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Computed Tomography – the Details

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Computed Tomography – the Details

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ABSTRACT

Computed Tomography (CT) is a well established technique, particularly in medical imaging, but also applied in Synthetic Aperture Radar (SAR) imaging. Basic CT imaging via back-projection is treated in many texts, but often with insufficient detail to appreciate subtleties such as the role of non-uniform sampling densities. Herein are given some details often neglected in many texts.

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FOREWORD

While researching another topic with similarities to tomography, several issues arose that challenged our understanding of computed tomography. This necessitated a detailed study of subtleties of the filtered back-projection algorithm and how it relates to Fourier reconstruction. A specific question arose "Why do we need to filter prior to back-projection?" This led to a study of non-uniform sampling and how to properly account for variations in sampling density. These subtleties seemed to often be overlooked in books on tomography. This report records our analysis of these issues.

1 Introduction & Background

Computed Tomography (CT) gained its initial utility in the field of medical imaging. The first commercially viable CT scanner was invented by Godfrey Newbold Hounsfield at Thorn EMI Central Research Laboratories in Hayes, England using X-rays. Hounsfield first conceived his idea in 1967, and the technique was publicly announced in 1972. The first US patent was issued to Hounsfield in 1973.[1] Many books and papers have since been written on the topic.[2,3,4]

Munson, et al.,[5] showed the relationship of CT to Synthetic Aperture Radar (SAR) image formation, and a number of researchers in the area have adopted this paradigm for SAR analysis if not for a viable image formation algorithm.

CT image formation is most often implemented via a technique known as "filtered back-projection", or "convolution back-projection". This is very related to Fourier reconstruction, one being just the other-domain version of its counterpart. However there are some subtleties related to the fact that the raw data is generally not uniformly sampled in either of the two dimensions. Properly relating Fourier reconstruction to back-projection requires accounting for this. While texts often present the correct result, they typically do not advance an explanation for why this is true.

This report develops the filtered back-projection algorithm and details the effects of non-uniform sampling. While the analysis herein is most certainly not novel, it may nevertheless prove useful to those wishing to explore the details of tomographic image reconstruction.

“One of the advantages of being disorderly is that one is constantly making exciting discoveries.”

A. A. Milne (1882 - 1956)

2 Filtered Back-projection Tomography

The following development is consistent with medical imaging.

Consider a two-dimensional object $g(x, y)$ defined with Cartesian coordinates x and y.

We can define a new coordinate frame as the old one rotated by an angle θ. The new coordinates are

$$\begin{aligned} t &= x\cos\theta + y\sin\theta \\ s &= -x\sin\theta + y\cos\theta \end{aligned} \tag{1}$$

Similarly, the original coordinated can be calculated from these as

$$\begin{aligned} x &= t\cos\theta - s\sin\theta \\ y &= t\sin\theta + s\cos\theta \end{aligned} \tag{2}$$

and we define

$$g'(t,s) = g(x,y) \tag{3}$$

This is illustrated in Figure 1.

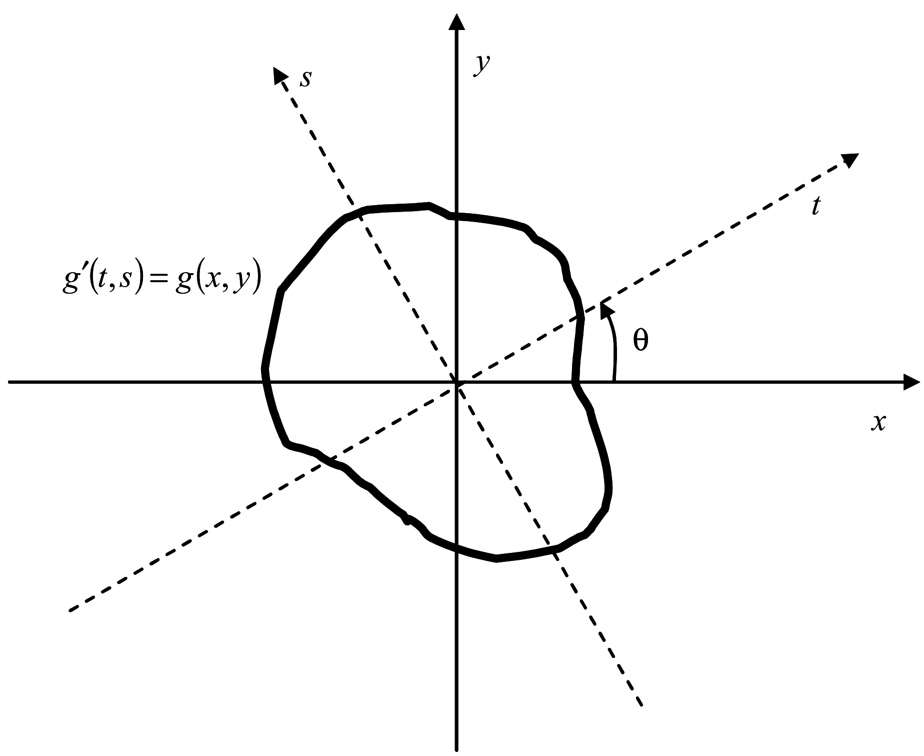

Figure 1. Coordinate geometry.

