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Although the concept of the photon as a quantum particle is sharpened by the quantization of the energy of the classical radiation field in a cavity, the photon's spin has remained a classical degree of freedom. The photon is considered a spin-1 particle, although only two classical polarization states transverse to its direction of propagation are allowed. Effectively therefore the photon is a spin-1/2 particle, although it still obeys Bose-Einstein statistics because the photon-photon interaction is zero.

Here we show that the two polarization states of the photon can be quantized using Pauli's spin vector, such that a suitable equation of motion for the photon is Dirac's relativistic wave equation for zero mass and zero charge. Maxwell's equations for a free photon are inferred from the Dirac-field formalism and thus provide proof of this claim.

For photons in the presence of electronic sources for electromagnetic fields we posit Lorentz-invariant inhomogeneous photonic equations of motion. Electrodynamic operator equations are inferred from this modified Dirac-field formalism which reduce to Maxwell's equations if spin-dependent terms in the radiation-matter interaction are dropped.

I. Introduction

It is generally believed that spin-1/2 particles obey Fermi-Dirac statistics. The photon on the other hand effectively has two polarization states transverse to its direction of propagation and yet is known to obey Bose-Einstein statistics. This phenonmenon is explained by saying that the photon is a spin-1 particle, not withstanding the supression of one of its three polarization states by transversality. Although the concept of the photon as a quantum particle is sharpened by the quantization of the energy of the classical radiation field in a cavity [1], it's polarization or "spin" states are treated classically, and in this sense the photon is not completely quantized as compared with other quantum particles.

Would the quantization of the photon's two polarization states as a spin-1/2 particle violate Bose-Einstein statistics? It would not because the photon-photon interaction is zero. Conversely would a theory ignoring the electron's spin violate Fermi-Dirac statistics? Obviously it would not: for example the Schroedinger description of the electronic problem does not depend on spin. The exclusion of more than two electrons from a single orbital in the many-electron problem arises, not from the spin property of an individual electron, but rather from the symmetry of the Hamiltonian with respect to particle interchange. It is only through a Zeeman or some other electron-polarization experiment that we learn about the spin degree of freedom of the electron.

On the other hand if the electron lost its charge, it would still be a spin-1/2 particle but, having zero interparticle interaction, it would obey Bose-Einstein statistics. In this paper we show that a suitable equation of motion for the photon, in which its two polarization or "spin" states are also quantized, is the Dirac equation for zero mass and zero charge. Maxwell's equations for a free photon can be inferred from the Dirac-field formalism and thus provide proof of this claim. From the new point of view the standard quantization of the classical radiation field [1] is in reality a second quantization of a relativistic quantum field for a mass-0, spin-1/2 particle.

The plan of the paper is as follows. In section II we show that each of Dirac's pair of first-order equations can be factorized into the scalar product of four-vectors for electron momentum and electron spin. Hence each first-order equation is a Lorentz invariant since the scalar product of four-vectors is always a Lorentz invariant. The generality of this result suggests that first-order Lorentz-invariant equations of motion can be written as the scalar product of momentum and spin four-vectors for any particle, including the photon. We assign Dirac's equation for zero mass and zero charge to the photon and show that Maxwell's equations for a free photon can be inferred from the Dirac-field formalism.

In section III we show that electrodynamic operator equations follow from the Dirac-field formalism once it is modified by positing Lorentz-invariant inhomogeneous electronic sources for electromagnetic fields. Maxwell's equations are inferred from this set by dropping all spin-dependent terms in the radiation-matter interaction.

In section IV we present our conclusions.

II. Lorentz-invariance of Dirac's first-order equations; photonic equations of motion from which Maxwell's equations for a free photon can be inferred

In this section we show -for the first time to our knowledge - that Dirac's pair of first-order equations can be written as the scalar products of a pair of four vectors and hence are themselves a pair of Lorentz invariants. Hitherto it was known that Dirac's Hamiltonian is a Lorentz invariant such that the Lorentz covariance of his equation required a separate proof [2].

Dirac's equation, or the Dirac-Fock equation as appropriate, is used for example to calculate high-Z atomic structures [3], high-Z materials' equations-of-state [4], or electron or positron inelastic cross sections from highly-stripped ions [5]. It is also used in particle physics as a field theory for spin-1/2 particles and their interactions.

