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Angularly Adaptive P_1-Double P_0 Flux-Limited Diffusion Solutions of Non-Equilibrium Grey Radiative Transfer Problems

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Abstract

The double spherical harmonics angular approximation in the lowest order, i.e. double P_0 (DP_0), is developed for the solution of time-dependent non-equilibrium grey radiative transfer problems in planar geometry. Although the DP_0 diffusion approximation is expected to be less accurate than the P_1 diffusion approximation at and near thermodynamic equilibrium, the DP_0 angular approximation can more accurately capture the complicated angular dependence near a non-equilibrium radiation wave front. In addition, the DP_0 approximation should be more accurate in non-equilibrium optically thin regions where the positive and negative angular domains are largely decoupled. We develop an adaptive angular technique that locally uses either the DP_0 or P_1 flux-limited diffusion approximation depending on the degree to which the radiation and material fields are in thermodynamic equilibrium. Numerical results are presented for two test problems due to Su and Olson and to Ganapol and Pomraning for which semi-analytic transport solutions exist. These numerical results demonstrate that the adaptive P_1-DP_0 diffusion approximation can yield improvements in accuracy over the standard P_1 diffusion approximation, both without and with flux-limiting, for non-equilibrium grey radiative transfer.

Key words: radiative transfer, non-equilibrium flux-limited radiation diffusion, spherical harmonics approximation, double spherical harmonics approximation

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1 Introduction

Time-dependent non-equilibrium radiative transfer problems in the transport description are challenging to solve numerically. As a result, low-order spherical harmonics angular approximations have been examined as less expensive computational alternatives [1]. The standard P_1 angular approximation represents the angular dependence of the radiation specific intensity at each point in space and time using a linear function in the angular domain $-1 \leq \mu \leq 1$ [2]. Thus, there are two angular moment unknowns, the radiation energy density (proportional to the zeroth moment) and the radiation flux (first moment), at each point in space and time. Under the assumption that the time derivative of the radiation flux is negligible, the two equations for the angular moment unknowns can be combined into a single P_1 diffusion equation for the radiation energy density. An equation for the energy balance of the background material through which the radiation is propagating and with which it is exchanging energy is also required.

In contrast, the double P_0 (DP_0) angular approximation represents the angular dependence of the radiation specific intensity as isotropic (constant) in each half angular range $-1 \leq \mu < 0$ and $0 < \mu \leq 1$, again resulting in two (half-range) angular moment unknowns at each point in space and time [2]. The DP_0 approximation of the radiation specific intensity can be written in a form that differs from the P_1 expression only in the linearly angular component [3–5]. The resulting DP_0 diffusion approximation (under the assumption of a negligible radiation flux time derivative) differs from the P_1 diffusion approximation only in the definition of the diffusion coefficient. The DP_0 diffusion approximation, with flux limiting [1], can be incorporated into a non-equilibrium planar geometry grey radiation diffusion code with only minor modifications.

The double P_0 approximation should be more accurate than the P_1 approximation in non-equilibrium optically thin regions where the positive and negative angular domains are largely decoupled. Also, as pointed out by Olson et al. [1], the angular dependence of the specific intensity near a radiation wave front is complicated. The sharp radiation wave front can act as an internal boundary that is difficult for the P_1 angular approximation to accurately model. At an internal boundary, the angular distribution of the specific intensity can be nearly discontinuous when viewed as a function of angle. Brantley [5] has recently investigated the application of a mixed P_1-DP_0 angular approximation for more accurately treating material interfaces for time-independent planar geometry neutronics problems. In this paper, we investigate the application of the DP_0 angular approximation to more accurately model the angular dependence of the radiation specific intensity near a wave front for non-equilibrium grey radiative transfer problems. Numerical results from two test problems proposed by Su and Olson [6] and Ganapol and Pomraning [7] for which semi-

analytic transport solutions are available demonstrate that the DP_0 diffusion approximation can produce more accurate results than the P_1 diffusion approximation, both without and with flux limiting, when the radiation and material are out of equilibrium.

When the radiation and material energies reach thermodynamic equilibrium, however, the radiative transfer and material energy balance equations asymptotically limit to first order to the equilibrium P_1 diffusion approximation [8]. Under these conditions, the DP_0 diffusion approximation gives an equation with a different (incorrect) diffusion coefficient. As such, we expect the DP_0 diffusion approximation to be less accurate than the P_1 diffusion approximation in and near thermodynamic equilibrium. This theoretical expectation is numerically confirmed in this paper using the two analytic test problems.

These physical and theoretical considerations motivated the development of an angularly adaptive P_1-DP_0 flux-limited diffusion approximation that is designed to use the most accurate angular approximation (either DP_0 or P_1) at any given point in space and time. The adaptivity criterion is simple, physically-based, and essentially prescribes the use of the DP_0 diffusion approximation when the radiation and material energy fields are away from thermodynamic equilibrium and the use of the P_1 diffusion approximation when the radiation and material energy fields are in or near equilibrium. Numerical solutions for the Su and Olson and the Ganapol and Pomraning semi-analytic test problems are presented to demonstrate that this adaptive angular approximation can yield overall accuracy improvements. This paper is an expanded version of a recent conference report [9].

A significant amount of previous research into the application of the double P_N approximation to radiative transfer problems has been reported in the literature. Within the engineering heat transfer, atmospheric physics, and astrophysics communities, the method is sometimes referred to as the two-flux method [10], the half-moment method [11], the double interval method [12], or the two-stream approximation [13]. The previous research has focused on often time-independent radiative transfer problems in which the radiation and material are in equilibrium, and the double P_N equations have typically been solved for these problems as a system of coupled first-order differential equations. In this paper, we apply the double P_0 approximation to time-dependent non-equilibrium radiative transfer problems. In addition, we illustrate how to formulate the double P_0 approximation such that it is easily implemented into an existing P_1 diffusion code. Finally, we develop an adaptive method of combining the P_1 and double P_0 flux-limited diffusion approximations for use in the same radiative transfer problem.

The remainder of this paper is organized as follows. In Section 2, we briefly outline the derivations of the P_1 and DP_0 diffusion approximations for the

case of non-equilibrium grey radiative transfer. We then describe how these two angular approximations can be readily combined into a single set of equations with a single parameter allowing the selection of the P_1 or the DP_0 flux-limited diffusion approximation. We conclude Section 2 with a description of the angularly adaptive P_1-DP_0 flux-limited diffusion approximation. In Section 3, we compare P_1, DP_0, and adaptive P_1-DP_0 diffusion numerical results for the Su and Olson and the Ganapol and Pomraning test problems with the semi-analytic transport solutions. Finally, we offer concluding remarks and suggestions for future work in Section 4.

2 Derivations of the P_1, DP_0, and Adaptive P_1-DP_0 Flux-Limited Diffusion Approximations

In this section, we briefly outline the derivations of the P_1 and DP_0 diffusion approximations for the case of non-equilibrium grey radiative transfer in planar geometry. We focus the description of the derivations on contrasting the two angular approximations. We then demonstrate how the two angular approximations can be combined into a single set of equations with a single parameter allowing the selection of the P_1 or the DP_0 flux-limited diffusion approximation. Finally, we theoretically motivate and describe a method for adaptively selecting the local angular approximation based on the degree to which the radiation and material energy density fields are in thermodynamic equilibrium.