The tomographic projection at some angle θ is a measure of the transmissivity of the object. While various illumination and collection geometries exist, we will presume the simplest, that is, a geometry as indicated in Figure 2.

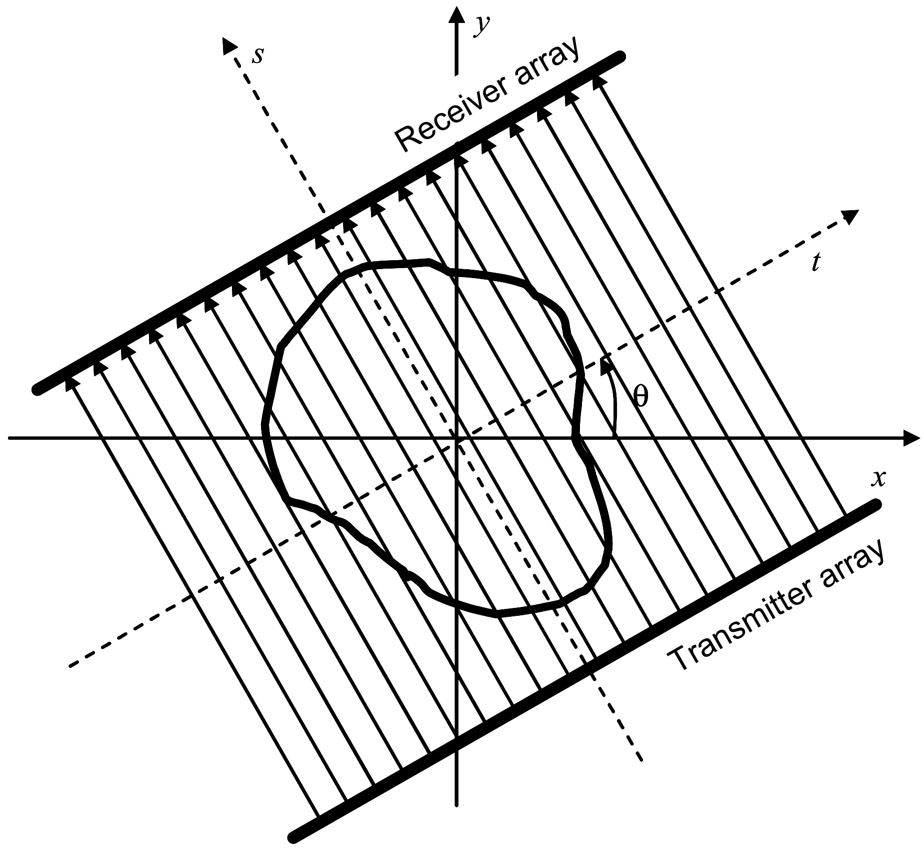

Figure 2. Data collection geometry.

The tomographic projections are then described by an integral for each position *t* along the transmission path, that it, an integration in the direction of the axis *s*.

$$p(t,\theta)=\int_{-\infty}^{\infty} g'(t,s)ds \tag{4}$$

Taking the Fourier transform over *t* yields

$$P(f,\theta)=\int_{-\infty}^{\infty} p(t,\theta)e^{-j2\pi ft}dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g'(t,s)e^{-j2\pi ft}dsdt \tag{5}$$

The characteristics of the Fourier transform for an individual tomographic projection is that we have the values of the Fourier transform along a line through the origin at an angel θ. This is illustrated in Figure 3.

