STRESS ANALYSIS OF LAMINATED COMPOSITE CYLINDERS UNDER NON-AXISYMMETRIC LOADING

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ABSTRACT

The use of thick-walled composite cylinders in structural applications has seen tremendous growth over the last decade. Applications include pressure vessels, flywheels, drive shafts, spoolable tubing, and production risers. In these applications, the geometry of a composite cylinder is axisymmetric but in many cases the applied loads are non-axisymmetric and more rigorous analytical tools are required for an accurate stress analysis. A closed-form solution is presented for determining the layer-by-layer stresses, strains, and displacements and first-ply failure in laminated composite cylinders subjected to non-axisymmetric loads. The applied loads include internal and external pressure, axial force, torque, axial bending moment, uniform temperature change, rotational velocity, and interference fits. The formulation is based on the theory of anisotropic elasticity and a state of generalized plane deformation along the axis of the composite cylinder. Parametric design trade studies can be easily and quickly computed using this closed-form solution. A computer program that was developed for performing the numerical calculations is described and results from specific case studies are presented.

KEY WORDS: anisotropic elasticity, composite materials, stress analysis

1. INTRODUCTION

In the design and analysis of laminated composite cylinders, axisymmetric loads and axisymmetric geometries are often assumed for developing closed-form analytic solutions. In addition, the cylinder is assumed to have an infinite length such that the stresses are not only independent of the circumferential coordinate but also independent of the axial coordinate. Solutions have been formulated based on both the theory of anisotropic elasticity (1,2) and the laminated shell theory (3,4). The laminated shell theory provides an accurate solution for thin-walled cylinders, whereas elasticity solutions are required for an accurate determination of the three-dimensional stress states that exist in thick-walled cylinders. In both of these analytical approaches, further simplifications are obtained by restricting the composite cylinder to be orthotropic.

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There are a limited number of closed-form solutions for the case of axisymmetric cylinder geometries with non-axisymmetric loads. Kollar and Springer (5) considered a laminated cylinder, or cylindrical segment, subjected to hygrothermal and mechanical loads that varied in the radial and circumferential, but not in the axial direction. The theory of elasticity was used to derive the solution for stresses, strains, and displacements in the cylinder without any restrictions on ply angle and lamination sequence. The length of the cylinder was assumed to be large compared to the wall thickness and inner and outer radii such that end effects could be neglected. The only restriction on the applied mechanical loads was that they had to be in equilibrium. Pagano (6) presented a general solution for a cylindrically anisotropic cylinder subjected to surface tractions that could be expressed by a Fourier series. The surface tractions had to be independent of the axial coordinate and consistent with overall equilibrium of the cylinder. On the end faces of the cylinder, the surface tractions were prescribed as statically equivalent force and moment resultants.

2. ANALYTICAL FORMULATION

2.1 Single Layer Solution The work of Pagano (6), as suggested by the author, was developed in a form that could be extended to analyze a laminated composite cylinder having anisotropic layers. This single layer solution is the foundation for the current work and the underlying assumptions and basic equations are briefly described in this section. A circular cylinder having an inner radius, r_1, and an outer radius, r_2, is considered where the stress field is independent of the axial coordinate. Traction boundary conditions are applied on the surfaces $r = r_1, r_2$, and on the end planes, independent of the axial coordinate, x, and expressed in the form of a Fourier series. The constitutive equations for a material having a single plane of symmetry (xθ) with respect to a cylindrical coordinate system (x, θ, r) are written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{\theta r} \\ \tau_{xr} \\ \tau_{x\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_\theta - \alpha_\theta \Delta T \\ \varepsilon_r - \alpha_r \Delta T \\ \gamma_{\theta r} \\ \gamma_{xr} \\ \gamma_{x\theta} - \alpha_{x\theta} \Delta T \end{Bmatrix} \tag{1}$$

The equilibrium equations in cylindrical coordinates are written as:

$$\begin{aligned} &\sigma_{r,r} + \frac{1}{r}\left(\tau_{\theta r,r} + \sigma_r - \sigma_\theta\right) + F_r = 0 \\ &\tau_{xr,r} + \frac{1}{r}\left(\tau_{x\theta,\theta} + \tau_{xr}\right) = 0 \\ &\tau_{\theta r,r} + \frac{1}{r}\left(\sigma_{\theta,\theta} + 2\tau_{\theta r}\right) = 0 \end{aligned} \tag{2}$$

where the components of stress are functions of the θ and r coordinates and the comma denotes differentiation. The F_r body force term is included and for rotational velocity is written as $\rho\omega^2 r$. The components of displacement are u, v, and w in the radial, circumferential, and axial directions, respectively, and are functions of x, θ, and r. The strain-displacement relationships are written as:

$$
\begin{aligned}
\varepsilon_r &= u_{,r} \\
\varepsilon_\theta &= \frac{1}{r}\left(u + v_{,\theta}\right) \\
\varepsilon_x &= w_{,x} \\
\gamma_{\theta r} &= v_{,r} + \frac{1}{r}\left(u_{,\theta} - v\right) \\
\gamma_{xr} &= w_{,r} + u_{,x} \\
\gamma_{x\theta} &= v_{,x} + \frac{1}{r} w_{,\theta}
\end{aligned}
\tag{3}
$$