In the following, we assume that hydrodynamic motion and heat conduction can be neglected, that scattering is isotropic, and that the background material is in local thermodynamic equilibrium. Under these assumptions, the equation of time-dependent grey radiative transfer in a one-dimensional planar geometry medium $0 \leq x \leq L$ with space- and temperature-dependent opacities is given by [14]

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial x} + \sigma_t I = \frac{1}{2}ac\sigma_a T_m^4 + \frac{1}{2}\sigma_s \int_{-1}^{+1} I d\mu' + \frac{1}{2}S \quad ,$$

$$0 < x < L \quad , \quad -1 \leq \mu \leq 1 \quad , \quad t > 0 \quad , \tag{1}$$

coupled with the material energy balance equation

$$C_v \frac{\partial T_m}{\partial t} = \sigma_a \left[\int_{-1}^{+1} I d\mu - acT_m^4 \right] \quad , \quad 0 \leq x \leq L \quad , \quad t > 0 \quad , \tag{2}$$

where x, μ, and t are the space, angular direction, and time variables, respectively; $I(x, \mu, t)$ [*energy/area-time*] denotes the specific intensity of radiation; $T_m(x, t)$ [*temperature*] is the material temperature; $\sigma_a(x, T_m)$, $\sigma_s(x, T_m)$, and

$\sigma_t(x, T_m)$ [*length*$^{-1}$] are the absorption, scattering, and total opacities, respectively, at material temperature $T_m(x,t)$; $S(x,t)$ [*energy/volume-time*] is an isotropic source of radiation; c [*length/time*] is the speed of light in a vacuum; a [*energy/temperature*^{4}*-volume*] is the radiation constant; and $C_v(x, T_m)$ [*energy/temperature-volume*] is the material heat capacity. The initial and boundary conditions for Eqs. (1) and (2) are given by

$$I(x,\mu,0) = I_0(x,\mu) \quad , \quad 0 \le x \le L \quad , \quad -1 \le \mu \le 1 \quad , \tag{3}$$

$$T_m(x,0) = T_0(x) \quad , \quad 0 \le x \le L \quad , \tag{4}$$

$$I(0,\mu,t) = \Gamma_0(\mu,t) \quad , \quad 0 < \mu \le 1 \quad , \quad t \ge 0 \quad , \tag{5}$$

$$I(L,\mu,t) = \Gamma_L(\mu,t) \quad , \quad -1 \le \mu < 0 \quad , \quad t \ge 0 \quad , \tag{6}$$

where I_0 and T_0 are the prescribed initial data and Γ_0 and Γ_L are the prescribed boundary data.

In preparation for deriving the P_1 and DP_0 diffusion approximations to Eqs. (1)–(6), we first define two angular moments of the radiation specific intensity. The zeroth angular moment of the specific intensity is the radiation energy density $E(x,t)$ [*energy/volume*] given by

$$E(x,t) \equiv \frac{1}{c}\int_{-1}^{+1} I(x,\mu,t)\, d\mu \quad . \tag{7}$$

The first angular moment of the specific intensity is the radiation flux $F(x,t)$ [*energy/area-time*] given by

$$F(x,t) \equiv \int_{-1}^{+1} \mu I(x,\mu,t)\, d\mu \quad . \tag{8}$$

In terms of the radiation energy density, the material energy balance equation is given by

$$C_v \frac{\partial T_m}{\partial t} = c\sigma_a \left(E - aT_m^4\right) \quad , \quad 0 \le x \le L \quad , \quad t > 0 \quad . \tag{9}$$

2.1 Derivation of the P_1 Diffusion Approximation

In this section, we briefly outline the derivation of the P_1 diffusion approximation. The P_1 representation of the radiation specific intensity for all x and t is given by [2]

$$I(x,\mu,t) = \frac{c}{2}E(x,t) + \frac{3}{2}\mu F(x,t) \quad , \quad -1 \le \mu \le 1 \quad . \tag{10}$$

Both $E(x,t)$ and $F(x,t)$ are continuous functions of space and time, so the P_1 representation of the specific intensity is linearly anisotropic in angle and

is continuous in space, angle, and time. To obtain equations for the unknowns $E(x,t)$ and $F(x,t)$, we insert Eq. (10) into Eq. (1) and integrate the result over $-1 \leq \mu \leq 1$ to obtain the radiation energy balance equation

$$\frac{1}{c}\frac{\partial E}{\partial t} + \frac{1}{c}\frac{\partial F}{\partial x} + \sigma_a E = a\sigma_a T_m^4 + \frac{1}{c}S \quad , \quad 0 < x < L \quad , \quad t > 0 \quad . \tag{11}$$

Next, we insert Eq. (10) into Eq. (1), multiply the result by μ, and integrate over $-1 \leq \mu \leq 1$ to obtain

$$\frac{1}{c}\frac{\partial F}{\partial t} + \frac{c}{3}\frac{\partial E}{\partial x} + \sigma_t F = 0 \quad , \quad 0 < x < L \quad , \quad t > 0 \quad . \tag{12}$$

Eqs. (11) and (12), coupled with the material energy balance equation given by Eq. (9), constitute the P_1 approximation for grey radiative transfer (along with corresponding initial and boundary conditions).

To obtain the P_1 *diffusion* approximation, we make the further assumption that the time derivative of the radiation flux is negligible. Using this assumption in Eq. (12), the radiation flux is given by

$$F = -\frac{c}{3\sigma_t}\frac{\partial E}{\partial x} \quad , \quad 0 < x < L \quad , \quad t > 0 \quad . \tag{13}$$

Then Eqs. (11) and (13) can be combined into a single equation for the radiation energy density given by

$$\frac{1}{c}\frac{\partial E}{\partial t} - \frac{\partial}{\partial x}\frac{1}{3\sigma_t}\frac{\partial E}{\partial x} + \sigma_a E = a\sigma_a T_m^4 + \frac{1}{c}S \quad , \quad 0 < x < L \quad , \quad t > 0 \quad . \tag{14}$$

Eq. (14) for the radiation energy density, coupled with the material energy balance equation Eq. (9) for the material temperature, constitutes the P_1 diffusion approximation for grey radiative transfer.

We now turn to the specification of initial and boundary conditions. Initial conditions for $E(x,t)$ and $F(x,t)$ are easily derived from Eq. (3) by successive angular moment integrations analogous to the above to obtain

$$E(x,0) = E_0(x) \equiv \frac{1}{c}\int_{-1}^{+1} I_0(x,\mu)\, d\mu \quad , \quad 0 \leq x \leq L \quad , \tag{15}$$

$$F(x,0) = F_0(x) \equiv \int_{-1}^{+1} \mu I_0(x,\mu)\, d\mu \quad , \quad 0 \leq x \leq L \quad . \tag{16}$$

Although there is some ambiguity surrounding which is the most accurate boundary condition for a P_N approximation [15], we focus on the widely-used Marshak or Milne boundary condition. This boundary condition is obtained, at $x = 0$ for example, by inserting the P_1 representation for the radiation

specific intensity Eq. (10) into the equation

$$\int_0^1 \mu \left[I\left(0,\mu,t\right) - \Gamma_0\left(\mu,t\right) \right] d\mu = 0 \quad , \tag{17}$$

and performing the angular integrations to obtain

$$\frac{c}{4}E + \frac{1}{2}F = \int_0^1 \mu \Gamma_0 d\mu \quad . \tag{18}$$

Inserting the expression for the P_1 radiation flux given by Eq. (13), we obtain the Marshak boundary condition at $x = 0$ for the P_1 diffusion approximation

$$\frac{c}{4}E - \frac{1}{6\sigma_t}\frac{\partial E}{\partial x} = \int_0^1 \mu \Gamma_0 d\mu \quad . \tag{19}$$

An analogous boundary condition holds at $x = L$. The P_1 diffusion approximation for non-equilibrium grey radiative transfer problems is now fully specified by Eqs. (14), (15), and (19), coupled with the material energy balance equation given by Eq. (9) and its initial condition given by Eq. (4).