We write a pair of Lorentz invariants first as the scalar product of the electron four-momentum $(\frac{1}{c}(i\hbar\frac{\partial}{\partial t} - e\,\Phi - mc^2, i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})$ and the wave-function four-vector $(\psi_L, \vec{\sigma}\psi_S)$ and second as the scalar product of the electron four-momentum $(\frac{1}{c}(i\hbar\frac{\partial}{\partial t} - e\,\Phi + mc^2, i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})$

and the wave-function four-vector $(\psi_S, \vec{\sigma}\psi_L)$, giving identically Dirac's pair of first-order equations,

$$\frac{1}{c}(i\hbar\frac{\partial}{\partial t} - e\Phi - mc^2)\,\psi_L + \vec{\sigma}\cdot(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})\,\psi_S = 0 \tag{1a}$$

$$\frac{1}{c}(i\hbar\frac{\partial}{\partial t} - e\Phi + mc^2)\,\psi_S + \vec{\sigma}\cdot(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})\,\psi_L = 0 \quad , \tag{1b}$$

where $\vec{\sigma}$ is Pauli's vector and the wave functions are the large (L) and small (S) two-component spinors.

This set of equations generally describes a spin-1/2 particle of arbitrary mass, although Dirac originally derived the set for the electron. For example the neutrino is described by Eqs. (1) with mass m and charge e set equal to zero.

The scalar product of two four-vectors is always a Lorentz invariant. Hence each equation of Eqs. (1) is a Lorentz invariant in which the Lorentz constant is zero. Next we reconcile our derivation of Eqs. (1) with the traditional presentation of Dirac's equation. Dirac's four-component Hamiltonian divided by c can be written by setting the scalar product of the four-vectors $(\gamma_0, -\vec{\gamma})$ and

$$(\frac{1}{c}(i\hbar\frac{\partial}{\partial t} - e\Phi), -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A})$$ equal to mc,

$$\frac{\gamma_0}{c}\left(i\hbar\frac{\partial}{\partial t} - e\Phi\right) + i\hbar\,\vec{\gamma}\cdot\left(\vec{\nabla} + \frac{e}{c}\vec{A}\right) = mc \quad , \tag{2}$$

where $\gamma_0 = \beta$ and,

$$\vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} . \tag{3}$$

It is understood that the four-component Hamiltonian given by Eq. (2) operates on the column vector $\begin{pmatrix} \Psi_L \\ \Psi_S \end{pmatrix}$ of large and small two-component spinors. Operating with Eq. (2) on the column vector we obtain identically Eqs. (1).

The generalized Lorentz conditions given by Eqs. (1) provides an independent proof of the covariance of Dirac's equation. If the quantum mechanical equations of motion are themselves a pair of Lorentz invariants, as well as the Hamiltonian, then Dirac's equation must surely be covariant. Dirac's procedure for deriving his equation, in which the Lorentz invariant is the Hamiltonian [Eq. (2)], requires a second step to prove covariance [2] of his equation. The Lorentz invariance of each of his pair of first-order equations as a product of four-vectors was not recognized at the time or to our knowledge in later studies.

In view of the factorizbility of each member of the set of equations into the product of a four vector for momentum and a four-vector for spin, one might ask if the set for mass-0, which is assumed to be the correct equation of motion for the neutrino, might also be suitable as an equation of motion for the photon, where Pauli's vector is used to represent the two polarization states of the photon transverse to its direction of propagation. The bosonic character of the photon however is preserved by its zero photon-photon interaction Hamiltonian. This would seem to be a point of confusion in the literature: fermionic character depends on the symmetry of the many-particle Hamiltonian with respect to the interchange of particle labels and not on the spin-1/2 nature of an individual particle. The many-fermion wave function is requied by the Pauli exclusion principle to be antisymmetric with respect to the interchange of particle labels including the spin label. However it is the symmetry of the Hamiltonian with respect to the exchange of identical particles which causes the spatial wave function to separate into either even or odd symmetry with respect to the interchange of particle labels. If the interparticle interaction is zero, as it is for the photon, then Pauli exclusivity with respect to orbital occupancy by many particles does not occur, regardless of the spin characterof an individual particle. In such cases one learns of the spin-property of the particle from polarization experiments, as in the case of the photon.

Here we pose Eqs.(1) with zero mass and zero charge,

$$\frac{i\hbar}{c}\frac{\partial\psi_L}{\partial t} + i\hbar\,\vec{\sigma}\cdot\vec{\nabla}\psi_S = 0 \tag{4a}$$

$$\frac{i\hbar}{c}\frac{\partial\psi_S}{\partial t} + i\hbar\,\vec{\sigma}\cdot\vec{\nabla}\psi_L = 0 \quad , \tag{4b}$$

as suitable equations of motion for the photon.

