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Multi-Phase-Center IFSAR

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ABSTRACT

We present new methods for resolving IFSAR ambiguities and SAR layover. The analytic properties of these techniques make them well suited for reliable, efficient computation.

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CONTENTS

1. Introduction

This investigation studies the advantages of using multiple phase center IFSAR (interferometric synthetic aperture radar) to resolve IFSAR phase ambiguities and SAR layover. For an approximate "point" target, the phase of the complex coherence factor, computed from a two-phase center antenna, equals the principal or wrapped value of the return signal's angle-of-arrival from the reflector. Since the terrain elevation at that location is proportional to the unwrapped phase, we must unwrap the phase to estimate the height. A number of algorithms for phase unwrapping have been proposed (Ghiglia [1]), but the most reliable methods require additional information. It is known that a three-phase-center IFSAR system can be used to remove, with fairly high probability, the ambiguity in phase (see Jakowatz et al. [2] or Bickel and Hensley [3]).We show that a fourth phase center can be used to significantly reduce the error probability for removing IFSAR phase ambiguities; in fact, for complex Gaussian noise, the error probability decreases approximately as the square of the error probability for the corresponding three-phase-center IFSAR. In addition, we show that a three-phase-center system can be used to detect and compute the angle-of-arrival for two point targets that are projected into a single range-azimuth resolution cell of the slant plane (SAR layover).

Roughly, three broad classes of IFSAR operation can be used to collect data for the methods presented in this study. The first class employs a three or four phase-center antenna placed orthogonal to the flight path of the aircraft (see Fig. 1). Here, we consider an IFSAR antenna combined with one or two monopulse antennas to be approximately equivalent to a three or four phase-center antenna, respectively. A second class employs a single IFSAR antenna and two separate passes of the aircraft with slightly different grazing angles (see Martinez et al. [4]). In this multi-pass approach, each of the IFSAR antennas act as a pair of phase centers, and assuming that the two images are sufficiently coherent, the two IFSAR antennas can be combined to form a third pair of phase centers (see Figure 1). A third class uses an IFSAR antenna operated at different frequencies. By operating at different frequencies, the baseline to wavelength ratio of the antenna changes, in effect producing a different pair of phase centers. We note that multi-phase-center IFSAR may also be viewed as multiple baseline IFSAR.

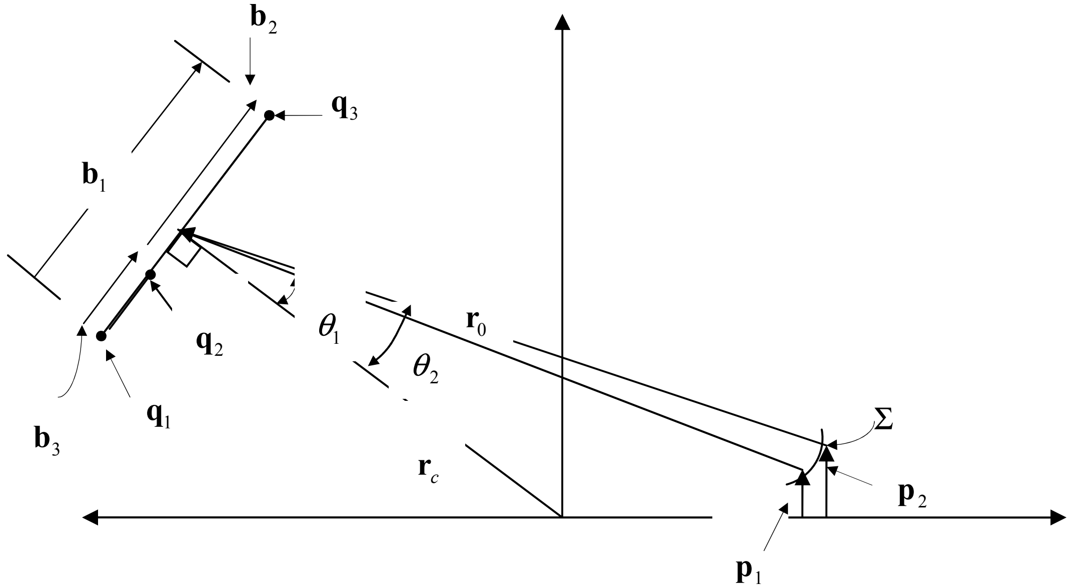

Figure 1. Three-Phase-Center IFSAR

The basic idea in SAR interferometry is to construct a phase difference map from the complex images produced by a pair of phase-centers. A mathematical description of the complex correlation or coherence of a pair of phase-centers in an interferometric imaging system is provided by the Van Cittert-Zernike Theorem (Goodman [5]). We show that, for two point targets projected into a single range-resolution cell, the Van Cittert-Zernike Theorem takes the form

$$\mu(k_l) \approx \exp(jk_l s)\left[\frac{I_1}{I_1+I_2}\exp(-jk_l d)+\frac{I_2}{I_1+I_2}\exp(jk_l d)\right] \tag{1.1}$$

$$\approx |\mu(k_l)|\exp\left[j(k_l s+t_l)\right],$$

where $\mu(k_l)$ ($l = 1,2,3$) equals the complex coherence factor for one pair of phase-centers, and

$$t_l = t_l\left(\frac{I_1}{I_1+I_2}, d\right) = \tan^{-1}\left(1 - 2\frac{I_1}{I_1+I_2}\tan(k_l d)\right). \qquad (1.2)$$

We assume that $k_1 > k_2 > k_3$, $k_1 = k_2 + k_3$, and we define:

$k_l = 2\pi\chi b_l / \lambda_c$,

b_l = baseline length

λ_c = center wavelength,

θ_i = signal's angle-of-arrival from point $\mathbf{p}_i$ (see Figure 1),

$I_i / (I_1 + I_2)$ = relative intensity of the point target at $\mathbf{p}_i$

$s = (\sin\theta_1 + \sin\theta_2)/2$

$d = (\sin\theta_2 - \sin\theta_1)/2$

$\chi = 1$ or 2 depending on whether the antenna is used in the standard mode or multiplex mode, respectively.

Here, we adopt the convention that the points are labeled in such a way that $2d = \sin\theta_2 - \sin\theta_1 \geq 0$ .The last expression in Eq. (1.1) is the coherence factor given in polar form. The key idea is to use the three nonlinear equations, derived from the arguments of three coherence factors, to solve for the three unknowns s, d and $I_1 / (I_1 + I_2)$.

In this report, we assume that the noise variates are uncorrelated random variables. Since, in addition to assuming that the terms are uncorrelated, we also assume that the noise is complex Gaussian (Goodman [5]); it follows that the noise variates are actually independent.

The problem of solving for the angle-of-arrival from arbitrary point targets has received considerable attention, see Stoica [6] and Kay [7]. It is known that the Maximum Likelihood Estimator provides a nearly optimal solution (Kay [7] and Ziskind [8]), but, because of its high computational cost, this approach is not considered to be practical. This has led to the introduction of a variety of suboptimal spectral techniques with reduced computational costs, Stoica [6]. These methods, however, require a large number of samples or “looks” to produce a variance that is close to the Cramer-Rao lower bound, Kay [7], Bickel and DeLaurentis [9]. We present a direct or phase method that may overcome both of these obstacles. Since our approach involves, essentially, estimation of parameters in an analytic expression, this method may provide an efficient technique for estimating the unknowns as well as providing a variance that may be closer to the Cramer-Rao lower bound.

In the next section, we derive the form of the Van Cittert-Zernike Theorem for two point targets projected into a single resolution cell. The third section presents our phase method for resolving two point targets by using the phase of three coherence factors. Also, we show that the solution is unique provided the difference values, d , are bounded below. The fourth section presents a method that uses the magnitude and phase of two coherence factors to resolve two point targets. In the fifth section we derive the reduced error probability for IFSAR phase ambiguities when a fourth phase-center is employed. Aside from these practical considerations, the methods presented may have intrinsic value in the analysis of multi-phase-center imaging and communication.

2. Complex Coherence

The mutual intensity in the observation region takes the form (Goodman [5])

$$J(\mathbf{q}_l,\mathbf{q}_m)=\frac{\kappa}{(\lambda_c r_c)^2}\iint_\Sigma I\exp\left(-j\frac{2\pi\chi}{\lambda_c}(r_m-r_l)\right)dS \tag{2.1}$$

where Σ represents a section of the imaging surface that is projected into a single resolution cell of the slant plane, I equals the intensity, dS denotes a surface element, λ_c is the center wavelength, r_c denotes the distance from the midpoint of the baseline to the scene center, and $\chi=1$ or 2 depending on whether the antenna is used in the standard mode or multiplex mode respectively, see Figure 1 (here and in the following we use boldface letters to denote a vector). For the baseline corresponding to $\mathbf{b}_1$, the paraxial approximation yields for the approximate point target at $\mathbf{p}_1$,

$$\begin{aligned} r_3-r_1&=\left|\mathbf{r}_0+\mathbf{b}_1/2\right|-\left|\mathbf{r}_0-\mathbf{b}_1/2\right|=r_0+\frac{\mathbf{r}_0}{2r_0}\bullet\mathbf{b}_1+....-r_0+\frac{\mathbf{r}_0}{2r_0}\bullet\mathbf{b}_1+.... \\ &\approx\frac{\mathbf{r}_0}{r_0}\bullet\mathbf{b}_1=-b_1\sin(\theta_1) \end{aligned} \tag{2.2}$$

where $\mathbf{r}_3=\mathbf{q}_3-\mathbf{p}_1$, $r_3=\left|\mathbf{r}_3\right|$, $\mathbf{r}_1=\mathbf{q}_1-\mathbf{p}_1$, $r_1=\left|\mathbf{r}_1\right|$, $\mathbf{b}_1=\mathbf{q}_3-\mathbf{q}_1$, $b_1=\left|\mathbf{b}_1\right|$, $\mathbf{r}_0=\mathbf{r}_1+\mathbf{b}_1/2=\mathbf{r}_2-\mathbf{b}_1/2$, $r_0=\left|\mathbf{r}_0\right|$, and θ_1 is the angle of arrival from the point reflector at $\mathbf{p}_1$ (see Figure 1). A similar result holds for the approximate point reflector or point target at $\mathbf{p}_2$. A paraxial approximation for the other baselines can be derived in a similar fashion; this allows us to replace b_1 in Eq. (2.2) by b_l. We adopt the convention that the points in a cell are labeled in such a way that $\sin\theta_2-\sin\theta_1\geq 0$.