2.2 Derivation of the DP_0 Diffusion Approximation

In this section, we briefly outline the derivation of the double P_0 diffusion approximation. The DP_0 representation of the radiation specific intensity for all x and t is given by [2]

$$I\left(x,\mu,t\right) = \begin{cases} I^+(x,t) \quad , & 0 < \mu \leq 1 \quad , \\ I^-(x,t) \quad , & -1 \leq \mu < 0 \quad , \end{cases} \tag{20}$$

where $I^+(x,t)$ and $I^-(x,t)$ are continuous functions of space and time. Thus, the DP_0 representation of the specific intensity is isotropic in angle in each of the half angular ranges $-1 \leq \mu < 0$ and $0 < \mu \leq 1$ and is continuous in space and time but potentially discontinuous in angle. Using Eq. (20) in Eqs. (7) and (8) [5], we find for the DP_0 radiation energy density the expression

$$E\left(x,t\right) = \frac{1}{c}\left[I^+(x,t) + I^-(x,t) \right] \quad , \tag{21}$$

and for the DP_0 radiation flux the expression

$$F\left(x,t\right) = \frac{1}{2}\left[I^+(x,t) - I^-(x,t) \right] \quad . \tag{22}$$

After algebraic manipulation, we can rewrite Eq. (20) in terms of Eqs. (21) and (22) as

$$I\left(x,\mu,t\right)=\frac{c}{2}E\left(x,t\right)+\frac{3}{2}f\left(\mu\right)F\left(x,t\right)\quad,\quad-1\leq\mu\leq1\quad, \tag{23}$$

where the function $f\left(\mu\right)$ is defined as

$$f\left(\mu\right)=\begin{cases}+\frac{2}{3}\quad, & 0<\mu\leq1\quad,\\ -\frac{2}{3}\quad, & -1\leq\mu<0\quad.\end{cases} \tag{24}$$

Comparing Eq. (23) to Eq. (10), the P_1 and DP_0 angular representations of the radiation specific intensity differ only in the linear angular component.

To obtain equations for the unknowns $E\left(x,t\right)$ and $F\left(x,t\right)$, we first insert Eq. (23) into Eq. (1) and integrate the result over $-1\leq\mu\leq1$ to obtain the radiation energy balance equation given by Eq. (11). Next, we insert Eq. (23) into Eq. (1), multiply the result by $f\left(\mu\right)$, and integrate over $-1\leq\mu\leq1$ to obtain

$$\frac{1}{c}\frac{\partial F}{\partial t}+\frac{c}{4}\frac{\partial E}{\partial x}+\sigma_tF=0\quad,\quad0<x<L\quad,\quad t>0\quad. \tag{25}$$

Eqs. (11) and (25), coupled with the material energy balance equation Eq. (9), constitute the DP_0 angular approximation for grey radiative transfer (along with corresponding initial and boundary conditions).

To obtain the DP_0 *diffusion* approximation, we make the further assumption that the time derivative of the radiation flux is negligible. Using this assumption in Eq. (25), the radiation flux is given by

$$F=-\frac{c}{4\sigma_t}\frac{\partial E}{\partial x}\quad,\quad0<x<L\quad,\quad t>0\quad. \tag{26}$$

Then Eqs. (11) and (26) can be combined into a single equation for the radiation energy density given by

$$\frac{1}{c}\frac{\partial E}{\partial t}-\frac{\partial}{\partial x}\frac{1}{4\sigma_t}\frac{\partial E}{\partial x}+\sigma_aE=a\sigma_aT_m^4+\frac{1}{c}S\quad,\quad0<x<L\quad,\quad t>0\quad. \tag{27}$$

Eq. (27) for the radiation energy density, coupled with the material energy balance equation Eq. (9) for the material temperature, constitutes the DP_0 diffusion approximation for grey radiative transfer. Comparing Eqs. (14) and (27), the DP_0 diffusion approximation clearly differs from the P_1 diffusion approximation only in the definition of the diffusion coefficient.

We now specify the initial and boundary conditions for the DP_0 diffusion approximation. Initial conditions for $E\left(x,t\right)$ and $F\left(x,t\right)$ for the DP_0 approximation are the same as for the P_1 approximation and are given by Eqs. (15)

and (16). The Marshak boundary condition at $x = 0$ is obtained by substituting the DP_0 representation for the radiation specific intensity, Eq. (23), into Eq. (17) and performing the angular integrations to again obtain Eq. (18). Inserting the representation for the DP_0 radiation flux given by Eq. (26), we obtain the Marshak boundary condition at $x = 0$ for the DP_0 diffusion approximation

$$\frac{c}{4}E - \frac{1}{8\sigma_t}\frac{\partial E}{\partial x} = \int_0^1 \mu\Gamma_0 d\mu \quad . \tag{28}$$

An analogous boundary condition holds at $x = L$. The DP_0 diffusion approximation for non-equilibrium grey radiative transfer problems is now fully specified by Eqs. (27), (15), and (28), coupled with the material energy balance equation given by Eq. (9) and its initial condition given by Eq. (4). We note that the form of the DP_0 diffusion approximation is only trivially different than the P_1 diffusion approximation.

2.3 The Adaptive P_1-DP_0 Flux-Limited Diffusion Approximation

In this section, we describe how the P_1 and DP_0 diffusion approximations for non-equilibrium grey radiative transfer in planar geometry can be combined into a single set of equations with a single parameter specifying the local angular approximation. We then theoretically motivate and describe a method for adaptively selecting the local angular approximation based on the degree to which the radiation and material energy density fields are in thermodynamic equilibrium.

Recalling that the only difference between the form of the two approximations is in the diffusion coefficient, we define a generalized diffusion coefficient as

$$D\left(x,t,T_m\right) = \frac{1}{3M\left(x,t\right)\sigma_t\left(x,T_m\right)} \quad , \tag{29}$$

where the multiplier function $M\left(x,t\right)$ is defined by

$$M\left(x,t\right) \equiv \frac{4}{3}\left[1-\delta\left(x,t\right)\right] + \delta\left(x,t\right) \quad . \tag{30}$$

Then setting $\delta\left(x,t\right) = 1$ at a given x and t yields the P_1 form of the diffusion coefficient and $\delta\left(x,t\right) = 0$ yields the DP_0 form. Thus, the DP_0 diffusion approximation can be incorporated into a P_1 diffusion code using only a multiplier on the diffusion coefficient.

