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Effective Medium Theories for Multicomponent Poroelastic Composites

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Abstract

In Biot's theory of poroelasticity, elastic materials contain connected voids or pores and these pores may be filled with fluids under pressure. The fluid pressure then couples to the mechanical effects of stress or strain applied externally to the solid matrix. Eshelby's formula for the response of a single ellipsoidal elastic inclusion in an elastic whole space to a strain imposed at a distant boundary is a very well-known and important result in elasticity. Having a rigorous generalization of Eshelby's results valid for poroelasticity means that the hard part of Eshelby's work (in computing the elliptic integrals needed to evaluate the fourth-rank tensors for inclusions shaped like spheres, oblate and prolate spheroids, needles and disks) can be carried over from elasticity to poroelasticity — and also thermoelasticity — with only relatively minor modifications. Effective medium theories for poroelastic composites such as rocks can then be formulated easily by analogy to well-established methods used for elastic composites. An identity analogous to Eshelby's classic result has been derived [*Physical Review Letters* 79:1142–1145 (1997)] for use in these more complex and more realistic problems in rock mechanics analysis. Descriptions of the application of this result as the starting point for new methods of estimation are presented, including generalizations of the coherent potential approximation (CPA), differential effective medium (DEM) theory, and two explicit schemes. Results are presented for estimating drained shear and bulk modulus, the Biot-Willis parameter, and Skempton's coefficient. Three of the methods considered appear to be quite reliable estimators, while one of the explicit schemes is found to have some undesirable characteristics.

1 INTRODUCTION

Naturally occurring porous media are typically heterogeneous, so methods designed to predict the elastic, poroelastic, and/or thermoelastic behavior of these materials need to treat this heterogeneity. One approach to dealing with inhomogeneity is to treat the medium as a composite material (Mura, 1987; Milton, 2002) and then try to bring various effective medium methods to bear on the problem by generalizing good approaches already in use (Thomsen, 1985; Berryman, 1986; 1992; 1998; Jakobsen and Hudson, 2003; Jacobsen *et al.*, 2003; Levin and Alvarez-Tostado, 2003; Tod, 2003; Levin *et al.*, 2004; Wilmanski, 2004; Levin and Markov, 2005).

Having an identity analogous to Eshelby's classic result (Eshelby, 1957) — for the response of a single ellipsoidal elastic inclusion in an elastic whole space to a strain imposed at infinity — available in still more complex problems in composites analysis (such as poroelastic or thermoelastic composites) is therefore of great practical interest for applications to poroelastic composites. In Biot's poroelasticity (Biot, 1941; Gassmann, 1951; Biot, 1962), elastic materials contain connected voids or pores and these pores may be filled with fluids under pressure. The fluid pressure then couples to the mechanical effects of an externally applied stress or strain. With a rigorous generalization of Eshelby's formula valid for poroelasticity, the hard part of Eshelby's work (in computing the elliptic integrals needed to evaluate the fourth-rank tensors for inclusions shaped like spheres, oblate and prolate spheroids, needles and disks) can then be carried over to these new results with relatively minor modifications. Then, effective medium theories for poroelastic composites like rocks can be formulated easily by analogy to well-established theories for elastic composites (Korringa *et al.*, 1979; Berryman, 1980; 1992).

The author discovered a simple mathematical trick (Berryman, 1997), applicable to media having isotropic constituents and based on a linear combination of results from two thought experiments, that makes the derivation of a generalization of Eshelby's formula to poroelasticity an elementary task. Unknown to the author, similar ideas had been published previously by Benveniste and Dvorak (1990) specifically for thermoelastic composites, and without any restriction to isotropic materials. Some other closely related ideas have also been presented by Willis (1981) and Mura (1987). In other earlier work by the author (Berryman, 1985; 1992), the problem of acoustical scattering by a *spherical* inhomogeneity of one poroelastic material imbedded in another was solved and the results then used to construct various single-scattering-based effective medium theories. The Eshelby generalization now permits incorporation of Eshelby's results for arbitrary ellipsoidal-shaped inclusions into both quasistatic formulations of effective medium theory and/or into formulas derived from scattering theory. The resulting improved estimates of poroelastic material properties has important applications in geothermal and oil reservoir modeling, nuclear waste disposal, CO_2 sequestration, and hydrology — among others.

Generalization of many effective medium theories in elasticity [see Berryman and Berge (1996) for a discussion] can now proceed more easily into the realm of poroelastic composites by making use of this generalization of Eshelby's results. This paper treats four well-known theories: Mori-Tanaka (MT), Kuster-Toksöz (KT), differential effective medium (DEM), and the coherent potential approximation (CPA) — which is closely related to some of the self-consistent (SC) effective medium theories.

Section 2 reviews the basic formulas of poroelasticity, and the generalization of Eshelby's formula in this context. Section 3 derives the main results of the paper. Section 4 provides some numerical examples. Section 5 summarizes our results. Appendix A reviews exact results in heterogeneous poroelastic media having just two constituents. Appendix B reviews the results of effective medium theory in poroelasticity for spherical inclusions. Appendix C collects the mathematical details of calculations for unjacketed pore bulk modulus.

2 POROELASTICITY AND ESHELBY

Our subject is the treatment of rocks and soils — and, especially, fluid-saturated and partially saturated rocks and soils — as composite poroelastic media. By this we mean to study and partially answer the question of how the elastic/poroelastic constants of a rock or soil can be estimated from a knowledge of its constituents, their volume fractions, and possibly the geometry of individual grains and/or pores — when that information is also available.

Equations of quasistatic poroelasticity — as presented, for example, by Rice and Cleary (1976) — may be written concisely in the form:

$$\varepsilon_{pq} = M_{pqrs} \langle \sigma_{rs} \rangle \,, \tag{1}$$

$$\zeta = (m - m_0)/\rho_0 = \frac{\alpha}{K_d} \left[\frac{\sigma_{qq}}{3} + \frac{p_f}{B} \right] . \tag{2}$$

Commonly understood terms appearing in these equations are the strains ε_{ij}, the solid stress σ_{ij}, the fluid pressure p_f, the elastic compliance tensor M_{ijkl} of the drained porous frame, and the increment of fluid content ζ (which is related to the initial m_0 and current m fluid mass contents, and to the initial density ρ_0 of the fluid). Applying well-known definitions from Biot and Willis (1957), the effective stress (for strain) appearing in (1) is

$$\langle \sigma_{pq} \rangle = \sigma_{pq} + \alpha p_f \delta_{pq}, \tag{3}$$

where the coefficient $\alpha = 1 - K_d/K_s'$ is the Biot-Willis parameter, K_d is the drained bulk modulus of the porous solid frame (jacketed modulus), and K_s' is the unjacketed solid modulus. The coefficient B is Skempton's pore-pressure buildup coefficient (Skempton, 1954; Green and Wang, 1986; Hart and Wang, 1995), given by

$$\frac{1}{B} = 1 + \frac{\phi_0 K_d}{\alpha} \left(\frac{1}{K_f} - \frac{1}{K_s''} \right) , \tag{4}$$

where ϕ_0 is the initial porosity, K_f is the bulk modulus of the pore fluid, and K_s'' is the unjacketed *pore* modulus. An equation for the change in porosity $\phi = V_\phi/V$ (where V_ϕ is pore volume and V is total volume) is

$$\frac{\delta V_\phi}{V} = \frac{\alpha}{K_d} \left[\frac{\sigma_{qq}}{3} + p_f \right] - \frac{\phi_0}{K_s''} p_f. \tag{5}$$

[In other work the present author (Berryman and Milton, 1991) has often used the alternative notation $K_s = K_s'$ and $K_\phi = K_s''$ for the two unjacketed bulk moduli. See Appendix A and also Brown and Korringa (1975).]

Starting from these basic equations of poroelasticity, we want to formulate methods of computing the effective coefficients in composite poroelastic media when these media are themselves composed of simpler (generally microhomogeneous) poroelastic media. The corresponding problem in elasticity has been studied extensively for almost 50 years — the key date being publication of Eshelby's paper in 1957. It is desirable to try to make the transition from composite elastic media to composite poroelastic media as elegantly as possible. One way in which this might be accomplished within effective medium theory is through the use of similar techniques applied to the full poroelastic equations such as those used in Berryman (1992). Another way to reach the same goal is to find new extensions to poroelasticity of some of the classic results like Eshelby (1957) that make the analysis virtually the same as that in the elastic case.

We restrict discussion here to poroelasticity, but the modifications necessary for application to thermoelasticity are not difficult and follow a well-known analogy between the two theories (Norris, 1992). In our notation, a superscript i refers to the inclusion phase, while superscripts h and $*$ refer to host and composite media, respectively. In some applications, the composite is a very simple one, being an infinite medium of host material with a single ellipsoidal inclusion of the i-th phase. The basic result of Eshelby (1957) is then of the form

$$\varepsilon^{(i)}_{pq} = T_{pqrs}\varepsilon^{*}_{rs}, \tag{6}$$

where $\varepsilon^{(i)}$ is the uniform induced strain in the ellipsoidal inclusion, ε^{*} is the uniform applied strain of the composite at a great distance, and T is the fourth-rank tensor relating these two strains. The summation convention for repeated indices is assumed in expressions such as (6).

