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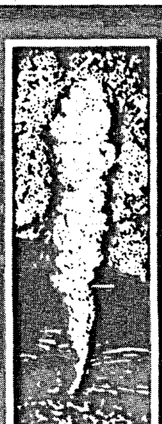

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A Convolution Boundary Element Method for Unsteady State Groundwater Flow in Homogeneous Aquifers

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Abstract

In this paper, Boundary Element (BEM) solutions were obtained for the transient flow of fluids through homogeneous, anisotropic porous media. The Green's function method with Euler method of forward time differencing and Laplace transform method have been used by previous authors. Unlike these methods, this paper uses the fundamental solution to the differential equation and the convolution behavior of the resulting integrals to obtain an implicit and stable solution. This allows large time steps to be taken without significant loss in accuracy. Comparison with the Laplace transform method and Green's function method with discrete time stepping, for two test cases, show that the method is very accurate. The computations however, become quite storage intensive owing to the dynamic increase in the number of stored matrices. It has been shown elsewhere that for certain problems with both Dirichlet and Neumann boundary conditions, asymptotic expression generated from exact solution is needed for starting the computational procedure. The present formulation alleviates this requirement.

These solutions are developed for use in the analysis of pressure transients in complex reservoir problems.

INTRODUCTION

Solution of reservoir engineering problems associated with fluid injection and fluid movement in reservoirs and its related pressure response at the well is of major importance in the exploitation of geothermal reservoirs. Usually these reservoirs are of complex geometries (shapes) and are produced by means of numerous wells. These problems are difficult to treat accurately and efficiently by numerical methods which suffer from dispersion and grid orientation effects. Analytical techniques are not available except for a very few regular geometries. In this paper we explore the use of the boundary element method to solve such problems.

The Boundary Integral Equation Method (BIEM), or Boundary Element Method (BEM) as it is often called, is gaining popularity in solving problems encountered in solid mechanics, heat transfer, groundwater hydrology and various other fields. The methodology of solving partial differential equations follows closely that of finite element method where the governing differential equation is cast in an integral form. Instead of choosing basis functions which approximate the differential equations in the domain as in the finite element method, fundamental solutions to the differential operator are used to reduce the problem to quadratures. The solution becomes a pure boundary procedure if inhomogeneities in the differential operators are removed.

The primary advantage which has encouraged people to explore and use this technique is that the dimensionality of the problem is reduced by one. A 3-D problem is reduced to a 2-D problem and so forth. Since it is a boundary procedure, it conforms well to boundaries. The results obtained with this method are usually more accurate than finite difference or finite element methods, as it is an analytic technique requiring only numerical evaluation of integrals. Integration is a smoothing procedure and for well behaved functions can be performed quite accurately. In BEM the governing differential equation is exactly satisfied in the domain of the problem, it is only on the boundaries that the approximations are made. The only bottleneck in this procedure is finding the fundamental solutions to the differential operators.

The efficiency and accuracy of the method has been proved for elliptic operators. *Numbere et al.*(1986), and *Masukawa et al.*(1986) have used BEM as a useful streamline generating method for Laplace's equations with odd shaped boundaries and with moving interfaces. A variety of problems arising in groundwater hydrology have been solved by *Liggett* and coworkers [*Liggett and Liu* (1983), *Lafe and Liggett* (1981), *Taigbenu and Liggett* (1985)] using Boundary Element Methods.

The present work involves solution of unsteady state (transient) problems with BEM. A slightly different formulation than the one presented by *Liggett et al.*(1979) has been used. Another similar problem, the solution of the diffusivity equation by integral equations, which govern the transient heat conduction in materials, was looked at by *Rizzo and Shippy* (1970). They removed the time derivative by converting the problem into Laplace space.

Taigbenu and Liggett (1985), use the fundamental solution to the diffusion equation and cast it in terms of an integral equation. Depending on the interpolating functions used between nodes, some of the integrals, in the discretized equations, can be performed analytically. They performed the analytic integrations in space first and then used Euler's method of time stepping to evaluate the solution at each time step. Use of this method in a mixed type problem [with both Neumann (flux) and Dirichlet (potential) boundary conditions] requires that the initial normal derivatives of the velocity potential be known. They derived an asymptotic expression from the exact solution and used it to start the computational procedure at early time.