Both the P_1 and DP_0 *diffusion* approximations propagate radiation with an infinite velocity in optically thin media. In the presence of steep gradients, both angular approximations can also propagate radiation faster than the speed of light. One somewhat ad-hoc method for correcting this incorrect propagation

velocity is through the use of a "flux limiter" [1]. While some standard flux limiters such as the "sum" flux limiter cause a loss of accuracy in the equilibrium diffusion limit [8], a flux limiter proposed by Larsen has been shown [1, 8] to retain first order accuracy in the equilibrium diffusion limit. Larsen's flux limiter and other standard flux limiters such as the "sum" and "max" flux limiters [1] are readily applied to the generalized diffusion coefficient given above. For the Larsen "square-root" flux limiter, the generalized flux-limited diffusion coefficient is given by

$$D\left(x,t,T_m\right)=\frac{1}{\left\{\left[3M\left(x,t\right)\sigma_t\left(x,T_m\right)\right]^2+\left[\frac{1}{E(x,t)}\left|\frac{\partial E}{\partial x}\right|\right]^2\right\}^{1/2}} \quad . \tag{31}$$

For the standard "sum" and "max" flux limiters, the generalized flux-limited diffusion coefficient is given by

$$D\left(x,t,T_m\right)=\frac{1}{3M\left(x,t\right)\sigma_t\left(x,T_m\right)+\frac{1}{E(x,t)}\left|\frac{\partial E}{\partial x}\right|} \quad , \tag{32}$$

and

$$D\left(x,t,T_m\right)=\frac{1}{max\left[3M\left(x,t\right)\sigma_t\left(x,T_m\right),\frac{1}{E(x,t)}\left|\frac{\partial E}{\partial x}\right|\right]} \quad , \tag{33}$$

respectively.

The arbitrary P_1-DP_0 flux-limited diffusion approximation is then fully specified as

$$\frac{1}{c}\frac{\partial E}{\partial t}-\frac{\partial}{\partial x}D\frac{\partial E}{\partial x}+\sigma_a E=a\sigma_a T_m^4+\frac{1}{c}S \quad , \quad 0<x<L \quad , \quad t>0 \quad , \tag{34}$$

with the initial condition

$$E\left(x,0\right)=E_0\left(x\right)\equiv\frac{1}{c}\int_{-1}^{+1}I_0\left(x,\mu\right)d\mu \quad , \quad 0\leq x\leq L \quad , \tag{35}$$

and the Marshak boundary condition at $x=0$

$$\frac{c}{4}E-\frac{1}{2}D\left(0,t,T_m\right)\frac{\partial E}{\partial x}=\int_0^1\mu\Gamma_0 d\mu \quad . \tag{36}$$

An analogous boundary condition exists at $x=L$, and these equations are coupled to the material energy balance equation given by Eq. (9) along with its initial condition Eq. (4). The desired angular approximation at any space and time point can be easily chosen by using the appropriate value of the function $\delta\left(x,t\right)$.

When the radiation and material energies reach thermodynamic equilibrium, the radiative transfer and material energy balance equations asymptotically limit to first order to the equilibrium P_1 diffusion approximation [8]. Under

these conditions, the DP_0 diffusion approximation gives an equation with a different (incorrect) diffusion coefficient. As such, we expect the DP_0 diffusion approximation to be less accurate than the P_1 diffusion approximation at and near thermodynamic equilibrium. However, we hypothesize that the DP_0 angular approximation can more accurately capture the behavior of the radiation specific intensity than the P_1 angular approximation when the radiation and material energy density fields are out of equilibrium (e.g. near a radiation wave front). Motivated by these considerations, we propose an "adaptive" P_1-DP_0 diffusion approximation in which the value of the function $\delta(x,t)$ for each x and t is chosen according to the criterion

$$\delta(x,t) = \begin{cases} 1 \quad , & |\frac{E(x,t)}{E_m(x,t)} - 1| \leq \beta \quad , \\ 0 \quad , & otherwise \quad , \end{cases} \tag{37}$$

where $E_m(x,t)$ [*energy/volume*] is the material energy density and we have used $\beta = 0.1$ in our simulations. This prescription for $\delta(x,t)$ ensures that the P_1 angular approximation is used at and near thermodynamic equilibrium [where $E(x,t) \approx E_m(x,t)$] and that the DP_0 angular approximation is used otherwise. To avoid spatial discontinuities in the derivative of the radiation specific energy caused by a spatially discontinuous diffusion coefficient [16], we have in practice used a continuous sigmoid approximation to Eq. (37) given by

$$\delta(x,t) = 1 - \frac{1}{1 + \exp\left[-100\left(|\frac{E(x,t)}{E_m(x,t)} - 1| - \beta\right)\right]} \quad . \tag{38}$$

The DP_0 angular approximation can more accurately model the angular discontinuity in the radiation specific intensity near external boundaries [2, 5, 13]. Setting the angular approximation parameter $\delta(x,t)$ to zero along external boundaries (to use the DP_0 approximation) independent of the degree to which the radiation and material are in equilibrium may therefore give improved accuracy. We do not examine this issue further in this paper.

3 Numerical Results

In the previous section, we derived the P_1 and DP_0 flux-limited diffusion approximations for the case of non-equilibrium grey radiative transfer in planar geometry. We showed that the P_1 and DP_0 diffusion approximations differ only in the definition of the diffusion coefficient. As a result, these approximations can be easily combined into one set of equations with a single parameter specifying the desired angular approximation. Finally, we described a simple, physically-based technique for adaptively choosing the local angular approximation at each point in space and time.

We have numerically implemented the angularly adaptive P_1-DP_0 flux-limited diffusion approximation using the spatial and temporal discretizations described in [17]. In particular, the radiation diffusion and material energy balance equations are spatially discretized using a second-order cell-centered differencing and temporally discretized implicitly using backward Euler differencing. The material properties are treated explicitly in time. No iteration on the temperature dependence of the material properties is performed, so extremely small timesteps are used to obtain the numerical results.

In this section, we apply the P_1, DP_0, and adaptive P_1-DP_0 diffusion approximations to the solution of two benchmark problems for which semi-analytic transport solutions exist: the Su and Olson semi-analytic benchmark problem [6] and the Ganapol and Pomraning non-equilibrium Marshak wave benchmark problem [7]. These test problems have a somewhat different character in that the Su and Olson problem is driven by an isotropic interior source while the Ganapol and Pomraning problem is driven by an isotropic incident specific intensity on a free surface. In addition, the Su and Olson problem focuses on spatial distributions of the radiation and material energy densities as a function of time while the Ganapol and Pomraning problem focuses on the spatially integrated material and energy densities as a function of time. We present results both without flux limiting and with flux limiting using the Larsen square-root flux limiter with the diffusion coefficient defined by Eq. (31).

3.1 The Su and Olson Non-Equilibrium Benchmark Problem

In this section, we describe the Su and Olson semi-analytic benchmark problem for which a semi-analytic transport solution exists [6]. Then we apply the P_1, DP_0, and angularly adaptive P_1-DP_0 diffusion approximations, both without and with flux limiting, to the numerical solution of this problem.

To linearize the radiative transfer equation and the material energy balance equation such that a semi-analytic solution can be obtained, Su and Olson [6] make two major assumptions: the material opacities are constant (independent of the material temperature) and the heat capacity of the material is proportional to the cube of the material temperature, $C_v = \alpha T_m^3$. While not physically realistic, these assumptions make the equations of radiative transfer linear and allow their semi-analytic solution.