After considering two thought experiments – one when there is no fluid present in the pores and another when a saturating fluid is present and both the confining and pore pressures are chosen so that a uniform expansion of the host medium and inclusion occur (Berryman and Milton, 1991), the author found previously (Berryman, 1997) that the final form of the generalization of Eshelby's formula to poroelasticity for isotropic media is given quite simply by

$$\varepsilon^{(i)}_{pq} - e_{pq}(p_f) = T_{pqrs}\left[\varepsilon^{*}_{rs} - e_{rs}(p_f)\right]. \tag{7}$$

The full analysis — which will not be repeated — shows that, if the pore fluid pressure vanishes (*e.g.*, $p_f = 0$ in the absence of a pore fluid), then the uniform strain e disappears from (7) and it reduces exactly to (6) as it should. In the other limiting case, if the pore pressure has been specified to be a nonzero constant, then the uniform strain e in (7) can be easily computed (Berryman and Milton, 1991). So, if the strain at infinity happens to be chosen to be equal to this uniform strain, then from (7) the inclusion strain also takes the same value — again as it should. Since the equation for $\varepsilon^{(i)}$ is necessarily linear, these two cases are enough to determine the behavior for arbitrary values of ε^* and p_f. In poroelasticity, the strain e_{pq} can be determined in advance from the applied fluid pressure p_f and the properties of the host and inclusion. In particular, we find from results of Berryman and Milton (1991) that

$$e_{pq} = \left(\frac{\alpha^{(h)} - \alpha^{(i)}}{K_d^{(h)} - K_d^{(i)}} \right) \frac{p_f}{3} \delta_{pq}. \tag{8}$$

The formulas presented in the following work form one set of useful applications of this generalization of Eshelby's formula in (7).

3 EFFECTIVE MEDIUM THEORIES

Analysis to follow will come in two main steps for each of the examples presented. The first step involves recovering the elastic result for the case when the pore pressure vanishes, *i.e.*, for the drained porous frame. In fact this step involves no new work, as Eqs. (1) and (3) imply, when $p_f = 0$, that

$$\varepsilon_{pq} = M_{pqrs} \sigma_{rs}. \tag{9}$$

Therefore, this step is completely equivalent to the analysis already presented in Berryman and Berge (1996). These results (along with quick derivations for the sake of completeness) are presented only because the results are needed to understand the next step of the analysis in each case. The second step is to derive the equivalent effective medium theory expression for K_s', or equivalently for the overall Biot-Willis parameter $\alpha^* = 1 - K_d^*/K_s'$.

The main result we use for the drained analysis takes the form [see Eqs. (19) and (20) of Berryman and Berge (1996)] for stiffness

$$(\mathbf{L}^* - \mathbf{L}^{(r)}) \sum v^{(i)} \mathbf{V}^{ri} \varepsilon_r = \sum v^{(i)} (\mathbf{L}^{(i)} - \mathbf{L}^{(r)}) \mathbf{V}^{ri} \varepsilon_r, \tag{10}$$

and for compliance

$$(\mathbf{M}^* - \mathbf{M}^{(r)}) \sum v^{(i)} \mathbf{W}^{ri} \sigma_r = \sum v^{(i)} (\mathbf{M}^{(i)} - \mathbf{M}^{(r)}) \mathbf{W}^{ri} \sigma_r, \tag{11}$$

where $\mathbf{L}^*$ is the overall stiffness tensor (inverse of the compliance tensor $\mathbf{M}^*$) to be determined, $\mathbf{L}^{(r)}$ is the stiffness tensor of some convenient (and also arbitrary) elastic reference material. Other parameters are: $v^{(i)}$, the volume fraction of the ith constituent; $\mathbf{L}^{(i)}$, the stiffness tensor of the ith constituent; ε_r is the strain in the reference material; and $\mathbf{V}^{ri}$ is the (exact and generally unknown) linear coefficient relating strains in material i to those in material r according to $\varepsilon_i = \mathbf{V}^{ri}\varepsilon_r$, and $\mathbf{W}^{ri} = \mathbf{M}^{(i)}\mathbf{V}^{ri}\mathbf{L}^{(r)}$. Eqs. (10) and (11) are completely general results that follow from an earlier analysis of the elastic problem by Hill (1963), and they are also completely consistent with each other.

3.1 Coherent potential approximation

First we consider a scheme sometimes called the Coherent Potential Approximation (CPA) (Soven, 1967; Gubernatis and Krumhansl, 1975; Milton, 1985; Berryman, 1992; Berryman and Berge, 1996) or the Self-Consistent (SC) Scheme (Korringa *et al.*, 1979; Berryman, 1980).

When there is no pore fluid present (or, equivalently, for drained frame conditions), the equations of poroelasticity reduce to those of elasticity (10) for the porous frame material. Within CPA, the idea is to treat all constituents on an equal footing, so no single material serves as host medium for the others. For this reason, the CPA is sometimes known as a symmetrical self-consistent scheme. To find the formulas for the CPA, we take the reference material to be the composite itself, so $r = *$. (See Figure 1.) The formula (10) then reduces to

$$\sum v^{(i)} (\mathbf{L}^{(i)} - \mathbf{L}^*) \mathbf{V}^{*i} = 0. \tag{12}$$

Although this equation is exact, the linear coefficient $\mathbf{V}^{*i}$ is still unknown. The usual approximation at this stage comes from substituting the Eshelby-Wu tensor $\mathbf{T}^{*i}$ (Wu, 1966), corresponding to inclusions of stiffness $\mathbf{L}^{(i)}$ in host material of stiffness $\mathbf{L}^{*}_{CPA}$, for $\mathbf{V}^{*i}$. When this is done, we have

$$\sum v^{(i)}(\mathbf{L}^{(i)} - \mathbf{L}^{*}_{CPA})\mathbf{T}^{*i} = 0. \tag{13}$$

(Note that we use the same symbol $\mathbf{L}^{*}$ for the exact and unknown effective stiffness and for the approximations to it, but the approximations will also have subscripts to keep track of which approximation is being discussed.) The resulting equations for drained bulk and shear moduli are

$$\sum v^{(i)}(K^{(i)} - K^{*}_{CPA})P^{*i} = 0 \tag{14}$$

$$\sum v^{(i)}(G^{(i)} - G^{*}_{CPA})Q^{*i} = 0, \tag{15}$$

where P^{*i}, Q^{*i} are the polarization factors for the bulk and shear moduli, respectively.

TABLE 1 displays three of the common results that are used for the polarization factors, namely spheres, needles, and penny-shaped cracks. Many other choices are possible, including oblate and prolate spheroids having arbitrary aspect ratios (Berryman, 1980). We will use just these three cases however in the examples to follow.

To make use of the generalization of Eshelby's formula for poroelasticity in the case when pore fluid and pore pressure are significant factors, we note that each inclusion is effectively imbedded in the composite material (*), so it makes sense to consider generalizing to a formula of the same form as (7). So the poroelastic version of the general heterogeneous problem is

$$\varepsilon^{(i)} - e^{*i}(p_f) = \mathbf{V}^{*i}\left[\varepsilon^{*} - e^{*i}(p_f)\right]. \tag{16}$$

For just two components, we know that the exact result is (7), with e determined by (8). So, for the more general heterogeneous problem, the strain corresponding to equal expansion or contraction of both materials i and $*$ is taken to be

$$e^{*i}_{pq} = \left(\frac{\alpha^{*} - \alpha^{(i)}}{K^{*} - K^{(i)}}\right)\frac{p_f}{3}\delta_{pq}. \tag{17}$$

and the final approximation is

$$\varepsilon^{(i)} - e^{*i}(p_f) = \mathbf{T}^{*i}\left[\varepsilon^* - e^{*i}(p_f)\right]. \tag{18}$$

If the mixture were composed only of the two materials i and $*$, then the uniform expansion result would apply exactly. In a composite poroelastic material, (18) should be viewed as an estimate of the true strain of the ith constituent. This estimate is conceptually on the same footing as that traditionally used when saying that $\varepsilon^{(i)} = \mathbf{T}^{*i}\varepsilon^*$ is a reasonable approximation of the strain in the ith constituent of an elastic composite, even though there may be many other types of materials present.

In both these situations, the underlying justification follows from implicitly assumed separation of scales. The constituents composing the effective $(*)$ materials are assumed to be of a significantly smaller size than those considered in Eq. (18). How important an approximation this assumed scale separation may be is clearly problem dependent. These issues are significant, but must be analyzed from a theoretical perspective outside of the effective medium theory itself.

