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Velocity Boundary Conditions for Vorticity Formulations of the Incompressible Navier-Stokes Equations

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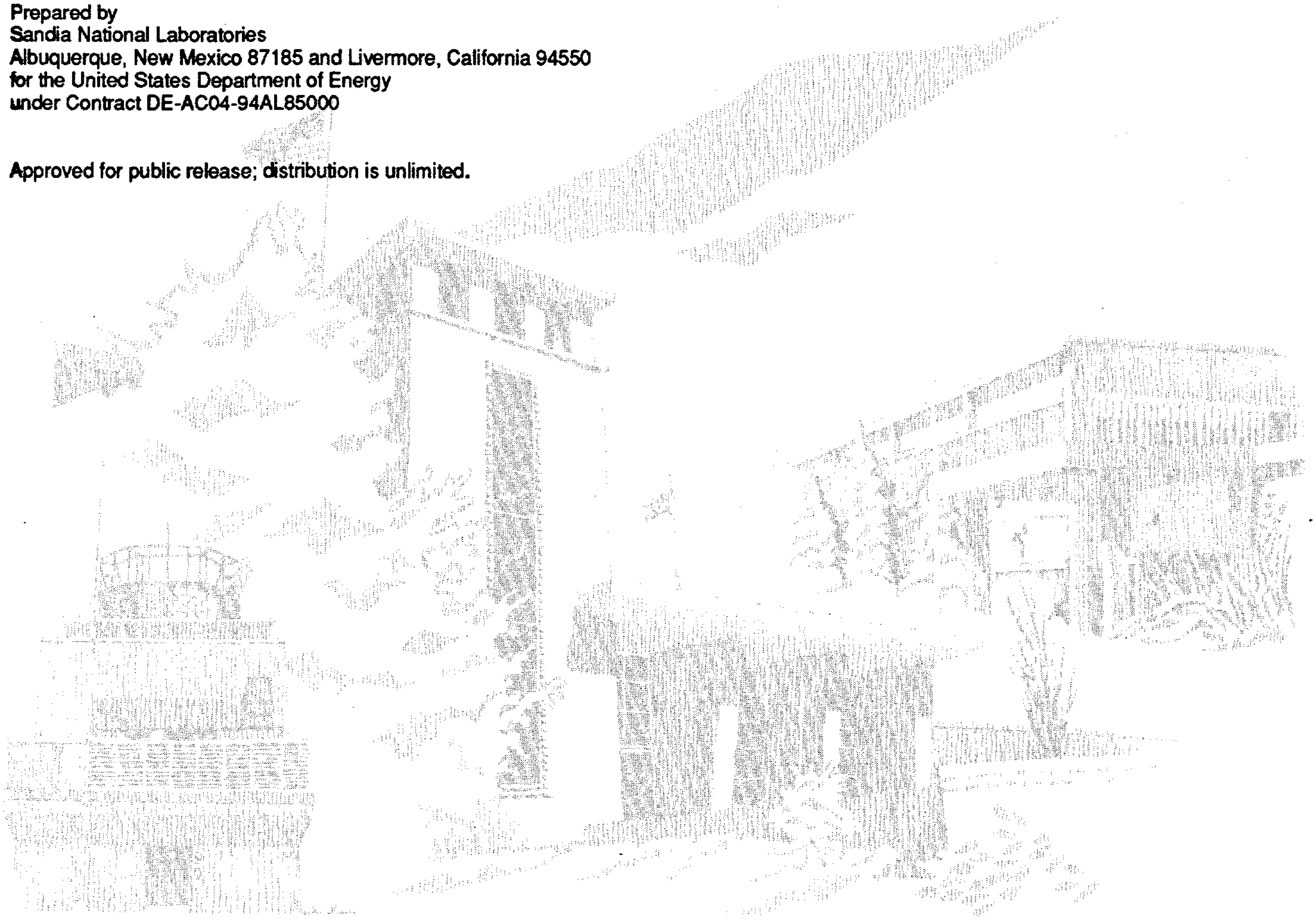

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Abstract

A formulation to satisfy velocity boundary conditions for the vorticity form of the incompressible, viscous fluid momentum equations is presented. The tangential and normal components of the velocity boundary condition are satisfied simultaneously by creating vorticity adjacent to boundaries. The newly created vorticity is determined using a kinematical formulation which is a generalization of Helmholtz' decomposition of a vector field. Related forms of the decomposition were developed by Bykhovskiy and Smirnov [5] in 1983, and Wu and Thompson [30] in 1973. Though it has not been generally recognized, these formulations resolve the over-specification issue associated with creating vorticity to satisfy velocity boundary conditions. The generalized decomposition has not been widely used, apparently due to a lack of a useful physical interpretation. An analysis is presented which shows that the generalized decomposition has a relatively simple physical interpretation which facilitates its numerical implementation.