Su and Olson rewrite the radiative transfer equation and the material energy balance equation in dimensionless form as follows. The spatial variable x is written in terms of the dimensionless optical depth, $z \equiv \sigma_t x$. A dimensionless parameter $\epsilon \equiv 4a/\alpha$ is defined, where α is the material heat

capacity proportionality constant. Then a dimensionless time variable is defined as $\tau \equiv \epsilon c \sigma_t t$. Dimensionless absorption and scattering opacities are defined as $c_a \equiv \sigma_a / \sigma_t$ and $c_s \equiv \sigma_s / \sigma_t$, respectively, such that $c_a + c_s = 1$. Defining a reference "hohlraum" temperature T_R, the dimensionless radiation specific intensity, radiation energy density, and material energy density are given by $U(z, \mu, \tau) \equiv I(x, \mu, t) / a T_R^4$, $W(z, \tau) \equiv \int_{-1}^{+1} U(z, \mu, \tau) \, d\mu$, and $V(z, \tau) \equiv [T_m(x, t) / T_R]^4$, respectively. Finally, the dimensionless radiation source is given by $Q(z, \tau) \equiv S(x, t) / a T_R^4$. Using these definitions, the dimensionless radiative transfer equation is given by

$$\left(\epsilon \frac{\partial}{\partial \tau} + \mu \frac{\partial}{\partial z} + 1 \right) U(z, \mu, \tau) = \frac{c_a}{2} V(z, \tau) + \frac{c_s}{2} W(z, \tau) + \frac{1}{2} Q(z, \tau) \quad ,$$

$$-\infty < z < \infty \quad , \quad -1 \leq \mu \leq 1 \quad , \quad \tau > 0 \quad , \tag{39}$$

coupled with the dimensionless material energy balance equation given by

$$\frac{\partial V(z, \tau)}{\partial \tau} = c_a [W(z, \tau) - V(z, \tau)] \quad , \quad -\infty < z < \infty \quad , \quad \tau > 0 \quad . \tag{40}$$

The boundary and initial conditions for Eq. (39) are given by

$$\lim_{z \to \pm \infty} U(z, \mu, \tau) = 0 \quad , \tag{41}$$

and

$$U(z, \mu, 0) = 0 \quad , \tag{42}$$

respectively. The initial condition for Eq. (40) is given by

$$V(z, 0) = 0 \quad . \tag{43}$$

The Su and Olson non-equilibrium grey radiative transfer problem consists of an initially cold, homogeneous, infinite, and isotropically scattering medium with an internal radiation source defined by

$$Q(z, \tau) = \frac{1}{2 z_0} [H(z + z_0) - H(z - z_0)] [H(\tau) - H(\tau - \tau_0)] \quad , \tag{44}$$

where H is the Heaviside step function. The source is thus defined on the dimensionless spatial domain $-z_0 \leq z \leq z_0$ and is nonzero for dimensionless times $0 \leq \tau \leq \tau_0$. Su and Olson obtained a semi-analytic transport solution to this problem by applying the Fourier transform to the spatial variable and the Laplace transform to the time variable [6].

We apply the P_1, DP_0, and angularly adaptive P_1-DP_0 diffusion approximations to the numerical solution of the Su and Olson benchmark problem. Su and Olson obtained semi-analytic results both with and without scattering; we present results for the absorption-only case here. We obtain a solution for the

case of the radiation source defined by Eq. (44) with $z_0 = 0.5$ and $\tau_0 = 10$ and the dimensionless parameter $\epsilon \equiv 4a/\alpha = 1$. Our numerical simulations were performed with an initial timestep of $\Delta\tau = 10^{-9}$ and a timestep control that restricts the maximum relative change in the radiation energy density in any zone during a timestep to 1%. The spatial zoning was uniform with a zone size of $\Delta z = 0.001$, i.e. one-thousandth of a mean free path. The material properties (σ_a and C_v) and the angular approximation parameter (δ) were treated explicitly in time. We have numerically solved this problem using the pure P_1 diffusion approximation, the pure DP_0 diffusion approximation, and the angularly adaptive P_1-DP_0 diffusion approximation with $\beta = 0.1$ (see Eq. (38)). We present results both without flux limiting and with flux limiting using the Larsen flux limiter.

The radiation energy density from the simulations without flux limiting is plotted in Fig. 1 as a function of space at several time values. The exact semi-analytic transport values are plotted as points. The P_1 diffusion solution for the radiation energy density has an error of approximately 20% early in time ($\tau \approx 0.1 - 10$) for small z. The extremely large errors for larger z are due to the fact that the P_1 *diffusion* approximation (as opposed to the true P_1 approximation) propagates radiation with an infinite velocity [1]. As a result, the radiation front propagates too quickly. Similar comments hold regarding the errors in the material energy density predicted by the P_1 diffusion approximation. Later in time, the radiation and material approach thermodynamic equilibrium and the magnitude of the errors decreases significantly. Because the radiative transfer equation asymptotically limits to the P_1 diffusion approximation at thermodynamic equilibrium, this is a theoretically expected result.

Early in time (up to $\tau = 10$), the DP_0 diffusion solution is significantly more accurate overall than the P_1 diffusion solution (note the plots are on log-log scales). The DP_0 diffusion approximation suffers from the same infinite propagation velocity as the P_1 diffusion approximation and hence exhibits large errors past the wave front. As the radiation and material approach thermodynamic equilibrium later in time ($\tau > 10$), however, the DP_0 solution becomes significantly less accurate than the P_1 solution. These numerical results confirm the theoretical prediction described above that the DP_0 diffusion approximation will be less accurate than the P_1 diffusion approximation near thermodynamic equilibrium.

The adaptive P_1-DP_0 diffusion solution is essentially identical to the DP_0 diffusion solution for $\tau \leq 3.16228$ and is therefore more accurate than the P_1 diffusion solution. For $\tau = 10$, the adaptive P_1-DP_0 solution is slightly less accurate than the DP_0 solution but still significantly more accurate overall than the P_1 solution. At $\tau = 31.6228$, the adaptive P_1-DP_0 solution is significantly more accurate overall than either the P_1 or the DP_0 solution alone. For

$\tau = 100$, the adaptive P_1-DP_0 solution is slightly less accurate than the P_1 solution but drastically more accurate than the DP_0 solution.

The radiation energy density for the simulations using the Larsen flux limiter is plotted in Fig. 2 as a function of space at several time values. In general, the flux limiter gives improved accuracy over the results without flux limiting for all angular approximations, particularly near the radiation wave front. The conclusions from the flux-limited results are largely the same as for the results without flux limiting. The DP_0 diffusion approximation is generally more accurate than the P_1 diffusion approximation early in time, while the P_1 diffusion approximation is more accurate late in time as the radiation and material reach equilibrium. The adaptive P_1-DP_0 diffusion solution follows the DP_0 diffusion solution early in time and transitions towards the P_1 diffusion solution as the problem approaches equilibrium.