We use the convolution character of the integral equation as observed by *Wrobel and Brebbia* (1981) and *Pina* (1984) to develop a boundary element code. The formulation is implicit in nature and thus very stable with respect to time step sizes. Performing the time integrations analytically removes the singularity of the fundamental solution in the time dimension. The space integrals are then evaluated analytically or by the use of accurate Gaussian quadrature techniques. This formulation does not require knowledge of the analytic solutions beforehand in order to develop asymptotic expressions at early times to start the computational procedure. *Wrobel and Brebbia* (1984) suggested a term by term integration of the series to find values of singular integrals obtained when the collocation point is on the same boundary element as the field point. The series converges very slowly for large arguments which are frequent due to the small time step sizes used. We provide here an analytic integration technique in terms of smoothly behaved functions which are easier to evaluate.

In the solution process with constant time step size, the left hand side coefficient matrix is generated once, inverted and stored. At every step, one additional matrix needs to be generated. However, all the previous coefficient matrices and solution vectors need to be stored because of the convolution character of the problem. But with high speed auxiliary memory access and large swap spaces in modern computers this extra storage requirement is not much of a problem.

Some simple test problems which have exact solutions were used to check the efficacy of the technique.

FORMULATION

The continuity equation for two-dimensional flow of a slightly compressible, single phase fluid in a homogeneous, anisotropic and confined reservoir is (*Aziz and Settari* (1979))

$$\nabla \bullet \left(\frac{K\rho}{\mu} \nabla P\right) = \frac{\partial(\phi\rho)}{\partial t} + Q^* \qquad (1)$$

where Darcy's law has been used. Q^* is the strength of a sink in mass per unit volume per unit time. Using the equation of state for small and constant compressibility fluid and assuming that the permeability tensor can be diagonalized and also that the viscosity of the fluid and the porosity of the medium are constant, we obtain in a cartesian coordinate system, with the coordinate axes aligned with the principal permeability directions, the following equation

$$\frac{\partial}{\partial x'}\left[\frac{k_{xx}}{\mu}\frac{\partial P}{\partial x'}\right] + \frac{\partial}{\partial y'}\left[\frac{k_{yy}}{\mu}\frac{\partial P}{\partial y'}\right] = \phi c \frac{\partial P}{\partial t} + \frac{Q^*}{\rho} \qquad (2)$$

Assuming k_{xx} and k_{yy} to be constants and performing a coordinate transformation given by

$$x = x' \; ; \qquad y = y'\left[\frac{k_{xx}}{k_{yy}}\right]^{1/2} \qquad (3)$$

we obtain

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{1}{\eta}\frac{\partial P}{\partial t} + \frac{Q^*\mu}{\rho k_{xx}} \qquad (4)$$

where, $\eta = \dfrac{k_{xx}}{\phi\mu c}$

Since the system geometry can be odd, the above system can be normalized with respect to the area (A) of the system. A description of a typical system geometry is shown in Fig. 1. Thus, defining

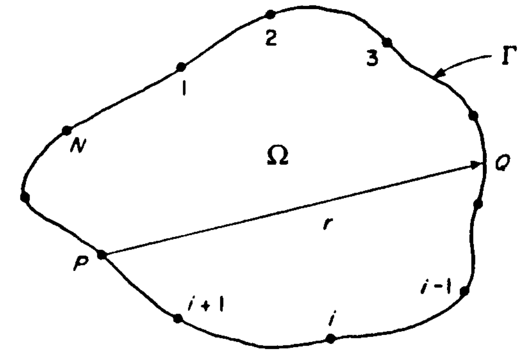

Figure 1. Typical computational domain

$$t_{DA} = \frac{k_{xx}\, t}{\phi\mu c A} = \frac{\eta t}{A} \; ; \qquad P_D = \frac{P_i - P}{P_i}$$

$$x_D = \frac{x}{\sqrt{A}} \; ; \qquad y_D = \frac{y}{\sqrt{A}} \qquad (5)$$

gives

$$\frac{\partial^2 P_D}{\partial {x_D}^2} + \frac{\partial^2 P_D}{\partial {y_D}^2} = \frac{\partial P_D}{\partial t_{DA}} + Q_D \qquad (6)$$

where,

$$Q_D = \frac{Q^*\mu A}{\rho k_{xx}}$$

and Q_D is non-dimensionalized with respect to a flow rate of unity.