For point reflectors, the intensity $I(\mathbf{p})$ takes the form

$$I(\mathbf{p})\approx I_1\delta(\mathbf{p}-\mathbf{p}_1)+I_2\delta(\mathbf{p}-\mathbf{p}_2) \tag{2.3}$$

This corresponds to the case in which the resolution cell contains one or two point targets. The mutual intensity may be rewritten as

$$J(\mathbf{q}_1, \mathbf{q}_3) \approx \frac{\kappa}{(\lambda_c r_c)^2}\left(I_1 \exp(k_1 \sin\theta_1) + I_2 \exp(k_1 \sin\theta_2)\right) \tag{2.4}$$

where $k_1 = 2\pi\chi b_1 / \lambda_c$.

Normalizing by $J(\mathbf{q}_1, \mathbf{q}_1)$, we obtain the complex coherence factor,

$$\gamma(k_1) = \frac{J(\mathbf{q}_1, \mathbf{q}_3)}{J(\mathbf{q}_1, \mathbf{q}_1)} = \frac{\iint_\Sigma I \, \exp\left(-j\frac{2\pi\chi}{\lambda_c}(r_3 - r_1)\right) dS}{\iint_\Sigma I \, dS} \tag{2.5}$$

$$= \frac{I_1}{I_1 + I_2}\exp(jk_1 \sin\theta_1) + \frac{I_2}{I_1 + I_2}\exp(jk_1 \sin\theta_2).$$

A similar argument applied to the other baselines yields

$$\gamma(k_l) = \frac{I_1}{I_1 + I_2}\exp(jk_l \sin\theta_1) + \frac{I_2}{I_1 + I_2}\exp(jk_l \sin\theta_2) \tag{2.6}$$

where $k_l = 2\pi\chi b_l / \lambda_c$ and $k_1 > k_2 > k_3$. For $\mathbf{p}_1 \neq \mathbf{p}_2$, this expression represents the Van Cittert-Zernike Theorem for two point targets projected into a single range-azimuth resolution cell (in Eq. (2.5) we replaced the approximately equals symbol by equality to simplify the notation).

We may rewrite $\gamma(k_l)$ as

$$\gamma(k_l) = \exp(jk_l s)\left[\frac{I_1}{I_1+I_2}\exp(-jk_l d) + \frac{I_2}{I_1+I_2}\exp(jk_l d)\right] \tag{2.7}$$

where $s = (\sin\theta_1 + \sin\theta_2)/2$ and $d = (\sin\theta_2 - \sin\theta_1)/2$. (We note that, for small angles, $s \approx (\theta_1 + \theta_2)/2$ and $d \approx (\theta_2 - \theta_1)/2$). In polar coordinates, the preceding, Eq. (2.7), becomes

$$\gamma(k_l) = |\gamma(k_l)|\exp\left[j(k_l s + t_l)\right] \tag{2.8}$$

where

$$t_l = t_l\left(\frac{I_1}{I_1+I_2}, d\right) = \tan^{-1}\left[\left(1 - 2\frac{I_1}{I_1+I_2}\right)\tan(k_l d)\right] \tag{2.9}$$

Setting $d = 0$ in either Eq. (2.7) or Eq. (2.8), we obtain the Van Cittert-Zernike Theorem for a single target in a resolution cell (see Jakowatz et al. [10]),

$$\gamma(k_l) = \exp(jk_l s) \tag{2.10}$$

where $s = \sin\theta$ and θ denotes the angle-of-arrival from the point target. In this way, we may use Eq. (2.7) or Eq. (2.8) as a representation of $\gamma(k_l)$ for a scene in which either one or two approximate point targets reside in a single resolution cell.

3. Phase Method

In this section we present a direct method for analyzing the SAR layover problem. The two-phase-center IFSAR assumes that each range-azimuth bin contains at most one target, but in scenes where the height changes abruptly, the return signal may be the superposition of two signals from different height targets which have been mapped into the same range bin; SAR layover, see Fig. 1. We propose a direct or phase method that uses a three-phase-center IFSAR to resolve this problem. The key idea is to show that the three unknowns (two angles and a relative intensity) may be derived from the intersection of a surface with a line.

We recall from Eq. (2.7) that the coherence factor, μ, is given by

$$\mu(k_l) = \frac{I_1}{I_1 + I_2} \exp(jk_l \sin\theta_1) + \frac{I_2}{I_1 + I_2} \exp(jk_l \sin\theta_2) \tag{3.1}$$

where $k_1 > k_2 > k_3$, $k_1 = k_2 + k_3$, and $\mu(k_l)$ ($l = 1,2,3$) equals the coherence factor for one pair of phase-centers with:

$k_l = 2\pi m b_l / \lambda_c$,

b_l = baseline length

λ_l = center wavelength,

θ_i = angle-of-arrival of the signal from point $\mathbf{p}_i$ (see),

$I_i / (I_1 + I_2)$ = relative intensity of the point target at $\mathbf{p}_i$

$\chi = 1$ or 2 depending on whether the antenna is used in the standard mode or multiplex mode respectively.

Here, we adopt the convention that the points in a cell are labeled in such a way that $2d = \sin\theta_2 - \sin\theta_1 \geq 0$.

In polar coordinates (see Eq. (2.8)) the coherence factor $\mu(k_l)$ becomes

$$\mu(k_l) = \exp(jk_l s)\left[\frac{I_1}{I_1+I_2}\exp(-jk_l d) + \frac{I_2}{I_1+I_2}\exp(jk_l d)\right] \tag{3.2}$$

$$= |\mu(k_l)|\exp\left[j\left(k_l s + t_l\left(\frac{I_1}{I_1+I_2}, d\right)\right)\right]$$

where

$$|\mu(k_l)|^2 \equiv \frac{I_1^{\,2}}{(I_1+I_2)^2} + \frac{I_2^{\,2}}{(I_1+I_2)^2} + 2\frac{I_1 I_2}{(I_1+I_2)^2}\cos(2k_k d) \tag{3.3}$$

$$t_l(\alpha,\beta) = \tan^{-1}\left[(1-2\alpha)\tan(k_l\beta)\right] \tag{3.4}$$

$$s = (\sin\theta_1 + \sin\theta_2)/2 \tag{3.5}$$

$$d = (\sin\theta_2 - \sin\theta_1)/2 \tag{3.6}$$

In this section we assume that the unwrapped phase of $\mu(k_l)$, given by

$$y_l = k_l s + t_l\left(I_1/(I_1+I_2), d\right), \qquad l = 1,2,3 \tag{3.7}$$

is known. We show that the preceding equations are sufficient to enable us to solve for the three unknowns: s, d and $I_1/(I_1+I_2)$. We begin by presenting a criterion for detecting the presence of two targets, next we consider the case $I_1/(I_1+I_2) = I_2/(I_1+I_2) = 1/2$, and finally we examine the more difficult case $I_1/(I_1+I_2) \neq I_2/(I_1+I_2)$.

From the expression for the magnitude squared, we see that $|\mu(k_l)| < 1$ if and only if two targets are present in the same range resolution cell. Here, we assume that there are at most two targets in the same range bin. This provides us with a criterion for detecting the presence of a second target, namely, if $|\mu(k_l)| < 1$, we assume that two targets are present (in this paper we only address the case for which at most two targets reside in the same range-azimuth bin).

Assuming that two targets have been detected, it can be shown that for, $k_l > k_m$,

$$\operatorname{sgn}\left\{\frac{k_m}{k_l} y_l - y_m\right\} = \operatorname{sgn}\left\{1 - 2\frac{I_1}{I_1 + I_2}\right\} \tag{3.8}$$

where $m, l \in \{1,2,3\}$ and

$$\operatorname{sgn}\{x\} = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ +1, & x > 0 \end{cases}$$

In particular, if $\operatorname{sgn}\left\{\frac{k_m}{k_l} y_l - y_m\right\} = 0$ it follows that $I_1/(I_1 + I_2) = I_2/(I_1 + I_2) = 1/2$. In this case, we can solve for d from the magnitude squared (see Eq. (3.3)) to obtain

$$d = \frac{1}{k_l}\cos^{-1}\left(2|\mu(k_l)|^2 - 1\right) \tag{3.9}$$

where $|\mu(k_l)|$ is a measured quantity. Since, the sum s is given by

$$s = \frac{1}{k_l} y_l \tag{3.10}$$

We turn now to the case $I_1/(I_1 + I_2) \neq I_2/(I_1 + I_2)$.

