^{\rm orb} = -\frac{27428707}{3^8}, \ldots$$

Let us now turn to higher genus amplitudes. The analytic continuation from the point at infinity to the orbifold point can be done with the Barnes integral, as also explained in [8]. This relates

$$\Pi = \begin{pmatrix} -\frac{1}{1-\alpha} c_2 & \frac{\alpha}{1-\alpha} c_1 & \frac{1}{3} \\ c_2 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_D \\ \sigma \\ 1 \end{pmatrix} \tag{6.22}$$

with the coefficients

$$c_1 = \frac{i}{2\pi} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma^2\left(\frac{2}{3}\right)}, \qquad c_2 = -\frac{i}{2\pi} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)}, \tag{6.23}$$

which are not integers. This is because the natural basis (σ, σ_D) diagonalizes the monodromy around the orbifold point, and this cannot be done in $SL(2,\mathbb{Z})$.[25] Note that $c_1 c_2 = \frac{\alpha(\alpha-1)}{(2\pi i)^3}$; correspondingly the change of basis *does not* preserve the symplectic form, we have rather that

$$dp \wedge dx = \frac{1}{\beta} d\sigma_D \wedge d\sigma$$

where

$$\beta = -(2\pi i)^3.$$

Because of this fact, the analysis of section 2 goes through, but one has to be careful with normalizations. More precisely, it implies that the effective string coupling at the orbifold $(g_s^{\rm orb})^2$ is renormalized relative to the large radius g_s^2 by $(g_s^{\rm orb})^2 = \beta g_s^2$.

Then, knowing the Gromov-Witten amplitudes at large radius, we can predict them at the orbifold:

$$\beta^{g-1} \mathcal{F}_g^{\rm orb} = \mathcal{F}_g^{\infty} + \Gamma_g(\Delta,\ \partial_{i_1} \dots \partial_{i_n} \mathcal{F}_{r<g}^{\infty}), \tag{6.24}$$

where the coefficient β comes from the renormalization of the string coupling, and

$$\Delta = \frac{3}{\tau + C^{-1}D}.$$

The coefficient 3 above comes from (6.6). The coefficients C and D are computed from (the inverse of) (6.22) as before, which gives

$$C^{-1}D = \frac{1}{1-\alpha}. \tag{6.25}$$

In order to extract the σ-expansion of $\mathcal{F}_g^{\rm orb}$ such as we presented for $\mathcal{F}_0^{\rm orb}$, we compute the right hand side of (6.24) in terms of the period t, and then use the relation between σ and t given in (6.22) to get expansions around $\sigma = 0$.

---

[25] We could have derived the change of basis in another way. There is another natural basis at the orbifold, (C_0, C_1, C_2), corresponding to the 3 fractional branes. This basis has monodromy around the orbifold point, which is the cyclic $\mathbb{Z}_3$ permutations of the branes,

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}.$$

The fractional brane basis is related to the large radius basis by an integral transformation — respecting the integrality of the D-brane charges — and the symplectic form. On the other hand, it is known [16] how the fractional branes couple to the twisted sectors: in particular, the i-th twisted sector corresponds to $\sum_j \alpha^{ij} C_j$. This reproduces (6.22).

Since $\tilde{\tau} = \frac{\partial \sigma_D}{\partial \sigma}$ vanishes at the orbifold point $\sigma = 0$, it follows from (6.22) that

$$\tau(\sigma = 0) = \frac{\alpha}{1-\alpha}. \tag{6.26}$$

Numerically, this corresponds to $q(\sigma = 0) = -e^{-\frac{\pi}{\sqrt{3}}} \sim -0.16$; at this value, the q-expansion of the modular expression (6.18) still converges rapidly. Indeed, since the coefficients of the σ-expansion of the topological string amplitude at the orbifold point are rational numbers, they can be easily recovered from their convergent q-expansion.

At genus 1, we get

$$\mathcal{F}_1^{\rm orb}(\sigma) = \sum_{n=1}^{\infty} \frac{N^{\rm orb}_{g=1,n}}{(3n)!} \sigma^{3n}$$

where, for instance,

$$N^{\rm orb}_{1,1} = 0, \quad N^{\rm orb}_{1,2} = \frac{1}{3^5}, \quad N^{\rm orb}_{1,3} = -\frac{14}{3^5},$$
$$N^{\rm orb}_{1,4} = \frac{13007}{3^8}, \quad N^{\rm orb}_{1,5} = -\frac{8354164}{3^{10}}, \ldots$$

It is good to note that simply expanding $\mathcal{F}_g^{\infty}(\tau)$ near $\tau(\sigma = 0)$, that is, doing only the analytic continuation of the amplitudes, would lead to non-rational coefficients in the σ-expansion.