To derive a formula within CPA for the Biot-Willis constant α^*, we want (and need) to make use of (18) somehow. For elasticity, the average stress equals the total stress, so $\sum v^{(i)}\sigma_i = \sigma$. This fact was actually used to derive (10). However, for poroelasticity with finite pore pressure p_f, it is no longer true that the average stress is equal to the total stress, *i.e.*, $\sum v^{(i)}\sigma^{(i)} \neq \sigma$. The correct relation for the effective stress is more complicated than this, since pore pressure effects must also be factored into the result. However, it is still true that the average strain is equal to the total strain, *i.e.*,

$$\sum v^{(i)}\varepsilon^{(i)} = \varepsilon^*. \tag{19}$$

Furthermore, this relation is just what is needed to make effective use of (18) possible. Substituting (18) into (19) and rearranging, we find that

$$\sum v^{(i)}(\mathbf{I} - \mathbf{V}^{*i})e^{*i}(p_f) = \sum v^{(i)}(\mathbf{I} - \mathbf{V}^{*i})\varepsilon^*, \tag{20}$$

where $\mathbf{I}$ is the identity matrix. Equation (20) is almost what is needed, but the right hand side seems problematic, because it depends explicitly on ε^*, which is arbitrary. It is known however that $\sum v^{(i)}(\mathbf{I}-\mathbf{V}^{*i}) \equiv 0$ (Hill, 1963; Berryman and Berge, 1996). This relationship can be checked in (20) by setting $p_f = 0$ on the left hand side of (20). Then, it is clear that the right hand side of (20) should in fact be set identically to zero. Thus, after substituting (17) into (20), the CPA for α^* is

$$\sum v^{(i)}(1-P^{*i})\frac{\alpha^*_{CPA}-\alpha^{(i)}}{K^*_{CPA}-K^{(i)}} = 0. \tag{21}$$

Eq. (21) reduces to (63) of Berryman (1992) in the case of two components and spherical inclusions, as it should. Some additional care should be taken however to check the degree of satisfaction of the implicit subsidiary condition $\sum v^{(i)}(\mathbf{I}-\mathbf{T}^{*i}) \simeq 0$, in order to make sure that it is at least approximately satisfied by the resulting estimate obtained for $\mathbf{L}^*_{CPA}$. It turns out that this condition is always satisfied exactly for spherical inclusions. Furthermore, in the case of spheres we have $P^{*i} = (K^* + \frac{4}{3}\mu^*)/(K^{(i)} + \frac{4}{3}\mu^*)$, so it is easy to show that (20) reduces in this case to

$$\sum v^{(i)}(\alpha^{(i)} - \alpha^*_{CPA})P^{*i} = 0, \tag{22}$$

which is formally quite similar to (but still distinct from) equations (13)–(15) for the bulk and shear moduli. Eq. (21) is a poroelastic result; there is no corresponding result in elasticity (but there is something similar in thermoelasticity for the thermal expansion coefficient).

Each theory considered must produce four constants: K_d, α, K''_s — and also G_d, but this one is not so difficult as it is usually determined simultaneously together with K_d. When there are only two components in the porous composite, the results of Appendix A [from results of Berryman and Milton (1991)] can be used directly. For multicomponent situations, we can use (70) from Appendix C to determine K''_s, which is the only constant still unknown:

$$\frac{\phi^*}{K''_s} = \frac{\alpha^*(1-\alpha^*)}{K^*_d} - \sum_{i=1} v^{(i)}(\alpha^{(i)} - \phi^{(i)}) \left[\chi^{*i}P^{*i} + \left(\frac{1}{K^{(i)}_m} - \chi^{*i}\right)\right]. \tag{23}$$

This formula can be used in (50) to determine D^*, from which Skempton's coefficient B^* is then easily found.

3.2 Average T-matrix/Kuster-Toksöz scheme

The second approximation scheme we will consider is sometimes called the Average T-Matrix Approximation (ATA) (Berryman, 1992; Jacobsen *et al.*, 2003) and for some of its variations it is called the Kuster-Toksöz (KT) Scheme (Kuster and Toksöz, 1974).

In the absence of a pore fluid, the poroelastic problem reduces again precisely to the elastic composite problem. Following the analysis of Berryman and Berge (1996), we find that the general result (10) is conveniently written as

$$(\mathbf{L}^* - \mathbf{L}^{(h)})\varepsilon = \sum v^{(i)}(\mathbf{L}^{(i)} - \mathbf{L}^{(h)})\mathbf{V}^{hi}\varepsilon_h. \tag{24}$$

We obtained this form from (10) by noting that $\varepsilon = \sum v^{(i)}\varepsilon_i = \sum v^{(i)}\mathbf{V}^{hi}\varepsilon_h$. The Kuster-Toksöz approximation includes the assumptions that $\varepsilon = \mathbf{V}^{h*}\varepsilon_h \simeq \mathbf{T}^{h*}\varepsilon_h$ and that $\mathbf{V}^{hi} \simeq \mathbf{T}^{hi}$. Then, the resulting formula for the approximation is

$$(\mathbf{L}^*_{KT} - \mathbf{L}^{(h)})\mathbf{T}^{h*} = \sum v^{(i)}(\mathbf{L}^{(i)} - \mathbf{L}^{(h)})\mathbf{T}^{hi}. \tag{25}$$

(Also see Figure 2.) One further assumption is normally made — especially for isotropic composites — that the tensor $\mathbf{T}^{h*}$ is always the one for spherical inclusions, while $\mathbf{T}^{hi}$ can be for arbitrary shapes of inclusions. With other (for nonspherical shapes) choices of $\mathbf{T}^{h*}$, we could have a suite of alternate approximations that differ somewhat from the original Kuster-Toksöz approach.

Formulas for drained bulk and shear moduli are

$$(K^*_{KT} - K^{(h)})P^{h*} = \sum v^{(i)}(K^{(i)} - K^{(h)})P^{hi}. \tag{26}$$

and

$$(G^*_{KT} - G^{(h)})Q^{h*} = \sum v^{(i)}(G^{(i)} - G^{(h)})Q^{hi}. \tag{27}$$

To derive a formula within ATA/KT for the Biot-Willis constant α^*, we need to use the Eshelby generalization again and make appropriate substitutions into the formula (19). The thought experiment for KT is a little more complex than that for CPA, however, so we actually need to do this in two steps. First, note that if we view the composite as a finite sphere and imbed this sphere in a host material (that may be and usually is chosen to be the same as one of the constituent materials), then the appropriate generalized Eshelby formula for the poroelastic case is

$$\varepsilon^{(i)} - e^{hi}(p_f) = \mathbf{T}^{hi}\left(\varepsilon - e^{hi}(p_f)\right), \tag{28}$$

where ε is the applied strain at infinity. Equation (28) can then be averaged to give

$$\sum v^{(i)}\varepsilon^{(i)} = \sum v^{(i)}\left(\mathbf{I} - \mathbf{T}^{hi}\right)e^{hi}(p_f) + \sum v^{(i)}\mathbf{T}^{hi}\varepsilon. \tag{29}$$

But now if we note that the composite has the effective properties $\mathbf{L}^*_{KT}$ and α^*_{KT} in the composite sphere imbedded in the host material, then we can also write — in place of (28):

$$\varepsilon^* = e^{h*}(p_f) + \mathbf{T}^{h*}\left[\varepsilon - e^{h*}(p_f)\right]. \tag{30}$$

And, since $\sum v^{(i)}\varepsilon^{(i)} = \varepsilon^*$ by construction, (30) should be equated to (29). The final result is

$$\left(\mathbf{I} - \mathbf{T}^{h*}\right)e^{h*}(p_f) = \sum v^{(i)}\left(\mathbf{I} - \mathbf{T}^{hi}\right)e^{hi}(p_f) + \ldots, \tag{31}$$

where the terms indicated by the ellipsis $(\ldots)$ are of the form $\sum v^{(i)}(\mathbf{T}^{hi} - \mathbf{T}^{h*})\varepsilon$ and should vanish identically for the same reasons as those discussed in the case of a similar term in the derivation for CPA. Thus, the KT formula for the Biot-Willis parameter α^* is

$$\left(1 - P^{h*}\right)\frac{\alpha^*_{KT} - \alpha^{(h)}}{K^*_{KT} - K^{(h)}} = \sum v^{(i)}\left(1 - P^{hi}\right)\frac{\alpha^{(i)} - \alpha^{(h)}}{K^{(i)} - K^{(h)}}. \tag{32}$$

Eq. (32) reduces to (56) of Berryman (1992) in the case of only two components and spherical inclusions as it should.

The remaining constant to be determined is K''_s. The result (70) is in the right format to be used directly by the KT theory, so we do not need to repeat any analysis. The constants such as α^* and K^*_d, etc., in (70) and in the polarization shape factors appearing there should be interpreted as values computed within the KT approximation as discussed here.

