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ANGULARLY ADAPTIVE P_1–DOUBLE P_0 DIFFUSION SOLUTIONS OF NON-EQUILIBRIUM GREY RADIATIVE TRANSFER PROBLEMS IN PLANAR GEOMETRY

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ABSTRACT

The double spherical harmonics angular approximation in the lowest order, i.e. double P_0 (DP_0), is developed for the solution of time-dependent non-equilibrium grey radiative transfer problems in planar geometry. The standard P_1 angular approximation represents the angular dependence of the radiation specific intensity using a linear function in the angular domain $-1 \leq \mu \leq 1$. In contrast, the DP_0 angular approximation represents the angular dependence as isotropic in each half angular range $-1 \leq \mu < 0$ and $0 < \mu \leq 1$. Neglecting the time derivative of the radiation flux, both the P_1 and DP_0 equations can be written as a single diffusion equation for the radiation energy density. Although the DP_0 diffusion approximation is expected to be less accurate than the P_1 diffusion approximation at and near thermodynamic equilibrium, the DP_0 angular approximation can more accurately capture the complicated angular dependence near the non-equilibrium wave front. We develop an adaptive angular technique that locally uses either the DP_0 or the P_1 diffusion approximation depending on the degree to which the radiation and material fields are in thermodynamic equilibrium. Numerical results are presented for a test problem due to Su and Olson for which a semi-analytic transport solution exists. The numerical results demonstrate that the adaptive P_1-DP_0 diffusion approximation can yield improvements in accuracy over the standard P_1 diffusion approximation for non-equilibrium grey radiative transfer.

KEYWORDS: radiative transfer, non-equilibrium radiation diffusion, spherical harmonics approximation, double spherical harmonics approximation

1. INTRODUCTION

Time-dependent non-equilibrium radiative transfer problems in the transport description are challenging to solve numerically. As a result, low-order spherical harmonics angular approximations have been examined as less expensive computational alternatives [1]. The standard P_1 angular approximation represents the angular dependence of the radiation specific intensity at each point in space and time using a linear function in the angular domain $-1 \leq \mu \leq 1$ [2]. Thus, there are two angular moment unknowns, the radiation energy density (proportional to the zeroth moment) and the radiation flux (first moment), at each point in space and time. Under the assumption that the time derivative of the radiation flux is negligible, the two

equations for the angular moment unknowns can be combined into a single P_1 diffusion equation for the radiation energy density. An equation for the energy balance of the background material through which the radiation is propagating and with which it is exchanging energy is also required.

In contrast, the double P_0 (DP_0) angular approximation represents the angular dependence of the radiation specific intensity as isotropic (constant) in each half angular range $-1 \leq \mu < 0$ and $0 < \mu \leq 1$, again resulting in two (half-range) angular moment unknowns at each point in space and time [2]. The DP_0 approximation of the radiation specific intensity can be written in a form that differs from the P_1 expression only in the linearly angular component [3–5]. The resulting DP_0 diffusion approximation (under the assumption of a negligible radiation flux time derivative) differs from the P_1 diffusion approximation only in the definition of the diffusion coefficient. The DP_0 diffusion approximation can be incorporated into a non-equilibrium planar geometry grey radiation diffusion code with only minor modifications.

As pointed out by Olson et al [1], the angular dependence of the specific intensity near a radiation wave front is complicated. The sharp radiation wave front can act as an internal boundary that is difficult for the P_1 angular approximation to accurately model. At an internal boundary, the angular distribution of the specific intensity can be nearly discontinuous when viewed as a function of angle. Brantley [5] has recently investigated the application of a mixed P_1-DP_0 angular approximation for more accurately treating material interfaces for time-independent planar geometry neutronics problems. In this paper, we investigate the application of the DP_0 angular approximation to more accurately model the angular dependence of the radiation specific intensity near a wave front for non-equilibrium grey radiative transfer problems. Numerical results from a test problem proposed by Su and Olson [6] for which a semi-analytic transport solution is available demonstrates that the DP_0 diffusion approximation can produce more accurate results than the P_1 diffusion approximation when the radiation and material are out of equilibrium.

When the radiation and material energies reach thermodynamic equilibrium, however, the radiative transfer and material energy balance equations asymptotically limit to first order to the equilibrium P_1 diffusion approximation [7]. Under these conditions, the DP_0 diffusion approximation gives an equation with a different (incorrect) diffusion coefficient. As such, we expect the DP_0 diffusion approximation to be less accurate than the P_1 diffusion approximation in and near thermodynamic equilibrium.