Instead of (6.24) , it is faster to use the recursion relations at the orbifold point directly in terms of the modular ambiguity (6.20) and the corresponding propagator,

$$E^{orb}(\tau) = \lim_{\bar{\tau} \to \bar{\tau}(\sigma=0)} \hat{E}(\tau, \bar{\tau})$$

where

$$\bar{\tau}(\sigma = 0) = -C^{-1}D$$

is just the complex conjugate of (6.26). This follows from the fact that $\hat{\mathcal{F}}_g$, on the one hand, satisfies the same recursion relations as $\mathcal{F}_g^{\infty}$ with E's and $\mathcal{F}_r^{\infty}$'s replaced by their hatted counterparts, and on the other hand $\hat{\mathcal{F}}_g(\tau, \bar{\tau})$ at $\bar{\tau}$ set to $\bar{\tau} = -C^{-1}D$ gives $\mathcal{F}_g^{\rm orb}$. In fact, the right hand side of (6.24) can be interpreted as computing just that. Either way, for $\mathcal{F}_g^{\rm orb}$, we find that

$$\mathcal{F}_g^{\rm orb}(\sigma) = \sum_{n=0}^{\infty} \frac{N^{\rm orb}_{g,n}}{(3n)!} \sigma^{3n}$$

with the numbers $N^{\rm orb}_{g,n\geq 1}$

g	$n=1$	2	3	4	5
0	$\frac{1}{3}$	$-\frac{1}{3^3}$	$\frac{1}{3^2}$	$-\frac{1093}{3^6}$	$\frac{119401}{3^7}$
1	0	$\frac{1}{3^5}$	$-\frac{14}{3^5}$	$\frac{13007}{3^8}$	$-\frac{8354164}{3^{10}}$
2	$\frac{1}{2^4\cdot 3^4\cdot 5}$	$-\frac{13}{2^4\cdot 3^6}$	$\frac{20693}{2^4\cdot 3^8\cdot 5}$	$-\frac{12803923}{2^4\cdot 3^{10}\cdot 5}$	$\frac{31429111}{2^4 3^{10}}$
3	$-\frac{31}{2^5 3^5 5\cdot 7}$	$\frac{11569}{2^5 3^9 5\cdot 7}$	$-\frac{2429003}{2^5 3^{10} 5\cdot 7}$	$\frac{871749323}{2^4 3^{11} 5\cdot 7}$	$-\frac{1520045984887}{2^5 3^{13} 5\cdot 7}$
4	$\frac{313}{2^7 3^9 5^2}$	$-\frac{1889}{2^7 3^9}$	$\frac{115647179}{2^6 3^{13} 5^2}$	$-\frac{29321809247}{2^8 3^{12} 5^2}$	$\frac{22766570703031}{2^7 3^{15} 5}$
5	$-\frac{519961}{2^9 3^{11} 5^2 7\cdot 11}$	$\frac{196898123}{2^9 3^{12} 5^2 7\cdot 11}$	$-\frac{339157983781}{2^9 3^{14} 5^2 7\cdot 11}$	$\frac{78658947782147}{2^9 3^{16} 5\cdot 7}$	$-\frac{1057430723091383537}{2^9 3^{17} 5^2 7\cdot 11}$
6	$\frac{14609730607}{2^{12} 3^{13} 5^3 7^2 11}$	$-\frac{258703053013}{2^{10} 3^{15} 5^1 7^2 11}$	$\frac{2453678654644313}{2^{12} 3^{14} 5^3 7^2 11}$	$-\frac{4001577419396960180 3}{2^{11} 3^{18} 5^3 7^2 11}$	$\frac{5342470197951654213739}{2^{12} 3^{19} 5\cdot 7^2 11}$

where we also included the genus 0 and 1 numbers obtained earlier for completeness.

The $n=0$ numbers, corresponding to untwisted maps for $g \geq 2$ (these are not well-defined for $g=0,1$), read

$$N^{\rm orb}_{2,0} = \frac{-1}{2160} + \frac{\chi}{5760}, \quad N^{\rm orb}_{3,0} = \frac{1}{544320} - \frac{\chi}{1451520}, \quad N^{\rm orb}_{4,0} = -\frac{7}{41990400} + \frac{\chi}{87091200},$$
$$N^{\rm orb}_{5,0} = \frac{3161}{77598259200} - \frac{\chi}{2554675200}, \quad N^{\rm orb}_{6,0} = -\frac{6261257}{317764871424000} + \frac{691\chi}{31384184832000}, \ldots$$

where χ is the "Euler number" of local $\mathbb{P}^2$. The natural value of χ is 3.