As in the CPA, we now have a subsidiary condition $\sum v^{(i)}(\mathbf{T}^{hi} - \mathbf{T}^{h*}) \simeq 0$ that should be checked for approximate satisfaction by $\mathbf{L}^*_{KT}$. Again, we find this condition is always satisfied exactly for spherical inclusions.

3.3 Differential effective medium approximation

The third scheme we consider is the Differential Effective Medium (DEM) Approximation (Cleary *et al.*, 1980; Norris, 1985; Avellaneda, 1987; Berryman and Berge, 1996). We limit the treatment here to the two-component case, as that is easiest to explain. This method is derived by assuming the composite is formed by successively mixing very small (infinitesimal) fractions dy of one inclusion material (i) in another host material. The host medium changes gradually during this process from material (h) at $y = 0$ into the desired composite material (*) at some finite y value. (See Figure 3.) Starting with (10), the resulting formula for the stiffness is the differential equation

$$(1-y)\frac{d}{dy}\mathbf{L}^*_{DEM}(y) = \left[\mathbf{L}^{(i)} - \mathbf{L}^*_{DEM}(y)\right]\mathbf{T}^{*i}, \tag{33}$$

where the initial value of the stiffness tensor is $\mathbf{L}^*_{DEM}(y = 0) = \mathbf{L}^{(h)}$. The Eshelby-Wu tensor $\mathbf{T}^{*i}$ is the one corresponding to inclusions of stiffness $\mathbf{L}^{(i)}$ imbedded in host material of stiffness $\mathbf{L}^*_{DEM}$. The resulting system of coupled equations may be integrated to any desired value of total inclusion volume fraction $y = v^{(i)}$ quite easily using (for example) a Runge-Kutta scheme (Hildebrand, 1956).

The formula for the Biot-Willis parameter is obtained in this scheme most easily by starting from (32) for the KT method, noting first that the sum on the right is reduced to a single term for the phase that is not the initial host phase, replacing the parameters for the host medium by their values evaluated at concentration y and the $*$ parameters by their values evaluated at concentration $y + dy$. The volume fraction is replaced by $v^{(i)} \to dy/(1-y)$ to account for the fact that more than the amount dy of the composite host material must be replaced in order to achieve the new desired volume fraction $y+dy$. Finally, taking the limit as $dy \to 0$ gives the desired formula. For ellipsoidal inclusions, the result is

$$(1-y)\frac{d}{dy}\alpha^*_{DEM}(y) = \left[\alpha^{(i)} - \alpha^*_{DEM}(y)\right]P^{*i}, \tag{34}$$

where $\alpha^*_{DEM}(0) = \alpha^{(h)}$. For spherical inclusions, (34) reduces to (70) of Berryman (1992), as it should.

The corresponding result for the drained bulk and shear moduli obtained directly from (33) are

$$(1-y)\frac{d}{dy}K^*_{DEM}(y) = \left[K^{(i)} - K^*_{DEM}(y)\right] P^{*i}, \tag{35}$$

and

$$(1-y)\frac{d}{dy}G^*_{DEM}(y) = \left[G^{(i)} - G^*_{DEM}(y)\right] Q^{*i}, \tag{36}$$

where $K_{DEM}(0) = K^{(h)}$ and $G_{DEM}(0) = G^{(h)}$. Both results were obtained previously for spherical inclusions (Berryman, 1992), but the present derivation is much more compact. The generalization to nonspherical inclusions is now straightforward, requiring only a reinterpretation of the polarization factors P^{*i} and Q^{*i} for different particle shapes.

This case is especially easy to use in poroelasticity because — as we have restricted the discussion here to the two-component case — the exact results of Appendix A apply for both α^* [*i.e.*, Eq. (43)] and K''_s [*i.e.*, Eq. (48)]. No further discussion is needed — just correct application of the results for $K_d(y)$ and $G_d(y)$ within these formulas.

3.4 Mori-Tanaka approximation

The final approximation we consider is the Mori-Tanaka (MT) Scheme of Mori and Tanaka (1973), as described by Weng (1984), Benveniste (1987), Ferrari (1991), Ferrari and Filipponi (1991), and others [see, for example, Berryman and Berge (1996)].

For the drained frame, the Mori-Tanaka approximation is obtained by assuming the composite has a host material with imbedded inclusions and then choosing the host to serve as the reference material, so $r = h$. (See Figure 4.) Making this choice in (10) and then substituting $\mathbf{V}^{hi} \simeq \mathbf{T}^{hi}$, we obtain

$$\sum v^{(i)}(\mathbf{L}^{(i)} - \mathbf{L}^*_{MT})\mathbf{T}^{hi} = 0. \tag{37}$$

The Mori-Tanaka results for drained bulk and shear moduli with arbitrary ellipsoidal inclusion shapes are

$$\sum v^{(i)}(K^{(i)} - K^*_{MT})P^{hi} = 0, \tag{38}$$

and

$$\sum v^{(i)}(G^{(i)} - G^*_{MT})Q^{hi} = 0. \tag{39}$$

Because the Mori-Tanaka scheme *cannot* be derived using any analogy to scattering theory – unlike the other three schemes considered so far [see Berryman and Berge (1996)], there is some ambiguity about how to apply the present method within Mori-Tanaka and different choices of formulas for the Biot-Willis parameter can result. One of the more straightforward approaches can be shown to lead to the formula

$$\sum v^{(i)}(\alpha^{(i)} - \alpha^*_{MT})P^{hi} = 0, \tag{40}$$

when the inclusions are all spherical in shape. We stress however that (40) is not the only possible formula that could also be considered fully consistent with the Mori-Tanaka scheme. In particular, another approach leads to a formula similar to (32). Note, however, that the known exact results for two-component composites provide very useful constraints on our choices. It is quite easy to see that (38) and (40) guarantee that (43) holds for two components. Also, the simplicity of all three formulas (38)-(40) is another clear advantage; each formula amounts to a weighted average, wherein the weights are either $v^{(i)}P^{hi}$ or $v^{(i)}Q^{hi}$.

The remaining constant to be determined is K_s''. The formula (70) is already in exactly the right format to be used directly by the MT theory, so again we do not need to repeat any analysis for this case. The constants such as α^* and K_d^*, etc., appearing in (70) and especially in the polarization shape factors need to use the values computed within the MT approximation as discussed above.

In fact, it is easy to show that both (40) and (34) have the advantage that they reproduce the known exact results (Berryman and Milton, 1991) for two component poroelastic media. This constraint appears to be a very useful selection criterion for choosing among various possibilities that arise when trying to identify the proper generalizations of these theories for the poroelastic case. We expand on this issue in Appendix A.

4 EXAMPLES

TABLE 2 provides examples of the results obtained using the methods discussed above. Two methods (CPA and DEM) are known to be realizable (Milton, 1985; Norris, 1985; Avellaneda, 1987). These particular examples were computed assuming spherical inclusions.

Input parameters are from Table 7 of Berryman (1992) for a clay and Kayenta sandstone mixture. Solid sandstone grains occupy 60% of the volume, porous clay occupies the remaining 40%, and the total porosity is 16%. DEM$^-$ assumes the weak component is the host, while DEM$^+$ assumes the strong component is the host. Since DEM$^-$ produces extremely weak results, we list only the DEM$^+$ results. For composites with only spherical inclusions, KT and MT are the same. Furthermore, both are equal to the upper Hashin-Shtrikman bounds (Hashin and Shtrikman, 1961). It is expected that the three methods considered in TABLE 2 should give differing results, because they implicitly assume different microstructures (Berge *et al.*, 1993).

General observations about the results in Figures 5 and 6 are these: As anticipated, the MT and KT results are identical for all the parameters in the case of sphere-sphere modeling (the host material and inclusion material are both modeled as being spherical in shape). These results are the stiffest found for G_d and K_d, but such results are unavoidable, since these two methods both reproduce the Hashin-Shtrikman upper bounds in this case. As is often observed, the DEM$^+$ model (assuming the stiffer component is host) is stiffer than the CPA results in all cases. We do not display DEM$^-$ (assuming the more compliant component is host), but the result will always be the opposite – DEM$^-$ computed stiffness always lies below the CPA results. As we change the shapes of the inclusion phase, we find that all of the results become more compliant, but three of the results maintain the same relationships with each other. We observe MT results for stiffness are always greater than DEM$^+$ results, which are in turn always greater the CPA results. The DEM$^-$ results (not shown here) would have always been found to lie below the CPA results as well.

On the other hand, the KT results do not maintain this relationship. The results from

KT are fairly volatile, and change from being stiffest for sphere-sphere to being most compliant (of those shown) for sphere-penny. Furthermore, the KT method displays some unphysical behavior (especially for the sphere-penny model), showing that the bulk and shear moduli both vanish, but for different values of the clay volume fraction. Furthermore, the Biot-Willis parameter should never exceed unity, but — although the other three models continue to predict reasonable values of α^* for all volume fractions — the KT model fails in this respect for values of clay volume fraction just above 10%. The author has taken some care to make certain that this result is not just a numerical or coding error, while earlier results (Berryman, 1980) have shown that KT is in fact unphysical for some models in the sense that it predicts moduli that exceed the Hashin-Shtrikman upper bounds for quite small values of the inclusion volume fraction. So it appears that these observations provide another reason to use this method only with considerable caution for larger volume fractions and alternative inclusion shapes that differ substantially from spherical.