We drop the subscripts hereafter for convenience. Now, three types of boundary conditions can be applied to Eqn. (6)

$$P = P_1 \qquad on\ \Gamma_1 \in \Gamma \quad (Dirichlet)$$

$$\frac{\partial P}{\partial n} = q \qquad on\ \Gamma_2 \in \Gamma \quad (Neumann) \qquad (7)$$

$$\alpha P + \beta\frac{\partial P}{\partial n} = \gamma \qquad on\ \Gamma_3 \in \Gamma \quad (Robin,\ Radiation\ or\ Mixed)$$

The free space Green's function for the above equation has been derived by *Greenberg* (1971), *Zauderer* (1983), *Morse and Feshbach* (1953). A two dimensional free space Green's function is defined [*Carslaw and Jaegar* (1959)] as the pressure at (x,y) at a time t due to an instantaneous line source of strength unity generated at the line P(ξ,η) at the time τ. The medium is initially at zero pressure and infinite in extent. The integral equation formulation of equation(6) using the divergence theorem of Gauss has been given by *Liggett and Liu* (1983). The integral equation is

$$2\alpha P(x,y,t) = \int_t d\tau \int_\Gamma \left(G\frac{\partial P}{\partial n} - P\frac{\partial G}{\partial n}\right)ds + \int_\Omega G_0 P_0\, dA$$

$$+ \int_t d\tau \int_\Omega G Q\, dA \qquad (8)$$

where,

$$G = \frac{1}{(t-\tau)}\, H(t-\tau)\, \exp\left[-\frac{r^2}{4(t-\tau)}\right] \qquad (9)$$

is the free space Green's function for the diffusion equation. and

H(t - τ) is the Heaviside step function.

also

$$r^2 = (x-\xi)^2 + (y-\eta)^2$$

where, ξ and η are the coordinates of a fictitious source point, and

$$\alpha = 2\pi \qquad \text{if } (x,y) \in \Omega \qquad (10a)$$

$$\alpha = \theta \qquad \text{if } (x,y) \in \Gamma \qquad (10b)$$

SOLUTION OF THE INTEGRAL EQUATION

If there is no inhomogeneity in the governing equation i.e.; Q_D is zero and also the equation is non-dimensionalized such that the initial condition is homogeneous, then the solution becomes strictly a boundary procedure. This is not a limitation because the forcing function on the right hand side of the differential equation is usually a source or a sink term. Since the differential operator is linear, we can use the concept of singularity programming to superpose the contribution due to sources and sinks separately onto the solutions free of sources and/or sinks.

Equation(8) can then be solved by choosing a finite number of elements on the boundary. An interpolating function for pressure and flux on a boundary element both in space and time dimensions is assumed. The integral equation could be solved by a collocation type technique by moving a fictitious source point to all the nodes and generating enough equations so that they match the number of unknowns. The problem then reduces to one of solving a matrix equation.

For simplicity we choose interpolating functions between boundary elements that are linear in space and constant in time. Higher dimensional elements could be chosen. *Wrobel and Brebbia* (1981) show some of the integrals arising from choosing higher dimensional interpolating functions in time. Pressure or the normal derivative of pressure at any point on the element is expressed in terms of the nodal values as follows :

$$P = [(P_{j+1} - P_j)\,\xi + (\xi_{j+1}P_j - \xi_j P_{j+1})]/(\xi_{j+1} - \xi_j) \qquad \xi_j < \xi < \xi_{j+1} \qquad (11)$$

$$\left(\frac{\partial P}{\partial n}\right) = \left\{\left[\left(\frac{\partial P}{\partial n}\right)_{j+1} - \left(\frac{\partial P}{\partial n}\right)_j\right]\xi + \left[\xi_{j+1}\left(\frac{\partial P}{\partial n}\right)_j - \xi_j\left(\frac{\partial P}{\partial n}\right)_{j+1}\right]\right\} \Big/ (\xi_{j+1} - \xi_j) \qquad \xi_j < \xi < \xi_{j+1} \qquad (12)$$

where, ξ is the local coordinate varying along the element. Integrations are performed after transferring every element in a moving local coordinate system. An illustration of a local coordinate system is given in Fig. 2. After performing all the