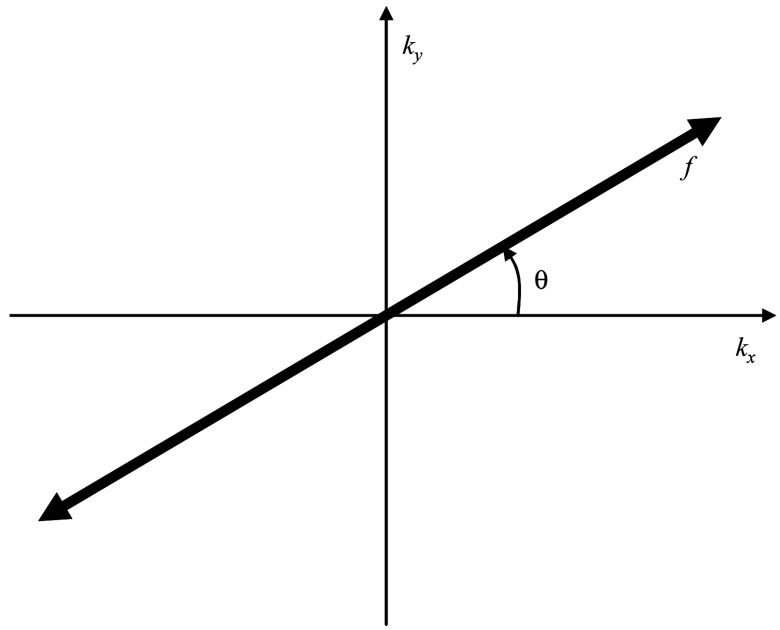

Figure 3. The Fourier space data corresponding to the data collected in Figure 2.

Collecting projections and transforming them at a set of angles for different θ_n yields a data set of Fourier transform (of the object) values along radial "spokes" in the Fourier plane, where the spokes are defined by the various angles θ_n of the projections. This is guaranteed by the "Fourier slice theorem." This is illustrated in Figure 4.

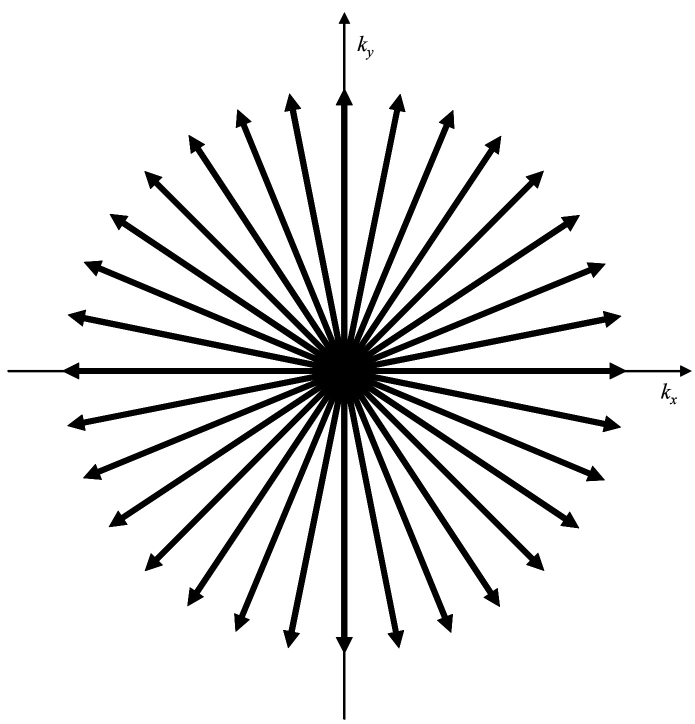

Figure 4. Fourier space of data from multiple collection angles.

Since at this point we have values of the Fourier transform of the object, we can reconstruct the object function by taking the 2-dimensional inverse Fourier transform of $P(f,\theta)$. The nature of $P(f,\theta)$ suggests a polar inverse Fourier transform.

$$g''(r,\phi) = \int_0^{2\pi}\int_0^{\infty} P(f,\theta) e^{j2\pi fr\cos(\theta-\phi)} f\, df\, d\theta \tag{6}$$

Where original Cartesian coordinates can be calculated as

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned} \tag{7}$$

But we only have data at discrete angles θ_n, so the integral over angle θ becomes a summation over a discrete set of θ_n. That is, for N angles evenly distributed and spanning the range 0 to 2π we have

$$g''(r,\phi) = \sum_{n=1}^{N}\int_0^{\infty} P(f,\theta_n) e^{j2\pi fr\cos(\theta_n-\phi)} f\, df \Delta\theta\ . \tag{8}$$

Tacit in this result is a subtlety that deals with the fact that data density in Fourier space decreases with radial distance from the origin. A discussion of the effects of data density is given in Appendix A. A discussion of the Polar Fourier Transform for data at sampled angles is given in Appendix B.