In some cases the magnification factor K_u/K_d is quite large. However, this result is found only when the estimated drained bulk modulus is quite small. The physical constraint that must always be satisfied is that $K_u \le K_m^+$, where K_m^+ is the bulk modulus of stiffest component of the poroelastic system. This condition is always satisfied by the CPA, DEM, and MT methods.

5 CONCLUSIONS

We have demonstrated that the generalized Eshelby formula (7) derived earlier by the author (Berryman, 1997) can be successfully used in various well-known effective medium theories to estimate the Biot-Willis parameter α when the inclusions are of arbitrary *ellipsoidal* shape. This generalizes earlier work of the author (Berryman, 1985; 1992) that provided means of computing these same constants, but only for the cases of *spherical* inclusions using the CPA, DEM, and KT methods. The new formulas are no more difficult to compute that the corresponding formulas for the bulk and shear (empty porous) frame moduli of these materials.

Our results also show how to compute the remaining poroelastic parameter B (Skempton's coefficient) for a general ellipsoidal inclusion within these various effective medium theories. For two-component media, the procedure for doing so is a straightforward extension of work published earlier by Berryman and Milton (1991) based on an analysis of (5), and in particular the unjacketed pore bulk modulus K_s''. For multicomponent porous media, the results of Appendix C provide good approximations, and these have been shown explicitly to reduce to the exact results when it is appropriate for them to do so.

The main conclusions are that three of the models considered provide physically reasonable results over a wide range of volume fractions. These three methods are the Mori-Tanaka (MT) explicit method, the differential effective medium (DEM) method, and the coherent potential approximation (CPA). Nevertheless, these methods do not produce the same answers. The reason for these differences is that each method implicitly assumes a different microstructure for the composite, and the nature of the microstructure does indeed matter to the results in real composites (Berge *et al.*, 1993). So the actual microstructure should be taken carefully into account when trying to decide which of these methods to apply in a given situation. The fourth method considered (the Kuster-Toksöz explicit method), although reproducing the Mori-Tanaka results for sphere-sphere composites, does not necessarily produce physical results for mixtures of other particle shapes that differ substantially from spheres; so we do not generally recommend this method for use when estimating parameters for poroelastic composites.

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APPENDIX A: EXACT RESULTS FOR COMPOSITES WITH TWO POROUS CONSTITUENTS

One particularly powerful means of checking the validity of any estimation scheme is to compare the results with those of various exact results that may be known for special cases. In the present class of problems, a result of Berryman and Milton (1991) provides a convenient check on all the formulas derived so far. We give a brief derivation for completeness.

The strain $e^{(i)}$ of the i-th inclusion type, and the increment of fluid content $\zeta^{(i)}$, are connected to the confining pressure p_c and the pore-fluid pressure p_f by

$$\begin{pmatrix} e^{(i)} \\ -\zeta^{(i)} \end{pmatrix} = \frac{1}{K_d^{(i)}} \begin{pmatrix} 1 & -\alpha^{(i)} \\ -\alpha^{(i)} & \alpha^{(i)}/B^{(i)} \end{pmatrix} \begin{pmatrix} -p_c \\ -p_f \end{pmatrix}, \tag{41}$$

where all the coefficients are defined as in (1) and (2). Then, a similar relationship holds for the overall behavior of a composite porous system:

$$\begin{pmatrix} e^{*} \\ -\zeta^{*} \end{pmatrix} = \frac{1}{K_d^{*}} \begin{pmatrix} 1 & -\alpha^{*} \\ -\alpha^{*} & \alpha^{*}/B^{*} \end{pmatrix} \begin{pmatrix} -p_c \\ -p_f \end{pmatrix}. \tag{42}$$

For an arbitrary two-component mixture of Gassmann materials, the Biot-Willis parameter α^* must satisfy the conditions

$$\frac{\alpha^* - \alpha^{(1)}}{K_d^* - K_d^{(1)}} = \frac{\alpha^{(2)} - \alpha^*}{K_d^{(2)} - K_d^*} = \frac{\alpha^{(2)} - \alpha^{(1)}}{K_d^{(2)} - K_d^{(1)}} \equiv -\chi. \tag{43}$$

This result follows after first noting that there exists a ratio $b = p_f/p_c$ such that

$$e^{(1)} = e^{(2)} = e^*, \tag{44}$$

and that this ratio is given by

$$b = \frac{1/K_d^{(1)} - 1/K_d^{(2)}}{\alpha^{(1)}/K_d^{(1)} - \alpha^{(2)}/K_d^{(2)}}. \tag{45}$$

Once $\alpha^* = 1 - K_d^*/K_s'$ has been determined from (43), then

$$K_s' = \frac{K_d^*}{1-\alpha^*}. \tag{46}$$

We always assume the value of K_d^* has been either measured directly or estimated using one of the effective medium theories.

Based on (5), Berryman and Milton (1991) show that

$$\phi^* \left(\frac{1}{K_s''} - \frac{1}{K_s'} \right) = \left[\sum_{i=1}^{2} v^{(i)} \frac{\alpha^{(i)} - \phi^{(i)}}{K_d^{(i)}} - \frac{\alpha^* - \phi^*}{K_d^*} \right] \frac{1-b}{b}. \tag{47}$$

The same result can be written alternatively as

$$\begin{aligned} \tfrac{\alpha^*}{K_s'} - \tfrac{\phi^*}{K_s''} &= \textstyle\sum_{i=1}^2 v^{(i)} \frac{\alpha^{(i)}-\phi^{(i)}}{K_m^{(i)}} + \left(\frac{\alpha^{(1)}-\alpha^{(2)}}{K_d^{(1)}-K_d^{(2)}} \right) \sum_{i=1}^2 v^{(i)} \left(\alpha^{(i)} - \alpha^* \right) \\ &= (\alpha^* - \phi^*)\chi + \textstyle\sum_{i=1}^2 v^{(i)} (\alpha^{(i)} - \phi^{(i)}) \left(\frac{1}{K_m^{(i)}} - \chi \right), \end{aligned} \tag{48}$$

where χ was defined in (43). In either form, these results can then be used to substitute directly for K_s'' into the equation for Skempton's overall coefficient

$$\frac{1}{B^*} = 1 + \frac{\phi^* K_d^*}{\alpha^*} \left(\frac{1}{K_f} - \frac{1}{K_s''} \right). \tag{49}$$

It is not hard to show that all the formulas presented satisfy these constraints as long as the side conditions that have been mentioned in text [*e.g.*, $\sum v^{(i)}(1 - P^{*i}) = 0$ for CPA or the corresponding side conditions for the other problems] are also satisfied. Checking for satisfaction (or approximate satisfaction) of this condition is especially easy to do for spherical inclusions, but is not strictly limited to that case. Thus, the theories presented here that do satisfy this important additional condition are presumably the best choices.

Another definition that will be convenient to use in the main text and also in the following Appendix B is

$$\frac{1}{D} = \frac{\phi}{K_f} + \left(\frac{\alpha}{K_s'} - \frac{\phi}{K_s''} \right) = \frac{\phi}{K_f} + \left(\frac{\alpha(1-\alpha)}{K_d} - \frac{\phi}{K_s''} \right). \tag{50}$$

Then the undrained bulk modulus is expressed as $K_u = K_d + \alpha^2 D = K_d/(1 - \alpha B)$, and D also satisfies

$$\alpha D = B K_u, \qquad \text{as well as} \qquad B = \frac{\alpha D}{K_d + \alpha^2 D}. \tag{51}$$

Scattering theory (Berryman, 1992) shows that D, or preferably the compliance $1/D$, is the most natural quantity to estimate when trying to study the behavior of undrained quantities in poroelasticity. The term ϕ/K_f averages simply since the pore fluid bulk modulus is either constant or (if not constant) is given by the volume fraction weighted harmonic mean of the fluid bulk moduli. The value of the next term α/K_s' is already known from the preceding analyses of K_d and α. So, the final unknown in our problem is the unjacketed pore bulk modulus K_s'', treated specifically in Appendix C.

APPENDIX B: EFFECTIVE MEDIUM RESULTS FOR POROUS COMPOSITES WITH SPHERICAL INCLUSIONS

This Appendix will summarize results from Berryman (1992) that are relevant to the analysis of the main text. Results for the Mori-Tanaka method for spherical inclusions are the same as those for Kuster-Toksöz. The parameter D was defined in (50), and is the quantity that needs to be estimated in order to determine effective unjacketed pore modulus K_s''. For simplicity of presentation, formulas are presented here for two components. Multicomponent versions, as well as some alternate formulas, may be found in Berryman (1992).