3.2 The Ganapol and Pomraning Non-Equilibrium Marshak Wave Benchmark Problem

The Ganapol and Pomraning non-equilibrium grey radiative transfer benchmark problem [7] consists of a semi-infinite homogeneous medium that is purely absorbing and isotropically irradiated at the free surface beginning at time zero. As in the Su and Olson problem described above, the opacity for this problem is assumed constant (independent of the material temperature) and the heat capacity of the material is assumed to be proportional to the cube of the material temperature. While not physically realistic, these assumptions make the equations of radiative transfer linear and allow their semi-analytic solution [7]. The equations describing the Ganapol and Pomraning radiative transfer problem are written in dimensionless form and are given by

$$\left(\epsilon\frac{\partial}{\partial\tau} + \mu\frac{\partial}{\partial z} + 1\right) U(z,\mu,\tau) = \frac{1}{2}V(z,\tau) \quad ,$$

$$0 \le z < \infty \quad , \quad -1 \le \mu \le 1 \quad , \quad \tau > 0 \quad , \tag{45}$$

coupled with the dimensionless material energy balance equation given by

$$\frac{\partial V(z,\tau)}{\partial\tau} = W(z,\tau) - V(z,\tau) \quad , \quad 0 \le z < \infty \quad , \quad \tau > 0 \quad , \tag{46}$$

where the various quantities are as defined for the Su and Olson problem described in Section 3.1. The boundary and conditions for Eq. (45) are given by

$$U(0,\mu,\tau) = \frac{1}{2} \quad , \quad \mu > 0 \quad , \tag{47}$$

$$\lim_{z\to\infty} U(z,\mu,\tau) = 0 \quad , \tag{48}$$

and the initial condition by

$$U(z,\mu,0) = 0 \quad . \tag{49}$$

The initial condition for Eq. (46) is given by

$$V(z,0) = 0 \quad . \tag{50}$$

Ganapol and Pomraning obtained a semi-analytic transport solution to this problem using a multiple collision series approach. They tabulated the radiation and material energy densities spatially integrated over the entire spatial domain as a function of the time variable τ.

Ganapol and Pomraning obtained semi-analytic results for several values of the parameter ϵ; we consider the $\epsilon = 1$ case in this paper. Our numerical simulations were performed with an initial timestep of $\Delta\tau = 10^{-9}$ and a timestep control that restricts the maximum relative change in the radiation energy density in any zone during a timestep to 1%. The spatial zoning was uniform with a zone size of $\Delta z = 0.001$, i.e. one-thousandth of a mean free path. We have numerically solved this problem using the pure P_1 diffusion approximation, the pure DP_0 diffusion approximation, and the angularly adaptive P_1-DP_0 diffusion approximation with $\beta = 0.1$ (see Eq. (38)). We present results both without flux limiting and with flux limiting using the Larsen square-root flux limiter.

The percent relative error (compared to the transport results [7]) in the radiation and material energy integrated over the spatial domain as computed by the P_1, DP_0, and adaptive P_1-DP_0 diffusion approximations is plotted in Fig. 3 versus the time variable τ. Considering first the results without flux limiting, the errors for all angular approximations are largest early in time and decrease as the problem evolves. The DP_0 diffusion approximation is more accurate than the P_1 diffusion approximation for $\tau \leq 10$. For $\tau \geq 20$, the DP_0 diffusion approximation becomes less accurate than the P_1 diffusion approximation as the radiation and material energy fields approach equilibrium. The adaptive P_1-DP_0 diffusion approximation yields essentially the DP_0 diffusion solution for $\tau \leq 4$ and is significantly more accurate than both the DP_0 and P_1 diffusion approximations for $\tau > 4$.

Considering next the flux limited results, we see that the error for all angular approximations is largest for $\tau \approx 0.1 - 0.2$. The flux-limited P_1 diffusion result is more accurate than the non-flux-limited P_1 diffusion result for all values of τ. The DP_0 flux-limited diffusion solution is of comparable or greater accuracy than the P_1 flux-limited diffusion solution for $\tau \leq 4$. For $\tau > 4$, the DP_0 flux-limited diffusion solution becomes less accurate than the P_1 flux-limited diffusion solution as the problem approaches thermodynamic equilibrium (as predicted theoretically). The adaptive P_1-DP_0 flux-limited diffusion solution

is more accurate than the P_1 flux-limited diffusion solution for $\tau \leq 7$ and is of comparable to slightly lower accuracy than the P_1 flux-limited diffusion solution for $\tau > 7$.

4 Conclusions

In this paper, we developed and applied the double P_0 flux-limited diffusion approximation for non-equilibrium grey radiative transfer problems in planar geometry. We described how the P_1 and DP_0 flux-limited diffusion approximations for non-equilibrium grey radiative transfer in planar geometry can be combined into a single set of equations with a single parameter specifying the local angular approximation to be used at any point in space and time. We then prescribed an adaptive method for choosing the local angular approximation such that the P_1 angular approximation is used at and near thermodynamic equilibrium and the DP_0 angular approximation is used otherwise.

We compared the accuracy of the DP_0 and the adaptive P_1-DP_0 diffusion approximations to the P_1 diffusion approximation, both without and with flux limiting, using two non-equilibrium semi-analytic test problems developed by Su and Olson [6] and Ganapol and Pomraning [7]. We found that the DP_0 diffusion approximation is generally more accurate than the P_1 diffusion approximation when the radiation and material are out of equilibrium but is less accurate near thermodynamic equilibrium. These results motivated the development of an adaptive P_1-DP_0 diffusion approximation in which the local angular approximation used is adaptively determined at each point in space and time by thermodynamic equilibrium considerations. This adaptive angular approximation provides improved accuracy for the Su and Olson test problem over the P_1 diffusion approximation during the non-equilibrium phase and essentially the same accuracy near thermodynamic equilibrium. Similar results are obtained for the Ganapol and Pomraning test problem. These numerical conclusions should be further verified on a broader range of problems with more realistic temperature-dependent opacities.

The full P_1 approximation, as opposed to the P_1 *diffusion* approximation, retains information regarding the time derivative of the radiation flux. As a result, the full P_1 approximation can give more accurate results than its corresponding diffusion approximation [1, 15]. One possible area of future work is to investigate whether the same holds true when comparing the full DP_0 and the DP_0 *diffusion* angular approximations. Another area of possible future work is to investigate whether the accuracy improvements demonstrated by the DP_0 diffusion approximation compared to the P_1 diffusion approximation remain when comparing the full P_1 and DP_0 approximations. If so, the full DP_0 and P_1 angular approximations could be combined into an adaptive angu-

lar approximation similar to the corresponding diffusion scheme presented in this paper. In addition, Olson et al [1] have recently proposed a so-called $P_{1/3}$ approximation that is an appropriately-weighted average of the full P_1 approximation and the P_1 diffusion approximation such that the correct radiation propagation velocity is obtained in the optically thin limit. This $P_{1/3}$ approximation retains first order accuracy in the equilibrium diffusion limit [8]. The same approach can be used with the full DP_0 approximation and the DP_0 diffusion approximation to develop a DP_0-based approximation with the correct radiation propagation velocity. These modified P_1- and DP_0-based angular approximations can also be combined in an angularly adaptive manner.