These physical and theoretical considerations motivated the development of an angularly adaptive P_1-DP_0 diffusion approximation that is designed to use the most accurate angular approximation (either DP_0 or P_1) at any given point in space and time. The adaptivity criterion is simple, physically-based, and essentially prescribes the use of the DP_0 diffusion approximation when the radiation and material energy fields are away from thermodynamic equilibrium and the use of the P_1 diffusion approximation when the radiation and material energy fields are in or near equilibrium. Numerical results from the Su and Olson test problem are presented to demonstrate that this adaptive angular approximation can yield overall accuracy improvements.

The remainder of this paper is organized as follows. In Section 2, we briefly outline the

derivations of the P_1 and DP_0 diffusion approximations for the case of non-equilibrium grey radiative transfer. We then describe how these two angular approximations can be readily combined into a single set of equations with a single parameter allowing the selection of the P_1 or the DP_0 approximation. We conclude Section 2 with a description of the angularly adaptive P_1-DP_0 diffusion approximation. In Section 3, we compare P_1, DP_0, and adaptive P_1-DP_0 diffusion numerical results for the Su and Olson test problem with the semi-analytic transport solution. Finally, we offer concluding remarks and suggestions for future work in Section 4.

2. DERIVATIONS OF THE P_1, DP_0, AND ADAPTIVE P_1-DP_0 DIFFUSION APPROXIMATIONS

In this section, we briefly outline the derivations of the P_1 and DP_0 diffusion approximations for the case of non-equilibrium grey radiative transfer in planar geometry. We focus the description of the derivations on contrasting the two angular approximations. We then demonstrate how the two angular approximations can be combined into a single set of equations with a single parameter allowing the selection of the P_1 or the DP_0 approximation. Finally, we theoretically motivate and describe a method for adaptively selecting the local angular approximation based on the degree to which the radiation and material energy density fields are in thermodynamic equilibrium.

In the following, we assume that hydrodynamic motion and heat conduction can be neglected, that scattering is isotropic, and that the background material is in local thermodynamic equilibrium. Under these assumptions, the equation of time-dependent grey radiative transfer in a one-dimensional planar geometry medium $0 \leq x \leq L$ with space- and temperature-dependent opacities is given by [8]

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial x} + \sigma_t I = \frac{1}{2}ac\sigma_a T_m^4 + \frac{1}{2}\sigma_s \int_{-1}^{+1} I d\mu' + \frac{1}{2}S \ ,$$

$$0 < x < L \ , \ -1 \leq \mu \leq 1 \ , \ t > 0 \ , \tag{1}$$

coupled with the material energy balance equation

$$C_v \frac{\partial T_m}{\partial t} = \sigma_a \left[\int_{-1}^{+1} I d\mu - acT_m^4 \right] \ , \ 0 \leq x \leq L \ , \ t > 0 \ , \tag{2}$$

where x, μ, and t are the space, angular direction, and time variables, respectively; $I(x,\mu,t)$ [*energy/area-time*] denotes the specific intensity of radiation; $T_m(x,t)$ [*temperature*] is the material temperature; $\sigma_a(x,T_m)$, $\sigma_s(x,T_m)$, and $\sigma_t(x,T_m)$ [$length^{-1}$] are the absorption, scattering, and total opacities, respectively, at material temperature $T_m(x,t)$; $S(x,t)$ [*energy/volume-time*] is an isotropic source of radiation; c [*length/time*] is the speed of light in a vacuum; a [*energy/temperature*^{4}*-volume*] is the radiation constant; and $C_v(x,T_m)$ [*energy/temperature-volume*] is the material heat capacity. The initial and boundary conditions for Eqs. 1 and 2 are given by

$$I(x,\mu,0) = I_0(x,\mu) \ , \ 0 \leq x \leq L \ , \ -1 \leq \mu \leq 1 \ , \tag{3}$$

$$T_m(x,0) = T_0(x) \ , \ 0 \leq x \leq L \ , \tag{4}$$

$$I(0,\mu,t) = \Gamma_0(\mu,t) \quad , \quad 0 < \mu \leq 1 \quad , \quad t > 0 \quad , \tag{5}$$

$$I(L,\mu,t) = \Gamma_L(\mu,t) \quad , \quad -1 \leq \mu < 0 \quad , \quad t > 0 \quad , \tag{6}$$

where I_0 and T_0 are the prescribed initial data and Γ_0 and Γ_L are the prescribed boundary data.