Now by eliminating the stresses and strains from Eqn. (1)-(3) a general solution for the displacements that satisfies the compatibility equations can be written as:

$$
\begin{aligned}
u &= U(r,\theta) + \frac{x^2}{2}\left(b_1 \cos\theta - b_2 \sin\theta\right) + x\left(b_5 \cos\theta - b_6 \sin\theta\right) \\
v &= V(r,\theta) - \frac{x^2}{2}\left(b_1 \sin\theta + b_2 \cos\theta\right) - x\left(b_5 \sin\theta + b_6 \cos\theta\right) + b_3 r x \\
w &= W(r,\theta) - r x\left(b_1 \cos\theta - b_2 \sin\theta\right) + b_4 x
\end{aligned}
\tag{4}
$$

where b_i are arbitrary constants that depend on the boundary conditions. The governing equations for U, V, and W are found by substituting Eqns. (1), (3), and (4) into Eqn. (2).

$$
\begin{aligned}
& C_{33}\left(U_{,rr} + \frac{1}{r}U_{,r}\right) + \frac{C_{44}}{r^2}U_{,\theta\theta} - \frac{C_{22}}{r^2}U + \left(\frac{C_{23} + C_{44}}{r}\right)V_{,\theta r} - \left(\frac{C_{22} + C_{44}}{r^2}\right)V_{,\theta} \\
& + \left(\frac{C_{36} + C_{45}}{r}\right)W_{,\theta r} - \frac{C_{26}}{r^2}W_{,\theta} = P_1(r,\theta) + T_1(r) \\
& \left(\frac{C_{45} + C_{36}}{r}\right)U_{,\theta r} + \frac{C_{26}}{r^2}U_{,\theta} + C_{45}V_{,rr} + \frac{C_{26}}{r^2}V_{,\theta\theta} + C_{55}\left(W_{,rr} + \frac{W_{,r}}{r}\right) + \frac{C_{66}}{r^2}W_{,\theta\theta} = P_2(r,\theta) \\
& \left(\frac{C_{23} + C_{44}}{r}\right)U_{,r\theta} + \left(\frac{C_{22} + C_{44}}{r^2}\right)U_{,\theta} + C_{44}\left(V_{,rr} + \frac{V_{,r}}{r} - \frac{V}{r^2}\right) + \frac{C_{22}}{r^2}V_{,\theta\theta} \\
& + C_{45}\left(W_{,rr} + \frac{2}{r}W_{,r}\right) + \frac{C_{26}}{r^2}W_{,\theta\theta} = P_3(r,\theta)
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
P_1(r,\theta) &= \left[\left(2C_{13} - C_{12}\right)b_1 + \left(\frac{C_{45} - C_{26} + C_{36}}{r}\right)b_6\right]\cos\theta + \\
&\left[\left(C_{12} - 2C_{13}\right)b_2 + \left(\frac{C_{45} - C_{26} + C_{36}}{r}\right)b_5\right]\sin\theta + \left(C_{26} - 2C_{36}\right)b_3 + \left(\frac{C_{12} - C_{13}}{r}\right)b_4 - \rho\omega^2 r \\
P_2(r,\theta) &= \left[\left(\frac{C_{66} - C_{55}}{r}\right)b_5 - C_{16}b_2\right]\cos\theta + \left[\left(\frac{C_{55} - C_{66}}{r}\right)b_6 - C_{16}b_1\right]\sin\theta \\
P_3(r,\theta) &= \left[\left(\frac{C_{26} - 2C_{45}}{r}\right)b_5 - C_{12}b_2\right]\cos\theta + \left[\left(\frac{2C_{45} - C_{26}}{r}\right)b_6 - C_{12}b_1\right]\sin\theta
\end{aligned}
\tag{6a}
$$

and

$$T_1(r) = \frac{\Delta T}{r}\left[(C_{13} - C_{12})\alpha_x + (C_{23} - C_{22})\alpha_\theta + (C_{33} - C_{23})\alpha_r + (C_{36} - C_{26})\alpha_{x\theta}\right] \tag{6b}$$

A solution to Eqn. (5) is sought subject to a set of boundary conditions that are expressed in terms of a Fourier series. Due to the rotational symmetry, the boundary conditions at the inner and outer radii are expressed in the following form:

$$\begin{aligned} \sigma_r(r_i,\theta) &= \sum_{n=0}^{\infty} p_{in}\cos n\theta \\ \tau_{\theta r}(r_i,\theta) &= q_{i0} + \sum_{n=1}^{\infty} q_{in}\sin n\theta \qquad (i = 1,2) \\ \tau_{xr}(r_i,\theta) &= t_{i0} + \sum_{n=1}^{\infty} t_{in}\sin n\theta \end{aligned} \tag{7}$$