The implementation of the generalized decomposition is discussed in detail. As an example the flow in a two-dimensional lid-driven cavity is simulated. The solution technique is based on a Lagrangian transport algorithm in the hydrocode ALEGRA. ALEGRA's Lagrangian transport algorithm has been modified to solve the vorticity transport equation and the generalized decomposition, thus providing a new, accurate method to simulate incompressible flows. This numerical implementation and the new boundary condition formulation allow vorticity-based formulations to be used in a wider range of engineering problems.

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Executive Summary

Vorticity formulations of the Navier-Stokes equations have distinct advantages over velocity-pressure formulations. These advantages remain largely unused, however, since appropriate boundary conditions for vorticity formulations are not resolved. The problem is that boundary conditions for the Navier-Stokes equations are in terms of velocities, but a boundary condition in terms of vorticity is required for vorticity formulations. Thus, it is necessary to deduce a vorticity boundary condition from a velocity boundary condition. A vorticity boundary condition could consist of a normal gradient of vorticity, or a value of vorticity. However, neither the vorticity on the boundary nor its gradient is generally known. Some recent efforts in this area are Koumoutsakos, *et al.*, [16], Wu, Wu, Ma, and Wu [34], and Anderson [1], based on the work by Quartapelle and Valz-Gris [25]. The objective of this discussion is to present and demonstrate a new generalized method to describe vorticity generation. This formulation allows the advantages of vorticity-based formulations to be applied to a wider range of flows that occur in engineering problems.

The main advantage of the proposed formulation is that all components of the velocity boundary conditions are imposed simultaneously. In classical formulations, normal and tangential components of the boundary velocity are imposed in separate steps. As a result, all components of the velocity boundary conditions are not always satisfied simultaneously. The proposed formulation allows all components of the velocity boundary condition to be satisfied simultaneously, as is desired for quantitative analysis.

The essential form of the boundary condition formulation was described previously by Wu [33] and Morino [19], [20]. Morino refers to the mathematical work of Bykhovskiy and Smirnov [5], and shows that it yields the same result as Wu and Thompson [30]. Details of this formulation are not well-understood, however, and as a result, it has not been widely used.

As will be shown, the formulations by Wu and Thompson, and Bykhovskiy and Smirnov are essentially generalizations of Helmholtz' decomposition of a vector field. To provide insight, and to facilitate its numerical implementation, the generalized decomposition is derived by applying limiting processes on Helmholtz' decomposition. This approach shows that velocity boundary conditions are simply vortex sheets and volume source distributions that lie *outside* the fluid, a feature not considered in classical formulations. To determine the vorticity which is created in the fluid, the vortex sheet outside the fluid is partitioned according to the tangential velocity boundary condition. This interpretation provides insight regarding the singular behavior on boundaries, thus providing confidence in the overall formulation, and helping to resolve long standing issues in the implementation of vorticity boundary conditions.

Introduction

A boundary condition for vorticity is required to solve the vorticity form of the Navier-Stokes equations for incompressible flows. Boundary conditions are typically specified in terms of velocities, however, so that vorticity boundary conditions must be deduced from velocity boundary conditions. The vorticity boundary condition essentially represents the creation of vorticity whenever the tangential velocity on the boundary is specified, as in viscous flows.

Over the last two decades, many formulations for vorticity boundary conditions have been formulated, as described in reviews by Gresho [9], and Puckett [23]. Accurate flow simulations have been obtained using a wide variety of approaches, yet several fundamental issues remain to be resolved. Several previous models are described below.