In preparation for deriving the P_1 and DP_0 diffusion approximations to Eqs. 1–6, we first define two angular moments of the radiation specific intensity. The zeroth angular moment of the specific intensity is the radiation energy density $E(x,t)$ [*energy/volume*] given by

$$E(x,t) = \frac{1}{c}\int_{-1}^{+1} I(x,\mu,t)\, d\mu \quad . \tag{7}$$

The first angular moment of the specific intensity is the radiation flux $F(x,t)$ [*energy/area-time*] given by

$$F(x,t) = \int_{-1}^{+1} \mu I(x,\mu,t)\, d\mu \quad . \tag{8}$$

In terms of the radiation energy density, the material energy balance equation is given by

$$C_v \frac{\partial T_m}{\partial t} = c\sigma_a \left(E - aT_m^4\right) \quad , \quad 0 \leq x \leq L \quad , \quad t > 0 \quad . \tag{9}$$

2.1. Derivation of the P_1 Diffusion Approximation

In this section, we briefly outline the derivation of the P_1 diffusion approximation. The P_1 representation of the radiation specific intensity for all x and t is given by [2]

$$I(x,\mu,t) = \frac{c}{2}E(x,t) + \frac{3}{2}\mu F(x,t) \quad , \quad -1 \leq \mu \leq 1 \quad . \tag{10}$$

Both $E(x,t)$ and $F(x,t)$ are continuous functions of space and time, so the P_1 representation of the specific intensity is linearly anisotropic in angle and is continuous in space, angle, and time. To obtain equations for the unknowns $E(x,t)$ and $F(x,t)$, we insert Eq. 10 into Eq. 1 and integrate the result over $-1 \leq \mu \leq 1$ to obtain the radiation energy balance equation

$$\frac{1}{c}\frac{\partial E}{\partial t} + \frac{1}{c}\frac{\partial F}{\partial x} + \sigma_a E = a\sigma_a T_m^4 + \frac{1}{c}S \quad , \quad 0 < x < L \quad , \quad t > 0 \quad . \tag{11}$$

Next, we insert Eq. 10 into Eq. 1, multiply the result by μ, and integrate over $-1 \leq \mu \leq 1$ to obtain

$$\frac{1}{c}\frac{\partial F}{\partial t} + \frac{c}{3}\frac{\partial E}{\partial x} + \sigma_t F = 0 \quad , \quad 0 < x < L \quad , \quad t > 0 \quad . \tag{12}$$

Eqs. 11 and 12, coupled with the material energy balance equation given by Eq. 9, constitute the P_1 approximation for grey radiative transfer (along with corresponding initial and boundary conditions).

To obtain the P_1 *diffusion* approximation, we make the further assumption that the time derivative of the radiation flux is negligible. Using this assumption in Eq. 12, the radiation flux is given by

$$F = -\frac{c}{3\sigma_t}\frac{\partial E}{\partial x} \ , \ 0 < x < L \ , \ t > 0 \ . \tag{13}$$

Then Eqs. 11 and 13 can be combined into a single equation for the radiation energy density given by

$$\frac{1}{c}\frac{\partial E}{\partial t} - \frac{\partial}{\partial x}\frac{1}{3\sigma_t}\frac{\partial E}{\partial x} + \sigma_a E = a\sigma_a T_m^4 + \frac{1}{c}S \ , \ 0 < x < L \ , \ t > 0 \ . \tag{14}$$

Eq. 14 for the radiation energy density, coupled with the material energy balance equation Eq. 9 for the material temperature, constitutes the P_1 diffusion approximation for grey radiative transfer.

We now turn to the specification of initial and boundary conditions. Initial conditions for $E(x,t)$ and $F(x,t)$ are easily derived from Eq. 3 by successive angular moment integrations analogous to the above to obtain

$$E(x,0) = E_0(x) = \frac{1}{c}\int_{-1}^{+1} I_0(x,\mu)\, d\mu \ , \ 0 \le x \le L \ , \tag{15}$$

$$F(x,0) = F_0(x) = \int_{-1}^{+1} \mu I_0(x,\mu)\, d\mu \ , \ 0 \le x \le L \ . \tag{16}$$

Although there is some ambiguity surrounding which is the most accurate boundary condition for a P_N approximation [9], we focus on the widely-used Marshak or Milne boundary condition. This boundary condition is obtained (at $x = 0$ for example) by inserting the P_1 representation for the radiation specific intensity Eq. 10 into the equation

$$\int_0^1 \mu[I(0,\mu,t) - \Gamma_0(\mu,t)]\, d\mu = 0 \ , \tag{17}$$

and performing the angular integrations to obtain

$$\frac{c}{4}E + \frac{1}{2}F = \int_0^1 \mu\Gamma_0 d\mu \ . \tag{18}$$

Inserting the expression for the P_1 radiation flux given by Eq. 13, we obtain the Marshak boundary condition at $x = 0$ for the P_1 diffusion approximation

$$\frac{c}{4}E - \frac{1}{6\sigma_t}\frac{\partial E}{\partial x} = \int_0^1 \mu\Gamma_0 d\mu \ . \tag{19}$$

An analogous boundary condition holds at $x = L$. The P_1 diffusion approximation for non-equilibrium grey radiative transfer problems is now fully specified by Eqs. 14, 15, and 19, coupled with the material energy balance equation given by Eq. 9 and its initial condition given by Eq. 4.