Generally in Gromov-Witten theory the denominators come from dividing by the finite automorphisms of the moduli space $\mathcal{M}_{g,n}$. In the $\mathbb{Z}_3$ orbifold case there are obviously various automorphisms of order 3, corresponding to the powers of 3 in the denominators. We note that all other prime factors in the denominators do not exceed the prime factors in $\frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!}$. Automorphism groups of this order arise already for the constant map Gromov-Witten invariant.

7. Local $\mathbb{P}^1 \times \mathbb{P}^1$

Our last example is the Gromov-Witten theory of the Calabi-Yau Y which is the total space of the canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. We will study this using modularity of the B-model topological string on the mirror manifold X.

To start with, let us review elementary facts about Y. Let A_1 and A_2 denote the classes of the two $\mathbb{P}^1$'s in $H_2(Y)$. There is also one compact four cycle – the $\mathbb{P}^1 \times \mathbb{P}^1$ itself, and denote by B the corresponding class in $H_4(Y)$. The intersection numbers of the cycles on Y are

$$\#(A_1 \cap B) = -2 = \#(A_2 \cap B).$$

The class $C = A_1 - A_2$ does not have a dual cycle in $H_4(Y)$, as it does not intersect B. From our discussion in section 2, C will correspond to a *non-normalizable* modulus of the theory. For the normalizable modulus A we can take A_2, for example, so let us define

$$A = A_2, \qquad C = A_1 - A_2.$$

The mirror manifold is a family of elliptic curves Σ, which is given by the following equation [10,20] in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$x_0^2 y_0^2 + z_1 x_1^2 y_0^2 + x_0^2 y_1^2 + z_2 x_1^2 y_1^2 + x_0 x_1 y_0 y_1 = 0, \tag{7.1}$$

where $[x_0 : x_1]$ and $[y_0 : y_1]$ are homogeneous coordinates of the two $\mathbb{P}^1$'s. The large radius point corresponds to $z_1 = 0 = z_2$.

Let t_1 and t_2 denote the periods of the one form λ around the 1-cycles mirror dual to A_1 and A_2 (which we also denote by A_1 and A_2):

$$t_1 = \int_{A_1} \lambda, \qquad t_2 = \int_{A_2} \lambda,$$

and let t_D be the period around the 1-cycle mirror dual to B:

$$t_D = \int_B \lambda.$$

The periods t_1 and t_2 compute the physical Kähler parameters, i.e. the masses of BPS D2-branes wrapping the two $\mathbb{P}_1$'s.[26] At large radius the complex structure parameters z_1 and z_2 are related to the Kähler parameters t_1, t_2 of Y by

$$z_{1,2} \sim e^{2\pi i t_{1,2}}.$$

More specifically, we can find the periods t_i in terms of the parameters z_i as the solutions of the Picard-Fuchs equations of X

$$\begin{aligned} \mathcal{L}_1 &= \Theta_1^2 - 2z_1(\Theta_1 + \Theta_2)(1 + 2\Theta_1 + 2\Theta_2), \\ \mathcal{L}_2 &= \Theta_2^2 - 2z_2(\Theta_1 + \Theta_2)(1 + 2\Theta_1 + 2\Theta_2), \end{aligned} \tag{7.2}$$

[26] The $\mathbb{P}^1$'s of the embedding space of the mirror will hopefully not be confused with the two $\mathbb{P}^1$'s generating $H_2(Y)$ on the A-model side.

where $\Theta_i = z_i \frac{\partial}{\partial z_i}$ for $i = 1, 2$. The solutions around the large radius point $z_1 = 0 = z_2$ can be determined by the Frobenius method from

$$\omega(z_1, z_2, r_1, r_2) := \sum_{m,n=1}^{\infty} \frac{z_1^{r_1+m} z_2^{r_2+n}}{\Gamma(-2(m+r_1) - 2(n+r_2) + 1)\Gamma^2(m+r_1+1)\Gamma^2(n+r_2+1)}$$

as

$$t_i = \frac{1}{(2\pi i)} \frac{\mathrm{d}}{\mathrm{d}r_i} \omega(z_1, z_2, r_1, r_2)\bigg|_{r_{1,2}=0}.$$