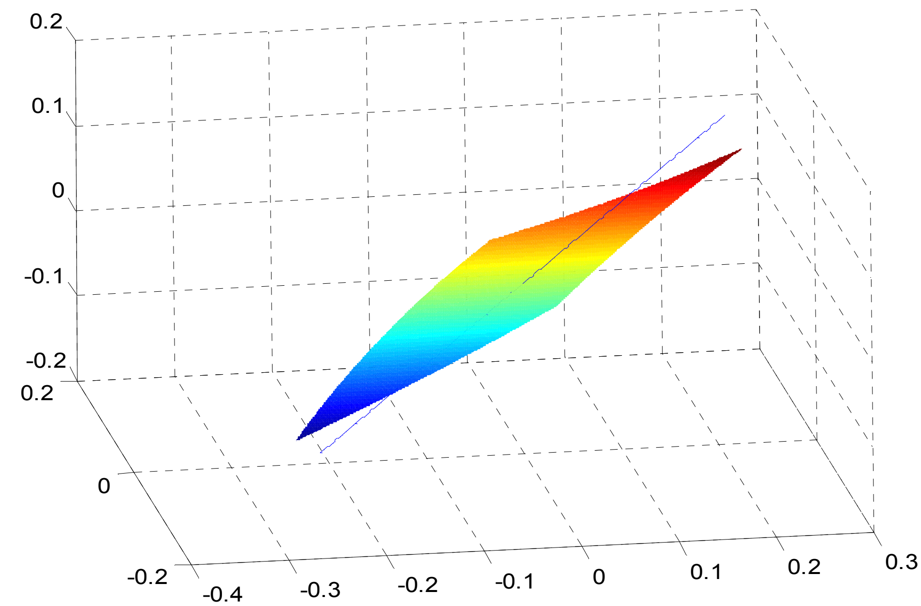

Figure 2 a

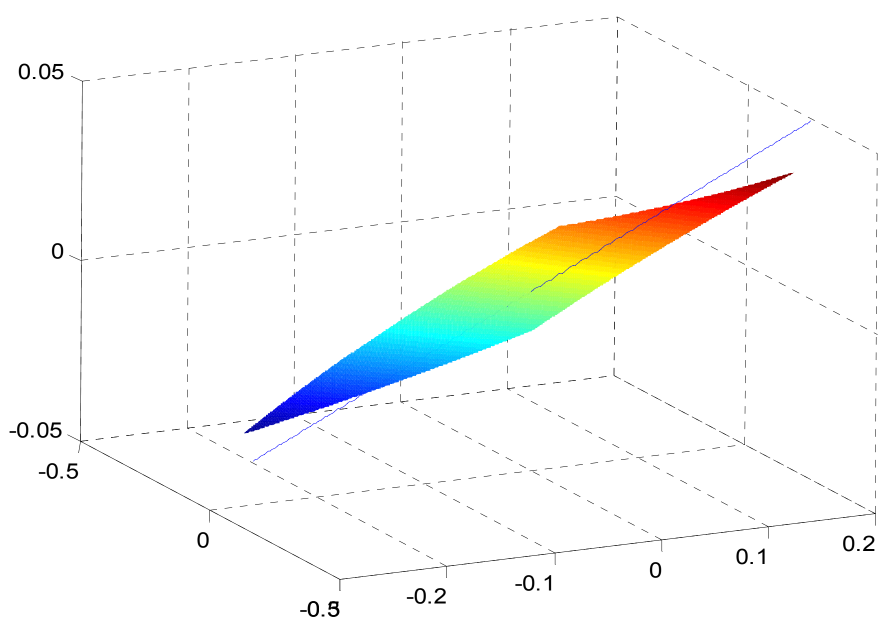

Figure 2 b: In both figures $I_1/(I_1+I_2)=.25$ $d=(\sin\theta_2-\sin\theta_1)/2\approx 1$ and $s=(\sin\theta_1+\sin\theta_2)/2=0$,

for Figure 2a $k_1=1.$, $k_2=.55$, $k_3=.45$ and in Fig. 1b $k_1=1.$, $k_2=.8$, $k_3=.2$.

Let us assume that $\operatorname{sgn}\left\{\left(k_m / k_l\right) y_l - y_m\right\} > 0$ so that $1 - 2I_1 / \left(I_1 + I_2\right) > 0$ or $0 < I_1 / \left(I_1 + I_2\right) < 1/2$. The case, $\operatorname{sgn}\left\{\left(k_m / k_l\right) y_l - y_m\right\} < 0$, is similar. We introduce the surface S defined by (see Figure 2a and Figure 2b)

$$S = \left\{\mathbf{Z}(\alpha, \beta) \mid 0 < \alpha < 1/2,\ 0 < \beta < \pi / \left(2k_1\right)\right\} \tag{3.11}$$

where

$$\begin{aligned}\mathbf{Z}(\alpha, \beta) &= \left(z_1(\alpha, \beta), z_2(\alpha, \beta), z_3(\alpha, \beta)\right) \\ &= \left(y_1 - t_1(\alpha, \beta), y_2 - t_2(\alpha, \beta), y_3 - t_3(\alpha, \beta)\right) = \mathbf{y} - \mathbf{t}\end{aligned} \tag{3.12}$$

and $t_l(\alpha, \beta)$ is defined by Eq. (3.4). (Since, by assumption $0 < I_1 / \left(I_1 + I_2\right) < 1/2$, we need only consider values for α between 0 and $1/2$. Also, we assume that the difference d satisfies the inequalities $0 < d < \pi / \left(2k_1\right)$, so that we only need to consider values for β between 0 and $\pi / \left(2k_1\right)$.) The unknown parameters can be determined from the unique intersection of the surface S with the line segment L defined by

$$L = \left\{\mathbf{r}(\eta) = \eta \mathbf{k} = \left(k_1 \eta, k_2 \eta, k_3 \eta\right) \mid |\eta| < a\right\} \tag{3.13}$$

where $\mathbf{k} = \left(k_1, k_2, k_3\right)$ and a is an upper bound for $|s|$. At the intersection of S and L, we have

$$k_l \eta = y_l - t_l(\alpha, \beta), \text{ for } l = 1, 2, 3 \tag{3.14}$$

Let us denote by $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\eta}$ the values of α, β, and η, respectively, at the intersection. Since the intersection is unique (see D & B []), it follows that

$$I_1/(I_1+I_2)=\hat{\alpha} \tag{3.15}$$

$$d=\hat{\beta} \tag{3.16}$$

and

$$s=\hat{\eta} \tag{3.17}$$

as desired. The figures (Figure 2a and Figure 2b) display the surface S, the line segment L and their intersection for two different choices of the parameters: k_1, k_2 and k_3.

To provide for a computationally efficient method of searching for the intersection, it is helpful to represent the surface in standard form. For this, we choose a coordinate system that enables us to represent S in the form

$$S=\left\{\left(z_1,z_2,g\left(z_1,z_2\right)\right)|\left(z_1,z_2\right)\in D\right\} \tag{3.18}$$

for some domain D and function $g(z_1,z_2)$. This is possible because the Jacobian matrix of $\mathbf{Z}$ has rank two (see [32]); in particular, we may use the coordinate system defined by

$$z_l = y_l - t_l(\alpha,\beta), \qquad l=1,2\,. \tag{3.19}$$

The functions $\alpha(z_1,z_2)$ and $\beta(z_1,z_2)$, defined implicitly by Eq. (3.19), satisfy

$$\alpha(z_1,z_2)=\frac{1}{2}\left[1-\frac{\tan(y_2-z_2)}{\tan(k_2\beta)}\right]=\frac{1}{2}\left[1-\frac{\tan(y_1-z_1)}{\tan(k_1\beta)}\right] \tag{3.20}$$

and

$$G(z_1,z_2,\beta)\equiv\frac{\tan(k_2\beta)}{\tan(k_1\beta)}-\frac{\tan(y_2-z_2)}{\tan(y_1-z_1)}=0 \tag{3.21}$$

for the domain
$D = \{(z_1, z_2) : 0 < y_l - z_l < \pi/2,\ l = 1,2,\ \mathit{and}\ \tan(y_2 - z_2)/\tan(y_1 - z_1) < k_2/k_1\}$. (Here, the function $G(z_1, z_2, \beta)$, used to define β, is obtained by rearranging the last two expressions in Eq.(3.20).) The function $\beta(z_1, z_2)$ is defined implicitly by Eq.(3.21), and, in turn, is used to define $\alpha(z_1, z_2)$ via Eq. (3.20). We note that $G(z_1, z_2, \beta)$ is a strictly decreasing function of β with $G \to k_2/k_1$ as β decreases to zero, and $G \to 0$ as β increases to $\pi/(2k_1)$. It follows that $\beta(z_1, z_2)$ is defined for every ordered pair (z_1, z_2) in the domain D. It can be shown that the function

$$\mathbf{u}(z_1, z_2) = (\alpha(z_1, z_2), \beta(z_1, z_2)) \tag{3.22}$$

is one-to-one on the domain D. In this coordinate system, we have

$$z_l = y_l - t_l \circ \mathbf{u}(z_1, z_2),\ l = 1,2, \tag{3.23}$$

and the function $g(z_1, z_2)$ is given by

$$g(z_1, z_2) = y_3 - t_3 \circ \mathbf{u}(z_1, z_2), \tag{3.24}$$

By inserting the map $\mathbf{u}$ into Eq. (3.12), we obtain

$$\mathbf{Z} \circ \mathbf{u}(z_1, z_2) = (z_1, z_2, g(z_1, z_2)). \tag{3.25}$$

We have produced the standard representation for the surface S.

We note that it is not necessary to compute the (z_1, z_2)-coordinate system. We may plot the surface S using the definition of $\mathbf{Z}(\alpha, \beta)$ given by Eq. (3.12) and then find the point $\mathbf{Z}(\alpha, \beta)$ that is closest to the line segment L. We have introduced the coordinate transformation as a means of providing for an alternative approach to finding the intersection. Also, this change of variables gives us an explicitly defined method for representing S in standard form.