In Eqn. (7), the constants p_{in}, q_{in}, and t_{in} for n=0,1 are not all independent as a result of global equilibrium for the cylinder. Direct integration of the equilibrium equations results in the following relationships between the constants in Eqn. (7).

$$\begin{aligned} &r_2 t_{20} = r_1 t_{10} \\ &r_2^2 q_{20} = r_1^2 q_{10} \\ &r_2\left[\sigma_r^*(r_2) - \tau_{r\theta}^*(r_2)\right] = r_1\left[\sigma_r^*(r_1) - \tau_{r\theta}^*(r_1)\right] \end{aligned} \tag{8}$$

where σ^* are the applied stress components corresponding to n = 1.

The remaining set of boundary conditions consists of the resultant axial force (F_x), torque (T), and moment (M) acting on any cross section of the cylinder.

$$\begin{aligned} F_x &= \int_0^{2\pi}\int_{r_1}^{r_2} \sigma_x r\,\partial r\,\partial\theta \\ T &= \int_0^{2\pi}\int_{r_1}^{r_2} \tau_{x\theta} r^2\,\partial r\,\partial\theta \\ M &= \int_0^{2\pi}\int_{r_1}^{r_2} \sigma_x r^2\cos\theta\,\partial r\,\partial\theta \end{aligned} \tag{9}$$

A general solution for U, V, and W is given by:

$$\begin{aligned} U(r,\theta) &= \sum_{n=0}^{\infty} U_n(r)\cos n\theta + \phi_1(r,\theta) \\ V(r,\theta) &= V_0(r) + \sum_{n=1}^{\infty} V_n(r)\sin n\theta + \phi_2(r,\theta) \\ W(r,\theta) &= W_0(r) + \sum_{n=1}^{\infty} W_n(r)\sin n\theta + \phi_3(r,\theta) \end{aligned} \tag{10}$$

Where $\phi_i(r,\theta)$ (i=1,2,3) correspond to the particular solution and the remaining terms are for the homogeneous solution when $P_i = T_1 = 0$. By substituting Eqn. (10) into Eqn. (5) the homogeneous solution is given by:

$$(U_n, V_n, W_n) = \sum_{s=1}^{6} (A_{ns}, B_{ns}, D_{ns}) r^{k_{ns}} \quad (n = 0,1,\ldots) \tag{11}$$

and

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} A_{ns} \\ B_{ns} \\ D_{ns} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{12}$$

$$\begin{aligned}
K_{11} &= \left[C_{33}k_{ns}^2 - C_{44}n^2 - C_{22}\right] & K_{12} &= \left[(C_{23} + C_{44})k_{ns} - C_{22} - C_{44}\right]n \\
K_{13} &= \left[(C_{36} + C_{45})k_{ns} - C_{26}\right]n & K_{21} &= \left[(C_{36} + C_{45})k_{ns} + C_{26}\right]n \\
K_{22} &= \left[C_{26}n^2 - C_{45}k_{ns}(k_{ns} - 1)\right] & K_{23} &= \left[C_{66}n^2 - C_{55}k_{ns}^2\right] \\
K_{31} &= \left[(C_{23} + C_{44})k_{ns} + C_{22} + C_{44}\right]n & K_{32} &= \left[C_{22}n^2 - C_{44}(k_{ns}^2 - 1)\right] \\
K_{33} &= \left[C_{26}n^2 - C_{45}k_{ns}(k_{ns} + 1)\right]
\end{aligned}$$

The determinant of the [K] matrix is set to zero and the result is a characteristic equation that is cubic in k_{ns}^2. The roots to this cubic equation provide the solution for the six constants, k_{ns}, for each value of n in the Fourier series. Special cases to the solution of Eqn. (11) occur for values of n equal to 0 and 1, where repeated roots are found for $k_{ns} = 0, 0$. For n = 0:

$$\begin{aligned}
U_0 &= A_{03}r^{\lambda} + A_{04}r^{-\lambda} \\
V_0 &= \frac{C_{45}}{C_{44}} D_{01} + B_{05}r + \frac{B_{06}}{r} \\
W_0 &= D_{01}\ln r + D_{02} - \frac{2C_{45}}{C_{55}r} B_{06}
\end{aligned} \tag{13a}$$

and

$$\lambda = \sqrt{\frac{C_{22}}{C_{33}}} \tag{13b}$$

For the case of n = 1:

$$\begin{aligned}
U_1 &= A_{11} + A_{12}\ln r \\
V_1 &= -A_{11} - A_{12}\left\{\left[\frac{C_{26}C_{36} - C_{66}(C_{23} + C_{44})}{C_{26}^2 - C_{66}(C_{22} + C_{44})}\right] + \ln r\right\} \\
W_1 &= -A_{12}\left[\frac{C_{26}(C_{23} + C_{44}) - C_{36}(C_{22} + C_{44})}{C_{26}^2 - C_{66}(C_{22} + C_{44})}\right]
\end{aligned} \tag{14}$$