B.1 Coherent Potential Approximation

The effective medium is also the background medium for CPA, so there is only one estimate available for each effective constant.

$$\sum_{i=1}^{2} v^{(i)} \frac{K^* - K^{(i)}}{K^{(i)} + 4G^*/3} = 0 \quad \text{or} \quad \frac{1}{K^* + 4G^*/3} = \sum_{i=1}^{2} \frac{v^{(i)}}{K^{(i)} + 4G^*/3}. \tag{52}$$

Similar expressions are then available also for the effective shear modulus G^*, but the equations for K^* and G^* are obviously coupled. Effective Biot-Willis parameter is determined by

$$\sum_{i=1}^{2} v^{(i)} \frac{\alpha^* - \alpha^{(i)}}{K^{(i)} + 4G^*/3} = 0, \tag{53}$$

where K^* is determined in (52) and G^* by the companion equation (not shown). The formula for D^* is

$$\begin{aligned} \frac{1}{D^*} &= \sum_{i=1}^{2} \frac{v^{(i)}}{D^{(i)}} + \sum_{i=1}^{2} v^{(i)} \frac{[\alpha^{(i)}-\alpha^*]^2}{K^{(i)}+4G^*/3} = \sum_{i=1}^{2} \frac{v^{(i)}}{D^{(i)}} + \frac{(\alpha^{(2)}-\alpha^*)(\alpha^*-\alpha^{(1)})}{K^*+4G^*/3} \\ &= \sum_{i=1}^{2} \frac{v^{(i)}}{D^{(i)}} + \left(\frac{\alpha^{(1)}-\alpha^{(2)}}{K^{(1)}-K^{(2)}}\right) \sum_{i=1}^{2} v^{(i)}(\alpha^{(i)} - \alpha^*), \end{aligned} \tag{54}$$

where K^* and G^* are determined as before, and α^* is determined by (53). The various forms are shown here to emphasize the similarities among the CPA and the other results to follow.

B.2 Kuster-Toksöz and/or Mori-Tanaka

The host material h can be either of the two constituents. So there are two KT (MT) approximations depending on which constituent plays host:

$$\sum_{i=1}^{2} v^{(i)} \frac{K^* - K^{(i)}}{K^{(i)} + 4G^{(h)}/3} = 0 \quad \text{or} \quad \frac{1}{K^* + 4G^{(h)}/3} = \sum_{i=1}^{2} \frac{v^{(i)}}{K^{(i)} + 4G^{(h)}/3}. \tag{55}$$

Similar expressions are then available also for the effective shear modulus G^*, but the equations for K^* and G^* are not coupled. Effective Biot-Willis parameter is determined by

$$\sum_{i=1}^{2} v^{(i)} \frac{\alpha^* - \alpha^{(i)}}{K^{(i)} + 4G^{(h)}/3} = 0, \tag{56}$$

where K^* is determined in (55). The effective unjacketed pore modulus K_s'' is determined by

$$\frac{1}{D^*} = \sum_{i=1}^{2} \frac{v^{(i)}}{D^{(i)}} + \frac{(\alpha^{(2)} - \alpha^*)(\alpha^* - \alpha^{(1)})}{K^* + 4G^{(h)}/3}, \tag{57}$$

where K^* and G^* are determined as before, and α^* is determined by (56). This particular expression for D^*_{KT} did not appear in Berryman (1992), but nevertheless follows from the same analysis.

B.3 Differential Effective Medium

The equation that determines the drained bulk modulus within the DEM approach is

$$(1-y)\frac{d}{dy}\left[K^*(y)\right] = \left[K^{(i)} - K^*(y)\right]\frac{K^*(y) + 4G^*(y)/3}{K^{(i)} + 4G^*(y)/3}, \tag{58}$$

where $K^*(0) = K^{(h)}$ with h and i being the host and inclusion phases, respectively. Like the KT approach, there are two distinct DEM approximations made possible here by different choices of the host material. There is also a similar equation for the shear modulus $G^*(y)$ and it is coupled to (58), thus requiring simultaneous integration of the two equations. Effective Biot-Willis parameter is determined by

$$(1-y)\frac{d}{dy}\left[\alpha^*(y)\right] = \left[\alpha^{(i)} - \alpha^*(y)\right]\frac{K^*(y) + 4G^*(y)/3}{K^{(i)} + 4G^*(y)/3}, \tag{59}$$

which, in light of the form of (58), can be integrated to yield

$$\frac{\alpha^*(y) - \alpha^{(i)}}{\alpha^{(h)} - \alpha^{(i)}} = \frac{K^*(y) - K^{(i)}}{K^{(h)} - K^{(i)}}, \tag{60}$$

where $K^*(y)$ is determined by (58) and the companion equation for $G^*(y)$ (not shown). Equation (60) is exact for a two-component medium (Berryman and Milton, 1991). The effective unjacketed pore modulus K''_s is determined by

$$\frac{1}{D^*} = \sum_{i=1}^{2}\frac{v^{(i)}}{D^{(i)}} + \left(\frac{\alpha^{(1)} - \alpha^{(2)}}{K^{(1)} - K^{(2)}}\right)\sum_{i=1}^{2} v^{(i)}(\alpha^{(i)} - \alpha^*), \tag{61}$$

where α^* was found in (60).

APPENDIX C: UNJACKETED PORE BULK MODULUS K''_S

The formula relating changes of porosity to changes of differential and pore pressure in poroelasticity is

$$-\delta\phi = \frac{\alpha - \phi}{K_d}\delta p_d - \phi\left(\frac{1}{K'_s} - \frac{1}{K''_s}\right)\delta p_f, \tag{62}$$

where δp_f is the change in pore pressure, and $\delta p_d = \delta p_c - \delta p_f$ is the change in differential pressure (some researchers call this the "effective pressure," but we reserve that term instead for quantities such as $\delta p_c - \alpha\delta p_f$).

Equation (62) as written is true for the composite medium as a whole, and also true for the individual constituents with the stipulation that the confining pressure to be used is the one actually experienced by that constituent. We further stipulate (this is not required, but merely a convenience for presentation) that each constituent is a Gassmann material, by which we mean that each constituent is composed of a single mineral. In this situation $K_s' = K_m^{(i)} = K_s''$ for the i-th constituent, and the formula for its change of porosity simplifies to

$$-\delta\phi^{(i)} = \frac{\alpha^{(i)} - \phi^{(i)}}{K_d^{(i)}}\delta p_d^{(i)}. \tag{63}$$

We are *not* assuming that all the solid minerals are the same, only that each identified porous constituent has uniform solid bulk modulus $K_m^{(i)}$.

Porosity averages in an especially simple manner (for linear poroelastic analysis) if we assume that the changes in volume fractions ($v^{(i)}$) of the constituents are negligible. For the thought experiment of Appendix A, this assumption is exactly right. For other situations, it is usually an approximation. We make this approximation here to achieve some simplicity, not out of necessity. With this approximation, we have

$$\delta\phi^* = \sum_{i=1} v^{(i)}\delta\phi^{(i)}. \tag{64}$$

Combining (64) with (62) and (63), we obtain a formula for K_s''. But to arrive at the formula, we need to use (7) to relate changes of global strain ε^* to changes of local confining pressure. We assume that the pore pressure has had time to equilibrate over the entire composite in order to determine the final constants of the composite.

First, note that

$$\delta p_c = -K_d^*\varepsilon^* + \alpha^*\delta p_f \qquad \text{and} \qquad \delta p_c^{(i)} = -K_d^{(i)}\varepsilon^{(i)} + \alpha^{(i)}\delta p_f, \tag{65}$$

and, from (7), we also have

$$\varepsilon_{pq}^{(i)} = -\chi^{hi}\frac{\delta p_f}{3}\delta_{pq} + T_{pqrs}[\varepsilon_{rs}^* + \chi^{hi}\frac{\delta p_f}{3}\delta_{rs}], \tag{66}$$

where χ^{hi} is defined as in (43). We can choose to treat ε^* and δp_f as the independent variables. Then, for the i-th constituent,

$$\begin{aligned}\delta\phi^{(i)} &= (\alpha^{(i)} - \phi^{(i)})\left(\varepsilon^{(i)} + \tfrac{\delta p_f}{K_m^{(i)}}\right) \\ &= (\alpha^{(i)} - \phi^{(i)})\left(P^{hi}[\varepsilon^* + \chi^{hi}\delta p_f] + \left[\tfrac{1}{K_m^{(i)}} - \chi^{hi}\right]\delta p_f\right).\end{aligned} \tag{67}$$

And, similarly, we have

$$\delta\phi^* = (\alpha^* - \phi^*)\varepsilon^* + \left(\frac{\alpha^*}{K_s'} - \frac{\phi^*}{K_s''}\right)\delta p_f. \tag{68}$$

Substituting (67) and (68) into (64), and making use of the linear independence of ε^* and δp_f, we find two equations for the coefficients:

$$\alpha^* - \phi^* = \sum_{i=1} v^{(i)}(\alpha^{(i)} - \phi^{(i)})P^{hi} \tag{69}$$

and

$$\left(\frac{\alpha^*}{K_s'} - \frac{\phi^*}{K_s''}\right) = \sum_{i=1} v^{(i)}(\alpha^{(i)} - \phi^{(i)})\left[\chi^{hi}P^{hi} + \left(\frac{1}{K_m^{(i)}} - \chi^{hi}\right)\right]. \tag{70}$$

In the case of just two components, (70) can be simplified by noting that $\chi^{hi} = \chi$ is constant, so that combining (69) with (70) recovers the exact result (48). Thus, for multicomponent media, (70) generalizes (albeit approximately) the exact result (48). The choice of generalization can also be tailored specifically to each type of effective medium approximation by making use of the dependence on the host medium in the factor χ^{hi} and on the particle shapes in P^{hi}.