The figures show the intersection of S and L for the case when the separation between the two targets is approximately two thirds the distance required by the Rayleigh criterion. Here, the intensity ratio $I_1/(I_1+I_2)$ equals $.25$. In Fig. 1a, the smaller baselines are nearly equal; $k_1 = 1,\ k_2 = .55$ and $k_3 = .45$. In Fig. 1b, the smallest baseline is one-fifth the length of the longest baseline; $k_1 = 1,\ k_2 = .8,\ \mathit{and}\ k_3 = .2$. In either case, the intersection is clearly discernable; however, the difference in the "slopes" at the intersection is greater for the case involving the longer baselines. This agrees with the intuitive conjecture which asserts that the longer baselines are less sensitive to noise.

We still need to show that the intersection is unique. To simplify this discussion we prove uniqueness only for the case $d \geq \beta_m$ where

$$\beta_m = \min\left\{\beta \mid \beta \geq \pi/(4k_2),\ \frac{\sin^2(k_2\beta)-\sin^2(k_3\beta)}{k_2-k_3} \geq \frac{\sin^2(k_1\beta)-\sin^2(k_3\beta)}{k_2}\right\};$$

that is, we assume $\beta_m \leq \beta < \pi/(2k_1)$. We explain our choice for β_m in the following. (The proof of uniqueness for the more general case is left as a subject for future study.)

To prove uniqueness for $\beta_m \leq \beta < \pi/(2k_1)$, we consider the curve Γ formed by the intersection of the surface S with the vertical plane P containing L.; that is, $\Gamma = S \cap P$ where P is orthogonal to $\mathbf{w} = (k_2/k_1, -1, 0)$, more precisely $P = \{\mathbf{z} = (z_1, z_2, z_3) \mid \mathbf{w} \bullet \mathbf{z} = 0\}$. The curve Γ is given parametrically by

$$\mathbf{Z}(\alpha, \beta(\alpha)) = \mathbf{y} - \mathbf{t}(\alpha, \beta(\alpha)) = (y_1 - t_1(\alpha,\beta), y_2 - t_2(\alpha,\beta), y_3 - t_3(\alpha,\beta)) \tag{3.26}$$

where $\beta(\alpha)$ is defined implicitly by the equation

$$H(\alpha,\beta(\alpha)) \triangleq \mathbf{w}\bullet(\mathbf{y}-\mathbf{t}) = \frac{k_2}{k_1}y_1 - y_2 - \left(\frac{k_2}{k_1}t_1(\alpha,\beta) - t_2(\alpha,\beta)\right) = 0\,. \tag{3.27}$$

Since $\partial H/\partial\beta < 0$, it follows that Eq. (3.27) has a unique solution for every α such that $H\left(\alpha,\left(\frac{\pi}{2k_1}\right)^-\right) < 0$. It can be shown that $\beta(\alpha)$ is defined over an interval so that Γ is connected. We note that Γ can also be defined by $(k_1\eta, k_2\eta, g(k_1\eta, k_2\eta))$, see Eq.(3.25)

We prove that the intersection is unique by showing that the tangent vectors to the curve Γ are never parallel to the line segment L. Since the curve Γ and L lie in the same plane, it follows, from the mean value theorem for derivatives, that if Γ crosses L twice then, at some point between the intersections, the tangent to the curve Γ must be parallel to L. If the tangent lines are never parallel to L then Γ can not cross L more than once.

Let us consider the tangent vectors $\mathbf{T}$ defined by

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3 = \mathbf{N}\times\mathbf{w} \tag{3.28}$$

where $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ and $\mathbf{N}$ is normal to the surface S. We define $\mathbf{N}$ by

$$\mathbf{N} = \frac{\partial\mathbf{Z}}{\partial\alpha}\times\frac{\partial\mathbf{Z}}{\partial\beta} = \frac{\partial(t_2,t_3)}{\partial(\alpha,\beta)}\mathbf{e}_1 - \frac{\partial(t_1,t_3)}{\partial(\alpha,\beta)}\mathbf{e}_2 + \frac{\partial(t_1,t_2)}{\partial(\alpha,\beta)}\mathbf{e}_3\,, \tag{3.29}$$

where

$$\frac{\partial(t_l,t_m)}{\partial(\alpha,\beta)} \triangleq \begin{vmatrix} \dfrac{\partial t_l}{\partial \alpha} & \dfrac{\partial t_l}{\partial \beta} \\ \dfrac{\partial t_m}{\partial \alpha} & \dfrac{\partial t_m}{\partial \beta} \end{vmatrix} = 2\beta(1-2\alpha)\frac{k_l k_m A_{l,m}}{B_l B_m}, \tag{3.30}$$

$$A_{l,m} = \operatorname{sinc}(2k_m\beta) - \operatorname{sinc}(2k_l\beta), \tag{3.31}$$

$$B_n = 1 - 4\alpha(1-\alpha)\sin^2(k_n\beta). \tag{3.32}$$

Here, the symbol $|\cdot|$ denotes the determinant and $\operatorname{sinc} x \triangleq \sin(x)/x$. For $l < m$, we have $k_m\beta < k_l\beta < \pi/2$, so that $A_{l,m} > 0$ and $\partial(t_l,t_m)/\partial(\alpha,\beta) > 0$.

The tangent vector $\mathbf{T}$ is not collinear with $\mathbf{k}$ if and only if $\mathbf{k}\times\mathbf{T} \neq 0$. Using the identity $\mathbf{a}\times\mathbf{b}\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ we have, since $\mathbf{k}$ is orthogonal to $\mathbf{w}$,

$$\mathbf{k}\times\mathbf{T} = \mathbf{k}\times\mathbf{N}\times\mathbf{w} = (\mathbf{k}\cdot\mathbf{w})\mathbf{N} - (\mathbf{k}\cdot\mathbf{N})\mathbf{w} = -(\mathbf{k}\cdot\mathbf{N})\mathbf{w}. \tag{3.33}$$

It follows from Eq. (3.33) that we need only show that $\mathbf{k}\cdot\mathbf{N} \neq 0$. We note that this result is independent of the curve Γ. We could have chosen any curve of the form $S \cap P$ (for some plane P) provided the intersection is connected and the resulting curve is differentiable.

We need to show that $\mathbf{k}\cdot\mathbf{N} \neq 0$, in fact, we show that

$$0 < \mathbf{k}\cdot\mathbf{N} = \frac{2\beta(1-2\alpha)k_1k_2k_3}{B_1B_2B_3}\left(A_{2,3}B_1 - A_{1,3}B_2 + A_{1,2}B_3\right). \tag{3.34}$$

It follows that we need only show that the expression inside the second set of parentheses is positive. Using the definition of B_l, Eq. (3.32), and the fact that $A_{1,3} = A_{1,2} + A_{2,3}$, we have

$$A_{2,3}B_1 - A_{1,3}B_2 + A_{1,2}B_3 = . \tag{3.35}$$

$$-4\alpha(1-2\alpha)\left(A_{2,3}\sin^2(k_1\beta) - A_{1,3}\sin^2(k_2\beta) + A_{1,2}\sin^2(k_3\beta)\right).$$

We have reduced the problem to showing that the expression inside the second set of parentheses is negative. Using the definition of $A_{l,m}$, Eq. (3.31), and rearranging terms, we obtain

$$A_{2,3}\sin^2(k_1\beta) - A_{1,3}\sin^2(k_2\beta) + A_{1,2}\sin^2(k_3\beta). \tag{3.36}$$

$$= C_{2,3}\operatorname{sinc}(2k_1\beta) - C_{1,3}\operatorname{sinc}(2k_2\beta) + C_{1,2}\operatorname{sinc}(2k_3\beta)$$

where

$$C_{l,m} = \sin^2(k_l\beta) - \sin^2(k_m\beta). \tag{3.37}$$

Now, we use the definition of β_m to obtain the inequality,

$$\gamma \triangleq \frac{C_{2,3}}{C_{1,3}} = \frac{\sin^2(k_2\beta) - \sin^2(k_3\beta)}{\sin^2(k_1\beta) - \sin^2(k_3\beta)} \geq \frac{k_2 - k_3}{k_2} = 1 - \frac{k_3}{k_2}. \tag{3.38}$$

for $\beta_m \leq \beta < \pi/(2k_1)$. The lower bound β_m was defined so that the slopes of the chords of the function $\sin^2 x$ would satisfy Eq.(3.38). Since $k_1 = k_2 + k_3$, the inequality given in Eq.(3.38) implies

$$\gamma k_1 + (1-\gamma)k_3 = \gamma k_2 + k_3 \geq \left(\frac{k_2 - k_3}{k_2}\right)k_2 + k_3 = k_2. \tag{3.39}$$

Finally, using Eq. (3.39)and the fact that $\operatorname{sinc}(2x)$ is a concave, decreasing function, we arrive at the desired result

$$\gamma \operatorname{sinc}(2k_1\beta) + (1-\gamma)\operatorname{sinc}(2k_3\beta) < \operatorname{sinc}(2\gamma k_1\beta + 2(1-\gamma)k_3\beta) \le \operatorname{sinc}(2k_2\beta) \tag{3.40}$$

provided $\beta_m \le \beta < \pi/(2k_1)$. This inequality (Eq. (3.40)) asserts that the right-hand side of Eq. (3.36) is negative which, in turn, implies that $0 < \mathbf{k} \cdot \mathbf{N}$. It follows that $\mathbf{T}$ is not collinear with $\mathbf{k}$ and the intersection is unique for $\beta_m \le \beta < \pi/(2k_1)$. The key idea in the proof is to use the fact that the function $\operatorname{sinc}(2x)$ is concave and decreasing over the interval $(0, \pi/2)$. This completes the proof that the intersection is unique provided the noise is negligible.

