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**A NOTE ON  
SEVEN ANALOGOUS PROPERTIES BETWEEN  
STIRLING NUMBERS OF THE FIRST KIND  
AND BINOMIAL COEFFICIENTS**

Tommy Wright

MANAGED BY  
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Computer Science and Mathematics Division  
Mathematical Sciences Section

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## TABLE OF CONTENTS

|  |   |
|--|---|
| ABSTRACT   | v |
| 1. INTRODUCTION  | 1 |
| 2. SOME ANALOGOUS PROPERTIES OF THE COEFFICIENTS $\begin{bmatrix} n \\ r \end{bmatrix}$ and $\begin{pmatrix} n \\ r \end{pmatrix}$ | 1 |
| ACKNOWLEDGMENT   | 5 |
| REFERENCES   | 5 |



**A NOTE ON  
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**ABSTRACT**

This notes gives seven analogous properties between Stirling numbers of the first kind and binomial coefficients.



## 1. INTRODUCTION

If  $n$  and  $r$  are both nonnegative integers where  $r \leq n$ , the **binomial coefficient**  $\binom{n}{r}$  is given by

$$\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}. \quad (1)$$

The symbol  $\binom{n}{r}$  is called a binomial coefficient because it is the coefficient of the  $(r+1)^{th}$  term in the expansion of  $(1+x)^n$  by the binomial theorem. Furthermore, these coefficients are the entries in Pascal's triangle. For a recent historical treatment of Pascal's *arithmetical triangle's* roots, which stretch backward before Christ, see Edwards(1987). The binomial coefficient plays a fundamental role in several areas including combinatorics, applied probability, and probability sampling (Knuth, 1981; Graham, Knuth, and Patashnik, 1989; Ross, 1989; Wilf, 1989; and Wright, 1989, 1991).

If  $n$  and  $r$  are both nonnegative integers where  $r \leq n$ , the **Stirling Number of the First Kind**  $\left[ \begin{smallmatrix} n \\ r \end{smallmatrix} \right]$  is defined as

$$\left[ \begin{smallmatrix} n \\ r \end{smallmatrix} \right] \equiv \text{the sum of all possible products of } n-r \text{ integers taken from the first } n \text{ positive integers.} \quad (2)$$

For  $r = n$ , we define  $\left[ \begin{smallmatrix} n \\ n \end{smallmatrix} \right] \equiv 1$ . Note that  $\left[ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right] \equiv n!$ . Thus if  $n = 4$  and  $r = 2$ ,  $\left[ \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \right] = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35$ . Also  $\left[ \begin{smallmatrix} 4 \\ 0 \end{smallmatrix} \right] = 4! = 24$  and  $\left[ \begin{smallmatrix} 4 \\ 4 \end{smallmatrix} \right] = 1$ . In general, the number of terms in the sum  $\left[ \begin{smallmatrix} n \\ r \end{smallmatrix} \right]$  is  $\binom{n}{r}$ .

In a result analogous to the binomial theorem, it can be shown that

$$\prod_{i=1}^k (i+x) = \sum_{r=0}^k \left[ \begin{smallmatrix} k \\ r \end{smallmatrix} \right] x^r. \quad (3)$$

The quantities  $\left[ \begin{smallmatrix} n \\ r \end{smallmatrix} \right]$  have a triangular arrangement which is similar to Pascal's triangle for the binomial coefficients. (Graham, Knuth, and Patashnik (1989); and Wright(in press))

## 2. SOME ANALOGOUS PROPERTIES OF THE COEFFICIENTS $\left[ \begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ AND $\binom{n}{r}$

In this section, we list several properties of the coefficients  $\left[ \begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ . For each property, an analogous result is noted for Pascal's triangle. The proofs of these properties are straightforward.