The particular solution in Eqn. (10) is found by direct substitution into Eqn. (5) and is:

$$\phi_1(r,\theta)=\left(a_1b_4+\frac{\Delta T\beta}{C_{33}-C_{22}}\right)r+a_2b_3r^2-\left(\frac{\rho\omega^2}{9C_{33}-C_{22}}\right)r^3+a_3r^2\left(b_1\cos\theta-b_2\sin\theta\right)$$

$$\phi_2(r,\theta)=a_4r^2\left(b_1\sin\theta+b_2\cos\theta\right) \tag{15}$$

$$\phi_3(r,\theta)=a_5r^2\left(b_1\sin\theta+b_2\cos\theta\right)-r\left(b_5\cos\theta+b_6\sin\theta\right)$$

where

$$a_1=\frac{C_{12}-C_{13}}{C_{33}-C_{22}}$$

$$a_2=\frac{C_{26}-2C_{36}}{4C_{33}-C_{22}} \tag{16}$$

$$\beta=\left(C_{13}-C_{12}\right)\alpha_x+\left(C_{23}-C_{22}\right)\alpha_\theta+\left(C_{33}-C_{23}\right)\alpha_r+\left(C_{36}-C_{26}\right)\alpha_{x\theta}$$

and a_3, a_4, and a_5 are found by solving the following set of three simultaneous linear equations.

$$\begin{bmatrix}\left(4C_{33}-C_{44}-C_{22}\right) & \left(2C_{23}+C_{44}-C_{22}\right) & \left(2C_{36}+2C_{45}-C_{26}\right)\\ \left(2C_{36}+2C_{45}+C_{26}\right) & \left(C_{26}-2C_{45}\right) & \left(C_{66}-4C_{55}\right)\\ \left(2C_{23}+3C_{44}+C_{22}\right) & \left(C_{22}-3C_{44}\right) & \left(C_{26}-6C_{45}\right)\end{bmatrix}\begin{Bmatrix}a_3\\ a_4\\ a_5\end{Bmatrix}=\begin{Bmatrix}2C_{13}-C_{12}\\ C_{16}\\ C_{12}\end{Bmatrix} \tag{17}$$

Finally, due to the form of the prescribed boundary conditions in Eqn. (7) $b_2 = 0$ and neglecting rigid body motions results in:

$$b_5=b_6=A_{11}=B_{05}=D_{02}=0 \tag{18}$$

The solution for a single layer, as describe by the above equations, is applicable for all fiber orientations with some minor changes to the equations for orthotropic ($C_{16} = C_{26} = C_{36} = C_{45} = 0$) and transversely-isotropic ($C_{12} = C_{13}$, $C_{22} = C_{33}$, $C_{55} = C_{66}$, $C_{44} = ½(C_{22} - C_{23})$ layers. Taking the highest index in the Fourier series to be M, the actual solution to the problem contains 6M + 6 unknowns and there are 6M + 6 independent equations. The unknown constants are b_1, b_3, b_4, A_{03}, A_{04}, A_{ij} (j = 2,3,…,6), B_{06}, D_{01}, and A_{ns} (s = 1,2,…,6, 2 ≤ n ≤ M). Some of the details have been omitted for brevity here but can be found in the original work of Pagano (6).

2.2 Laminate Solution For a laminated cylinder, the solution described in Section. 2.1 is applied to each layer and interfacial continuity is invoked between neighboring layers. The boundary conditions at the inner and outer radii of the cylinder are applied to the inner radius of the first layer and the outer radius of the last layer, respectively. Let R_1 and R_2 be the cylinder inner and outer radii, respectively, $r_1^{(k)}$ and $r_2^{(k)}$ be the inner and outer radii of the k^{th} layer, and t_k be the thickness of the k^{th} layer. For N layers there are N-1 interfaces and by using the following notation:

$$r_1^{(1)}=R_1 \qquad r_2^{(N)}=R_2 \qquad r_1^{(k)}=r_2^{(k-1)} \qquad r_k=r_2^{(k)}=r_1^{(k)}+t_k \tag{19}$$

the continuity equations for k = 1,2,…N-1 are written as:

$$
\begin{aligned}
&\sigma_r^{(k)}\left(\theta, r_2^{(k)}\right)=\sigma_r^{(k+1)}\left(\theta, r_1^{(k+1)}\right) \\
&\tau_{\theta r}^{(k)}\left(\theta, r_2^{(k)}\right)=\tau_{\theta r}^{(k+1)}\left(\theta, r_1^{(k+1)}\right) \\
&\tau_{xr}^{(k)}\left(\theta, r_2^{(k)}\right)=\tau_{xr}^{(k+1)}\left(\theta, r_1^{(k+1)}\right) \\
&u^{(k)}\left(\theta, r_2^{(k)}\right)=u^{(k+1)}\left(\theta, r_1^{(k+1)}\right)-\delta_k\left(\theta, r_k\right) \\
&v^{(k)}\left(\theta, r_2^{(k)}\right)=v^{(k+1)}\left(\theta, r_1^{(k+1)}\right) \\
&w^{(k)}\left(\theta, r_2^{(k)}\right)=w^{(k+1)}\left(\theta, r_1^{(k+1)}\right)
\end{aligned} \tag{20}
$$