The introduction of noise into the system causes a translation in the surface and, in turn, produces a shift in the location of the intersection. This effect is a subject for future study. In particular, it would be interesting to compare the variance of the direct method with the Rao-Cramer lower bound. Further study may also provide insights that would enable us to optimize the design parameters of a multi-phase-center IFSAR. In addition to providing a practical method for resolving the layover problem, this method may prove to be a useful tool for analyzing the IFSAR resolution limits in the presence of noise.

4. Magnitude of the Complex Coherence

In this section we show that all three unknowns, two angles and a relative intensity (assuming that the IFSAR phase ambiguity has been removed), can be determined, at least in principle, by using both the magnitude and phase of *two* complex coherence factors. The coherence factors may arise from, say, the same pair of phase-centers operated at two different frequencies. The basic idea is to use the magnitudes to determine the unknown difference, d, and relative intensity, $I_1/(I_1+I_2)$; the sum, s, can then be derived from the phase.

As in the preceding section, we use the criterion $|\gamma(k_l)|<1$ to determine when two targets reside in the same resolution cell. Also, we assume that the unwrapped phase of $\mu(k_l)$ is known; that is, we are given,

$$y_l = k_l s + t_l\left(I_1/\left(I_1+I_2\right), d\right) \tag{4.1}$$

for $l=1,2$. As before, to determine if $I_1/\left(I_1+I_2\right)=1/2$, $I_1/\left(I_1+I_2\right)\in\left(0,1/2\right)$, or $I_1/\left(I_1+I_2\right)\in\left(1/2,1\right)$, we use the fact that

$$\operatorname{sgn}\left\{\frac{k_2}{k_1}y_1-y_2\right\}=\operatorname{sgn}\left\{1-2\frac{I_1}{I_1+I_2}\right\} \tag{4.2}$$

If $\operatorname{sgn}\left\{\frac{k_2}{k_1}y_1-y_2\right\}=0$ it follows that $I_1/\left(I_1+I_2\right)=I_2/\left(I_1+I_2\right)=1/2$, and we may compute d and s as in the preceding section (see Eqs. (3.9) and (3.10)). Since the case $\operatorname{sgn}\left\{\left(k_2/k_1\right)y_1-y_2\right\}<0$ is similar, we need only consider the case $\operatorname{sgn}\left\{\frac{k_2}{k_1}y_1-y_2\right\}>0$, in other words, we assume $0<I_1/\left(I_1+I_2\right)<1/2$.

We show that the unknowns d and $I_1/(I_1+I_2)$ can be completely determined from the magnitudes; in turn, the remaining unknown, s, can be computed from the phase of one of the coherence factors. We note that, in this approach, we use the phase from *both* coherence factors only to determine the sign of $\frac{k_2}{k_1}y_1 - y_2$.

The magnitude squared of the complex coherence factor $\gamma(k_l)$ equals (see Eq. (3.3))

$$|\mu(k_l)|^2 = \frac{I_1^2}{(I_1+I_2)^2} + \frac{I_2^2}{(I_1+I_2)^2} + 2\frac{I_1 I_2}{(I_1+I_2)^2}\cos(2k_l d) \qquad (4.3)$$

For each fixed k, let us consider the curve, $\beta(\alpha,k)$, defined implicitly by the equation

$$f(\alpha,\beta,k) = \alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)\cos(2k\beta) - \xi(k) = 0 \qquad (4.4)$$

where $\xi(k)$ is defined for $k_1 \geq k \geq k_2$ by

$$\xi(k) = |\mu(k)|^2 \triangleq \frac{I_1^2}{(I_1+I_2)^2} + \frac{I_2^2}{(I_1+I_2)^2} + 2\frac{I_1 I_2}{(I_1+I_2)^2}\cos(2kd) \qquad (4.5)$$

By assumption (see Eq. (4.3)), the point $(\alpha,\beta) = (I_1/(I_1+I_2),d)$ lies on both of the curves $f(\alpha,\beta,k)=0$, $k = k_1,k_2$. The key step, in this approach, is to recover $(I_1/(I_1+I_2),d)$ by solving for the intersection of the two curves defined by $f(\alpha,\beta,k)=0$, $k = k_1,k_2$.

Towards this end we introduce an explicit expression for the function, $\beta(\alpha,k)$,

$$\beta(\alpha,k)=\frac{1}{2k}\cos^{-1}\left(\frac{\xi(k)-\alpha^2-(1-\alpha)^2}{2\alpha(1-\alpha)}\right) \tag{4.6}$$

where the range of $\cos^{-1}(z)$ is $[o,\pi]$. Here, we solved $f(\alpha,\beta,k)=0$ for β, assuming that α lies in the domain of $\beta(\alpha,k)$. We recall that the labels for the points $\mathbf{p}_1$ and $\mathbf{p}_2$ (see Figure 1 and Eq. (3.6)) are chosen so that $d \geq 0$; this allows us to use the branch of $\cos^{-1}(z)$ with range $[o,\pi]$.

The next lemma shows that in order for the argument in Eq. (4.6) to remain in the interval $[-1,1]$ we must restrict α to the interval $\left[\frac{1}{2}-\frac{1}{2}|\mu(k)|,\frac{1}{2}+\frac{1}{2}|\mu(k)|\right]$. We see, from Eq.(4.6) that $\beta(\alpha,k)$ is symmetric about $\alpha=1/2$. Also, we note that at $\alpha=1/2$,

$$\beta\left(\frac{1}{2},k\right)=\frac{1}{2k}\cos^{-1}\left(2\xi(k)-1\right)=\frac{1}{2k}\cos^{-1}\left(2|\mu(k)|^2-1\right) \tag{4.7}$$

It follows from Eq. (4.7) that the values $\beta(1/2,k_1)$ and $\beta(1/2,k_2)$ can be computed from the magnitude of the coherence factors $|\mu(k_1)|$ and $|\mu(k_2)|$, respectively. Also, since $\xi(k)$ is a decreasing function of k ($\xi'(k)<0$), we have $\xi(k_1)\leq\xi(k)\leq\xi(k_2)$ and $|\mu(k_1)|\leq|\mu(k)|\leq|\mu(k_2)|$.

The basic idea is to find the roots of the equation

$$\beta(\alpha,k_1)-\beta(\alpha,k_2)=0 \tag{4.8}$$

The following lemma shows that if $\beta(1/2,k_1)=\beta(1/2,k_2)$, Eq. (4.8) has a unique solution, but if $\beta(1/2,k_1)<\beta(1/2,k_2)$, Eq. (4.8) has exactly two solutions. We defer the proof of the lemma to an appendix.

Lemma 1. Let us assume that $\operatorname{sgn}\left\{\frac{k_2}{k_1}y_1 - y_2\right\} > 0$, $1 > \left|\mu(k_l)\right|$, $l = 1,2$, $0 < 2k_1 d < \pi$, $k_1 \ge k \ge k_2$, and $k_1 > k_2 > 0$. If we set the domain of $\beta(\alpha,k)$ equal to the interval $\left[1/2 - \left|\mu(k)\right|/2, 1/2 + \left|\mu(k)\right|/2\right]$, the function $\beta(\alpha,k)$ is well defined and has range $[\beta(1/2,k), \pi/(2k)]$. The function $\beta(\alpha,k)$ is strictly decreasing on $\left(\frac{1}{2} - \frac{1}{2}\left|\mu(k)\right|, \frac{1}{2}\right)$, strictly increasing on $\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2}\left|\mu(k)\right|\right)$, and has a unique minimum at $\alpha = 1/2$. We have that Eq. (4.8) has exactly two solutions α^* and $1-\alpha^*$ with $\frac{1}{2} - \frac{1}{2}\left|\mu(k_1)\right| \le \alpha^* < \frac{1}{2}$. The solution satisfies

$$\beta(\alpha^*, k_1) = d = \beta(\alpha^*, k_2) \tag{4.9}$$

where $\alpha^* = I_1/(I_2 + I_2)$.

The Lemma asserts that d can be determined from two pairs of phase centers; but, an ambiguity in the intensity ratio $I_1/(I_1 + I_2)$ occurs when $\alpha^* \ne 1/2$. By using phase information we have removed this ambiguity. The following Theorem summarizes our combined magnitude, phase method for solving the SAR layover problem.

Theorem 1. Let us assume that $\operatorname{sgn}\left\{\frac{k_2}{k_1}y_1 - y_2\right\} > 0$, $1 > \left|\mu(k_l)\right|$, $l = 1,2$, $0 < 2k_1 d < \pi$, and $k_1 > k_2 > 0$. The curves $(\alpha, \beta(\alpha,k_1))$ and $(\alpha, \beta(\alpha,k_2))$ intersect exactly once on the interval $\left[\frac{1}{2} - \frac{1}{2}\left|\mu(k_1)\right|, \frac{1}{2}\right)$; that is, Eq. (4.8) has a unique solution, α^*, such that $\frac{1}{2} - \frac{1}{2}\left|\mu(k_1)\right| \le \alpha^* < \frac{1}{2}$. We have

$$d = \beta(\alpha^*, k_1) = \beta(\alpha^*, k_2) \tag{4.10}$$

$$I_1 / (I_2 + I_2) = \alpha^* \tag{4.11}$$

and

$$s = y_1 - \mathrm{t}_1\left(\frac{I_1}{I_1 + I_2}, d\right) \tag{4.12}$$

where $\mathrm{t}_1\left(\frac{I_1}{I_1 + I_2}, d\right)$ is defined by Eq. (2.9).

