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Hydrodynamic test problems

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We present test problems that can be used to check the hydrodynamic implementation in computer codes designed to model the implosion of a National Ignition Facility (NIF) capsule. The problems are simplified, yet one of them is three-dimensional. It consists of a nearly-spherical incompressible imploding shell subjected to an exponentially decaying pressure on its outer surface. We present a semi-analytic solution for the time-evolution of that shell with arbitrary small three-dimensional perturbations on its inner and outer surfaces. The perturbations on the shell surfaces are intended to model the imperfections that are created during capsule manufacturing.

Introduction

Codes that model the complicated hydrodynamics of an imploding National Ignition Facility (NIF) capsule are an important element in the design of capsules. A persistent concern for code developers and users is how well these codes model reality. Two kinds of errors may exist in the simulation: Those due to code design and those due to code implementation. Design errors occur because the input parameters or the equations being solved do not accurately reflect the physical processes to be modeled. The process of confirming that the code is solving the correct equations is called code validation.

Implementation errors occur because the equations are not solved correctly. A modern hydrodynamic code with three-dimensional (3D) capability has more than 500,000 lines of coding and 5,000 modules. Any mistake or bug in the coding whether caused by a typographical error or insufficient computing resources can affect the model's accuracy. The process of confirming that the code is solving the equations correctly is called code verification. This paper addresses code verification through comparison with analytic and semi-analytic solutions of imploding shells with and without surface perturbations.

Problem description and early time solution

Consider the implosion of a thin spherical shell of outer radius a, inner radius b, density ρ_0, longitudinal sound speed c, subjected to a pressure $P(t) = P_0 e^{-\alpha t} \ll$ bulk modulus K, which is applied to the outer boundary. Sharp 1942, Blake 1952, and Larson 1979, worked out the solution of the wave motion produced when a pressure is applied to the interior surface of a spherical cavity. Their interest was in geophysical applications

and seismic decoupling. Sharp restricted his studies to the case where the Lamé constants are equal, $\lambda = \mu$, or equivalently, that Poisson's ratio $\sigma = 1/4$. Blake did not make that restriction and found the general solution in the frequency domain. We can adapt these results to our problem if we consider a convergent (instead of a divergent) spherical wave and specialize to the case of a strengless shell ($\sigma = 1/2$). The expressions for the displacement and velocity at any location R reduce to:

$$u(R,\tau) = \frac{P_0 a}{\rho_0 \alpha^2}\left[\frac{\alpha}{Rc}(e^{-\alpha\tau}-1) - \frac{1}{R^2}(1 - e^{-\alpha\tau} - \alpha\tau)\right] \tag{1}$$

$$V(R,\tau) = -\frac{P_0 a}{\rho_0 \alpha}\left[\frac{\alpha}{Rc}e^{-\alpha\tau} + \frac{e^{-\alpha\tau}}{R^2} - \frac{1}{R^2}\right] \tag{2}$$

Where τ, the retarded time, is given by t-(a-R)/c. The jump-off velocity, which is the magnitude of the instantaneous velocity at the inner surface of radius b at the time of arrival ($\tau = 0$) of the first shock wave, is given by:

$$V_j = 2P_0 a / bc\rho_0 \tag{3}$$

The jump-off velocity under the restriction that the applied pressure is much smaller than the bulk modulus is thus linearly proportional to the initial pressure P_0. We now address the effects of small compressibility on the jump-off velocity.

Correction for small compressibility

To first order, we can account for the effect of small compressibility on the jump-off velocity of a thin ((a - b) / a << 1) shell by replacing the sound speed c with the shock speed $Us = [(K + P_0) / \rho_0]^{1/2}$. In this approximation, $V_j \sim P_0 / (P_0 + K)^{1/2}$. We can further improve the estimate by taking into account the following two effects: First, the outer radius, a, decreases as the shock propagates into the shell. Second, the peak pressure of a non-uniform shock wave decreases as a result of nonlinear propagation in the rarefaction following it. This second effect is sometimes referred to as "hydrodynamic attenuation" and has been worked out by Duvall 1962. The hydrodynamic attenuation depends on the spatial derivative of the pressure behind the shock front, which we linearize.

Late-time solution of the incompressible spherical shell

For an incompressible shell, the velocity and displacement at the inner surface can be reduced to two coupled first-order ordinary differential equations (ODE). These can be effectively solved to machine accuracy for any applied pressure using standard quadrature techniques. For the exponentially decaying pressure, the acceleration at the outer surface of the shell is given by the simple general expression:

$$a_o(t) = 0.5\frac{\dfrac{-2P_0 e^{-\alpha t}}{\rho_0} + 3V_o^2(t) + rV_o(t)[rV_i(t) - 4V_o(t)]}{r(t)[R_o(t) - R_i(t)]} \tag{4}$$

Where the subscript *o* refers to values at the outer surface, subscript *i* refers to values at the inner surface, V(t) is the velocity, R(t) is the radius, and r(t) is the ratio $R_o(t)/R_i(t)$. Equation (4) can be derived by equating the rate of change of kinetic energy

of the shell $\frac{d}{dt}\int_{R_i(t)}^{R_o(t)}\frac{1}{2}4\pi\rho_0 R^2[V(r)]^2 dR$ with the rate of work done by the pressure

applied to the outside surface $4\pi[R_o(t)]^2 P_0 e^{-\alpha t}V_o(t)$. Conservation of volume implies that the particle velocity within the shell at location R is proportional to $1/R^2$ and the kinetic energy can be expressed in a closed form. Differentiation and the chain rule then yields Eq. (4). The two ODE's that need to be solved simultaneously are then $dV_o(t)/dt = a_o(t)$ where a_o is given by Eq. (4), and $dR_o(t)/dt = V_o(t)$.

