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A NOTE ON SEVEN ANALOGOUS PROPERTIES BETWEEN STIRLING NUMBERS OF THE FIRST KIND AND BINOMIAL COEFFICIENTS

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ABSTRACT

This notes gives seven analogous properties between Stirling numbers of the first kind and binomial coefficients.

1. INTRODUCTION

If n and r are both nonnegative integers where $r \leq n$, the **binomial coefficient** $\binom{n}{k}$ is given by

$$\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}. \tag{1}$$

The symbol $\binom{n}{r}$ is called a binomial coefficient because it is the coefficient of the $(r+1)^{th}$ term in the expansion of $(1+x)^n$ by the binomial theorem. Furthermore, these coefficients are the entries in Pascal's triangle. For a recent historical treatment of Pascal's *arithmetical triangle's* roots, which stretch backward before Christ, see Edwards(1987). The binomial coefficient plays a fundamental role in several areas including combinatorics, applied probability, and probability sampling (Knuth, 1981; Graham, Knuth, and Patashnik, 1989; Ross, 1989; Wilf, 1989; and Wright, 1989, 1991).

If n and r are both nonnegative integers where $r \leq n$, the **Stirling Number of the First Kind** $\begin{bmatrix} n \\ r \end{bmatrix}$ is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix} \equiv \text{the sum of all possible products of } n-r \text{ integers taken from the first } n \text{ positive integers.} \tag{2}$$

For $r = n$, we define $\begin{bmatrix} n \\ n \end{bmatrix} \equiv 1$. Note that $\begin{bmatrix} n \\ 0 \end{bmatrix} \equiv n!$. Thus if $n = 4$ and $r = 2$, $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35$. Also $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 4! = 24$ and $\begin{bmatrix} 4 \\ 4 \end{bmatrix} = 1$. In general, the number of terms in the sum $\begin{bmatrix} n \\ r \end{bmatrix}$ is $\binom{n}{r}$.

In a result analogous to the binomial theorem, it can be shown that

$$\prod_{i=1}^{k}(i+x) = \sum_{r=o}^{k} \begin{bmatrix} k \\ r \end{bmatrix} x^r. \tag{3}$$

The quantities $\begin{bmatrix} n \\ r \end{bmatrix}$ have a triangular arrangement which is similar to Pascal's triangle for the binomial coefficients. (Graham, Knuth, and Patashnik (1989); and Wright(in press))

2. SOME ANALOGOUS PROPERTIES OF THE COEFFICIENTS $\begin{bmatrix} n \\ r \end{bmatrix}$ AND $\binom{n}{r}$

In this section, we list several properties of the coefficients $\begin{bmatrix} n \\ r \end{bmatrix}$. For each property, an analogous result is noted for Pascal's triangle. The proofs of these properties are straightforward.

Property 1.

$$\begin{bmatrix} n \\ r \end{bmatrix} = n \begin{bmatrix} n-1 \\ r \end{bmatrix} + (n-1) \begin{bmatrix} n-2 \\ r-1 \end{bmatrix} + \cdots + (n-r) \begin{bmatrix} n-(r+1) \\ 0 \end{bmatrix}.$$

Example 1.

$$\begin{bmatrix}5\\2\end{bmatrix} = 5\begin{bmatrix}4\\2\end{bmatrix} + 4\begin{bmatrix}3\\1\end{bmatrix} + 3\begin{bmatrix}2\\0\end{bmatrix}.$$

Analogous Property and Example:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-2}{r-1} + \binom{n-3}{r-2} + \cdots + \binom{n-(r+1)}{0}.$$

$$\binom{5}{2} = \binom{4}{2} + \binom{3}{1} + \binom{2}{0}.$$

Property 2.

$$\begin{bmatrix}n\\r\end{bmatrix} = \begin{bmatrix}n-1\\r-1\end{bmatrix} + n\begin{bmatrix}n-2\\r-1\end{bmatrix} + n(n-1)\begin{bmatrix}n-3\\r-1\end{bmatrix} + \cdots + n(n-1)\cdots(r+1)\begin{bmatrix}r-1\\r-1\end{bmatrix}.$$