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TABLE 1. Three examples of coefficients P and Q for spherical and nonspherical scatterers. The superscripts h and i refer to host and inclusion phases, respectively. Special characters are defined by $\beta = \mu[(3K+\mu)/(3K+4\mu)]$, $\gamma = \mu[(3K+\mu)/(3K+7\mu)]$, and $\zeta = (\mu/6)[(9K+8\mu)/(K+2\mu)]$. Expressions for spheres and needles were derived by Wu (1966) and Walpole (1969). Expressions for penny-shaped cracks were derived by Walsh (1965) and assume $K^{(i)}/K^{(h)} << 1$ and $\mu^{(i)}/\mu^{(h)} << 1$. The aspect ratio of the cracks is α_c.

Inclusion shape	P^{hi}	Q^{hi}
Spheres	$\frac{K^{(h)}+\frac{4}{3}\mu^{(h)}}{K^{(i)}+\frac{4}{3}\mu^{h}}$	$\frac{\mu^{(h)}+\zeta^{(h)}}{\mu^{(i)}+\zeta^{(h)}}$
Needles	$\frac{K^{(h)}+\mu^{(h)}+\frac{1}{3}\mu^{(i)}}{K^{(i)}+\mu^{(h)}+\frac{1}{3}\mu^{(i)}}$	$\frac{1}{5}\left(\frac{4\mu^{(h)}}{\mu^{(h)}+\mu^{(i)}}+2\frac{\mu^{(h)}+\gamma^{(h)}}{\mu^{(i)}+\gamma^{(h)}}+\frac{K^{(i)}+\frac{4}{3}\mu^{(h)}}{K^{(i)}+\mu^{(h)}+\frac{1}{3}\mu^{(i)}}\right)$
Penny cracks	$\frac{K^{(h)}+\frac{4}{3}\mu^{(i)}}{K^{(i)}+\frac{4}{3}\mu^{(i)}+\pi\alpha_c\beta^{(h)}}$	$\frac{1}{5}\left(1+\frac{8\mu^{(h)}}{4\mu^{(i)}+\pi\alpha_c(\mu^{(h)}+2\beta^{(h)})}+2\frac{K^{(i)}+\frac{2}{3}(\mu^{(i)}+\mu^{(h)})}{K^{(i)}+\frac{4}{3}\mu^{(i)}+\pi\alpha_c\beta^{(h)}}\right)$

TABLE 2. Three examples of computed values of the drained frame shear and bulk moduli G_d^* and K_d^*, the Biot-Willis parameter α^*, and the magnification factor $K_u^*/K_d^* = 1/(1-\alpha B)$ using the CPA, DEM$^+$, and MT theories for spherical inclusions and 40% volume fraction of clay.

Model	G_d^* (GPa)	K_d^* (GPa)	α^*	K_u^*/K_d^*
CPA	5.82	7.79	0.795	2.2
DEM$^+$	10.43	13.81	0.636	1.4
MT	12.42	16.38	0.568	1.3

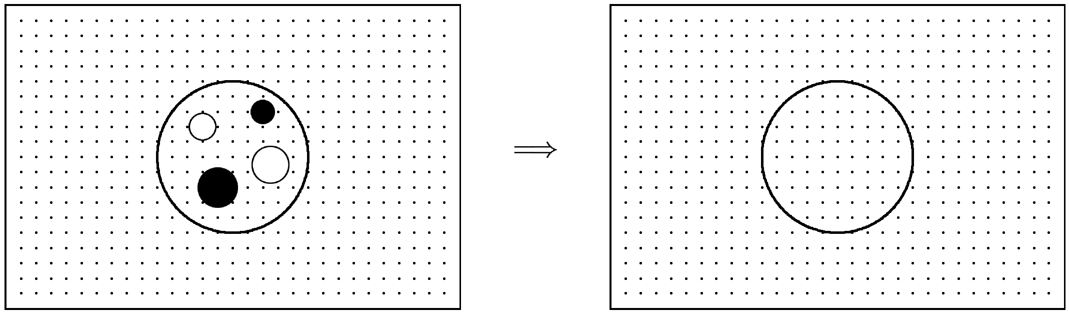

FIG. 1: The coherent potential approximation (CPA) treats the composite itself (regions of small dots) as the host and sets the single-scattering contributions from all the inclusions (shown as white and black circles on the left) equal to zero.

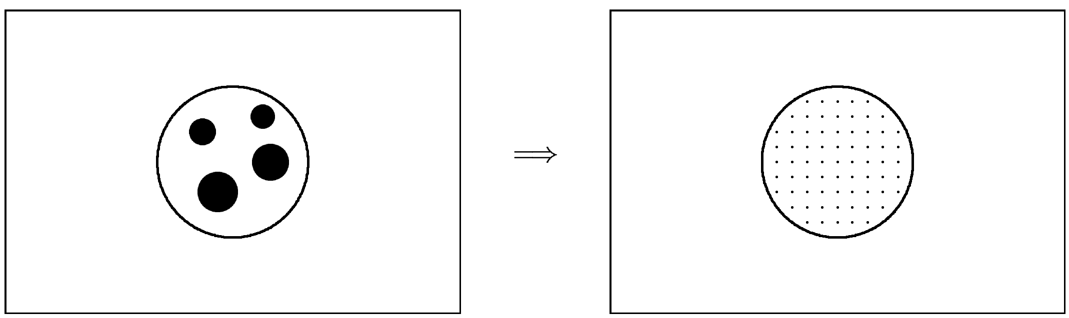

FIG. 2: The average T-matrix approximation (ATA) or Kuster-Toksöz (KT) method treats one of the constituents (shown as white background here) as the host and sets the single-scattering contributions from the inclusions equal to the scattering from a sphere of the effective composite (shown here as the cluster of dots inside the sphere).

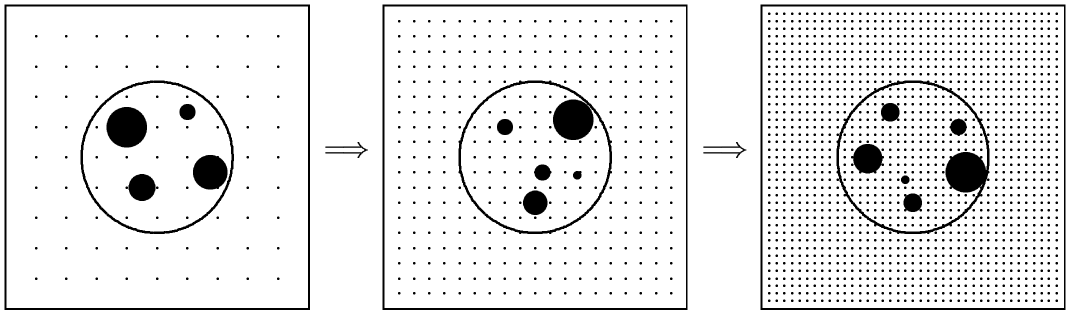

FIG. 3: The differential effective medium (DEM) approach treats the last computed effective constant $K(y)$ (shown here as the sparse dots) as the host and sets the single-scattering contributions for infinitesimal concentrations of inclusions equal to the scattering from a sphere of the next effective constant $K(y + dy)$ (shown here as the denser dots). After taking the limit $dy \to 0$, a differential equation for the effective constants is obtained (shown here as the densest dots).