Lighthill [17] proposed the basis for most approaches to describe vorticity creation. To begin, it is noted that the velocity induced by an arbitrary vorticity field (via the Biot-Savart law) will not, in general, satisfy either the normal or tangential velocity boundary condition. A new velocity field which satisfies the normal velocity boundary condition can be obtained by adding a potential velocity field, which does not change the vorticity field. The tangential velocity boundary condition is not generally satisfied by the new velocity field, and the deviation from the boundary condition is generally referred to as a slip velocity, u_{slip}. Lighthill indicated that the slip velocity is actually a vortex sheet with strength $-u_{slip}$, and that this vortex sheet represents the vorticity created on the boundary.

An approach for solving the Prandtl boundary layer equations for motionless boundaries was originated by Chorin (see Chorin and Marsden [6]). A brief description of the method follows. An approximate solution of the inviscid equations is advanced one time step. The velocity field induced by the vorticity field generally differs from the tangential velocity boundary condition by u_{slip}. To cancel this slip velocity, vortex sheets of cumulative strength $-2u_{slip}$ are created on each segment of the boundary. A Gaussian random walk is then applied to the sheets as a means to describe viscous diffusion. As a result of the random walk, half of the sheets leave the fluid domain, so that on average, the cumulative strength of the vortex sheets remaining in the fluid is $-u_{slip}$. This result is in agreement with Lighthill's work, although Chorin does not add the potential velocity field to satisfy the normal velocity boundary condition. In fact, the normal velocity boundary condition is not considered explicitly, although zero normal velocity can be shown to be satisfied in the half-plane. Wu [31] notes that for arbitrary geometries, it is not clear that the normal and tangential velocity boundary conditions are satisfied simultaneously.

Wu [33] proposed a formulation based on an equation derived by Wu and Thompson [30], which prescribes an integral relationship between a vorticity field and all components of the velocity boundary conditions. A similar formula was developed independently by Bykhovskiy and Smirnov [5], and is discussed by Morino [19], [20]. In their formulation, the velocity at one point depends on all other points, and all components of the velocity boundary conditions are considered simultaneously. Either the normal or tangential component of the formulation can be used to form a set of linear equations to be solved for the unknown vorticity. Wu indicates that the linear system of equations is rank deficient, and must be supplemented with the additional constraint that the integral of the vorticity field over the domain be zero. (Wu, *et al.*, [31], [32], [33])

Kinney, *et al.* [14], [13] and Hung and Kinney [10] use the tangential component of the Navier-Stokes equations on the boundary as the basis for determining a vorticity flux. (The Laplacian of the viscous term in the primitive variable equations is replaced by the curl of the vorticity using a vector identity.) Essentially, the vorticity flux is determined by specifying that the slip velocity vanishes over a timestep.

Anderson [1] combines the vorticity form of the Navier-Stokes equations with the time-derivative of an integral constraint on vorticity (developed by Quartapelle, *et al.*, [24], [25]). The resulting formulation is used to determine the vorticity boundary condition for streamfunction-vorticity formulations. This approach yields vortex sheets with the same strength as found by Lighthill and Chorin.

Koumoutsakos, *et al.* [16], use a streamfunction solution to determine the vortex sheet strengths on boundaries, from which a vorticity flux is determined and is distributed to vortex blobs which already exist near the boundary. The vortex sheet strengths are constrained by an integral based on Kelvin's theorem. In their model, creation of vortex sheets on the boundary results in *"the nullification of (the) spurious vortex sheet at the body surface so as to enforce the no-slip condition,"* where the "spurious" sheet refers to the sheet associated with the slip velocity.

Wu, Wu, Ma, and Wu [34] assert that kinematics are incapable of determining vorticity creation or any necessary constraints. (All the work described above is based on kinematics, except that of Kinney, *et al.*) They use two integral solutions. The first solution is for the Navier-Stokes equations, based on the fundamental solution to the heat equation in free space. This solution includes boundary integrals containing the vorticity and its normal gradient on the boundary. Since both the vorticity on the boundary and its normal gradient are unknown, an additional equation is needed. The second solution is for a pressure Poisson equation based on the fundamental solution of the Poisson equation in free space. (The pressure poisson equation is obtained by taking the divergence of the velocity-pressure form of the Navier-Stokes equations.) The pressure solution contains boundary integrals for the pressure and the vorticity on the boundary. Thus, the vorticity on the boundary is common to both solutions, but the normal gradient of vorticity and pressure on the boundary occur in only one equation, so there are three unknowns and only two equations. To form a closed system, Wu, *et al.* [34] show that the normal gradient of vorticity can be cast in terms of the pressure and vorticity on the boundary. Thus, the vorticity solution now has unknowns of the vorticity and pressure on the boundary, just as the pressure solution. After solving the equations simultaneously for vorticity and pressure, the normal gradient can be calculated directly.