Example 2.

$$\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix} + 5\begin{bmatrix}3\\1\end{bmatrix} + 5\cdot 4\begin{bmatrix}2\\1\end{bmatrix} + 5\cdot 4\cdot 3\begin{bmatrix}1\\1\end{bmatrix}.$$

Analogous Property and Example:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \binom{n-3}{r-1} + \cdots + \binom{r-1}{r-1}.$$

$$\binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}.$$

Where x is a real number, define $[x] \equiv$ the greatest integer less than or equal to x. Property 3 is a *symmetry* property.

Property 3.

$$\sum_{r=0}^{[n/2]} \begin{bmatrix}n\\2r\end{bmatrix} = \sum_{r=0}^{[n/2]} \begin{bmatrix}n\\2r+1\end{bmatrix}.$$

Example 3.

$$\begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}5\\2\end{bmatrix} + \begin{bmatrix}5\\4\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix} + \begin{bmatrix}5\\3\end{bmatrix} + \begin{bmatrix}5\\5\end{bmatrix}.$$

Analogous Property and Example:

$$\sum_{r=0}^{[n/2]} \binom{n}{2r} = \sum_{r=0}^{[n/2]} \binom{n}{2r+1}.$$

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = \binom{5}{1} + \binom{5}{3} + \binom{5}{5}.$$

Property 4.

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} = (n+1) \sum_{r=0}^{n-1} \begin{bmatrix} n-1 \\ r \end{bmatrix}.$$

Example 4.

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 5 \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}.$$

Analogous Property and Example:

$$\sum_{r=0}^{n} \binom{n}{r} = 2 \sum_{r=0}^{n-1} \binom{n-1}{r}.$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2\left\{ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right\}.$$

Property 5 follows from Property 4.

Property 5.

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} = (n+1)!.$$

Example 5.

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 5!.$$

Analogous Property and Example:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4.$$

For the nonnegative integers x and y where $x \leq y$, define P_x^y to be

$$P_x^y \equiv \frac{y!}{(y-x)!}. \tag{4}$$

Property 6.

$$\sum_{m=0}^{n} P_{n-m}^{n+1} \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} = \sum_{m=0}^{n} P_{n-m}^{n+1}(m+1)! = (n+1)(n+1)!.$$

Example 6. For $n = 3$,

$$P_3^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + P_2^4 \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} + P_1^4 \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} + P_0^4 \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$$

$$= 4 \cdot 3 \cdot 2(1!) + 4 \cdot 3(2!) + 4(3!) + (4!) = 4(4!).$$

Analogous Property and Example:

$$\sum_{m=0}^{n} \sum_{r=0}^{m} \binom{m}{r} = \sum_{m=0}^{n} 2^m = 2^{n+1} - 1.$$

For n = 3,

$$\binom{0}{0} + \left\{ \binom{1}{0} + \binom{1}{1} \right\} + \left\{ \binom{2}{0} + \binom{2}{1} + \binom{2}{2} \right\} + \left\{ \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \right\} = 2^4 - 1.$$

Property 7.

$$\sum_{i=0}^{r} \begin{bmatrix} n \\ i \end{bmatrix} = (n+1) \sum_{i=0}^{r-1} \begin{bmatrix} n-1 \\ i \end{bmatrix} + n \begin{bmatrix} n-1 \\ r \end{bmatrix}.$$

Example 7. For $n = 6$ and $r = 4$,

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = (6+1) \left\{ \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\} + 6 \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Analogous Property and Example:

$$\sum_{i=0}^{r} \binom{n}{i} = 2 \sum_{i=0}^{r-1} \binom{n-1}{i} + \binom{n-1}{r}.$$

For $n = 6$ and $r = 4$,

$$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 2 \left\{ \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} \right\} + \binom{5}{4}.$$

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