Nevertheless we also note that

$$r\cos(\theta_n - \phi) = r\cos\phi\cos\theta_n + r\sin\phi\sin\theta_n \tag{9}$$

Which using our previously defined coordinates becomes

$$r\cos(\theta_n - \phi) = x\cos\theta_n + y\sin\theta_n = t_n\ . \tag{10}$$

We have added the subscript n to t to indicate a specific oriented t at an angle θ_n

Then our reconstruction becomes

$$g''(r,\phi) = \sum_{n=1}^{N}\int_0^{\infty} P(f,\theta_n) e^{j2\pi ft_n} f\, df \Delta\theta \tag{11}$$

Or rearranging a bit

$$g''(r,\phi) = \sum_{n=1}^{N} \left[\int_{-\infty}^{\infty} P(f,\theta_n) f\, u(f) e^{j2\pi f t_n} df \right] \Delta\theta \tag{12}$$

Where $u(f)$ is the unit step function.

The entity in the square brackets is the inverse Fourier transform of $P(f,\theta_n) f\, u(f)$, which is simply a filtered version of $p(t_n,\theta_n)$. The Filter has transfer function

$$H(f) = f\, u(f) \tag{13}$$

allowing us to write

$$g(x,y) = \sum_{n=1}^{N} \left[\int_{-\infty}^{\infty} P(f,\theta_n) H(f) e^{j2\pi f t_n} df \right] \Delta\theta \ . \tag{14}$$

Consequently the algorithm becomes to collect a set of projections $p(t_n,\theta_n)$ and filter them with $H(f)$. In the raw data domain this is accomplished via convolution. Then the filtered projections are back-projected, and the resulting set of back-projections appropriately summed. This forms the image.

“All truths are easy to understand once they are discovered; the point is to discover them.”

Galileo Galilei (1564 - 1642)

3 Summary and Conclusions

The ‘filtering’ part of the Filtered-Back-projection algorithm is what makes it equal to an inverse polar Fourier transform of the Fourier data. This is due to the non-uniform sampling density that results from equal-angle sampling, where 2-dimensional sampling density decreases with radial distance from the spectral origin.

"The most exciting phrase to hear in science, the one that heralds new discoveries, is not 'Eureka!' (I found it!) but 'That's funny ...' "

Isaac Asimov (1920 - 1992)

Appendix A - Nonuniform Sampling

In calculating the Fourier Transform of a signal with variable-rate sampling, effectively the samples need to be weighted by the inverse of their local sampling density.

Two sampling rates

The purpose here is to calculate the spectrum of a function from sampled data where different parts of a function are sampled at one of two separate sampling rates.

Consider a function $g(x)$. We identify its Fourier transform as $G(f)$ where

$$G(f) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx \tag{A1}$$

and consequently identify the Fourier transform pair

$$g(x) \Leftrightarrow G(f). \tag{A2}$$

We now identify a masking function $h_1(x)$ which has unit amplitude for some range of x, and is zero everywhere else. Furthermore, we define its complement as

$$h_2(x) = 1 - h_1(x) \tag{A3}$$

and stipulate that $h_1(x)$ and $h_2(x)$ are non-overlapping, that is, $h_1(x)h_2(x) = 0$ for any and all x.

Their Fourier transforms are identified as

$$\begin{aligned} h_1(x) &\Leftrightarrow H_1(f) \\ h_2(x) &\Leftrightarrow H_2(f) = \delta(f) - H_1(f). \end{aligned} \tag{A4}$$

We now identify two parsed signals

$$\begin{aligned} w_1(x) &= h_1(x)g(x) \\ w_2(x) &= h_2(x)g(x). \end{aligned} \tag{A5}$$

We shall assume that both of these signals are effectively band-limited. They are not strictly band-limited, but for all intents and purposes may be considered as such.

Each of these is sampled at different rates, f_{s1} and f_{s2} respectively. We shall furthermore assume that all sampling rates are adequate to prevent significant aliasing. The sampled signals are then

$$w_{s1}(x) = w_1(x)\sum_{n_1}\delta\left(x - \frac{n_1}{f_{s1}}\right)$$
$$w_{s2}(x) = w_2(x)\sum_{n_2}\delta\left(x - \frac{n_2}{f_{s2}}\right). \tag{A6}$$

We identify the Fourier transform pair for the sampling function as

$$\sum_n \delta\left(x - \frac{n}{f_s}\right) \Leftrightarrow f_s \sum_u \delta(f - f_s u). \tag{A7}$$

This allows us to identify the Fourier transform of our sampled signals as

$$W_{s1}(f) = W_1(f) * f_{s1}\sum_{u_1}\delta(f - f_{s1}u_1) = G(f) * H_1(f) * f_{s1}\sum_{u_1}\delta(f - f_{s1}u_1)$$
$$W_{s2}(f) = W_2(f) * f_{s2}\sum_{u_2}\delta(f - f_{s2}u_2) = G(f) * H_2(f) * f_{s2}\sum_{u_2}\delta(f - f_{s2}u_2). \tag{A8}$$