2.2. Derivation of the DP_0 Diffusion Approximation

In this section, we briefly outline the derivation of the double P_0 diffusion approximation. The DP_0 representation of the radiation specific intensity for all x and t is given by [2]

$$I(x,\mu,t) = \begin{cases} I^+(x,t) \ , & 0 < \mu \leq 1 \ , \\ I^-(x,t) \ , & -1 \leq \mu < 0 \ , \end{cases} \tag{20}$$

where $I^+(x,t)$ and $I^-(x,t)$ are continuous functions of space and time. Thus, the DP_0 representation of the specific intensity is isotropic in angle in each of the half angular ranges $-1 \leq \mu < 0$ and $0 < \mu \leq 1$ and is continuous in space and time but potentially discontinuous in angle. Using Eq. 20 in Eqs. 7 and 8 [5], we find that $E(x,t) = \frac{1}{c}\left[I^+(x,t) + I^-(x,t)\right]$ and $F(x,t) = \frac{1}{2}\left[I^+(x,t) - I^-(x,t)\right]$. After algebraic manipulation, we can rewrite Eq. 20 as

$$I(x,\mu,t) = \frac{c}{2}E(x,t) + \frac{3}{2}f(\mu)F(x,t) \ , \ -1 \leq \mu \leq 1 \ , \tag{21}$$

where the function $f(\mu)$ is defined as

$$f(\mu) = \begin{cases} +\frac{2}{3} \ , & 0 < \mu \leq 1 \ , \\ -\frac{2}{3} \ , & -1 \leq \mu < 0 \ . \end{cases} \tag{22}$$

Comparing Eq. 21 to Eq. 10, the P_1 and DP_0 angular representations of the radiation specific intensity differ only in the linear component.

To obtain equations for the unknowns $E(x,t)$ and $F(x,t)$, we first insert Eq. 21 into Eq. 1 and integrate the result over $-1 \leq \mu \leq 1$ to obtain the radiation energy balance equation given by Eq. 11. Next, we insert Eq. 21 into Eq. 1, multiply the result by $f(\mu)$, and integrate over $-1 \leq \mu \leq 1$ to obtain

$$\frac{1}{c}\frac{\partial F}{\partial t} + \frac{c}{4}\frac{\partial E}{\partial x} + \sigma_t F = 0 \ , \ 0 < x < L \ , \ t > 0 \ . \tag{23}$$

Eqs. 11 and 23, coupled with the material energy balance equation Eq. 9, constitute the DP_0 angular approximation for grey radiative transfer (along with corresponding initial and boundary conditions).

To obtain the DP_0 *diffusion* approximation, we make the further assumption that the time derivative of the radiation flux is negligible. Using this assumption in Eq. 23, the radiation flux is given by

$$F = -\frac{c}{4\sigma_t}\frac{\partial E}{\partial x} \ , \ 0 < x < L \ , \ t > 0 \ . \tag{24}$$

Then Eqs. 11 and 24 can be combined into a single equation for the radiation energy density given by

$$\frac{1}{c}\frac{\partial E}{\partial t} - \frac{\partial}{\partial x}\frac{1}{4\sigma_t}\frac{\partial E}{\partial x} + \sigma_a E = a\sigma_a T_m^4 + \frac{1}{c}S \ , \ 0 < x < L \ , \ t > 0 \ . \tag{25}$$

Eq. 25 for the radiation energy density, coupled with the material energy balance equation Eq. 9 for the material temperature, constitutes the DP_0 diffusion approximation for grey radiative transfer. Comparing Eqs. 14 and 25, the DP_0 diffusion approximation clearly differs from the P_1 diffusion approximation only in the definition of the diffusion coefficient.

We now specify the initial and boundary conditions for the DP_0 diffusion approximation. Initial conditions for $E(x,t)$ and $F(x,t)$ for the DP_0 approximation are the same as for the P_1 approximation and are given by Eqs. 15 and 16. The Marshak boundary condition at $x = 0$ is obtained by substituting the DP_0 representation for the radiation specific intensity, Eq. 21, into Eq. 17 and performing the angular integrations to again obtain Eq. 18. Inserting the representation for the DP_0 radiation flux given by Eq. 24, we obtain the Marshak boundary condition at $x = 0$ for the DP_0 diffusion approximation

$$\frac{c}{4}E - \frac{1}{8\sigma_t}\frac{\partial E}{\partial x} = \int_0^1 \mu\Gamma_0 d\mu \ . \tag{26}$$

An analogous boundary condition holds at $x = L$. The DP_0 diffusion approximation for non-equilibrium grey radiative transfer problems is now fully specified by Eqs. 25, 15, and 26, coupled with the material energy balance equation given by Eq. 9 and its initial condition given by Eq. 4. We note that the DP_0 diffusion approximation is only trivially different in form than the P_1 diffusion approximation.