Now using the identity and definitions ,

$$\vec{\sigma}\ (\vec{\sigma}\cdot\vec{\nabla}) = \vec{\nabla} + i\,(\vec{\nabla}\times\vec{\sigma}) \tag{5a}$$

$$\vec{\sigma}\,\psi_L = \vec{H} \tag{5b}$$

$$i\,\vec{\sigma}\,\psi_S = \vec{E} \quad , \tag{5c}$$

two operator Maxwell equations follow from each of Eqs. (4) by operating on each from the left with $\vec{\sigma}$ and then again operating on the result with $\vec{\sigma}$. From Eq. (4a) we obtain the equation,

$$\vec{\sigma}\cdot\left(\frac{1}{c}\frac{\partial\vec{H}}{\partial t}+\vec{\nabla}\times\vec{E}\right)-i\,\vec{\nabla}\cdot\vec{E}=0 \quad . \tag{6}$$

Two of Maxwell's equations can be inferred from Eq. (6),

$$\vec{\nabla}\cdot\vec{E}=0 \tag{7a}$$

$$\frac{1}{c}\frac{\partial\vec{H}}{\partial t}+\vec{\nabla}\times\vec{E}=0 \quad , \tag{7b}$$

where we have recognized that Eq. (7a) is the gauge condition appropriate for vanishing electronic charge; Eq. (7b) follows by equating the vector coefficient of $\vec{\sigma}$ equal to zero.

From Eq. (4b) we obtain the equation,

$$\vec{\sigma}\cdot\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}-\vec{\nabla}\times\vec{H}\right)+i\,\vec{\nabla}\cdot\vec{H}=0 \quad . \tag{8}$$

The other two Maxwell's equations can be inferred from Eq. (8),

$$\vec{\nabla}\cdot\vec{H}=0 \tag{9a}$$

$$\frac{1}{c}\frac{\partial\vec{E}}{\partial t}-\vec{\nabla}\times\vec{H}=0 \quad , \tag{9b}$$

where we have recognized that Eq. (9a) is the gauge condition appropriate for the magnetic field since $\vec{H} = \vec{\nabla} \times \vec{A}$, where $\vec{A}$ is the vector potential; Eq. (9b) follows by equating the vector coefficient of $\vec{\sigma}$ equal to zero.

Eqs. (7) and (9) are identically Maxwell's equations in a vacuum. The operator character of the equations is no longer explicit, such that the result of classical electrodynamics is recovered. In other words one may evaluate the fields in the standard way by using a unit vector in the direction of polarization.

We regard the recovery of Maxwell's equations as proof of our claim that Dirac's equation for zero mass and zero charge is a suitable equation of motion for the photon. Dirac's original motivation was to derive a Lorentz-invariant equation of motion for the electron which was first-order in space and time. Now it appears that his result has a generality which he probably did not anticipate. The spin-1/2 nature of the Dirac particle is a sufficient condition for the Lorentz invariance of the equations of motion such that it contains implicitly the laws of electrodynamics. Finite particle mass is associated with the branching of the energy and with matter and antimatter states. Finite particle mass and charge are associated with the particle's magnetic moment.

The interpretation of classical versus nonclassical behavior in electrodynamics would seem to be impacted by our result. Obviously one can expand the vector potential $\vec{A}$ in a mode series whose operator coefficients are specified by quantization of the electromagnetic energy in a cavity [1]. As we have shown however the mass-0, spin-1/2 relativistic quantum field is already quantized, and Eqs. (4) are the quantum equations of motion. Planck's constant factors from the equations of motion only for zero mass and zero charge, such that the standard interpretation of the electrodynamic equations is a classical one. From the new point of view however the electrodynamic equations can be evaluated as operator equations whose eigenstates are the two-component spinors. Thus the standard representation of $\vec{A}$ in terms of field operators [1] can be re-interpreted as the second quantization of the relativistic quantum field for a mass-0, spin-1/2 particle.