where δ_k is a prescribed interference between layers. The resultant force and moment given by Eqn. (9) are modified to be a summation over all the layers and are rewritten as:

$$
\begin{aligned}
F_x &= \sum_{k=1}^{N} \int_0^{2\pi} \int_{r_1^k}^{r_2^k} \sigma_x^k r \partial r \partial \theta \\
T &= \sum_{k=1}^{N} \int_0^{2\pi} \int_{r_1^k}^{r_2^k} \tau_{x\theta}^k r^2 \partial r \partial \theta \\
M &= \sum_{k=1}^{N} \int_0^{2\pi} \int_{r_1^k}^{r_2^k} \sigma_x^k r^2 \cos\theta \partial r \partial \theta
\end{aligned} \tag{21}
$$

Recall that each layer has 6M + 6 unknown constants and therefore, the solution to the problem of a cylinder having N layers has 6N(M + 1) unknown constants. There are 6M + 6 independent equations from the boundary conditions and 6(N – 1)(M + 1) from the continuity equations for a total of 6N(M+1) equations.

The solution procedure is divided into three separate parts that depend on the number of terms in the Fourier series. For n = 0 in the Fourier series there are 4N simultaneous equations that are used to solve for the 4N unknowns of $b_3^{(k)}$, $b_4^{(k)}$, $A_{03}^{(k)}$, and $A_{04}^{(k)}$. There are 4(N – 1) equations from continuity of radial stress and continuity of the three displacement components. The remaining 4 equations are from the radial stress boundary conditions at the inner and outer radii of the cylinder (p_{10} and p_{20}) and from the resultant axial force and torque conditions. The 2N unknowns of $D_{01}^{(k)}$ and $B_{06}^{(k)}$ are solved from continuity of the two shear stress components (2N – 2 equations) and the two boundary conditions for shear stress at the inner radius of the cylinder (q_{10} and t_{10}). For n = 1, there are 6N unknowns with 6(N – 1) equations from continuity of the three stress and three displacement components and 6 equations from the boundary conditions corresponding to the terms of p_{11}, p_{21}, q_{11}, t_{11}, and t_{21}, and the resultant moment. The unknowns are $b_1^{(k)}$, $A_{12}^{(k)}$, and $A_{1s}^{(k)}$ (s = 3,4,5,6). Finally, for each n = 2,3,,,M a 6N X 6N system of equations is solved for the 6N(M – 1) unknowns of $A_{ns}^{(k)}$ (s = 1,2,…,6). There are 6(N – 1)(M – 1) equations from the 6 continuity equations and there are 6(M – 1) equations from the boundary conditions terms of p_{1n}, p_{2n}, q_{1n}, q_{2n}, t_{1n}, and t_{2n}.

A FORTRAN program has been developed for performing the calculations described in the above solution procedure. The program is general in the sense that monoclinic, orthotropic, and transversely-isotropic layers can be analyzed, and multiple material systems are acceptable. The layer-by-layer stresses, strains, and displacements are calculated in both the global cylindrical coordinate system and in the layer principal material directions. Failure criteria for determining first-ply failure are implemented in the program based on using Hashin's criteria (7) and the Tsai-Wu criterion (8).

3. DESIGN CASES

3.1 Antenna Mast Composite materials may be used to reduce the weight and life-cycle costs, as well as minimize deflections, of antenna masts. The masts are often subjected to wind conditions that subject the mast to non-axisymmetric loads. The wind velocity causes an external pressure that is well estimated by:

$$p_{30} = 0.0026V_{f30}^2 \tag{22}$$

where V_{f30} is the velocity measured at 30 feet above ground in open country for the fastest mile of wind, i.e., the mean wind velocity over time that corresponds to a movement of air 1 mile (9). At the top of the cylinder, the pressure distribution is given by:

$$p = 2.64\left(\frac{H}{z_g}\right)^{2\alpha} p_{30}\sum_{n=0}^{7} A_n \cos n\theta \tag{23}$$

where H is the height of the cylinder, z_g ,the gradient height above which the wind velocity is practically constant, and α depend on the terrain. For open country z_g equals 274.4 meters and α equals 1/7. The constants A_n are given in Table 1.