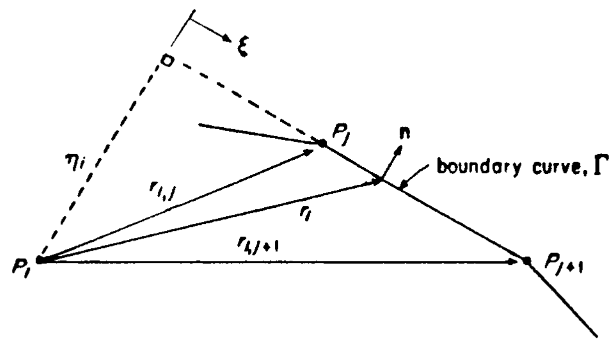

Figure 2. Local co-ordinate system

integrations which can be done analytically we obtain,

$$\sum_{j=1}^{N} 2\alpha_i\, \delta_{ij}\, P_j = A_i + B_i + \sum_{j=1}^{N}\left[(AA)_{I_{ij}} + (AA)_{II_{ij}}\right]; \quad i = 1,N \quad (13)$$

where

$$A_i = \int_{\Omega} \exp\left[\frac{-\frac{r^2}{4(t-\tau)}}{(t-\tau)}\right] P_0\, dA \qquad (14)$$

$$B_i = \int_{t} d\tau \int_{\Omega} \exp\left[\frac{-\frac{r^2}{4(t-\tau)}}{(t-\tau)}\right] Q\, dA \qquad (15)$$

$$(AA)_{I_{ij}} = \frac{\eta_i}{(\xi_{j+1} - \xi_j)}(P_{j+1} - P_j)\left[E_1\left[\frac{\xi_j^2 + \eta_i^2}{4(t-\tau)}\right] - E_1\left[\frac{\xi_{j+1}^2 + \eta_i^2}{4(t-\tau)}\right]\right]$$
$$+ \frac{2\eta_i}{(\xi_{j+1} - \xi_j)} \exp\left[-\frac{\eta_i^2}{4(t-\tau)}\right](\xi_{j+1}P_j - \xi_j P_{j+1})$$
$$\left[\int_{\xi_j}^{\xi_{j+1}} \frac{d\xi}{(\xi^2 + \eta_i^2)} \exp\left[-\frac{\xi^2}{4(t-\tau)}\right]\right] \qquad (16)$$

and

$$(AA)_{II_{ij}} = \frac{(P_{n_{j+1}} - P_{n_j})}{(\xi_{j+1} - \xi_j)}\left[\frac{1}{2}\left\{(\xi_{j+1}^2 + \eta_i^2)\, E_1\left[\frac{\xi_{j+1}^2 + \eta_i^2}{4(t-\tau)}\right]\right.\right.$$
$$\left. - (\xi_j^2 + \eta_i^2)\, E_1\left[\frac{\xi_j^2 + \eta_i^2}{4(t-\tau)}\right]\right\} + 2(t-\tau)\exp\left[-\frac{\eta_i^2}{4(t-\tau)}\right]$$
$$\left.\left\{\exp\left[-\frac{\xi_j^2}{4(t-\tau)}\right] - \exp\left[-\frac{\xi_{j+1}^2}{4(t-\tau)}\right]\right\}\right] + \frac{(\xi_{j+1}P_{n_j} - \xi_j P_{n_{j+1}})}{(\xi_{j+1} - \xi_j)}$$
$$\left[\int_{\xi_j}^{\xi_{j+1}} E_1\left[\frac{\xi^2 + \eta_i^2}{4(t-\tau)}\right] d\xi\right] \qquad (17)$$

where,

$$E_1(x) = \int_x^{\infty} \frac{e^{-u}}{u}\, du \qquad (18)$$

Singular Integral Evaluation

When the collocation point is on the same boundary element as the field point we obtain integrals which are singular at one of the limits. This can be seen easily from Fig. 2. This happens when η is zero and ξ_j or ξ_{j+1} is zero. The usual techniques for numerical integration are unsuitable and special care needs to be taken. Such a functions is $E_1(z)$, which is singular at $z = 0$. *Pina*(1984) used series expansion of $E_1(z)$ which is uniformly and absolutely convergent [*Abramowitz and Stegun* (1964) page 229] and integrated it term by term. For arguments greater than 1.0 the series converges very slowly. Since the time step sizes used are usually very small thus the argument of the function tends to be large. We present here a simple closed form integration in terms of very smoothly behaved functions.