The double P_0 angular approximation has seen decreased use in recent years, due at least in part to its perceived inapplicability to multi-dimensional geometries. However, Paveri-Fontana and Amster [4] theoretically extended the DP_0 approximation for time-independent neutronics problems to multi-dimensional geometries using both a generalization of the typical angular moment derivation of the DP_0 approximation as well as a variational analysis. The diffusion equation they derived for the DP_0 approximation differed from the standard P_1 diffusion equation only in the value of the diffusion coefficent. However, their paper does not present numerical results to examine the accuracy of the method. Recently, Ripoll and Wray [18] proposed a multidimensional half-moment model for time-dependent grey equilibrium radiative transfer problems using a moments-based derivation. These authors make the same assumption about the direction of angular discontinuity as Paveri-Fontana and Amster (although they were apparently unaware of the previous work by Paveri-Fontana and Amster), but they close the system of equations using the more complicated maximum entropy approach. We plan to investigate the multidimensional extension of the work presented in the present paper using the variational approach of Paveri-Fontana and Amster.

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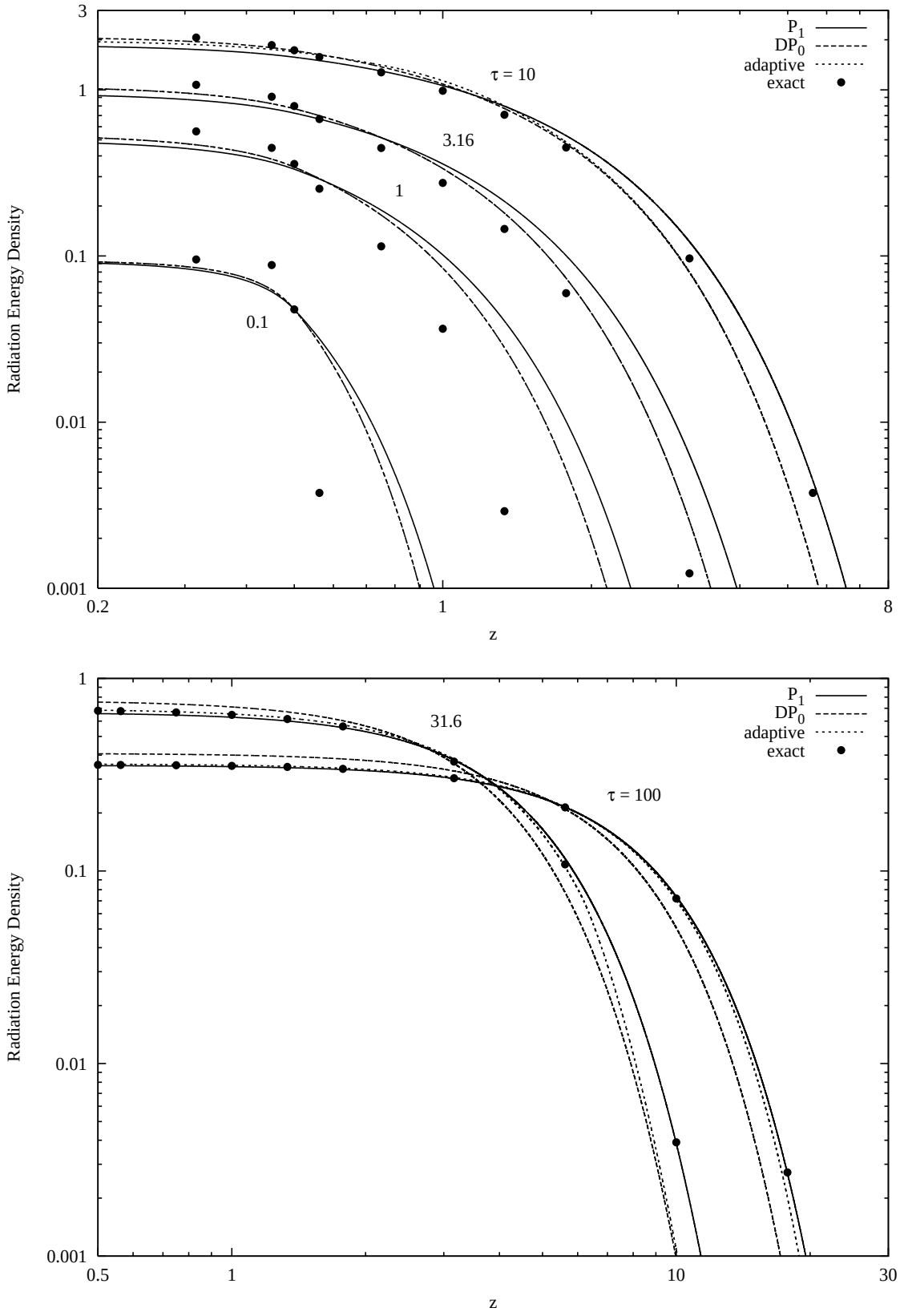


Fig. 1. Su and Olson radiation energy density as a function of space at several time values (no flux limiting)