Table 1. Constants for circumferential distribution of wind velocity

Angle from windward meridian	n	A_n
0	0	-0.2636
15	1	0.3419
30	2	0.5418
45	3	0.3872
60	4	0.0525
75	5	-0.0771
90	6	-0.0039
105	7	0.0341

Consider the case of a quasi-isotropic graphite/epoxy cylinder 10 meters in height subjected to a V_{f30} wind velocity of 40 km/hr. The prescribed boundary conditions are written as:

$$\sigma_r(R_2,\theta) = \sum_{n=0}^{7} p_{2n}\cos n\theta\ , \quad \sigma_r(R_1,\theta) = \tau_{r\theta}(R_i,\theta) = \tau_{xr}(R_i,\theta) = 0 \quad (i = 1,2) \tag{24}$$

where

$$p_{2n} = 2.64\left(\frac{H}{z_g}\right)^{2\alpha} p_{30}A_n \tag{25}$$

Carpet plots in the rθ-plane of the mast are shown in Figs. 1-4 for the radial displacement, radial stress, hoop stress, and xθ-shear stress, respectively. The results in Fig. 1 indicate that the maximum radial displacement occurs at the circumferential coordinates of 0-, 90-, and 180-degrees. Also, these results show that the tube undergoes radial expansion at 0 and 180-degrees, whereas at 90-degress the tube experiences radial contraction. Figs. 2-4 show the radial stress to be a continuous function but the hoop and shear stresses have discontinuities at the layer interfaces. This is a direct result of the continuity conditions given by Eqn. (20). In addition, the shear stress results, as expected, are zero in the 0-degree and 90-degree plies.

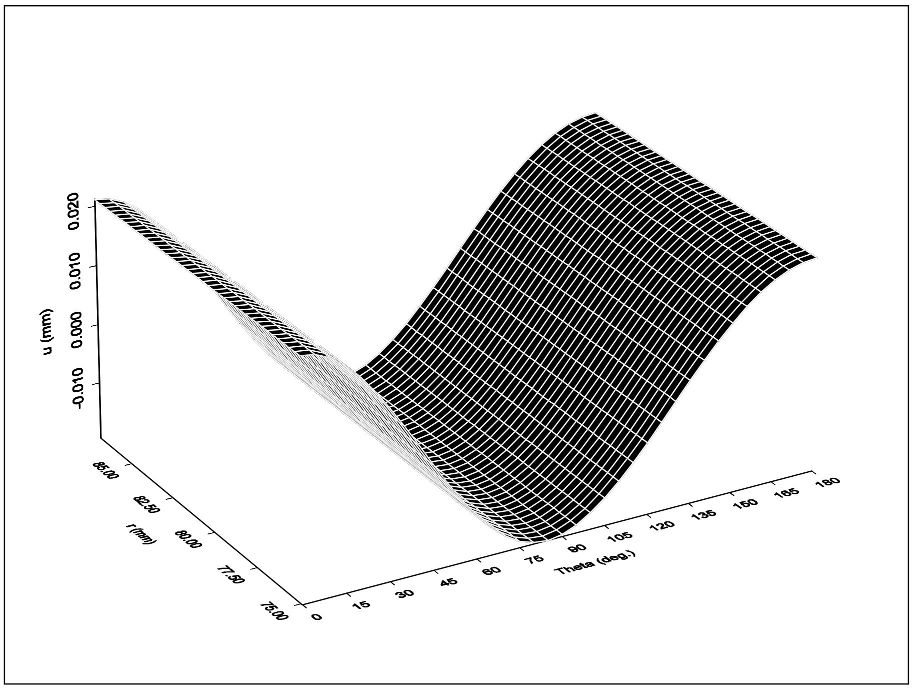

Figure 1. Radial displacement contour for composite mast subjected to wind loads.