The singular integrals that are encountered are of the form :

$$\int_0^c E_1(\frac{x^2}{4t})\, dx$$

We obtain the following result;

$$\int_0^c E_1(x^2/4t)\, dx = cE_1(\frac{c^2}{4t}) + \sqrt{\frac{\pi}{\alpha}}\, erf(\sqrt{\alpha}c) \qquad (19)$$

The details of the derivation are given in Appendix-A.

Solution of Matrix Equations

At every node, one piece of information is prescribed by the boundary condition, and the other is unknown. Thus, one could move the fictitious source point to all the boundary nodes. This will generate N equations for N boundary nodes. The system of N equations in N unknowns can then be solved, in principle. The matrix equation generated is

$$\sum_{\alpha=1}^{t} H^{n\alpha} u^{\alpha} = b^{\alpha} \qquad (20)$$

Note that the integrals that are obtained have time both in the limit and in the integrand and thus are of the form,

$$\int_0^t f(\tau)\, G(t-\tau)\, d\tau \qquad (21)$$

The convolution character of the above equation is evident. If we assume a constant time step size then, $t = t_0 + n\Delta t$, where Δt is the step size. If the solution up to $(n-1)^{th}$ time step is known then the solution at the n^{th} time level could be found from the above equations as

$$H^{n1}u^1 + H^{n2}u^2 + ... + H^{nn}u^n = b^n \qquad (22a)$$

which on transposing becomes

$$H^{nn}u^n = b^n - \sum_{\alpha=1}^{n-1} H^{n\alpha}u^{\alpha} \qquad (22b)$$

The matrix $H^{n\alpha}$ depends entirely on the geometry of the system and the step size. At the n^{th} time level for example, Equation(22b) suggests that the unknown vector u^n at time step n is multiplied by a coefficient matrix H^{nn}. Because of the convolution character of the matrices $H^{nn} = H^{11}$, if the time step size is constant. Similarly $H^{(n)(n-1)} = H^{21}$. Similar relations could be derived between various matrices. What this provides is the need to create only one extra matrix at every time step. Another way to solve the same problem would be to start the time integrations from zero at every time step. This becomes quite time consuming, whereas the above method needs extra storage space. *Taigbenu and Liggett* (1985) have used the solution obtained after the first time step as the initial condition to advance in time. This too reduces the storage requirement but introduces a domain integral in the integral equation. To evaluate such an integral the entire domain has to be discretized once every time step in a finite element type subdivision. This reduces the charm of the method as being a boundary procedure, though it may improve the efficiency of calculations.

VERIFICATION

A computer code has been developed on the foregoing lines. The program has been tested on four simple problems until now. These problems have closed form analytical solutions to compare with.

A Neumann Problem

The first example is a solution to the diffusivity equation in a porous medium with a step change in the flux at the inlet end. All the boundary conditions are of Neumann type and the initial condition is homogeneous. Figure 3 shows the problem domain. The governing equation and the boundary conditions are as follows :

$$\nabla^2 P = \frac{\partial P}{\partial t}$$

$$P(x,y,0) = 0$$

$$\frac{\partial}{\partial n}P(0,y,t) = 1 \qquad \frac{\partial}{\partial n}P(1,y,t) = -1$$

$$\frac{\partial}{\partial n}P(x,0,t) = \frac{\partial}{\partial n}P(x,1,t) = 0$$

The results are shown in Figs. 4 and 5. Pressure at the inlet end is plotted as a function of time. Smaller step sizes give more accurate results but the implicit nature of the solution procedure allows us to take large step sizes.