Each of these signals exhibits a replicated spectrum, with copies centered at multiples of their respective sampling frequencies. However, we are interested only in the baseband components, that is, we select via filtering during reconstruction the components

$$W'_{s1}(f) = G(f) * H_1(f) * f_{s1}\delta(f)$$
$$W'_{s2}(f) = G(f) * H_2(f) * f_{s2}\delta(f). \tag{A9}$$

These then become

$$W'_{s1}(f) = G(f) * f_{s1}H_1(f)$$
$$W'_{s2}(f) = G(f) * f_{s2}H_2(f). \tag{A10}$$

Our goal is to recover $G(f)$ from $W'_{s1}(f)$ and $W'_{s2}(f)$ via a linear combination of these component spectral pieces. Consequently we equate

$$G(f) = k_1 W'_{s1}(f) + k_2 W'_{s2}(f). \tag{A11}$$

This can be expanded to

$$G(f) = G(f) * \left(k_1 f_{s1} H_1(f) + k_2 f_{s2} H_2(f)\right) \tag{A12}$$

or

$$G(f) = G(f) * \left(k_2 f_{s2}\delta(f) + (k_1 f_{s1} - k_2 f_{s2}) H_1(f)\right) \tag{A13}$$

or furthermore to

$$G(f) = k_2 f_{s2} G(f) + (k_1 f_{s1} - k_2 f_{s2}) G(f) * H_1(f). \tag{A14}$$

It becomes readily apparent that to accomplish the equality for arbitrary $H_1(f)$, and arbitrary sampling frequencies f_{s1} and f_{s2}, we need to choose scaling factors

$$\begin{aligned} k_1 &= 1/f_{s1} \\ k_2 &= 1/f_{s2} \,. \end{aligned} \tag{A15}$$

In summary, to properly compute the spectrum from the sampled data, the spectral contribution from the two sections of data needs to be scaled inversely proportional to the sampling rate used for those two sections of data.

This can be accomplished by weighting each sample inversely proportional to the sampling rate at which it was collected. We need not wait to scale the Fourier transforms themselves.

<u>More than two sampling rates</u>

The purpose here is to extend the results of the previous section and calculate the spectrum of a function from sampled data where multiple different sections of a function are sampled at different sampling rates.

We now identify *N* masking functions $h_n(x)$ each of which have unit amplitude for some range of *x*, and is zero everywhere else. Furthermore, we again assume that the $h_n(x)$ are non-overlapping, and that one and only one $h_n(x)$ is non-zero for any value *x*. That is

$$\sum_{n=1}^{N} h_n(x) = 1 \,. \tag{A16}$$

Their Fourier transforms are identified as

$$h_n(x) \Leftrightarrow H_n(f) \tag{A17}$$

and

$$\sum_{n=1}^{N} H_n(f) = \delta(f) \,. \tag{A18}$$

We identify the parsed signals

$$w_n(x) = h_n(x) g(x) \,. \tag{A19}$$

We shall assume that all of these signals are effectively band-limited. Again, they are not strictly band-limited, but for all intents and purposes may be considered as such.

Each of these is sampled at different rates, f_{sn} respectively. We shall furthermore assume that all sampling rates are adequate to prevent significant aliasing. The sampled signals are then

$$w_{sn}(x) = w_n(x) \sum_{m_n} \delta\left(x - \frac{m_n}{f_{sn}} \right). \tag{A20}$$

This allows us to identify the Fourier transform of our sampled signals as

$$W_{sn}(f) = G(f) * H_n(f) * f_{sn} \sum_{u_n} \delta(f - f_{sn} u_n). \tag{A21}$$

Each of these signals exhibits a replicated spectrum, with copies centered at multiples of their respective sampling frequencies. However, we are interested only in the baseband components, that is, we select via filtering during reconstruction the components

$$W'_{sn}(f) = G(f) * H_n(f) * f_{sn} \delta(f) = G(f) * f_{sn} H_n(f). \tag{A22}$$

Our goal is to recover $G(f)$ from the $W'_{sn}(f)$ via a linear combination of these component spectral pieces. Consequently we equate

$$G(f) = \sum_{n=1}^{N} k_n W'_{sn}(f). \tag{A23}$$

This can be expanded to

$$G(f) = G(f) * \sum_{n=1}^{N} k_n f_{sn} H_n(f) = G(f) * \left(\sum_{n=1}^{N-1} k_n f_{sn} H_n(f) + k_N f_{sN} \left(\delta(f) - \sum_{n=1}^{N-1} H_n(f) \right) \right) \tag{A24}$$

or

$$G(f) = G(f) * \left(k_N f_{sN} \delta(f) + \sum_{n=1}^{N-1} (k_n f_{sn} - k_N f_{sN}) H_n(f) \right) \tag{A25}$$

or furthermore to

$$G(f) = k_N f_{sN} G(f) + \sum_{n=1}^{N-1} (k_n f_{sn} - k_N f_{sN}) G(f) * H_n(f). \tag{A26}$$