2.3. The Adaptive P_1-DP_0 Diffusion Approximation

In this section, we describe how the P_1 and DP_0 diffusion approximations for non-equilibrium grey radiative transfer in planar geometry can be combined into a single set of equations with a single parameter specifying the local angular approximation. We then theoretically motivate and describe a method for adaptively selecting the local angular approximation based on the degree to which the radiation and material energy density fields are in thermodynamic equilibrium.

Recalling that the only difference between the form of the two approximations is in the diffusion coefficient, we define a generalized diffusion coefficient as

$$D(x,t,T_m) = \frac{1}{3M(x,t)\,\sigma_t(x,T_m)} \ , \tag{27}$$

where the multiplier function $M(x,t)$ is given by

$$M(x,t) = \frac{4}{3}\left[1-\delta(x,t)\right] + \delta(x,t) \ . \tag{28}$$

Then choosing $\delta(x,t) = 1$ at a given x and t yields the P_1 form of the diffusion coefficient and $\delta(x,t) = 0$ yields the DP_0 form.

The arbitrary P_1-DP_0 radiation diffusion approximation is then fully specified as

$$\frac{1}{c}\frac{\partial E}{\partial t} - \frac{\partial}{\partial x}D\frac{\partial E}{\partial x} + \sigma_a E = a\sigma_a T_m^4 + \frac{1}{c}S \ , \ 0 < x < L \ , \ t > 0 \ , \tag{29}$$

with the initial condition

$$E(x,0) = E_0(x) = \frac{1}{c}\int_{-1}^{+1} I_0(x,\mu)\,d\mu \ , \ \ 0 \le x \le L \ , \tag{30}$$

and the Marshak boundary condition at $x = 0$

$$\frac{c}{4}E - \frac{1}{2}D(0,t,T_m)\frac{\partial E}{\partial x} = \int_0^1 \mu\Gamma_0 d\mu \ . \tag{31}$$

An analogous boundary condition exists at $x = L$, and these equations are coupled to the material energy balance equation given by Eq. 9 along with its initial condition Eq. 4. The desired angular approximation at any space and time point can be easily chosen by the appropriate selection of $\delta(x,t)$.

When the radiation and material energies reach thermodynamic equilibrium, the radiative transfer and material energy balance equations asymptotically limit to first order to the equilibrium P_1 diffusion approximation [7]. Under these conditions, the DP_0 diffusion approximation gives an equation with a different (incorrect) diffusion coefficient. As such, we expect the DP_0 diffusion approximation to be less accurate than the P_1 diffusion approximation at and near thermodynamic equilibrium. However, we hypothesize that the DP_0 angular approximation can more accurately capture the behavior of the radiation specific intensity than the P_1 angular approximation when the radiation and material energy density fields are out of equilibrium. Motivated by these considerations, we propose an "adaptive" P_1-DP_0 diffusion approximation in which the value of the parameter $\delta(x,t)$ for each x and t is chosen according to the criterion

$$\delta(x,t) = \begin{cases} 1 \ , & |\frac{E(x,t)}{E_m(x,t)} - 1| \le \beta \ , \\ 0 \ , & otherwise \ , \end{cases} \tag{32}$$

where $E_m(x,t)$ [*energy/volume*] is the material energy density and we have used $\beta = 0.1$ in our simulations. This prescription for $\delta(x,t)$ ensures that the P_1 angular approximation is used at and near thermodynamic equilibrium [where $E(x,t) \approx E_m(x,t)$] and that the DP_0 angular approximation is used otherwise.

3. NUMERICAL RESULTS

In the previous section, we derived the P_1 and DP_0 diffusion approximations for the case of non-equilibrium grey radiative transfer in planar geometry. We showed that the P_1 and DP_0 diffusion approximations differ only in the definition of the diffusion coefficient. As a result, these approximations can be easily combined into one set of equations with a single parameter specifying the desired angular approximation. Finally, we described a simple, physically-based technique for adaptively choosing the local angular approximation at each point in space and time.

We have numerically implemented the angularly adaptive P_1-DP_0 diffusion approximation using the spatial and temporal discretizations described in reference [10]. In particular, the radiation diffusion and material energy balance equations are spatially discretized using a second-order

cell-centered differencing and temporally discretized implicitly using backward Euler differencing. The material properties are treated explicitly in time. No iteration on the temperature dependence of the material properties is performed, so extremely small timesteps are used to obtain the numerical results.