A Mixed Problem

The problem domain is the same as the previous one. The governing equation and the initial condition is also the

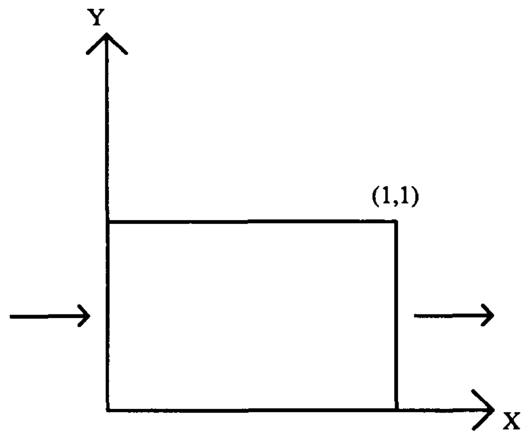

Figure 3. Domain used for example calculations

same. The boundary conditions are

$$P(0,y,t) = 1 \qquad P(1,y,t) = 0$$

$$\frac{\partial}{\partial n}P(x,0,t) = \frac{\partial}{\partial n}P(x,1,t) = 0$$

In this problem the inner and the outer boundaries are held at constant pressures and the other two boundaries at a no flux condition. Figures 6 through 9 show the comparisons with the analytical solution. Pressure solutions match very well even for large time steps. But flux is infinite at early times and thus poses a problem for large time steps. On taking sufficiently small time steps the fluxes match well. The effect of time step size on the solutions for pressures are shown in Figs. 6 and 7 whereas Figs. 8 and 9 show the effect of time step size on the calculation of fluxes at the inner boundary.