It becomes readily apparent that to accomplish the equality for arbitrary $H_n(f)$, and arbitrary sampling frequencies f_{sn}, we need to choose scaling factors

$$k_n = 1/f_{sn} \ . \tag{A27}$$

In summary, to properly compute the spectrum from the sampled data, the spectral contribution from the various sections of data need to be scaled inversely proportional to the sampling rate used for those sections of data.

Again, this can be accomplished by weighting each sample inversely proportional to the sampling rate at which it was collected.

<u>Sampling in two dimensions</u>

The purpose here is to extend the results of the first section to two dimensions.

Consider a function $g(x,y)$. We identify its Fourier transform as $G(f_x, f_y)$ where

$$G(f_x, f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) e^{-j2\pi(f_x x + f_y y)} dxdy \tag{A28}$$

and consequently identify the Fourier transform pair

$$g(x,y) \Leftrightarrow G(f_x, f_y). \tag{A29}$$

We now identify a masking function $h_1(x,y)$ which has unit amplitude for some range of x, and some range of y, and is zero everywhere else. Furthermore, we define its non-overlapping complement as

$$h_2(x,y) = 1 - h_1(x,y). \tag{A30}$$

Their Fourier transforms are identified as

$$\begin{aligned} h_1(x,y) &\Leftrightarrow H_1(f_x, f_y) \\ h_2(x,y) &\Leftrightarrow H_2(f_x, f_y) = \delta(f_x)\delta(f_y) - H_1(f_x, f_y). \end{aligned} \tag{A31}$$

We now identify two parsed signals

$$\begin{aligned} w_1(x,y) &= h_1(x,y) g(x,y) \\ w_2(x,y) &= h_2(x,y) g(x,y). \end{aligned} \tag{A32}$$

We shall assume that both of these signals are effectively band-limited in all dimensions. We shall furthermore assume that all sampling rates are adequate to prevent significant aliasing in any dimension. The sampled signals are then

$$w_{s1}(x,y) = w_1(x,y)\sum_{n_{y1}}\sum_{n_{x1}}\delta\left(x-\frac{n_{x1}}{f_{sx1}}\right)\delta\left(y-\frac{n_{y1}}{f_{sy1}}\right)$$
$$w_{s2}(x,y) = w_2(x,y)\sum_{n_{y2}}\sum_{n_{x2}}\delta\left(x-\frac{n_{x2}}{f_{sx2}}\right)\delta\left(y-\frac{n_{y2}}{f_{sy2}}\right). \quad \text{(A33)}$$

There is no correlation between, f_{sx1}, f_{sy1}, f_{sx2} and f_{sy2}.

We identify the Fourier transform pair for the 2-dimensional sampling function as

$$\sum_{n_x}\sum_{n_y}\delta\left(x-\frac{n_x}{f_{sx}}\right)\delta\left(y-\frac{n_y}{f_{sy}}\right) \Leftrightarrow f_{sx}f_{sy}\sum_{v}\sum_{u}\delta(f_x - f_{sx}u)\delta(f_y - f_{sy}v). \quad \text{(A34)}$$

This allows us to identify the Fourier transform of our sampled signals as

$$W_{s1}(f_x,f_y) = G(f_x,f_y) * H_1(f_x,f_y) * f_{sx1}f_{sy1}\sum_{v_1}\sum_{u_1}\delta(f_{x1} - f_{sx1}u_1)\delta(f_{y1} - f_{sy1}v_1)$$
$$W_{s2}(f_x,f_y) = G(f_x,f_y) * H_2(f_x,f_y) * f_{sx2}f_{sy2}\sum_{v_2}\sum_{u_2}\delta(f_{x2} - f_{sx2}u_2)\delta(f_{y2} - f_{sy2}v_2).$$
$$\text{(A35)}$$

Each of these signals exhibits a replicated spectrum, with copies centered at multiples of their respective sampling frequencies in the two dimensions. However, we are interested only in the baseband components, that is, we select via filtering during reconstruction the baseband components

$$W'_{s1}(f_x,f_y) = G(f_x,f_y) * H_1(f_x,f_y) * f_{sx1}f_{sy1}\delta(f_{x1})\delta(f_{y1})$$
$$W'_{s2}(f_x,f_y) = G(f_x,f_y) * H_2(f_x,f_y) * f_{sx2}f_{sy2}\delta(f_{x2})\delta(f_{y2}). \quad \text{(A36)}$$