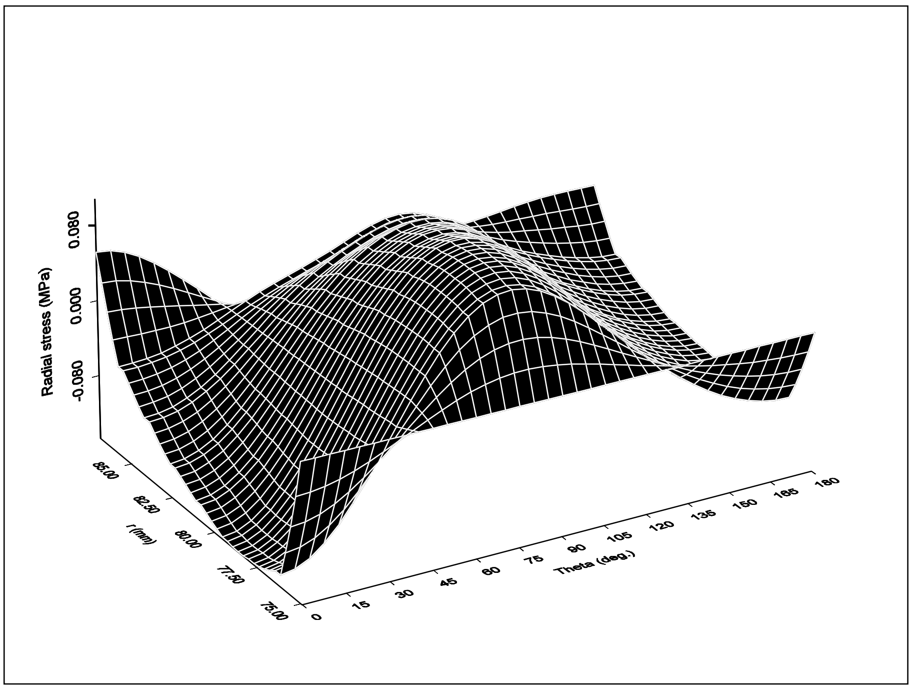

Figure 2. Radial stress contour for composite mast subjected to wind loads.

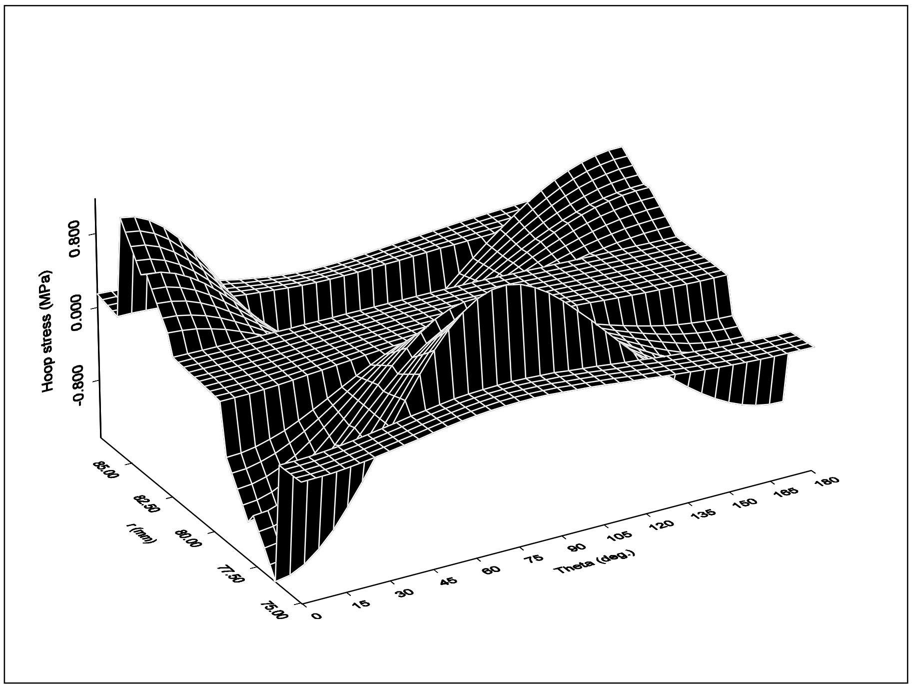

Figure 3. Hoop stress contour for composite mast subjected to wind loads.

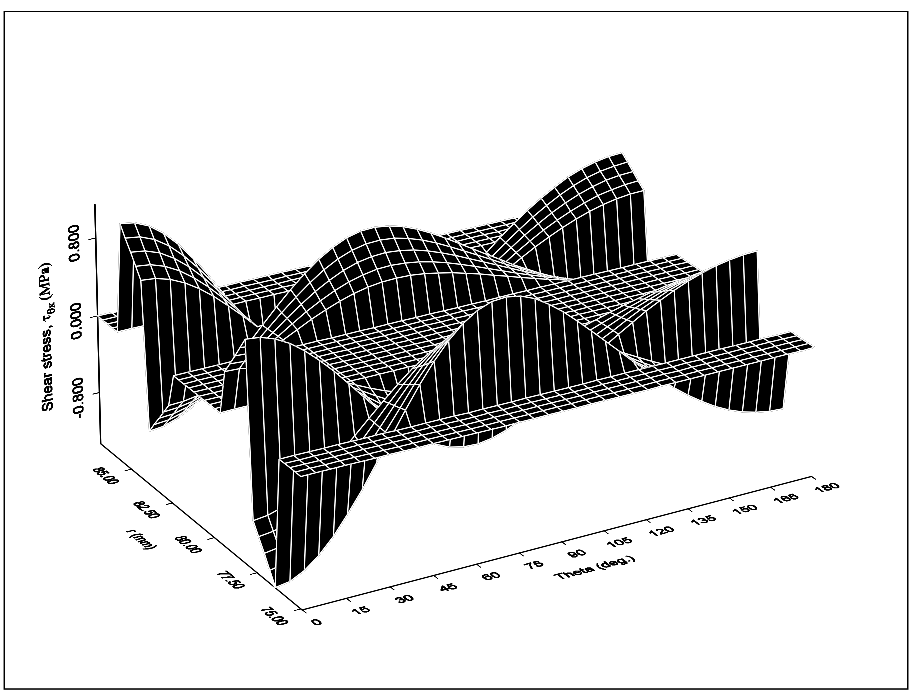

Figure 4. Shear stress contour for composite mast subjected to wind loads.