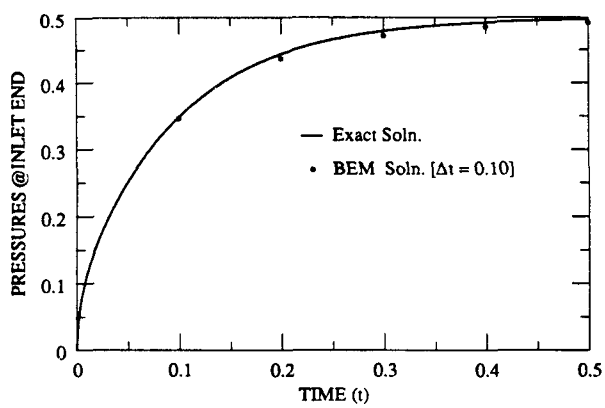

Figure 4. Neumann problem

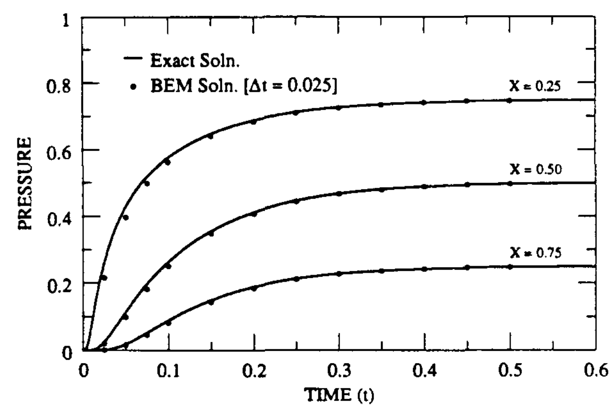

Figure 6. Mixed problem

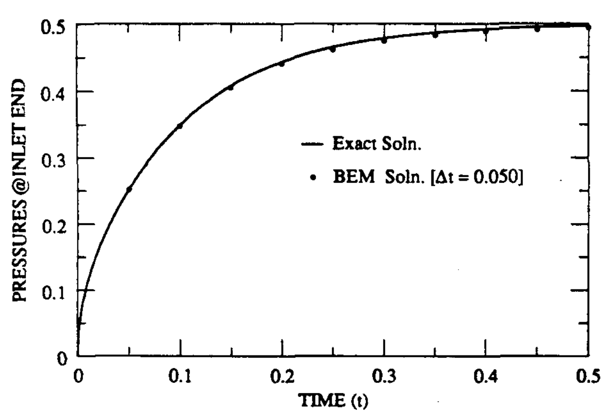

Figure 5. Effect of step size on the solution for Neumann problem

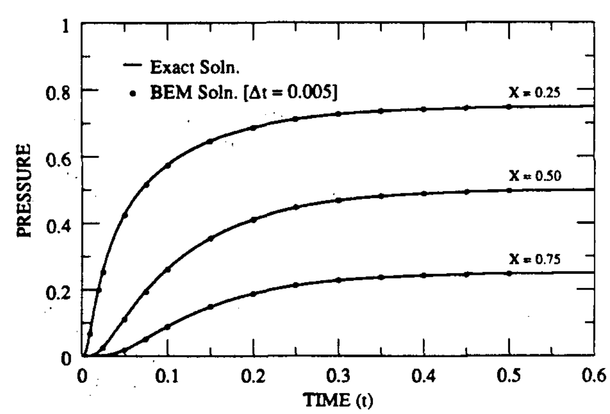

Figure 7. Effect of step size on the solution for mixed problem

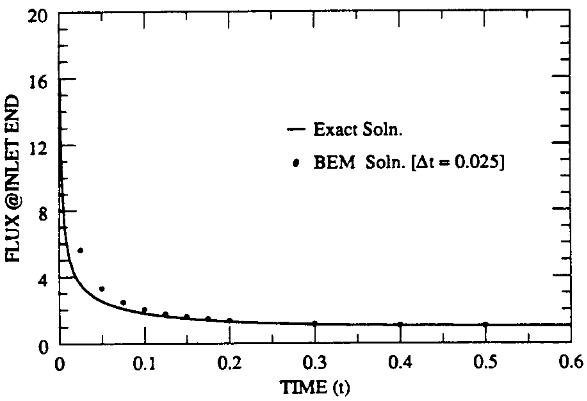

Figure 8. Mixed problem: Matching of flux singularity at the inlet

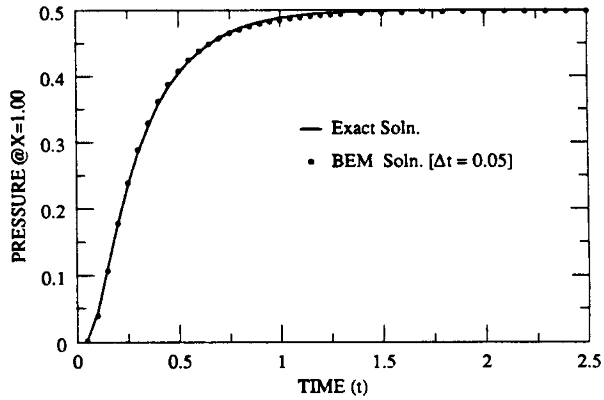

Figure 10. Constant pressure inner boundary and radiation outer boundary

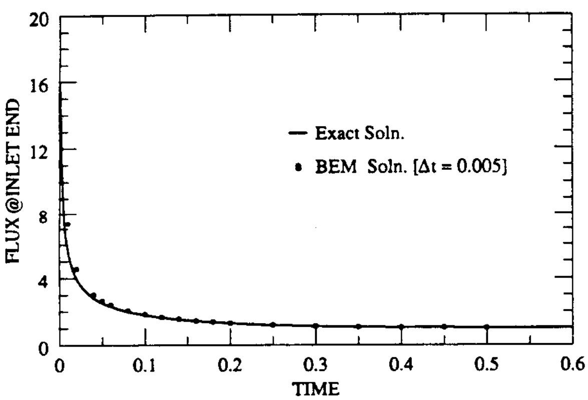

Figure 9. Effect of step size on flux for mixed problem

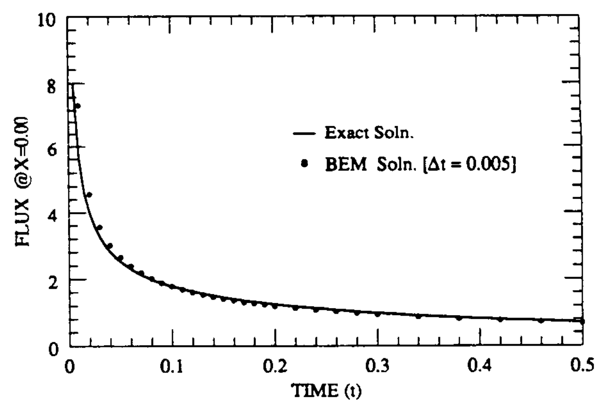

Figure 11. Matching fluxes for constant pressure IBC and radiation OBC

Problems with Radiation Boundary Conditions

The problem domain remains the same as the previous problems. Two different inner boundary conditions were used. The outer boundary conditions are the same in both the examples considered below. The outer boundary conditions for the two problems are