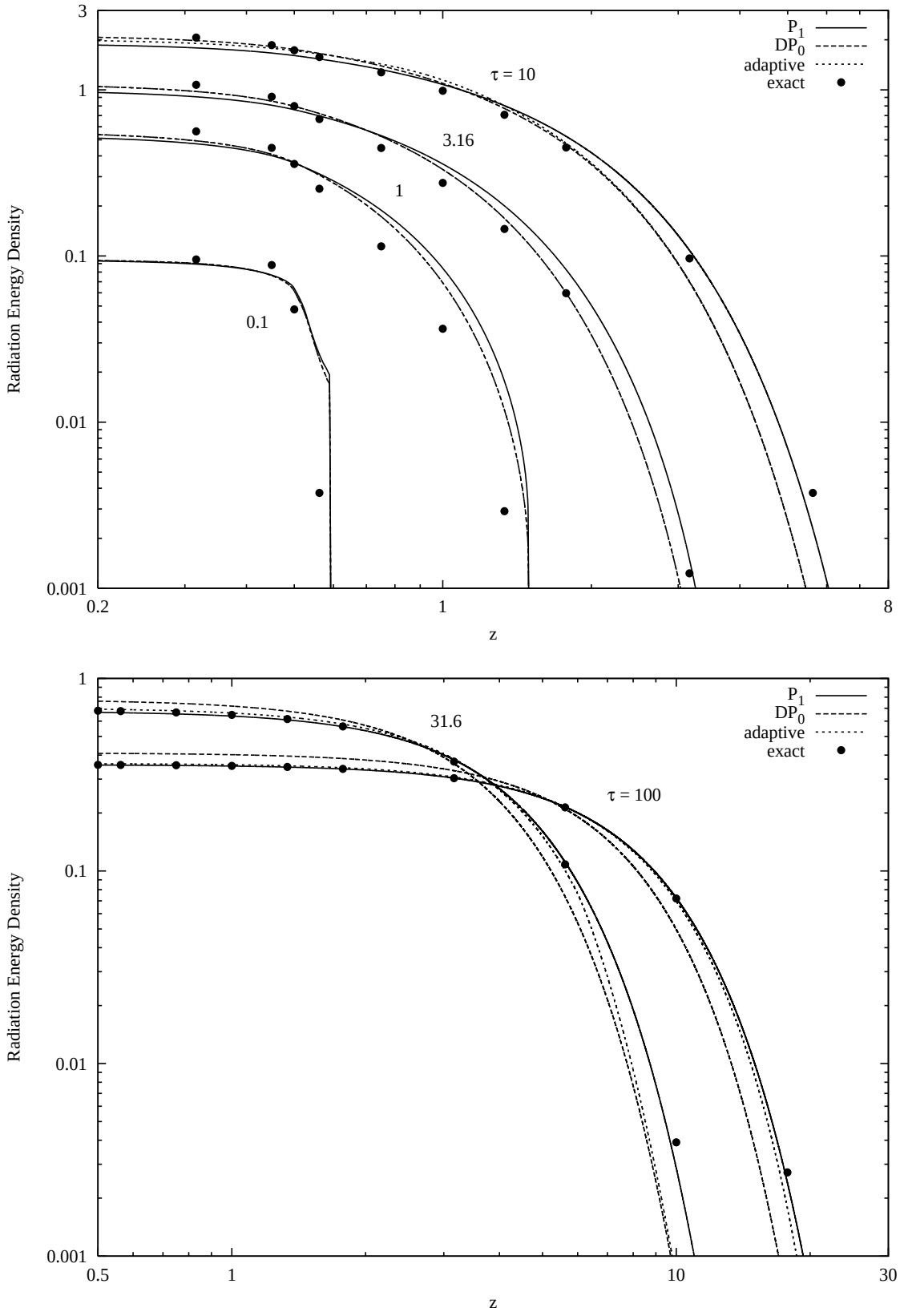
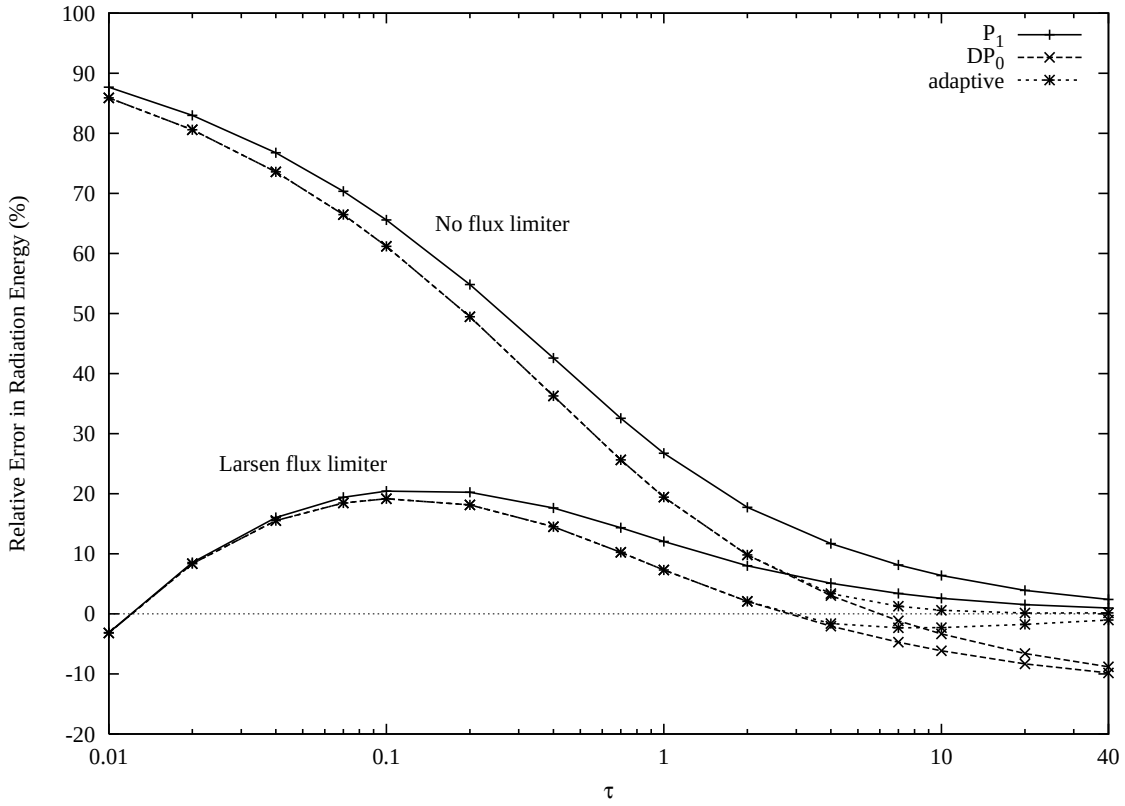


Fig. 2. Su and Olson radiation energy density as a function of space at several time values (Larsen flux limiter)



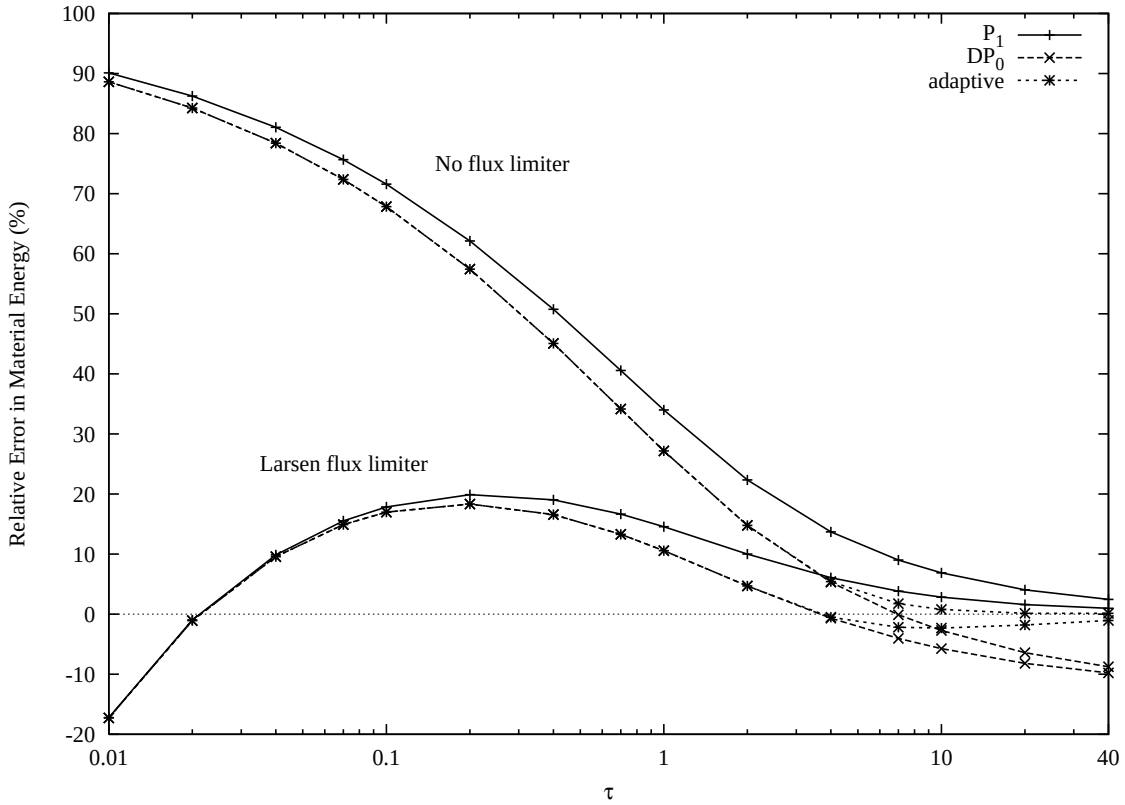


Fig. 3. Relative errors in Ganapol and Pomraning integrated radiation and material energy