Thus

$$t_1(z_1, z_2) = \log(z_1) + 2z_1 + 2z_2 + 3z_1^2 + 12z_1 z_2 + 3z_2^2 + \dots$$

and similarly for t_2 with z_1 and z_2 exchanged. By inverting the above, we get the mirror maps:

$$\begin{aligned} z_1 &= q_1 - 2(q_1 + q_1 q_2) + 3(q_1^3 + q_1 q_2^2) - 4(q_1^4 + q_1^3 q_2 + q_1^2 q_2^2 + q_1 q_2^3) + \dots \\ z_2 &= q_2 - 2(q_2 + q_1 q_2) + 3(q_2^3 + q_2 q_1^2) - 4(q_2^4 + q_2^3 q_1 + q_1^2 q_2^2 + q_2 q_1^3) + \dots \end{aligned} \tag{7.3}$$

where $q_i = \exp(2\pi i t_i)$ for $i = 1, 2$.

In addition to this there are two other solutions to the Picard-Fuchs equations. First, there is a double logarithmic solution, which is the period t_D introduced previously. Second, there is a constant solution, corresponding to the period mirror to the D0 brane in the A-model. This constant period, together with

$$m = t_1 - t_2 = \int_C \lambda,$$

where C is the 1-cycle of the curve mirror dual to the class C of Y (again we use the same letter to denote mirror dual objects), should be regarded as constant parameters that enter in specifying the model. In fact, it is easy to see that the period m does not receive instanton corrections, i.e. $q_m = \exp(2\pi i m)$ satisfies

$$q_m = q_1/q_2 = z_1/z_2,$$

which is consistent with the interpretation of m as an auxiliary parameter.

In the following we will denote the physical modulus by T

$$T = t_2 = \int_A \lambda,$$

and define $Q = \exp(2\pi i T)$.

In order to find the modularity properties of the amplitudes, we now study in more detail the family of elliptic curves Σ.

7.1. The Family of Elliptic Curves

The family of elliptic curves Σ in (7.1) can be brought into Weierstrass form,[27]

$$y^2 = 4x^3 - g_2 x - g_3$$

with

$$\begin{aligned} g_2 &= \frac{2^{2/3}}{3}(16z_1^2 + (1-4z_2)^2 + 8z_1(-1+28z_2)), \\ g_3 &= \frac{2}{27}(64z_1^3 + (-1+4z_2)^3 - 48z_1^2(1+44z_2) + z_1(12+480z_2-2112z_2^2). \end{aligned}$$

Its j-function reads

$$j(\tau) = \frac{(16z_1^2 + (1-4z_2)^2 + 8z_1(-1+28z_2))^3}{z_1 z_2(16z_1^2 + (1-4z_2)^2 - 8z_1(1+4z_2))^2}. \tag{7.6}$$

As usual, by $j(\tau)$ we mean that the j-function is a function of the standard complex parameter τ of the family of elliptic curves $\Sigma = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$.

As it turns out, we have met this curve before! Recall that the j-function of the $\Gamma(2)$ modular curve, the $SU(2)$ Seiberg-Witten curve, is (5.13)

$$j(\tau) = \frac{64(3+u^2)^3}{(u^2-1)^2}. \tag{7.7}$$

[27] To do so, we first use the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ given by the map

$$([x_0 : x_1], [y_0 : y_1]) \mapsto [X_0 : X_1 : X_2 : X_3] = [x_0 y_0, x_1 y_0, x_0 y_1, x_1 y_1],$$

where $[x_0 : x_1]$ and $[y_0 : y_1]$ are homogeneous coordinates of the two $\mathbb{P}^1$'s and X_i, $i = 0, \ldots, 3$ are homogeneous coordinates of $\mathbb{P}^3$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the hypersurface

$$X_0 X_3 - X_1 X_2 = 0 \tag{7.4}$$

in $\mathbb{P}^3$. The family of elliptic curves Σ is now given by the complete intersection of (7.4) and the hypersurface defined by

$$X_0^2 + z_1 X_1^2 + X_2^2 + z_2 X_3^2 + X_0 X_3 = 0. \tag{7.5}$$

After a linear change of variable, (7.5) becomes linear in X_3, so X_3 can be eliminated from (7.5) and (7.4) to get a cubic equation in $\mathbb{P}^2$. Then, given any cubic in $\mathbb{P}^2$ we can use Nagell's algorithm [9,11] to transform it into its Weierstrass form.

If we make the substitution

$$u = \frac{q_m^{-1/2}}{8z_2} - \frac{1}{2}(q_m^{1/2} + q_m^{-1/2}) \tag{7.8}$$

in (7.7), we get exactly the j-function (7.6), using the fact that $q_m = z_1/z_2$. Since the j-function captures all the coordinate-invariant data of the elliptic curve, the curves in the family mirror to local $\mathbb{P}^1 \times \mathbb{P}^1$ are in fact isomorphic to the curves in the $SU(2)$ Seiberg-Witten family, through reparameterization of the moduli space as in (7.8). In particular, it follows immediately that the curves in the family Σ have monodromy group $\Gamma(2)$.