These then become

$$W'_{s1}(f_x,f_y) = G(f_x,f_y) * f_{sx1}f_{sy1}H_1(f_x,f_y)$$
$$W'_{s2}(f_x,f_y) = G(f_x,f_y) * f_{sx2}f_{sy2}H_2(f_x,f_y). \quad \text{(A37)}$$

Our goal is to recover $G(f_x,f_y)$ from $W'_{s1}(f_x,f_y)$ and $W'_{s2}(f_x,f_y)$ via a linear combination of these component spectral pieces. Consequently we equate

$$G(f_x,f_y) = k_1W'_{s1}(f_x,f_y) + k_2W'_{s2}(f_x,f_y). \quad \text{(A38)}$$

This can be expanded to

$$G(f_x,f_y) = G(f_x,f_y) * \left(k_1 f_{sx1}f_{sy1}H_1(f_x,f_y) + k_2 f_{sx2}f_{sy2}H_2(f_x,f_y)\right) \quad \text{(A39)}$$

which can be manipulated to

$$G(f_x, f_y) = k_2 f_{sx2} f_{sy2} G(f_x, f_y) + (k_1 f_{sx1} f_{sy1} - k_2 f_{sx2} f_{sy2}) G(f_x, f_y) * H_1(f_x, f_y). \tag{A40}$$

It becomes readily apparent that to accomplish the equality for arbitrary $H_1(f_x, f_y)$, and arbitrary sampling frequencies in both dimensions we need to choose scaling factors

$$\begin{aligned} k_1 &= 1/(f_{sx1} f_{sy1}) \\ k_2 &= 1/(f_{sx2} f_{sy2}). \end{aligned} \tag{A41}$$

In summary, to properly compute the spectrum from the sampled data, the spectral contribution from the two sections of data needs to be scaled inversely proportional to the 2-dimensional sampling density used for those two sections of data.

This can be accomplished by weighting each sample inversely proportional to the local sampling density at which it was collected. We need not wait to scale the Fourier transforms themselves.

<u>Extensions</u>

In calculating the multi-dimensional Fourier transform from sampled data, the sample weighting by the inverse of the local sampling density is expected to hold for multiple sampling rates in multiple dimensions.

"There is much pleasure to be gained from useless knowledge."

Bertrand Russell (1872 - 1970)

Appendix B - Fourier Transform of function with samples at equally spaced angular spokes

If a function is sampled at evenly spaced angles, then the samples need to be weighted (scaled) by the radial distance from the origin in order to estimate the Fourier transform of the original function.

Rectangular coordinates

Consider a 2-dimensional function $g(x,y)$ with Fourier transform given by

$$G(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(ux+vy)}dxdy\,. \tag{B1}$$

Thus, $g(x,y)$ and $G(u,v)$ constitute a Fourier transform pair, which we illustrate by writing

$$g(x,y)\Leftrightarrow G(u,v)\,. \tag{B2}$$

We wish to identify the function along some line passing through the origin. We do this by forming the angular sample

$$g_s(x,y,\theta_n)=g(x,y)\delta(y\cos\theta_n - x\sin\theta_n) \tag{B3}$$

If we allow a composite angular sampled function to contain many such individual lines, all passing through the origin, and equally spaced in angle, then we can define our composite sampled function as

$$g_s(x,y)=\sum_n g_s(x,y,\theta_n)=\sum_n g(x,y)\delta(y\cos\theta_n - x\sin\theta_n)\,. \tag{B4}$$

Since we are dealing with lines, we can completely sweep the angular space by restricting θ_n to the range 0 to π.

Polar coordinates

We define the corresponding polar-coordinate function and its transform as

$$g'(r,\theta)\Leftrightarrow G'(s,\phi) \tag{B5}$$

where

$$G'(s,\phi) = \int_0^{2\pi}\int_0^{\infty} g'(r,\theta) e^{-j2\pi sr\cos(\phi-\theta)} r\,dr d\theta . \tag{B6}$$

The angular sampled function becomes

$$g_s'(r,\theta) = \sum_n g'(r,\theta)\left(\delta\left(r(\theta-\theta_n)\right) + \delta\left(r(\theta-\theta_n-\pi)\right)\right) \tag{B7}$$

noting that we are limiting $r \geq 0$. However, we still have restricted θ_n to the range 0 to π. If we allow θ_n to range all the way to 2π, then we can write

$$g_s'(r,\theta) = \sum_n g'(r,\theta)\delta\left(r(\theta-\theta_n)\right) \tag{B8}$$

where θ_n is limited to values corresponding to equal increments of $\Delta\theta$ and

$$\Delta\theta = \frac{2\pi}{N} \tag{B9}$$

where N is the total number of radial 'spokes' to the sampled function. Consequently

$$g_s'(r,\theta) = \sum_{n=1}^{N} g'(r,\theta)\delta\left(r(\theta-\theta_n)\right). \tag{B10}$$