III. Electrodynamics equations for photons in the presence of electronic sources for electromagnetic fields; reduction to Maxwell's equations ignoring spin in the radiation-matter interaction

In this section we show that electrodynamic equations can be derived by positing Lorentz-invariant inhomogeneous terms, which represent electronic sources for electromagnetic fields, in the photonic equations of motion. We posit the equations,

$$\frac{i\hbar}{c}\frac{\partial\psi_L}{\partial t}+i\hbar\vec{\sigma}\cdot\vec{\nabla}\,\psi_S=\frac{4\pi e\hbar}{3}\rho_g \tag{10a}$$

$$\frac{i\hbar}{c}\frac{\partial\psi_S}{\partial t}+i\hbar\vec{\sigma}\cdot\vec{\nabla}\,\psi_L=4\pi e\hbar\,\rho_g\quad, \tag{10b}$$

where ρ_g is the Lorentz-invariant density formed from the scalar product of the four-spin $(1,-\vec{\sigma})$ and the electronic four-density $(\rho,\vec{J}/c)$,

$$\rho_g=\rho-\sigma\cdot\vec{J}/c\quad, \tag{11}$$

where ρ is the electronic density and $\vec{J}$ is the electronic current. Each one of Eqs. (10) is a conservation equation which reads, Momentum times photonic field strength =

Momentum times electronic field strength, (12)

where the left-hand-side and the right-hand-side of each equation is a Lorentz invariant.

Now using the identities given by Eqs. (5) augmented by $\vec{\sigma}\cdot\vec{\sigma} = 3$ two operator Maxwell equations follow from each of Eqs. (10) by operating on each from the left with $\vec{\sigma}$ and then again on the result with $\vec{\sigma}$. From Eq. (10a) we obtain the equation,

$$\vec{\sigma}\cdot\left(\frac{1}{c}\frac{\partial\vec{H}}{\partial t} + \vec{\nabla}\times\vec{E}\right) - i\,\vec{\nabla}\cdot\vec{E} = -\,4\pi i e\,\rho_g \quad . \tag{12}$$

Two of Maxwell's equations can be inferred from Eq. (12),

$$\vec{\nabla}\cdot\vec{E} = 4\pi e\,\rho_g\,, \tag{13a}$$

$$\frac{1}{c}\frac{\partial\vec{H}}{\partial t} + \vec{\nabla}\times\vec{E} = 0 \quad . \tag{13b}$$

where we have recognized that Eq. (13a) is the gauge condition appropriate for finite electronic charge; Eq. (13b) follows by equating the coefficient of $\vec{\sigma}$ equal to zero.

From Eq. (10b) we obtain the equation,

$$\vec{\sigma}\cdot\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}-\vec{\nabla}\times\vec{H}+\frac{4\pi e}{c}\vec{J}_g\right)+i\,\vec{\nabla}\cdot\vec{H}=0 \quad . \tag{14}$$

The other two Maxwell's equations can be inferred from Eq. (14)

$$\vec{\nabla}\cdot\vec{H}=0 \tag{15a}$$

$$\frac{1}{c}\frac{\partial\vec{E}}{\partial t}-\nabla\times\vec{H}=-\frac{4\pi e}{c}\vec{J}_g \tag{15b}$$

where we have recognized that Eq. (15a) is the gauge condition appropriate for the magnetic field; Eq. (15b) follows by equating the coefficient of $\vec{\sigma}$ equal to zero. The generalized current is defined,

$$\vec{J}_g=-c\,\vec{\sigma}\,\rho_g \quad . \tag{16}$$

These equations reduced identically to the classical Maxwell equations by dropping all spin-dependent terms on the right-hand-sides of Eqs. (13a) and (15b). To show this result for Eq. (15b) one must use Eqs. (11) and (16) and the identity,

$$\vec{\sigma}\,(\vec{\sigma}\cdot\vec{J})=\vec{J}+i\,(\vec{J}\times\vec{\sigma}) \quad . \tag{17}$$

IV. Conclusions

The operator electrodynamical equations derived here follow from Dirac's original ansatz [Eqs. (1)] and from our conservation ansatz given by Eq. (10). We have been guided by our discovery that each of Dirac's pair of first-order equations is factorizable into the scalar product of a four-vector for momentum and a four-vector for spin such that each equation is a Lorentz invariant.

The correctness of Dirac's set of homogeneous equations for the motion of a spin-1/2 particle seems beyond question. The correctness of our set of inhomogeneous equations [Eqs. (10)] for photonic motion in the presence of electronic sources for electromagnetic fields is supported at this point only by the correct form of the operator electrodynamical equations which follow and from their reduction to Maxwell's equations by ignoring all spin-dependent terms in the radiation-matter interaction.

It should be noted that the reduction to Maxwell's equations destroys the Lorentz invariance of the electrodynamical equations. Maxwell's equations are Lorentz invariant only for free photons, and one may wonder if the absence of Lorentz invariance in Maxwell's equations for photons in the presence of electronic sources for electromagnetic fields underlies the controversial nature of all so-called neoclassical or self-field theories used to describe the Lamb shift of the atom.

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