We could also have found the monodromy transformations of the periods directly from the Picard-Fuchs equations, as we did for local $\mathbb{P}^2$, but it requires more work. The j-function approach, when the mirror geometry is a family of elliptic curves, provides a simpler way to determine the monodromy group, at least the part of it restricted to the physical periods. Fortunately, this is all that is relevant for our purposes.

Using this result, we can borrow heavily the results from the $SU(2)$ theory. In particular, using the expression for u in terms of modular forms of $\Gamma(2)$ in (5.14) and relating z_2 to the period T, we find[28]

$$Q(q_m, q) = q_m^{-1/2} q^{1/2} - (2 + 2q_m^{-1})\, q + q_m^{-3/2}(5 - 4q_m + 5q_m^2)\, q^{3/2} + \dots \tag{7.9}$$

where $q = e^{2\pi i\tau}$, $q_m = e^{2\pi m}$ and $Q = e^{2\pi iT}$. From this expansion, we see that the period T does not only depend on τ; the coefficients of the power series in q depend explicitly on the auxiliary parameter m (or q_m).

7.2. Genus 0, 1 *and Yukawa Coupling*

Let us start by finding the partition function at genus 1. Recall that $\mathcal{F}_1$ is fixed by its modular properties and its behavior at the discriminant of the family of elliptic curves Σ. In the local $\mathbb{P}^1 \times \mathbb{P}^1$ case, we can show that

$$\mathcal{F}_1 = -\log \eta(\tau) \tag{7.10}$$

[28] Note that we could invert the series because q_m is just a parameter, i.e. it must be τ-independent.

transforms as required and has precisely the good behavior at the discriminant — this is the same expression as in $SU(2)$ Seiberg-Witten theory. As a consistency check, if we expand (7.10) using the expansion of q in terms of q_m and Q we get

$$\mathcal{F}_1 = -\frac{1}{24}\log(q_m Q^2) - \frac{1}{6}(1+q_m)Q - \frac{1}{12}(1+4q_m+q_m^2)Q^2 \\ - \frac{1}{18}(1+9q_m+9q_m^2+q_m^3)Q^3 + \mathcal{O}(Q^4),$$

which reproduces precisely the genus one partition function of local $\mathbb{P}^1 \times \mathbb{P}^1$.

Now consider the Yukawa coupling, i.e. the third derivative of $\mathcal{F}_0(m,T)$ with respect to T, which we will need to compute higher genus amplitudes. Using

$$\frac{\partial^3}{\partial T^3}\mathcal{F}_0(m,T) = -\frac{1}{2}\frac{\partial}{\partial T}\tau(m,T)$$

and the expansion for τ in terms of q_m and Q we get the following expansion

$$\frac{\partial^3}{\partial T^3}\mathcal{F}_0(m,T) = -1 - 2(1+q_m)Q - 2(1+16q_m+q_m^2)Q^2 + \mathcal{O}(Q^3). \tag{7.11}$$

However, what we would like to obtain is a modular expression for $\frac{\partial^3}{\partial T^3}\mathcal{F}_0$ defined globally over the moduli space of complex structures, such as our expression (7.10) for $\mathcal{F}_1$, not just an expansion in the large complex structure limit.

To identify the modular form we make use of the change of variable (7.8), which relates the usual $\Gamma(2)$ curve to our curve with the auxiliary parameter q_m. Through this change of variable, we identify the j-functions of the two curves, and correspondingly the parameter τ, via the q-expansion of the j-function. In particular, this implies a relation between the periods

$$a = a(T,m),$$

where a is the usual Seiberg-Witten period, coming from the identification of the j-functions, which we write schematically as

$$j(a) = j(\tau) = j(T,m).$$

As a result, acting on any function of τ (at fixed m), we get

$$\frac{\partial}{\partial T} = \frac{\partial a}{\partial T}\frac{\partial}{\partial a}.$$

For instance, we can write

$$\frac{\partial^3 \mathcal{F}_0}{\partial T^3} = -\frac{1}{2}\frac{\partial \tau}{\partial T} = -\frac{1}{2}\frac{\partial a}{\partial T}\frac{\partial \tau}{\partial a}.$$