Proof. According to Lemma 1 the intersection is unique, and since the point $\left(I_1/(I_1+I_2), d\right)$ must lie on both curves, it follows that this is the point of intersection. Given $I_1/(I_1+I_2)$ and d, we solve for s from Eq. (4.1).

Since there are three unknowns, two angles and a relative intensity (assuming that the IFSAR phase ambiguity has been removed), we expect to need both the magnitude and phase of the two complex coherence factors to be able to solve for the unknowns. The theorem asserts that, in principle, the magnitude and phase of the two complex coherence factors are indeed sufficient to determine all three unknowns. The drawback to this approach, however, is that the magnitude measurements may be unreliable (see Bickel and Hensley [3]).

5. IFSAR Phase Ambiguities

In this section we treat the case in which the phase wraps have not been removed; but, only one target resides in a range-azimuth resolution cell. More precisely, we assume the vector $\mathbf{t}$ to be negligible, the longer baseline has phase wraps, and the noise terms can not be neglected. We show that the error probabilities associated with resolving IFSAR phase ambiguities can be significantly reduced if an additional phase center is used. We illustrate this point by comparing the error probabilities for two pairs of phase-centers with the error probabilities for three pairs of phase-centers. This occurs in the case of an IFSAR antenna with one or two monopulse antennas, respectively. In the four phase-center case, we are given three pairs of complex coherence factors, $\gamma(k_l)$ $l = 1,2,3$, where the error in the longer baseline (k_1) is assumed negligible, but the error terms for the smaller baselines (k_2 and k_3) can not be neglected. For the three-phase-center antenna, we are given only the complex coherence factor for the longer baseline and one of the smaller baselines. The longer baseline may involve phase wraps; but, the smaller baselines are assumed small enough that no phase wraps occur, see Figure 3. At the end of this section we indicate a technique for removing the phase wraps when all the baselines have phase wraps.

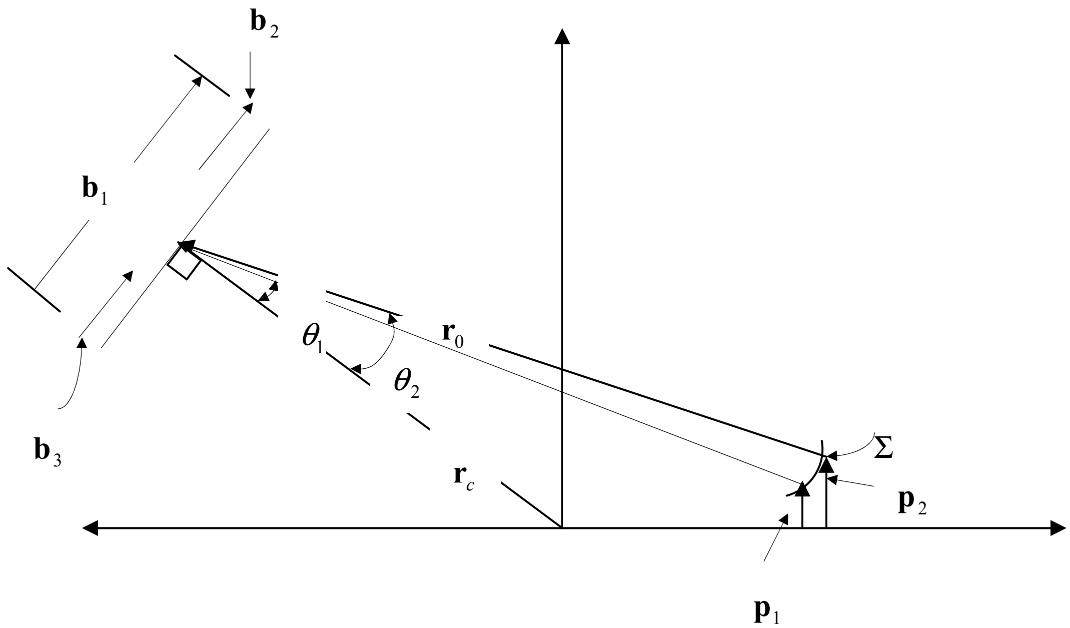

Figure 3. Multi-Phase-Center IFSAR

The image of the phase plus error term, $\mathbf{y}(s)=\mathbf{x}(s)+\boldsymbol{\varepsilon}$, defined by Eq. (2.8) with $\mathbf{t}=(t_1,t_2,t_3)=\mathbf{0}$, is a three dimensional map $\mathbf{y}(s)$ from $[-a,a]$ into $[-\pi,\pi)^3$ given by,

$$\mathbf{y}(s)=\mathbf{x}(s)+\boldsymbol{\varepsilon}=s\mathbf{k}-2\pi n\mathbf{e}_1+\boldsymbol{\varepsilon} \tag{5.1}$$

where $\mathbf{x}=\mathbf{x}(s)\triangleq s\mathbf{k}-2\pi n\mathbf{e}_1$, $n=\lfloor(k_1s+\pi)/(2\pi)\rfloor$ ($\lfloor r\rfloor$ denotes the greatest integer less than or equal to r), $\mathbf{k}=(k_1,k_2,k_3)$, $\mathbf{e}_1=(1,0,0)$, and $\boldsymbol{\varepsilon}=(0,\varepsilon_2,\varepsilon_3)$. For the baseline $\mathbf{b}_i$, we have

$$k_l=2\pi\chi b_l/\lambda_c \tag{5.2}$$

where $b_i=|\mathbf{b}_i|$, $k_1>k_2+k_3$ and χ equals 1 or 2 depending on whether the IFSAR antenna is operated in the standard or multiplex mode, respectively ($\chi=1$ for a monopulse antenna), see Figure 3. The map, $\mathbf{y}(s)$, consists of an error vector, $\boldsymbol{\varepsilon}$, plus the "wrapped" line segments $s\mathbf{k}-2\pi n\mathbf{e}_1$ (here s is treated as a parameter) that lie on the plane orthogonal to $\mathbf{w}=\mathbf{k}\times\mathbf{e}_1$, see Figure 4. By assumption, the error term ε_1 is zero. Also by assumption, we have $[k_ls+\varepsilon_l]_{-\pi,\pi}=k_ls+\varepsilon_l$ for $l=2,3$ (here $[z]_{-\pi,\pi}$ denotes the number z modulo $[-\pi,\pi)$). Our goal is to estimate the number of wraps, n, in Eq. (5.1). Using this estimate we can solve for $s=(\sin\theta_1+\sin\theta_2)/2$ (the parameter $d=(\sin\theta_2-\sin\theta_1)/2$ is computed by using the magnitude, see Eq. (3.9)).

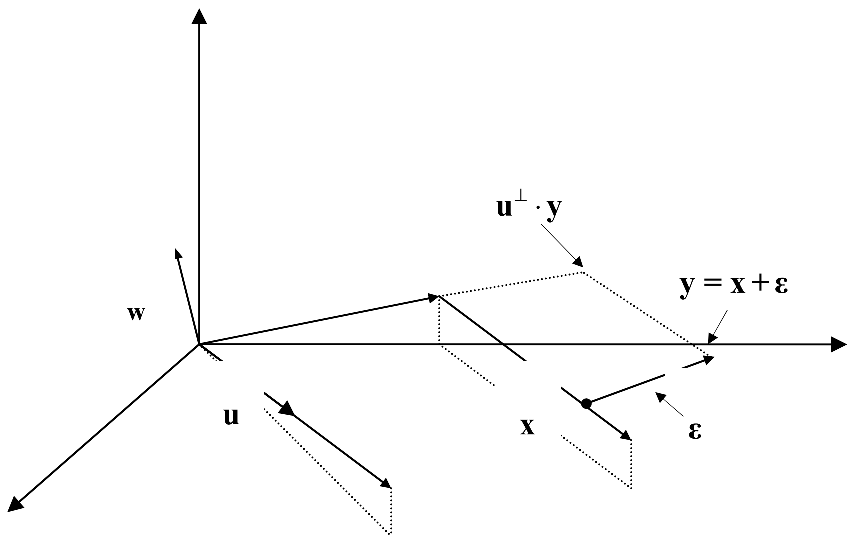

Figure 4 The graph of $\mathbf{x}(s)$

The main idea is to set n equal to the number of wraps associated with the line segment that is closest to $\mathbf{y}$. This is accomplished by projecting $\mathbf{y}$ onto the line orthogonal to the line segments $s\mathbf{k} - 2\pi n\mathbf{e}_1$, see Figure 4. For this we set

$$\mathbf{v} = -2\pi\mathbf{e}_1 - (-2\pi\mathbf{e}_1 \bullet \mathbf{u})\mathbf{u} = 2\pi\left(\frac{k_1}{|\mathbf{k}|}\mathbf{u} - \mathbf{e}_1\right) = \frac{2\pi}{|\mathbf{k}|^2}\left(-(k_2 + k_3), k_1k_2, k_1k_3\right) \qquad (5.3)$$

where $\mathbf{u} = \mathbf{k}/|\mathbf{k}|$, $\mathbf{k} = (k_1, k_2, k_3)$ and $|\mathbf{k}|$ is the norm of $\mathbf{k}$. We note that vector, $\mathbf{v}$, is orthogonal to $\mathbf{k}$ and equals the vector emanating from the origin to the closest point on the line segment with one wrap. We define

$$\mathbf{u}^{\perp} = \mathbf{v}/|\mathbf{v}| = \left(-\left(k_2^2 + k_3^2\right), k_1k_2, k_1k_3\right)/\left(|\mathbf{k}|\sqrt{k_2^2 + k_3^2}\right) \qquad (5.4)$$

where $|\mathbf{v}| = 2\pi\sqrt{k_2^2 + k_3^2} / |\mathbf{k}|$ equals the distance between the line segments. Also, we define an estimator for the number of wraps by,

$$\hat{n} = \text{int}\{\frac{1}{|\mathbf{v}|}\mathbf{u}\bullet\mathbf{y}\} \tag{5.5}$$

where $\text{int}\{\varsigma\}$ denotes the integer closest to ς. It can be shown that this estimator chooses the number of wraps associated with the line segment that is closest to the point $\mathbf{y}$.