The aforementioned models for vorticity creation are similar in many respects, but they also differ in several fundamental respects. One issue is that the proper type of boundary condition (vorticity on the boundary or its normal derivative) is not clear; nor is it clear that there is a proper type of boundary condition.

Another issue is whether vorticity creation should be determined from formulations based on kinematics or dynamics. Kinematic formulations are generally based on the relationship between vorticity, velocity, and streamfunctions. Dynamic formulations generally use the tangential component of the Navier Stokes equations on the boundary.

Most investigators indicate that a linear set of equations must be solved to determine the vorticity created on the boundary. Most investigators also indicate that the linear set of equations must be supplemented with an integral constraint, although the precise mathematical justification for such constraints is not always clear. For example, Wu [31] indicates that the linear system of equations is rank deficient. For closure, Wu specifies that the volume integral of the vorticity field must be zero. Wu, Wu, Ma, and Wu [34] claim that a constraint is needed to exclude spurious solutions that arise due to the fact that the vorticity equation contains higher order derivatives of velocity. They use a pressure-based constraint which requires a pressure poisson equation to be solved. Sarpkaya [27] uses a constraint based on the requirement that the pressure be single valued on the boundary. Koumoutsakos *et al.* also indicate that an integral constraint is needed to obtain a unique solution; they use a constraint based on Kelvin's theorem. Quartapelle, *et al.* [24], [25] indicate that, in order to satisfy both normal and tangential velocity boundary conditions for stream-function vorticity methods, vorticity created on the boundary must satisfy an integral constraint they derived. Thus, it is seen that numerous constraints have been proposed for numerous reasons. By specifying an additional constraint, the problem becomes over-specified. Using various techniques such as Lagrange multipliers, a solution can be obtained, but it is only a least squares solution so that the boundary conditions are satisfied only approximately (Wu, Wu, Ma, and Wu [34]).

A related issue concerns the well-posedness of the mathematical problems associated with vorticity creation, which appears to be (but is not actually) over-specified. For example, in two-dimensional flows only one component of vorticity is created, but there are two velocity boundary conditions (normal and tangential components of velocity). Each of the issues described above remains to be resolved.

Our point of view is that vorticity creation can be specified from purely kinematic considerations. The point of departure for the present analysis is the formula by Wu and Thompson [30], which will be shown to be a generalization of Helmholtz decomposition. It will be shown that the vortex sheets have the same strength as indicated by Chorin and Anderson.

Wu and Thompson's [30] formulation is not well-understood, and as a result, it has not been widely used. One issue of interest is that the generalized decomposition is a vector equation, and both Wu and Morino assert that only a single component of the equation should be used to calculate the vorticity generated on a boundary. (Morino states that only the normal velocity boundary condition is needed, whereas Wu allows for specification of normal or tangential components.) The implication is that the components of the velocity boundary condition depend on each other, which appears to contradict the general notion that the components of velocity boundary conditions are independent. It is shown that all components are in fact independent, but are coupled by a jump in velocity on the boundary, which is the key feature of the generalized decomposition.

The objective of this investigation is to implement the generalized Helmholtz decomposition. The implementation is facilitated by showing that several important features are implicit in the formulation. In particular, the reason that vorticity creation is not over-specified becomes clear. In addition, the necessity for integral constraints are shown to arise from discretization of the governing equations. That is, the analytical form of the generalized Helmholtz decomposition satisfies the constraints implicitly, but discrete representations do not. This suggests future work to formulate a discrete representation that satisfies the constraint identically.

An important feature of the formulation is that it contains singular boundary integrals, and by making use of certain physical interpretations, the nature of the singular behavior becomes clear, which further facilitates its numerical implementation. The formulation can then be used to describe vorticity creation on boundaries.

This manuscript is organized as follows. First, the generalized Helmholtz decomposition is presented including a physically-based derivation, and a description of how it resolves the over-specification problem. Boundary integrals in the generalized decomposition are shown to represent vortex sheets and volume sources *outside* the fluid. This interpretation facilitates the formulation of the boundary conditions, which are described in detail. As an example, the flow field in a lid-driven cavity is described using the new boundary condition formulation.