Now, we saw in section 5 that

$$\frac{\partial \tau}{\partial a} = -2\frac{\sqrt{b}}{cd}(\tau)$$

and we can compute that

$$f := \frac{\partial a}{\partial T} = -\frac{1}{2}\left(q_m^{1/2} + q_m^{-1/2} + 2\frac{d+c}{b}(\tau)\right)^{1/2} \tag{7.12}$$

using (7.11) and (7.9). In the above equations we used the modular forms b, c and d as defined in the $\Gamma(2)$ part of Appendix A. Putting all this together, we get

$$\frac{\partial^3 \mathcal{F}_0}{\partial T^3} = -\frac{1}{2}\frac{\sqrt{b}}{cd}\left(q_m^{1/2} + q_m^{-1/2} + 2\frac{d+c}{b}(\tau)\right)^{1/2}$$

which is a modular form of $\Gamma(2)$ of weight (-3), as expected. Note that f itself has weight zero.

To summarize, given the function $f = \frac{\partial a}{\partial T}$ in (7.12), which relates the a-period of the $\Gamma(2)$ curve to the T and m periods of the $\mathbb{P}^1 \times \mathbb{P}^1$ curve, we directly obtain modular expressions for the higher genus amplitudes in terms of the modular expressions already obtained for $SU(2)$ Seiberg-Witten theory.

7.3. Higher genus amplitudes

First, we can take the propagator to be

$$E(\tau) = -\frac{2\pi i}{6}E_2(\tau),$$

which is the same propagator as in $SU(2)$ Seiberg-Witten theory, up to a sign (see section 5). The sign comes from the different conventions for the relative orientation of the A and the B-cycles.

To get higher genus amplitudes, we use the by now familiar Feynman expansions with the above propagator. To relate the expansions to the $SU(2)$ Seiberg-Witten expansions, we simply use the chain rule for derivatives: whenever we need to take derivatives with respect to T in the Feynman expansions, we use the function f given in (7.12) to write

$$\frac{\partial}{\partial T} = f\frac{\partial}{\partial a}.$$

This relates the amplitudes on the local $\mathbb{P}^1 \times \mathbb{P}^1$ to those in the $SU(2)$ Seiberg-Witten theory, up to an exactly modular form. Plugging all these results in the Feynman expansion for the genus 2 partition function $\mathcal{F}_2$ we get the nice and simple expression for the modular function $h_2^{(0)}$ in terms of the partition functions $\mathcal{F}_g^{\rm SW}$, $g \leq 2$ of $SU(2)$ Seiberg-Witten theory:

$$h_2^{(0)} = \mathcal{F}_2 + \frac{1}{4}\mathcal{F}_2^{\rm SW}\left(q_m^{1/2} + q_m^{-1/2} + 2\frac{c+d}{b}\right) - \frac{1}{576}\frac{E_2^2}{cd}.$$

This is an interesting result. Through our modular formalism, we can express higher genus amplitudes of local Calabi-Yau manifolds in a very simple way in terms of higher genus amplitudes of the corresponding theory with no auxiliary parameters — in this case $SU(2)$ Seiberg-Witten theory. More precisely, given two theories governed by elliptic curves with j-functions related by a change of variables (that generically also involves the auxiliary parameters), all one needs to do is to determine the function $f = \frac{\partial a}{\partial T}$ relating the physical periods, and everything else follows from the formalism.

Finally, by plugging in the known expansion for $\mathcal{F}_2$ (obtained for instance through the topological vertex formalism) we could determine $h_2^{(0)}$, and show that it is a modular form of weight 0, as we did for local $\mathbb{P}^2$. We could also go to higher genera, and relate the expressions to the Seiberg-Witten expressions; we will not present the explicit formulae here, but it is straightforward to calculate them.

7.4. Seiberg-Witten Limit

Let us end this section by showing that the double scaling limit to recover $SU(2)$ Seiberg-Witten theory from the local $\mathbb{P}^1 \times \mathbb{P}^1$ topological string amplitude is consistent with our results above. Since we know the j-function of the mirror family of elliptic curves in terms of the complex moduli z_1 and z_2, we first express the limit in these parameters, and then show that taking the limit gives the j-function of the $SU(2)$ Seiberg-Witten curve.

The double scaling limit was explained in details in [22,24]. Define first new parameters x and y satisfying $z_1 = 1/4x^2$ and $z_2 = y/4$, and then parameters x_1 and x_2 such that

$$x_1 = (1-x), \qquad x_2 = \frac{\sqrt{y}}{1-x}.$$

The double scaling limit is given by letting $x_1 = \epsilon^2 u$ and $x_2 = 1/u$, and then sending $\epsilon \to 0$. Taking this limit in our j-function (7.6) for the elliptic curve mirror to local $\mathbb{P}^1 \times \mathbb{P}^1$, we get

$$j(\tau) = \frac{64(3+u^2)^3}{(u^2-1)^2},$$

which is indeed exactly the j-function of the $SU(2)$ Seiberg-Witten curve.