In this section, we describe the Su and Olson semi-analytic benchmark problem for which a semi-analytic transport solution exists [6]. Then we apply the P_1, DP_0, and adaptive P_1-DP_0 diffusion approximations to the numerical solution of this problem.

3.1. Description of the Su and Olson Benchmark Problem

To linearize the radiative transfer equation and the material energy balance equation such that a semi-analytic solution can be obtained, Su and Olson [6] make two major assumptions: the material opacities are constant (independent of the material temperature) and the heat capacity of the material is proportional to the cube of the material temperature, $C_v = \alpha T_m^3$. While not physically realistic, these assumptions make the equations of radiative transfer linear and allow their semi-analytic solution.

Su and Olson write the radiative transfer equation and the material energy balance equation in dimensionless forms as follows. The spatial variable x is written in terms of the dimensionless optical depth, $z \equiv \sigma_t x$. A dimensionless parameter $\varepsilon \equiv 4a/\alpha$ is defined, where α is the material heat capacity proportionality constant. Then a dimensionless time variable is defined as $\tau \equiv \varepsilon c \sigma_t t$. Dimensionless absorption and scattering opacities are defined as $c_a \equiv \sigma_a/\sigma_t$ and $c_s \equiv \sigma_s/\sigma_t$, respectively, such that $c_a + c_s = 1$. Defining a reference "hohlraum" temperature T_R, the dimensionless radiation specific intensity, radiation energy density, and material energy density are given by $U(z,\mu,\tau) \equiv I(x,\mu,t)/aT_R^4$, $W(z,\tau) \equiv \int_{-1}^{+1} U(z,\mu,\tau)\,d\mu$, and $V(z,\tau) \equiv [T_m(x,t)/T_R]^4$, respectively. Finally, the dimensionless radiation source is given by $Q(z,\tau) \equiv S(x,t)/aT_R^4$. Using these definitions, the dimensionless radiative transfer equation is given by

$$\left(\varepsilon\frac{\partial}{\partial\tau} + \mu\frac{\partial}{\partial z} + 1\right) U(z,\mu,\tau) = \frac{c_a}{2}V(z,\tau) + \frac{c_s}{2}W(z,\tau) + \frac{1}{2}Q(z,\tau) \quad ,$$

$$-\infty < z < \infty \quad , \quad -1 \le \mu \le 1 \quad , \quad \tau > 0 \quad , \tag{33}$$

coupled with the dimensionless material energy balance equation given by

$$\frac{\partial V(z,\tau)}{\partial\tau} = c_a\left[W(z,\tau) - V(z,\tau)\right] \quad , \quad -\infty < z < \infty \quad , \quad \tau > 0 \quad . \tag{34}$$

The boundary and initial conditions for Eq. 33 are given by

$$\lim_{z\to\pm\infty} U(z,\mu,\tau) = 0 \quad , \tag{35}$$

and

$$U(z,\mu,0) = 0 \quad , \tag{36}$$

respectively. The initial condition for Eq. 34 is given by

$$V(z,0) = 0 \quad . \tag{37}$$

The Su and Olson non-equilibrium grey radiative transfer problem consists of an initially cold, homogeneous, infinite, and isotropically scattering medium with an internal radiation source defined by

$$Q(z,\tau) = \frac{1}{2z_0}\left[H(z+z_0) - H(z-z_0)\right]\left[H(\tau) - H(\tau-\tau_0)\right] \quad , \tag{38}$$

where H is the Heaviside step function. The source is thus defined on the dimensionless spatial domain $-z_0 \leq z \leq z_0$ and is turned on for dimensionless times $0 \leq \tau \leq \tau_0$. Su and Olson obtained semi-analytic transport solutions for this problem by applying the Fourier transform to the spatial variable and the Laplace transform to the time variable [6].

3.2. Su and Olson Problem Numerical Results

In this section, we apply the P_1, DP_0, and angularly adaptive P_1-DP_0 diffusion approximations to the numerical solution of the Su and Olson benchmark problem. Su and Olson obtained semi-analytic results both with and without scattering; we present results for the absorption-only case here. We obtain a solution for the case of the radiation source defined by Eq. 38 with $z_0 = 0.5$ and $\tau_0 = 10$ and the dimensionless parameter $\varepsilon \equiv 4a/\alpha = 1$. Our numerical simulations were performed with an initial timestep of $\Delta\tau = 10^{-9}$ and a timestep control that restricts the maximum relative change in the radiation energy density in any zone during a timestep to 1%. The spatial zoning was uniform with a zone size of $\Delta z = 0.001$, i.e. one-thousandth of a mean free path. The material properties (σ_a and C_v) and the angular approximation parameter (δ) were lagged in time. We have numerically solved this problem using the pure P_1 diffusion approximation, the pure DP_0 diffusion approximation, and the angularly adaptive P_1-DP_0 diffusion approximation with $\beta = 0.1$ (see Eq. 32). The radiation energy density from these simulations is plotted in Fig. 1 as a function of space at several time values.