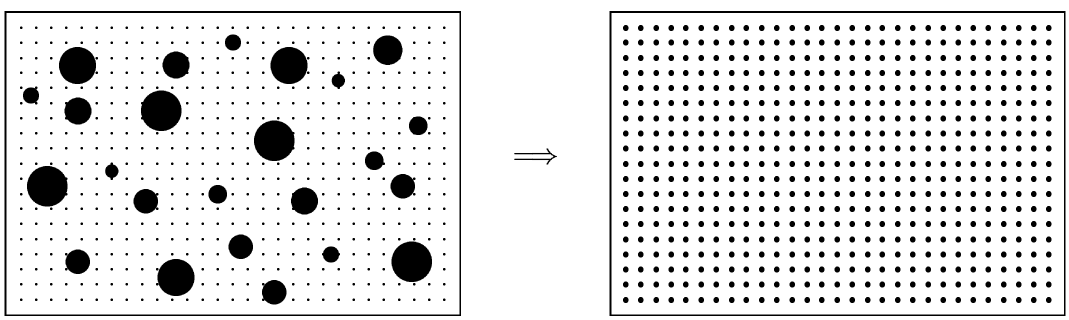

FIG. 4: The Mori-Tanaka (MT) method is a type of weighted averaging scheme that does not use a scattering analogy in formulating the estimator. The host material (shown here as the region filled with small dots on the left) will necessarily be one of the constituents. Inclusions of one other or many other materials are then imbedded in the host. The result of the averaging scheme is the effective medium represented schematically by the region of fatter dots on the right.

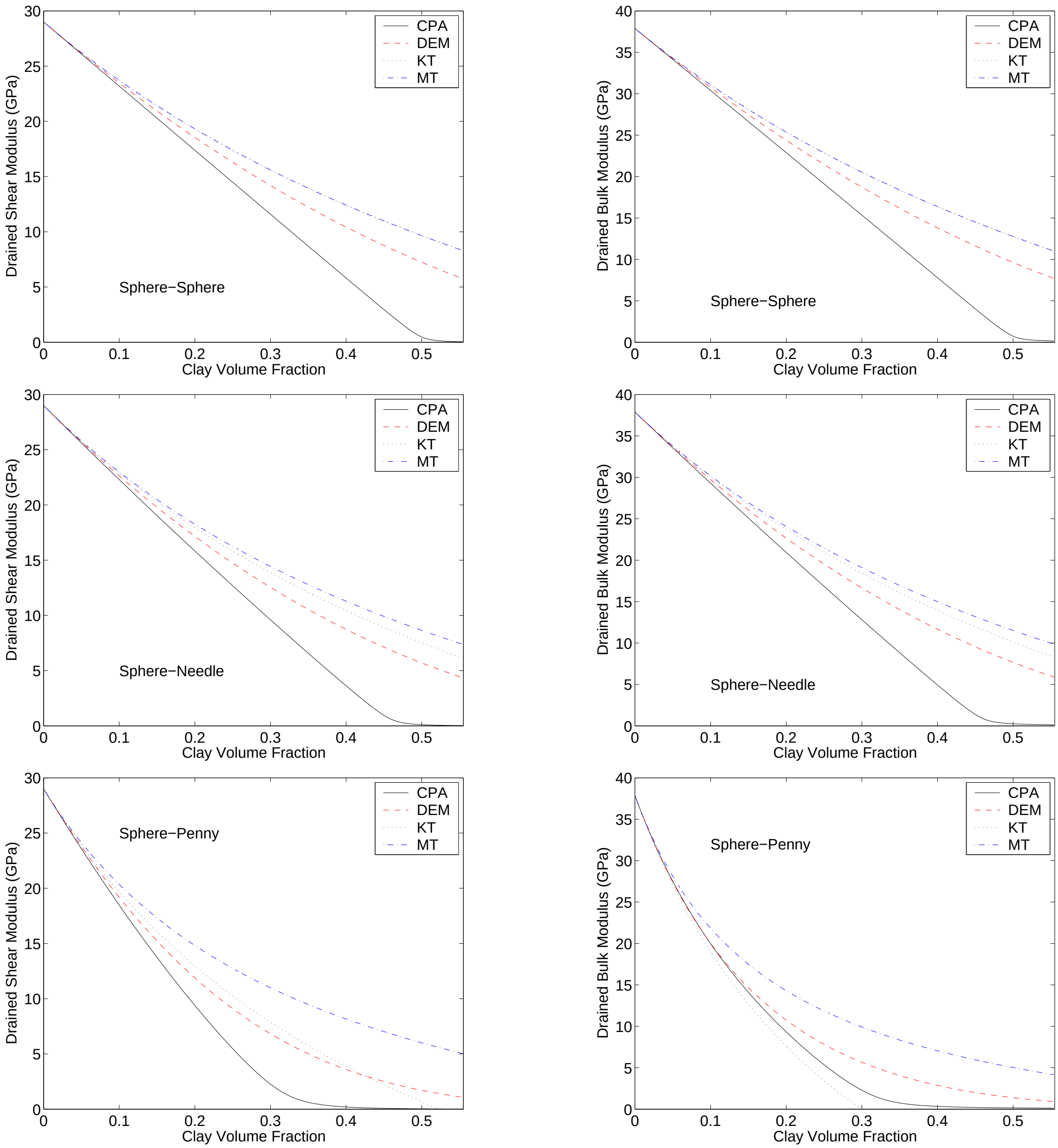

FIG. 5: Drained shear modulus G_d^* and bulk modulus K_d^* variation as a function of clay volume fraction for various models.

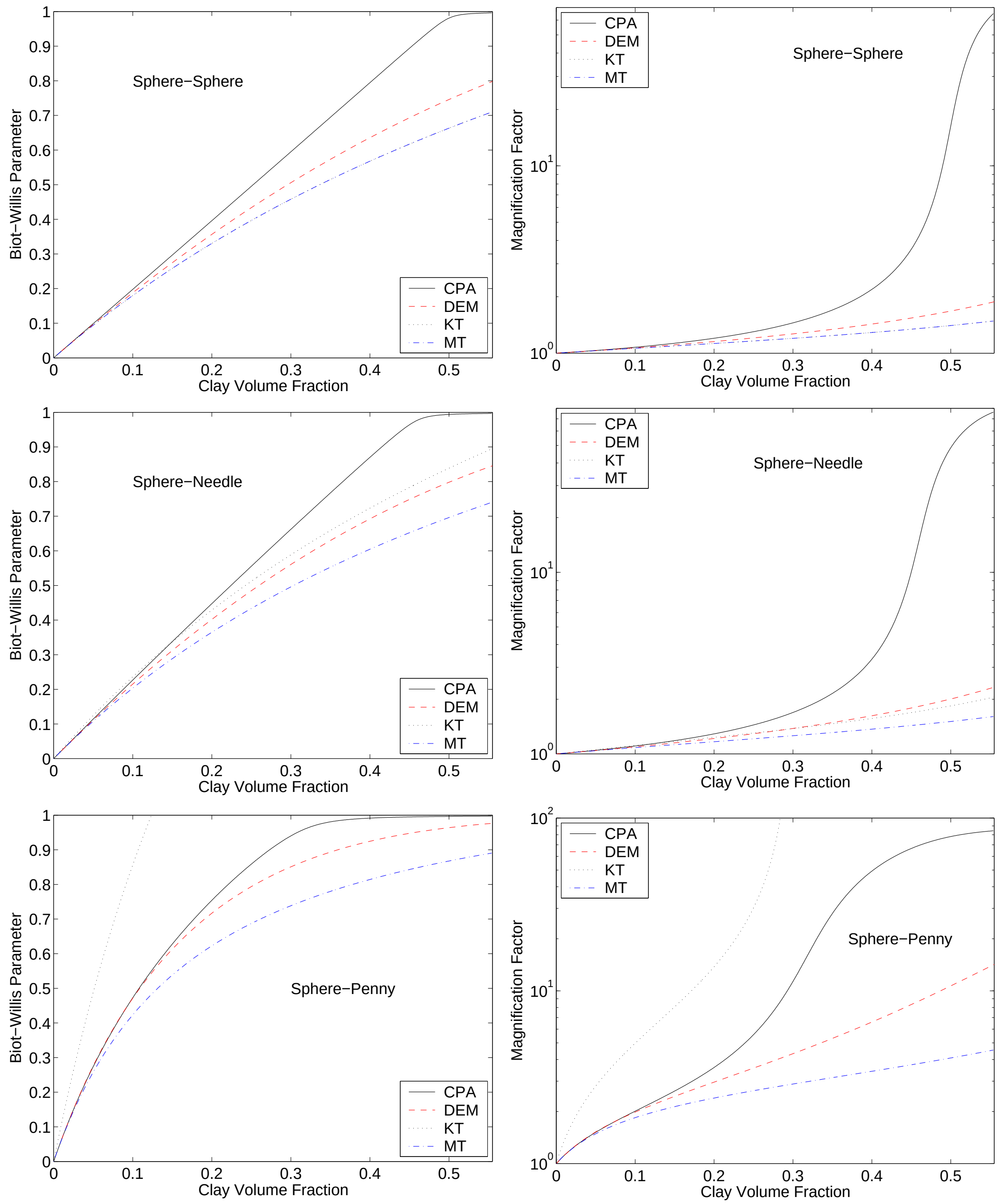

FIG. 6: Variation of the Biot-Willis parameter α and the magnificiation factor $K_u/K_d = 1/(1-\alpha B)$ as a function of clay volume fraction for various models.