8. Open Questions and Speculations

In this paper we showed how to use symmetries to constrain the topological string amplitudes. As a result, we obtained nice expressions for the amplitudes in terms of (almost) holomorphic modular forms. However, various open questions remained, and new ideas for future research emerged.

i. Compact case. Our formalism is completely general, and applies to both compact and non-compact Calabi-Yau threefolds. However, all the examples that we worked out explicitly consisted in non-compact target spaces. As explained in section 4.1, the reason is that in the compact case the period matrix τ_{IJ} does not have positive definite imaginary part. It would be interesting to understand how to get modular expressions in this case, perhaps using the closely related matrix $\mathcal{N}_{IJ}$, as also explained in section 4.1.

ii. Full group of symmetries. In this paper, we considered the group of symmetries of the topological string generated by monodromies of the periods. However, as explained in the introduction, this is just a subgroup of the full group of symmetries, which consists in the group of ω-preserving diffeomorphisms. In the local case, the ω preserving diffeomorphisms were used in [1] to solve completely the topological string. It would be very interesting to see if this generalizes to compact Calabi-Yau manifolds.

iii. Away from the weak coupling. In this work we obtained nice modular expressions for the topological string amplitudes genus by genus. However, the main object of study was the topological string wave function $Z(g_s, x)$, which should make sense at any value of the string coupling. It would be interesting to use the symmetries to constrain the topological string amplitude for all values of the string coupling. This would correspond to solving the equations (3.1) away from the weak coupling regime. However, this may be hard, as one has to pick the correct non-perturbative definition of (3.1).

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Appendix A. Modular Forms and Quasi-Modular Forms

In this appendix we review essential facts in the theory of modular forms and quasi-modular forms, mainly in order to fix our conventions.

Denote by $\mathcal{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$ the complex upper half-plane, and let $\Gamma \subset SL(2, \mathbb{Z})$ be a subgroup of finite index.

The action of the modular group Γ on $\mathcal{H}$ is given by

$$\tau \mapsto \frac{A\tau + B}{C\tau + D}, \text{ for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

A *modular form* of weight k on Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$f(\gamma\tau) = (C\tau + D)^k f(\tau) \quad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,$$

and growing at most polynomially in $1/\mathrm{Im}(\tau)$ as $\mathrm{Im}(\tau) \to 0$.

We can also define an *almost holomorphic modular form* of weight k on Γ as a function $\hat{f} : \mathcal{H} \to \mathbb{C}$ satisfying the same transformation property and growth condition as above, but with the form

$$\hat{f}(\tau, \bar{\tau}) = \sum_{m=0}^{M} f_m(\tau) \mathrm{Im}(\tau)^{-m},$$

for some integer $M \geq 0$, where the functions $f_m(\tau)$'s are holomorphic. The constant term in the series, $f_0(\tau)$, is a *quasi-modular form* of weight k; it is holomorphic, but not quite modular. It has the form

$$f_0(\tau) = \sum_{m=0}^{M} h_m(\tau) E_2(\tau)^m,$$

where the $h_m(\tau)$'s are holomorphic modular forms and we defined the second Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{(1 - q^n)},$$

which is itself quasi-modular of weight 2. Its almost holomorphic counterpart is defined as

$$E_2^*(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}.$$

Note that there is an isomorphism between the ring of almost holomorphic modular forms and the ring of quasi-modular forms.

A.1. Modular Forms of $\Gamma(2)$

Our conventions for the theta functions with characteristics are as follows:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau) = \sum_n q^{\frac{1}{2}(n+a)^2} e^{2\pi i (n+a)(b+z)}.$$

As usual, we denote the $\Gamma(2)$ theta constants by

$$\theta_2 = \theta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (0|\tau), \qquad \theta_3 = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0|\tau), \qquad \theta_4 = \theta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (0|\tau)$$

We also define the fourth powers

$$b := \theta_2^4(\tau), \qquad c := \theta_3^4(\tau), \qquad d := \theta_4^4(\tau),$$

which satisfy the identity $c = b+d$. Also, $\eta^{12} = 2^{-4}bcd$, where η is the Dedekind η-function.