3.2 Spoolable Tubing In oilfield applications, spoolable composite tubing offers advantages over traditional tubing materials in terms of reduced weight and improved fatigue life. The many different loading scenarios that can arise include bending strains, axial forces, internal and external pressure, and elevated temperatures. Wrapping the tubing around spool diameters that are typically around 2 meters in diameter induces the bending strain (10) and the bending of the tubing results in a non-axisymmetric stress distribution. To analyze the spooling load scenario, an applied resultant end moment is used.

For typical tube outer diameters of 38 mm (10) on a 2-meter spool diameter the bending strain, ε_0, is equal to 1.86%. This is calculated by dividing the distance from the neutral axis by the radius of curvature. The applied moment necessary to produce this bending strain is a function of the tube geometry and lamination sequence and is given by:

$$M = \frac{\varepsilon_0}{R_2} \sum_{k=1}^{N} E_{xk} I_k = \frac{\pi \varepsilon_0}{4R_2} \sum_{k=1}^{N} \left(r_2^4 - r_1^4 \right)_k E_{xk} \tag{26}$$

where E_{xk} and I_k are the k^{th} layer axial stiffness and layer moment of inertia (11).

Consider a graphite/epoxy (±15°) angle-ply laminated tube having a 38-mm outer diameter and a 6-mm wall thickness. The applied end moment is calculated to be 3,392 N-m. The principal stress in the fiber direction is plotted in Fig. 5 at the inner and outer radii of the tube as a function of the circumferential coordinate. The results illustrate the bending of the tube associated with spooling where the stresses are of equal magnitude but of opposite sign symmetrically about the 90-degree circumferential coordinate.

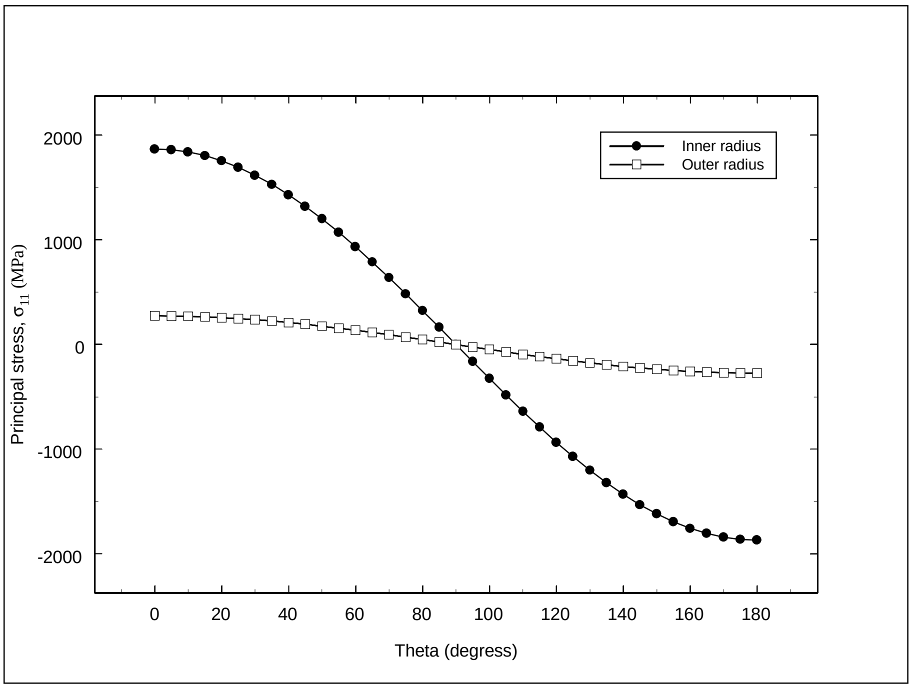

Figure 5. Principal stress distribution in the fiber direction for spooled load case.

4. CONCLUSIONS

A closed-form solution has been presented for design and analysis of an axisymmetric laminated cylinder subjected to non-axisymmetric loads. The solution is based on the theory of anisotropic elasticity and an assumed generalized plane deformation state of stress. The prescribed boundary conditions are expressed in terms of a Fourier series and are independent of the axial coordinate. Using zero terms in the Fourier series expansion treats the special case of axisymmetric loads. The solution is general in that monoclinic, orthotropic, and transversely-isotropic layers are considered in the formulation and multiple material systems may be used. The prescribed loadings include axial force, moment, torque, internal and external pressure, uniform temperature change, and rotational velocity with interference fits. A FORTRAN code was developed for performing the necessary calculations and the code can be executed from a desktop computer. This permits a computationally efficient method for conducting numerous design trade studies. The utilization of the code was demonstrated by two design cases for composite laminated cylinders subjected to non-axisymmetric loads.

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