<u>Lemma 2</u>. We obtain for the projection of $\mathbf{y}(s) = \mathbf{x}(s) + \boldsymbol{\varepsilon}$ onto $\mathbf{u}^{\perp}$

$$\mathbf{u}^{\perp}\bullet\mathbf{y} = |\mathbf{v}|n + \mathbf{u}^{\perp}\bullet\boldsymbol{\varepsilon} \tag{5.6}$$

where $\mathbf{u}^{\perp} = \mathbf{v}/|\mathbf{v}|$, $\mathbf{v} = 2\pi\left(\left(k_1/|\mathbf{k}|\right)\mathbf{u} - \mathbf{e}_1\right)$, $|\mathbf{v}| = 2\pi\sqrt{k_2^2 + k_3^2} / |\mathbf{k}|$ equals the distance between line segments and $\mathbf{u}^{\perp}$ is orthogonal to $\mathbf{u} = (u_1, u_2, u_3) = \mathbf{k}/|\mathbf{k}|$. If $|\mathbf{u}^{\perp}\bullet\boldsymbol{\varepsilon}|/|\mathbf{v}| < 1/2$, we have

$$n = \hat{n} \equiv \text{int}\left\{\frac{1}{|\mathbf{v}|}\mathbf{u}^{\perp}\bullet\mathbf{y}\right\} \tag{5.7}$$

where n equals the number of wraps.

<u>Proof.</u> We have

$$\mathbf{u}^{\perp}\bullet\mathbf{y} = s\mathbf{u}^{\perp}\bullet\mathbf{k} - 2\pi n\mathbf{u}^{\perp}\bullet\mathbf{e}_1 + \mathbf{u}^{\perp}\bullet\boldsymbol{\varepsilon} = |\mathbf{v}|n + \mathbf{u}^{\perp}\bullet\boldsymbol{\varepsilon} \tag{5.8}$$

The vector $\mathbf{v}$ is orthogonal to $\mathbf{u}$ since it is the projection of $2\pi\mathbf{e}_1$ onto the line orthogonal to $\mathbf{u}$ so that $\mathbf{u}^{\perp}$ is orthogonal to $\mathbf{u}$. It follows from Eq. (5.8), since $\left|\mathbf{u}^{\perp}\bullet\boldsymbol{\varepsilon}\right|/\left|\mathbf{v}\right|<1/2$,

$$\operatorname{int}\left\{\frac{1}{|\mathbf{v}|}\mathbf{u}^{\perp}\bullet\mathbf{y}\right\}=\operatorname{int}\left\{\frac{1}{|\mathbf{v}|}\mathbf{u}^{\perp}\bullet\boldsymbol{\varepsilon}+n\right\}=n \tag{5.9}$$

as desired.

The following theorem gives a formula for the probability that the estimator equals the actual number of wraps (see Eq.(5.7)).

<u>Theorem 2</u>. The probability that $\hat{n}$ equals n (the number of wraps) is given by,

$$P(\hat{n}=n)=P\left(\left|k_2\varepsilon_2+k_3\varepsilon_3\right|<\pi\left(k_2^2+k_3^2\right)/k_1\right) \tag{5.10}$$

where ε_2 and ε_3 are independent identically distributed random variables with continuous distributions. Given that $\hat{n}=n$, we obtain for the parameter s,

$$s=\hat{s}\triangleq\frac{x_1+2\pi\hat{n}}{k_1} \tag{5.11}$$

where $n=\hat{n}\equiv\operatorname{int}\left\{\mathbf{u}^{\perp}\bullet\mathbf{y}/|\mathbf{v}|\right\}$

<u>Proof.</u> By Lemma 2 we have

$$\mathbf{u}^{\perp}\bullet\mathbf{y} = |\mathbf{v}|n + \mathbf{u}^{\perp}\bullet\boldsymbol{\varepsilon} = |\mathbf{v}|n + \frac{k_1}{|\mathbf{k}|\sqrt{k_2^2 + k_3^2}}\left(k_2\varepsilon_2 + k_3\varepsilon_3\right) \tag{5.12}$$

Now using the fact that $|\mathbf{v}| = 2\pi\sqrt{k_2^2 + k_3^2} / |\mathbf{k}|$, we obtain from Eq. (5.12),

$$\frac{1}{|\mathbf{v}|}\mathbf{u}^{\perp}\bullet\mathbf{y} = n + \frac{k_1}{2\pi\left(k_2^2 + k_3^2\right)}\left(k_2\varepsilon_2 + k_3\varepsilon_3\right) \tag{5.13}$$

This shows that $\mathbf{u}^{\perp}\bullet\mathbf{y} / |\mathbf{v}|$ equals the number of wraps n plus an error term. We have from Eq. (5.13) and the definition of our estimator,

$$P\left(\hat{n} = n\right) = P\left(\left|\frac{1}{|\mathbf{v}|}\mathbf{u}^{\perp}\bullet\mathbf{y} - n\right| < \frac{1}{2}\right) = P\left(\left|\frac{k_1}{2\pi\left(k_2^2 + k_3^2\right)}\left(k_2\varepsilon_2 + k_3\varepsilon_3\right)\right| < \frac{1}{2}\right) \tag{5.14}$$

$$= P\left(\left|k_2\varepsilon_2 + k_3\varepsilon_3\right| < \pi\frac{k_2^2 + k_3^2}{k_1}\right),$$

We use that assumption that ε_2 and ε_3 are independent identically distributed random variables with continuous distributions in the first step of Eq. (5.14).Given $\hat{n} = n$, we obtain, from the definition of y_1, namely $y_1 = x_1 = sk_1 - 2\pi n$ (see Eq. (5.1)),

$$s = \frac{y_1 + 2\pi\hat{n}}{k_1} \tag{5.15}$$

as desired.

The theorem provides a formula for estimating the probability that the projection $\mathbf{u}^{\perp} \bullet \mathbf{y}$ is closest to the correct number of multiples of $\mathbf{v}$ (see Figure 4). Using Theorem 2, we obtain for the probability that $\hat{s}$ differs from the true value of s, the expression,

$$1 - P\left(\left|k_2\varepsilon_2 + k_3\varepsilon_3\right| < \pi\left(k_2^2 + k_3^2\right)/k_1\right) = P\left(\left|k_2\varepsilon_2 + k_3\varepsilon_3\right| > \pi\left(k_2^2 + k_3^2\right)/k_1\right) \qquad (5.16)$$

This is the error probability for the number of wraps in an IFSAR antenna.

For ε_2 and ε_3 independent Gaussian variables with mean zero and variance $\sigma^2 = (N_L - snr)^{-1}$, we have that $k_2\varepsilon_2 + k_3\varepsilon_3$ is Gaussian with mean zero and variance $\sigma^2 = (k_2^2 + k_3^2)(N_L - snr)^{-1}$, where N_L equals the number of looks and snr equals the signal-to-noise ratio. It follows that error probability is given by

$$P\left(\left|k_2\varepsilon_2 + k_3\varepsilon_3\right| > \pi\left(k_2^2 + k_3^2\right)/k_1\right) = P\left(\left|\varepsilon_0\right| > \pi\frac{\sqrt{k_2^2 + k_3^2}}{\sigma k_1}\right) \qquad (5.17)$$

where ε_0 is normally distributed with mean zero and variance one. For sufficiently large σ^{-1} we have (Feller [11])

$$P\left(\left|k_2\varepsilon_2 + k_3\varepsilon_3\right| > \pi\left(k_2^2 + k_3^2\right)/k_1\right) = P\left(\left|\varepsilon_0\right| > \pi\frac{\sqrt{k_2^2 + k_3^2}}{\sigma k_1}\right) \qquad (5.18)$$

$$\approx \frac{1}{\sqrt{2\pi}}\exp\left(-\pi^2\frac{k_2^2 + k_3^2}{2\sigma^2 k_1^2}\right)\left(\frac{\sigma k_1}{\pi\sqrt{k_2^2 + k_3^2}} - \left(\frac{\sigma k_1}{\pi\sqrt{k_2^2 + k_3^2}}\right)^3\right).$$

For $k_2 \approx k_3$, this expression simplifies to

$$P\left(\left|k_2\varepsilon_2 + k_3\varepsilon_3\right| > \pi\left(k_2^2 + k_3^2\right)/k_1\right) \approx \frac{1}{\sqrt{2\pi}} \exp\left(-\pi^2 \frac{k_2^2}{\sigma^2 k_1{}^2}\right)\left(\frac{\sigma k_1}{\sqrt{2}\pi k_2} - \left(\frac{\sigma k_1}{\sqrt{2}\pi k_2}\right)^3\right) \quad (5.19)$$

A similar calculation for two pairs of phase centers, say k_1 and k_2, yields for the probability of an IFSAR error

$$P\left(|\varepsilon| > \pi \frac{k_2}{\sigma k_1}\right) \approx \frac{1}{\sqrt{2\pi}} \exp\left(-\pi^2 \frac{k_2^2}{2\sigma^2 k_1{}^2}\right)\left(\frac{\sigma k_1}{\pi k_2} - \left(\frac{\sigma k_1}{\pi k_2}\right)^3\right) \quad (5.20)$$

where ε is normally distributed with mean zero and variance one. We notice that the exponent in Eq. (5.20) contains a factor of $1/2$ that is missing from Eq. (5.19). It follows that the exponential term in Eq. (5.19) decreases as the square of the exponential term in Eq. (5.20). Roughly, the error probability for resolving phase ambiguities, in the four-phase-center IFSAR, decreases approximately as the square of the error probability for the corresponding three-phase-center IFSAR.