Here are some useful formulae involving derivatives of modular forms:

$$24q\frac{d}{dq}\log(\eta) = E_2,$$

$$6q\frac{d}{dq}\log(d) = E_2 - b - c,$$

$$6q\frac{d}{dq}\log(c) = E_2 + b - d,$$

$$6q\frac{d}{dq}\log(b) = E_2 + c + d.$$

A.2. Modular Forms of $\Gamma(3)$

For the congruence subgroup $\Gamma(3)$, the relevant theta constants (taking their third powers) are[29]

$$a := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (0,\tau), \quad b := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \end{bmatrix} (0,\tau), \quad c := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix} (0,\tau), \quad d := \theta^3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} (0,\tau),$$

satisfying the identities

$$b = a + c, \qquad d = a + \alpha b,$$

[29] We use the same variables to denote the fourth powers of the $\Gamma(2)$ theta constants and the third powers of the $\Gamma(3)$ theta constants, but it should always be clear from the context which subgroup we are considering.

with $\alpha = e^{\frac{2\pi i}{3}}$. Moreover, the Dedekind η-function is given by $\eta^{12} = \frac{i}{3^{3/2}} abcd$.

We need derivative formulae for these theta constants as well. Let us first define the six following modular forms of weight 2:

$$t_1 = \frac{ac}{\eta^2}, \qquad t_2 = \frac{ab}{\eta^2}, \qquad t_3 = \frac{bc}{\eta^2},$$
$$t_4 = \frac{bd}{\eta^2}, \qquad t_5 = \frac{ad}{\eta^2}, \qquad t_6 = \frac{cd}{\eta^2}.$$

Then we found the relations:

$$8q\frac{d}{dq}\log a = \frac{1}{3}E_2\left(\frac{\tau+1}{3}\right) = E_2(\tau) - \frac{2}{3}(t_4 + t_6 + \alpha t_3),$$
$$8q\frac{d}{dq}\log b = \frac{1}{3}E_2\left(\frac{\tau}{3}\right) = E_2(\tau) + \frac{2}{3}(t_1 - t_5 + t_6),$$
$$8q\frac{d}{dq}\log c = \frac{1}{3}E_2\left(\frac{\tau+2}{3}\right) = E_2(\tau) + \frac{2}{3}(t_4 + t_5 - \alpha^2 t_2),$$
$$8q\frac{d}{dq}\log d = 3E_2(3\tau) = E_2(\tau) + \frac{2}{3}(-t_1 + \alpha^2 t_2 + \alpha t_3).$$

Note that the second equality in each line are 'triple' analogs of the doubling identities for the Eisenstein series $E_2(\tau)$.

Appendix B. Siegel modular forms

A good reference on Siegel modular forms is Ghitza's elementary introduction [19] and the more complete textbook [28].

Let Γ be a subgroup of finite index of the symplectic group $Sp(2r, \mathbb{Z})$ defined by

$$Sp(2r,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2r,\mathbb{Z}) | A^TC = C^TA, B^TD = D^TB, A^TD - C^TB = I \right\},$$

where I is the $r \times r$ identity matrix. Define the *Siegel upper half space*

$$\mathcal{H}_r = \{\tau \in \mathrm{Mat}_{\mathrm{r\times r}}(\mathbb{C}) | \tau^{\mathrm{T}} = \tau, \mathrm{Im}(\tau) > 0\};$$

this is the space of $r \times r$ symmetric matrices with positive definite imaginary part. The action of Γ on $\mathcal{H}_r$ is given by

$$\tau \mapsto (A\tau + B)(C\tau + D)^{-1} \ \text{ for } \ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

A weight k *(scalar-valued) Siegel modular form* of Γ is a holomorphic function $f : \mathcal{H}_r \to \mathbb{C}$ satisfying

$$f(\gamma\tau) = \det(C\tau + D)^k f(\tau) \quad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

Note that for $r > 1$ we do not need to impose the condition of holomorphicity at infinity in the definition of a modular form, as was the case for $r = 1$.

Moreover, for $r > 1$ one can define more general objects, which transform under irreducible representations of $GL(r,\mathbb{C})$. Given such a representation $\rho : GL(r,\mathbb{C}) \to GL(V)$, where V is a finite-dimensional vector space, we say that a function transforming under ρ is a Siegel modular form of weight ρ — see for instance [19].

We can also defined 'generalized' theta functions as

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z_i, \tau) = \sum_{n \in \mathbb{Z}^r} \exp\left(\pi i \sum_{ij} (n^i + a^i)\tau_{ij}(n^j + a^j) + 2\pi i \sum_i (z_i + b_i) n^i \right) ,$$

where a, b and z are vectors of length r.

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