Correction for small compressibility

Considerable progress in developing the late-time solution for the compressible shell is achieved if the shell is accelerated shocklessly. We use the exponentially decaying pressure as the driving force and search for initial conditions that lead to shockless acceleration and hence analytic or semi analytic solutions. We would like the solution using these initial conditions to closely match the time averaged solution for the shock accelerated case. We thus require that the total mass and thickness of the shell in the shockless case be identical to that in the shock accelerated case.

The key to finding these initial conditions is to have smoothly varying pressure and velocity across the shell thickness at time zero. These smoothly varying functions must also be consistent with the equation-of-state (EOS) of the shell and the exponentially decaying pressure on the outer boundary.

Initial conditions leading to a shockless compression

The equations of momentum and mass conservation in Eulerian spherical coordinates lead to:

$$\frac{\partial P}{\partial R} = -\rho\frac{DV}{Dt} \quad \text{and} \quad \frac{\partial V}{\partial R} + \frac{2V}{R} = -\frac{1}{\rho}\frac{D\rho}{Dt} \qquad (5)$$

Where P is the pressure, V is the velocity, ρ is the density and $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + V\frac{\partial}{\partial R}$. We restrict ourselves to the case of a simple yet useful EOS in which the pressure P is related to the compression through a bulk modulus K as follows: $P = K(\rho/\rho_0 - 1)$. In this case, the initial conditions for the pressure, and velocity, leading to a shockless acceleration, are given by the solution of the following 3 coupled ordinary differential equations:

$$\frac{dP}{dR} = -\rho_0 a(R)[K + P(r)]/K \qquad (6)$$

$$\frac{dV}{dR} = -\frac{2V}{R} + \frac{\alpha P(R)}{K + P(R)} \qquad (7)$$

$$\frac{da}{dR} = -\frac{2a(R)}{R} - \frac{\alpha^2 KP(R)}{[K + P(R)]^2} + 2\frac{V^2}{R^2} + \left(\frac{dV}{dR}\right)^2 \qquad (8)$$

Subject to the boundary conditions that the pressure at the inner boundary be identically zero, the total momentum be identically zero, and the total mass be identical to the shock accelerated case.

Effect of small surface perturbations on the sphericity of the implosion

Although 2D experiments (Weir et al., 1998) and calculations (Bakharakh et al.,

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1997) of imploding perturbed shells have been performed, we focus here on the 3D aspect of the problem. Analytic equations for the evolution of the perturbations were given by Mikalelian,1990 and applied to the case of a constant acceleration followed by a constant deceleration. We have extended that work to include an exponentially decaying pressure applied to the outer surface of the shell. The perturbed radius R_{per} is given as a function of the unperturbed radius R_{unp} by the following equation:

$$R_{per} = R_{unp} + \eta(t,n,m)Y_{n,m}(\theta,\phi) \tag{9}$$

Where t is the time, $\eta(t,n,m)$ is the amplitude of the perturbation, and $Y_{n,m}(\theta,\phi)$ is the spherical harmonic. The value of the unperturbed radius is given by the late-time solution of the incompressible shell described in this paper. The analytic solution describe the evolution of each node, $\eta(t,n,m)$, in terms of two coupled ordinary second-order differential equations (Eq. 20 in Mikaelian 1990). The amplitude of a given mode on the inner surface as a function of time depends on the amplitude of that mode on the outer surface as well as the position, velocity, and acceleration at the inner and outer surfaces. The analytic solution, which is exact in the limit of small perturbations, takes into account the feed-through effect, i.e. the interactions between perturbations on the inner and outer surfaces.

A question of practical value is the following: Which surface, if any, is more critical in manufacturing the capsule to reduce perturbations ? To address this question, we applied the perturbation to only one surface at a time. We found that for thin shells (such as the ones proposed for NIF), both surfaces are critical. This means that perfecting only one surface will not stop the perturbations from feeding through to the other surface. To account for the effect of compressibility, finely zoned 1D calculations can provide the average position, velocity, and acceleration at the inner and outer radii for a compressible shell. These can in turn be used in a modified Bell-Plesset equation (Goncharov *et al*. 2000, Amendt et al., 2003, Lin et al., 2002) to predict the growth of perturbations and compare with 3D code calculations with various resolutions.

Conclusions

The solution of the problems presented in this paper provide an independent check on the ability of newly developed 3D codes to model the hydrodynamics of a simplified implosion system. These analytic and semi analytic solutions are well suited to quantify numerical errors due to mesh resolution. Another benefit of these test problems is providing a way to compare the efficiency of various algorithms such as adaptive mesh refinement (Rendleman et al., 2000), or mesh free methods (Dilts, 2001).

Acknowledgements

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