We note that if phase wraps occur in the smaller baselines then the wrapped line segments $s\mathbf{k} - 2\pi n\mathbf{e}_1$ lie on a set of *affine* planes that are orthogonal to $\mathbf{w} = \mathbf{k} \times \mathbf{e}_1$, see Figure 4. To remove the phase wraps, we determine which affine plane is closest to the point $\mathbf{y}$ (this determines the number of wraps in the smaller baselines) and then apply the method presented in this section.

6. Summary

We have presented new methods for resolving IFSAR ambiguities and SAR layover. The key idea in the phase method for solving the SAR layover problem is to find the intersection between a surface and a line segment. The basic idea in removing IFSAR phase ambiguities or in determining the number of wraps is to find the wrapped line segment that is closest to the measured phase. The analytic characteristics of these methods make them well suited for efficient, reliable computation. A more general treatment of the effect of noise on these methods requires additional study.

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8. Appendix

Proof of Theorem 1

We begin by showing that the domain of β is given by $\left[1/2-\left|\mu(k)\right|/2, 1/2+\left|\mu(k)\right|/2\right]$. The partial derivative of the argument in Eq. (4.6) is given by

$$\frac{\partial}{\partial\alpha}h(\alpha,k)=\frac{\partial}{\partial\alpha}\frac{\xi(k)-\alpha^2-(1-\alpha)^2}{2\alpha(1-\alpha)}=\frac{(1-2\alpha)(1-\xi(k))}{2\alpha^2(1-\alpha)^2} \tag{A1}$$

The right hand side of Eq. (A1) changes from positive to negative at $\alpha=1/2$ for $0<\alpha<1$. Since $\xi(k)$ is strictly decreasing ($\xi'(k)<0$) and $\xi(k)\le\left|\gamma(k_2)\right|^2<1$, it follows that the argument $h(\alpha,k)$ is strictly increasing on (0,1/2), strictly decreasing on (1/2,1), and has a maximum at $\alpha=1/2$. The function $h(\alpha,k)$ equals –1 at $\alpha=1/2\pm\left|\gamma(k)\right|/2$ and $2\left|\gamma(k)\right|^2-1$ at $\alpha=1/2$. To restrict the range of $h(\alpha,k)$ to the interval $[-1,1]$, so that $\cos^{-1}z$ is defined, we must restrict α to the interval $\left[1/2-\left|\gamma(k)\right|/2, 1/2+\left|\gamma(k)\right|/2\right]$; this is the domain of $\beta(\alpha,k)$. Now, since $\cos^{-1}z$ is strictly decreasing it follows that $\beta(\alpha,k)$ is strictly decreasing for $\left(\frac{1}{2}-\frac{1}{2}\left|\gamma(k)\right|,\frac{1}{2}\right)$, strictly increasing for $\left(\frac{1}{2},\frac{1}{2}+\frac{1}{2}\left|\gamma(k)\right|\right)$, and has a minimum at $\alpha=1/2$. The range of $\beta(\alpha,k)$ is given by $\left(\beta(1/2,k),\beta\left(1/2+\left|\gamma(k)\right|/2,k\right)\right)=\left(\beta(1/2,k),\pi/(2k)\right)$.

We need to show that $I_1/(I_2+I_2)$ lies in the domain of $\beta(\alpha,k)$; this implies tha t d belongs to the range of β. For this we notice

$$\left|\gamma(k)\right|^2=\frac{I_1^{\,2}}{(I_1+I_2)^2}+\frac{I_2^{\,2}}{(I_1+I_2)^2}+2\frac{I_1I_2}{(I_1+I_2)}\cos(2kd)>\left(2\frac{I_1}{I_1+I_2}-1\right)^2 \tag{A2}$$

Here, we used the fact that $I_1/(I_1+I_2)=1-I_2/(I_1+I_2)$, the assumption $0<2k_1d<\pi$ and the inequality $\cos(2kd)>-1$. It follows from Eq. (A2) that $1/2-|\gamma(k)|/2<I_1/(I_1+I_2)<1/2+|\gamma(k)|/2$ and since, by assumption $0<2kd<2k_1d<\pi$, we have that d lies in the range of $\beta(\alpha,k)$; that is, $\beta(I_1/(I_1+I_2),k_1)=d=\beta(I_1/(I_1+I_2),k_2)$. This shows that Eq. (4.8) has at least one solution.

We turn now to the problem of determining the number of solutions to Eq. (4.8). The partial derivative of $\beta(\alpha,k)$ with respect to α is given by

$$\frac{\partial\beta}{\partial\alpha}=-\frac{\partial f/\partial\alpha}{\partial f/\partial\beta}=\frac{2\alpha-1}{2\alpha(1-\alpha)}g(\beta,k) \tag{A3}$$

where

$$g(\beta,k)=\frac{1-\cos(2k\beta)}{k\sin(2k\beta)} \tag{A4}$$

The second derivative of $\beta(\alpha,k)$ at $\alpha=1/2$ is given by

$$\frac{\partial^2\beta(1/2,k)}{\partial\alpha^2}=4g(\beta(1/2,k),k)=\frac{4\left(1-1/(2k)\left(2|\gamma(k)|^2-1\right)\right)}{k\sin\left(\cos^{-1}\left(2|\gamma(k)|^2-1\right)\right)}>0 \tag{A5}$$

From the above we have $0<\beta(\alpha,k)<\pi/(2k)$ for $\alpha\in\left(\frac{1}{2}-\frac{1}{2}|\gamma(k)|,\frac{1}{2}+\frac{1}{2}|\gamma(k)|\right)$ and $k_1\geq k\geq k_2$, so that

$$\frac{\partial g(\beta,k)}{\partial k}=\frac{(2k\beta-\sin(2k\beta))(1-\cos(2k\beta))}{k^2\sin^2(2k\beta)}>0 \tag{A6}$$

Here, we also use the fact that $\sin z < z$ for $z > 0$. It follows from Eq. (A6) that g is a strictly increasing function of k.

Let us suppose that the curves $\beta(\alpha, k_1)$ and $\beta(\alpha, k_2)$ intersect at $\alpha = \alpha^*$. If $\alpha^* > 1/2$, we have since $\beta(\alpha^*, k_1) = \beta(\alpha^*, k_2)$ and g is strictly increasing

$$\frac{\partial\beta(\alpha^*, k_1)}{\partial\alpha} = \frac{(2\alpha^*-1)\,g\left(\beta(\alpha^*, k_1), k_1\right)}{2\alpha^*(1-\alpha^*)} > \frac{(2\alpha^*-1)\,g\left(\beta(\alpha^*, k_2), k_2\right)}{2\alpha^*(1-\alpha^*)} = \frac{\partial\beta(\alpha^*, k_2)}{\partial\alpha} \quad \text{(A7)}$$

Similarly, if $\alpha^* < 1/2$, we have

$$\frac{\partial\beta(\alpha^*, k_1)}{\partial\alpha} < \frac{\partial\beta(\alpha^*, k_2)}{\partial\alpha} \quad \text{(A8)}$$

This shows that if the curves touch at a point α^* such that $\alpha^* \neq 1/2$ then $\beta(\alpha, k_1)$ must cross $\beta(\alpha, k_2)$ from below. It follows that if $\beta(1/2, k_1) > \beta(1/2, k_2)$ then the curves $\beta(\alpha, k_1)$ and $\beta(\alpha, k_2)$ do not intersect. Since Eq. (4.8) has at least one root, we must have that $\beta(1/2, k_1) \leq \beta(1/2, k_2)$.

The fact that $\beta(\alpha, k_1)$ must cross $\beta(\alpha, k_2)$ from below, implies that if $\beta(1/2, k_1) < \beta(1/2, k_2)$ then Eq. (4.6) has a unique solution on each of the intervals $\left(\frac{1}{2} - \frac{1}{2}|\gamma(k)|, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2}|\gamma(k)|\right)$, namely α^* and $1-\alpha^*$.

Let us suppose that $\beta(1/2, k_1) = \beta(1/2, k_2)$ so that Eq. (4.8) has a root at $\alpha^* = 1/2$. We want to show that this solution is unique. From Eq. (A.8) we obtain,

$$\frac{\partial^2 \beta(1/2, k_1)}{\partial \alpha^2} = 4g(\beta(1/2, k_1), k_1) > 4g(\beta(1/2, k_2), k_2) = \frac{\partial^2 \beta(1/2, k_1)}{\partial \alpha^2}. \qquad \text{(A.9)}$$

This inequality, together with the fact that $\partial\beta(1/2, k_1)/\partial\alpha = \partial\beta(1/2, k_2)/\partial\alpha = 0$, asserts that $\beta(\alpha, k_1) > \beta(\alpha, k_2)$ in some deleted neighborhood of $\alpha = 1/2$. Since $\beta(\alpha, k_1)$ can cross $\beta(\alpha, k_2)$ only from below for $\alpha \neq 1/2$, it follows that $\alpha^* = 1/2$ is the only root of Eq. (4.8). Now, by assumption $\alpha^* = I_1/(I_1 + I_2) \neq 1/2$ is a solution, so we must have that $\beta(1/2, k_1) < \beta(1/2, k_2)$. Using the uniqueness of the solutions we obtain Eq. (4.9).

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