The P_1 diffusion solution for the radiation energy density has an error of approximately 20% early in time ($\tau \approx 0.1 - 10$) for small z. The extremely large errors for larger z are due to the fact that the P_1 *diffusion* approximation (as opposed to the true P_1 approximation) propagates radiation with an infinite velocity [1]. As a result, the radiation front propagates too quickly. Similar comments hold regarding the errors in the material energy density predicted by the P_1 diffusion approximation. Later in time, the radiation and material approach thermodynamic equilibrium and the magnitude of the errors decreases significantly. Because the radiative transfer equation asymptotically limits to the P_1 diffusion approximation at thermodynamic equilibrium, this is a theoretically predicted result.

Early in time (up to $\tau = 10$), the DP_0 solution is significantly more accurate overall than the P_1 solution (note the plots are on log-log scales). The DP_0 diffusion approximation suffers from the same infinite propagation velocity as the P_1 diffusion approximation and hence exhibits large errors past the wave front. As the radiation and material approach thermodynamic equilibrium later in time ($\tau > 10$), however, the DP_0 solution becomes significantly less accurate than the P_1 solution. These numerical results confirm the theoretical prediction described above that the DP_0 diffusion approximation will be less accurate than the P_1 diffusion approximation near thermodynamic equilibrium.

The adaptive P_1-DP_0 solution is essentially identical to the DP_0 solution for $\tau \leq 3.16228$ and is

therefore more accurate than the P_1 solution. For $\tau = 10$, the adaptive P_1-DP_0 solution is slightly less accurate than the DP_0 solution but still significantly more accurate overall than the P_1 solution. At $\tau = 31.6228$, the adaptive P_1-DP_0 solution is significantly more accurate overall than either the P_1 or the DP_0 solution alone. For $\tau = 100$, the adaptive P_1-DP_0 solution is slightly less accurate than the P_1 solution but drastically more accurate than the DP_0 solution.

4. CONCLUSIONS

In this paper, we have developed and applied the double P_0 diffusion approximation for non-equilibrium grey radiative transfer problems in planar geometry. We described how the P_1 and DP_0 diffusion approximations for non-equilibrium grey radiative transfer in planar geometry can be combined into a single set of equations with a single parameter specifying the local angular approximation to be used at any point in space and time. We then prescribed an adaptive method for choosing the local angular approximation such that the P_1 angular approximation is used at and near thermodynamic equilibrium and the DP_0 angular approximation is used otherwise.

We compared the accuracy of the DP_0 and the adaptive P_1-DP_0 diffusion approximations to the P_1 diffusion approximation using a semi-analytic test problem developed by Su and Olson [6]. We found that the DP_0 diffusion approximation is generally more accurate than the P_1 diffusion approximation when the radiation and material are out of equilibrium but is less accurate near thermodynamic equilibrium. These results motivated the development of an adaptive P_1-DP_0 diffusion approximation in which the local angular approximation used is adaptively determined at each point in space and time by thermodynamic equilibrium considerations. This adaptive angular approximation provides improved accuracy for the Su and Olson test problem over the P_1 diffusion approximation during the non-equilibrium phase and essentially the same accuracy near thermodynamic equilibrium. These numerical conclusions should be further verified on a broader range of problems with more realistic opacities.

The full P_1 approximation, as opposed to the P_1 *diffusion* approximation, retains information regarding the time derivative of the radiation flux. As a result, the P_1 approximation can give more accurate results than its corresponding diffusion approximation [1, 9]. One possible area of future work is to investigate whether the same holds true when comparing the full DP_0 and the DP_0 *diffusion* angular approximations. Another area of possible future work would be to investigate whether the accuracy improvements demonstrated by the DP_0 diffusion approximation compared to the P_1 diffusion approximation remain when comparing the full P_1 and DP_0 approximations. If so, the full DP_0 and P_1 angular approximations could be combined into an adaptive angular approximation similar to the corresponding diffusion scheme presented in this paper. In addition, Olson et al [1] have recently proposed the $P_{1/3}$ approximation which is an appropriately weighted average of the full P_1 approximation and the P_1 diffusion approximation such that the correct radiation propagation velocity is obtained in the optically thin limit. This $P_{1/3}$ approximation retains first order accuracy in the equilibrium diffusion limit [7]. The same approach can be used with the full DP_0 approximation and the DP_0 diffusion approximation to develop a DP_0-based approximation with the correct radiation propagation velocity.