$$P + \frac{\partial}{\partial n} P(1,y,t) = 0$$

Figure 10 shows the pressure response at the outer boundary with time for the case of a constant pressure inner boundary condition. It is the flux at the inner boundary which has a singular behavior and is difficult to match. With refinement of step sizes we could match the fluxes well at early times. Figure 11 shows the flux at the inner boundary as a function of time, for a step size of 0.005.

Figure 12 shows the pressure response at inner and outer boundaries with time. This is for the case of a prescribed flux inner boundary condition and radiation outer boundary condition. The BEM solution matches the analytical solution very closely.

These problems are very simple but they do show the efficacy of the method which can now be used on complicated and odd shaped domains with accuracy.

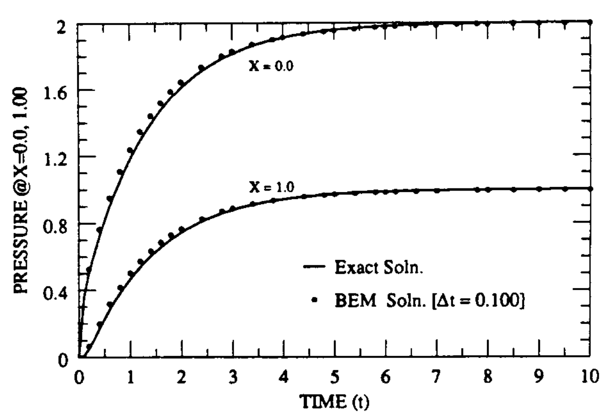

Figure 12. Constant flux inner boundary and radiation outer boundary

CONCLUSIONS

Problems governed by diffusivity equation in odd shaped boundaries can be solved efficiently and accurately by the Boundary Element Method. A slightly different formulation compared to other authors has been discussed and implemented. Simple problems with analytical solutions have been tested to check the validity of the method. All combinations of the three boundary conditions *viz;* Dirichlet, Neumann and Radiation (mixed) have been used. The solution methodology holds promise for solution of problems leading to pressure transient testing for single phase flow.

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APPENDIX-A

We evaluate here the singular integral obtained when the collocation point is on the same boundary element as the field point.

$$I = \int_0^c E_1(\alpha x^2)\, dx \qquad \text{(A1)}$$

where,

$$\alpha = \frac{1}{4t} \qquad \text{(A2)}$$

Integrating once by parts, we obtain

$$I = x\, E_1(\alpha x^2)\Big|_0^c - \int_0^c \frac{d}{dx}\Big[E_1(\alpha x^2)\Big]\, dx \qquad \text{(A3)}$$

$$= cE_1(\alpha c^2) - xE_1(\alpha x^2)\Big|_0 + 2\int_0^c \exp\,(-\alpha x^2)\, dx \qquad \text{(A4)}$$

Since $E_1(\alpha x^2)$ as $x \to 0$ grows logarithmically and hence grows slower than any polynomial, thus $x \to 0$ faster than E_1 grows.

$$=> \qquad x\, E_1(\alpha x^2)\Big|_0 \to 0 \;\; as \;\; x \to 0 \qquad \text{(A5)}$$

thus,

$$\int_0^c E_1(\alpha x^2)\, dx = cE_1(\alpha c^2) + \sqrt{\frac{\pi}{\alpha}}\, erf\,(\sqrt{\alpha} c) \qquad \text{(A6)}$$

where

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\,(-x^2)\, dx \qquad \text{(A7)}$$

For large arguments $E_1(\alpha c^2)$ goes to zero asymptotically and $erf\,(\sqrt{\alpha} c)$ goes to 1 asymptotically. Thus a very stable and easy to evaluate form is obtained.