We make use of the identity

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{B11}$$

and note that allows us to equate

$$g_s'(r,\theta) = \sum_{n=1}^{N} g'(r,\theta)\left(\frac{1}{r}\right)\delta(\theta-\theta_n). \tag{B12}$$

With malice of forethought, we define but leave unspecified two scaling parameters, k_r and k_θ, such that a new function is formed

$$g_s''(r,\theta) = k_r k_\theta g_s'(r,\theta) = \sum_{n=1}^{N} k_r k_\theta g'(r,\theta)\left(\frac{1}{r}\right)\delta(\theta-\theta_n). \tag{B13}$$

Now we calculate the Fourier transform of $g_s''(r,\theta)$ as

$$G''(s,\phi)=\int_0^{2\pi}\int_0^{\infty} g_s''(r,\theta)e^{-j2\pi sr\cos(\phi-\theta)} r\,drd\theta \tag{B14}$$

which becomes in turn

$$G''(s,\phi)=\int_0^{2\pi}\int_0^{\infty}\left(\sum_{n=1}^{N} k_r k_\theta g'(r,\theta)\left(\frac{1}{r}\right)\delta(\theta-\theta_n)\right)e^{-j2\pi sr\cos(\phi-\theta)} r\,drd\theta \tag{B15}$$

$$G''(s,\phi)=\sum_{n=1}^{N}\int_0^{\infty}\int_0^{2\pi} k_r k_\theta g'(r,\theta)e^{-j2\pi sr\cos(\phi-\theta)}\delta(\theta-\theta_n)\,d\theta dr \tag{B16}$$

$$G''(s,\phi)=\sum_{n=1}^{N}\int_0^{\infty} k_r k_\theta g'(r,\theta_n)e^{-j2\pi sr\cos(\phi-\theta_n)}\,dr \tag{B17}$$

We are driving towards making this look like an integration over θ so we can equate it to $G'(s,\phi)$. As is, unless compensated, as *N* increases the summation takes in more and more angular samples and consequently gets bigger. Prudence dictates that a scale factor that compensates for increasing *N* would allow meaningful comparison of $G''(s,\phi)$. Consequently, at this point we choose the scale factor

$$k_\theta=\Delta\theta=\frac{2\pi}{N}. \tag{B18}$$

Note that k_θ is a constant for a constant *N*. This allows

$$G''(s,\phi)=\sum_{n=1}^{N}\int_0^{\infty} k_r g'(r,\theta_n)e^{-j2\pi sr\cos(\phi-\theta_n)}\,dr\Delta\theta\,. \tag{B19}$$

Now as *N* increases, $\Delta\theta$ gets smaller. In the limit

$$\lim_{\Delta\theta\to 0} G''(s,\phi)=\int_0^{2\pi}\int_0^{\infty} g'(r,\theta)e^{-j2\pi sr\cos(\phi-\theta)}\,k_r drd\theta \tag{B20}$$

Since we desire this to be the 2-dimensional spectrum of our original function, we equate

$$\lim_{\Delta\theta\to 0} G''(s,\phi)=G'(s,\phi) \tag{B21}$$

which forces

$$\int_0^{2\pi}\int_0^{\infty} g'(r,\theta)e^{-j2\pi sr\cos(\phi-\theta)}\,k_r dr d\theta = \int_0^{2\pi}\int_0^{\infty} g'(r,\theta)e^{-j2\pi sr\cos(\phi-\theta)} r\, dr d\theta \qquad \text{(B22)}$$

This can only be true if we equate

$$k_r = r\,. \qquad \text{(B23)}$$

Bottom line:

Consequently, if we have angular sampled functions $g_s'(r,\theta) = \sum_n g'(r,\theta)\delta(r(\theta-\theta_n))$, then the Fourier transform of the un-sampled function $g'(r,\theta)$ is estimated by actually calculating the Fourier transform of $g_s''(r,\theta) = \frac{2\pi}{N} r\, g_s'(r,\theta)$.

Furthermore, weighting the sampled data as a function of radial distance from the origin would be required in any coordinate frame.

Note:

The weighting by *r* goes away (is not needed) if the data were resampled to a uniform density, e.g. on a rectangular grid with uniform sample spacing in each dimension.

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