The double P_0 angular approximation has seen little use in recent years, due at least in part to its

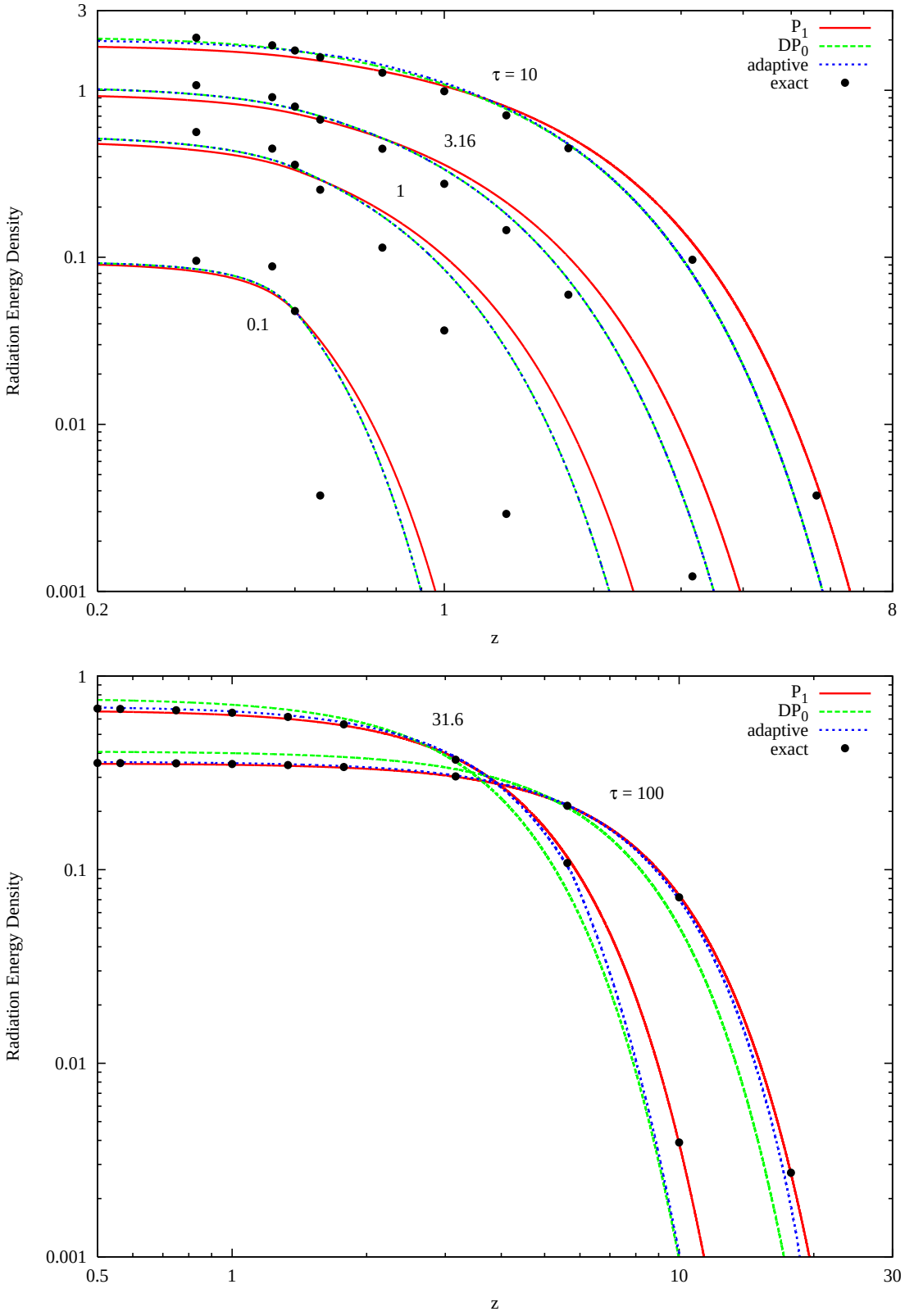

Figure 1. Radiation energy density as a function of space at several time values

perceived inapplicability to multi-dimensional geometries. However, Paveri-Fontana and Amster [4] theoretically extended the DP_0 approximation for steady-state neutronics problems to multi-dimensional geometries using both a generalization of the typical angular moments-based derivation of the DP_0 approximation as well as a variational analysis. The diffusion equation they derived for the DP_0 approximation differed from the standard P_1 diffusion equation only in the value of the diffusion coefficent. However, their paper does not present numerical results to examine the accuracy of the method. We plan to investigate this multidimensional extension for radiative transfer applications.

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