### Property 1.

$$\left[ \begin{smallmatrix} n \\ r \end{smallmatrix} \right] = n \left[ \begin{smallmatrix} n-1 \\ r \end{smallmatrix} \right] + (n-1) \left[ \begin{smallmatrix} n-2 \\ r-1 \end{smallmatrix} \right] + \cdots + (n-r) \left[ \begin{smallmatrix} n-(r+1) \\ 0 \end{smallmatrix} \right].$$

*Example 1.*

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

*Analogous Property and Example:*

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-2}{r-1} + \binom{n-3}{r-2} + \cdots + \binom{n-(r+1)}{0}.$$

$$\binom{5}{2} = \binom{4}{2} + \binom{3}{1} + \binom{2}{0}.$$

**Property 2.**

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + n \begin{bmatrix} n-2 \\ r-1 \end{bmatrix} + n(n-1) \begin{bmatrix} n-3 \\ r-1 \end{bmatrix} + \cdots + n(n-1)\cdots(r+1) \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}.$$

*Example 2.*

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 5 \cdot 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5 \cdot 4 \cdot 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

*Analogous Property and Example:*

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \binom{n-3}{r-1} + \cdots + \binom{r-1}{r-1}.$$

$$\binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}.$$

Where  $x$  is a real number, define  $[x] \equiv$  the greatest integer less than or equal to  $x$ . Property 3 is a *symmetry* property.

**Property 3.**

$$\sum_{r=0}^{[n/2]} \begin{bmatrix} n \\ 2r \end{bmatrix} = \sum_{r=0}^{[n/2]} \begin{bmatrix} n \\ 2r+1 \end{bmatrix}.$$

*Example 3.*

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

*Analogous Property and Example:*

$$\sum_{r=0}^{[n/2]} \binom{n}{2r} = \sum_{r=0}^{[n/2]} \binom{n}{2r+1}.$$

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = \binom{5}{1} + \binom{5}{3} + \binom{5}{5}.$$

**Property 4.**

$$\sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} = (n+1) \sum_{r=0}^{n-1} \begin{bmatrix} n-1 \\ r \end{bmatrix}.$$

*Example 4.*

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 5 \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}.$$

*Analogous Property and Example:*

$$\sum_{r=0}^n \binom{n}{r} = 2 \sum_{r=0}^{n-1} \binom{n-1}{r}.$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2 \left\{ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right\}.$$

Property 5 follows from Property 4.

**Property 5.**

$$\sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} = (n+1)!.$$

*Example 5.*

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 5!.$$

*Analogous Property and Example:*

$$\sum_{r=0}^n \binom{n}{r} = 2^n.$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4.$$

For the nonnegative integers  $x$  and  $y$  where  $x \leq y$ , define  $P_x^y$  to be

$$P_x^y \equiv \frac{y!}{(y-x)!}. \quad (4)$$

**Property 6.**

$$\sum_{m=0}^n P_{n-m}^{n+1} \sum_{r=0}^m \begin{bmatrix} m \\ r \end{bmatrix} = \sum_{m=0}^n P_{n-m}^{n+1} (m+1)! = (n+1)(n+1)!.$$

*Example 6.* For  $n = 3$ ,

$$\begin{aligned} P_3^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + P_2^4 \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} + P_1^4 \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} + P_0^4 \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\} \\ = 4 \cdot 3 \cdot 2(1!) + 4 \cdot 3(2!) + 4(3!) + (4!) = 4(4!). \end{aligned}$$

*Analogous Property and Example:*

$$\sum_{m=0}^n \sum_{r=0}^m \binom{m}{r} = \sum_{m=0}^n 2^m = 2^{n+1} - 1.$$

For  $n = 3$ ,

$$\binom{0}{0} + \left\{ \binom{1}{0} + \binom{1}{1} \right\} + \left\{ \binom{2}{0} + \binom{2}{1} + \binom{2}{2} \right\} + \left\{ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right\} = 2^4 - 1.$$

**Property 7.**

$$\sum_{i=0}^r \begin{bmatrix} n \\ i \end{bmatrix} = (n+1) \sum_{i=0}^{r-1} \begin{bmatrix} n-1 \\ i \end{bmatrix} + n \begin{bmatrix} n-1 \\ r \end{bmatrix}.$$

*Example 7.* For  $n = 6$  and  $r = 4$ ,

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = (6+1) \left\{ \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\} + 6 \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

*Analogous Property and Example:*

$$\sum_{i=0}^r \binom{n}{i} = 2 \sum_{i=0}^{r-1} \binom{n-1}{i} + \binom{n-1}{r}.$$

For  $n = 6$  and  $r = 4$ ,

$$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 2 \left\{ \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} \right\} + \binom{5}{4}.$$

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