Mathematical Formulation

Vorticity is defined as the curl of the velocity field, $\underset{\sim}{u}$,

$$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u}. \tag{1}$$

Transport of vorticity in a constant density and constant viscosity fluid is described by the vorticity form of the Navier-Stokes equations, *e.g.*, Batchelor [2]

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + (\underset{\sim}{u} \bullet \nabla)\, \underset{\sim}{\omega} = (\underset{\sim}{\omega} \bullet \nabla)\, \underset{\sim}{u} + \nu \nabla^2 \underset{\sim}{\omega} \qquad \text{in the domain } R. \tag{2}$$

The kinematic viscosity is ν. Velocity boundary conditions are denoted as

$$\underset{\sim}{u} = \underset{\sim}{u}_b \text{ on the boundary, } S. \tag{3}$$

In the course of solving Eq. (2), the velocity field, $\underset{\sim}{u}$, must be determined from the vorticity field, $\underset{\sim}{\omega}$, by solving the coupled equations,

$$\nabla \bullet \underset{\sim}{u} = 0,\ \nabla \times \underset{\sim}{u} = \underset{\sim}{\omega} \text{ in the domain } R, \tag{4}$$

with the velocity boundary conditions, Eq. (3). It is also necessary to describe the creation of vorticity on boundaries. The formulation proposed herein which performs these two operations is

$$\underset{\sim}{c}(\underset{\sim}{x}) \bullet \underset{\sim}{u}(\underset{\sim}{x}) = \tag{5}$$

$$\nabla \times \int_R \underset{\sim}{\omega}(\underset{\sim}{x}')\, G(\underset{\sim}{x}, \underset{\sim}{x}')\, dR(\underset{\sim}{x}') + \nabla \times \int_S [\underset{\sim}{\gamma}_c(\underset{\sim}{x}_b') - \hat{n}(\underset{\sim}{x}_b') \times \underset{\sim}{u}_b(\underset{\sim}{x}_b')]\, G(\underset{\sim}{x}, \underset{\sim}{x}_b')\, dS(\underset{\sim}{x}_b')$$

$$- \nabla \int_R D(\underset{\sim}{x}')\, G(\underset{\sim}{x}, \underset{\sim}{x}')\, dR(\underset{\sim}{x}') - \nabla \int_S (-[\hat{n}(\underset{\sim}{x}_b') \bullet \underset{\sim}{u}_b(\underset{\sim}{x}_b')])\, G(\underset{\sim}{x}, \underset{\sim}{x}_b')\, dS(\underset{\sim}{x}_b')$$

where $D = \nabla \bullet \underset{\sim}{u}$ (which is zero for the incompressible flows of interest here), $\hat{n}$ is the outward pointing unit normal vector on the boundary, and $G(\underset{\sim}{x}, \underset{\sim}{x}')$ is the infinite domain Green's function. In two-dimensions,

$$G(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{1}{2\pi} \log\left[\frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\right] \tag{6}$$

and in three-dimensions,

$$G(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{1}{4\pi} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} . \qquad (7)$$

$\underset{\sim}{c}(\underset{\sim}{x})$ is a tensor which arises from the singular behavior of the boundary integrals, whose components depend on the location of the evaluation point $\underset{\sim}{x}$. The domain is denoted as R (two- or three-dimensional), S is the boundary of the domain, $\underset{\sim}{x}$ is a location in the domain, and a prime superscript denotes a variable of integration. Locations on the boundary are denoted as $\underset{\sim}{x}_b$.

- In the domain, $\underset{\sim}{c}(\underset{\sim}{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

- outside the domain, $\underset{\sim}{c}(\underset{\sim}{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

- on the boundary, the components of c take on the value of the internal angle, divided by 2π for two-dimensional flows, and the internal solid angle divided by 4π in three-dimensional flows. In any orthogonal, right-handed coordinate system,

$$\underset{\sim}{c}(\underset{\sim}{x}) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

  For example, on a smooth boundary of a two-dimensional domain, the internal angle is π, so that $\alpha = 1/2$. $c(\underset{\sim}{x}_b)$ can be determined as described in APPENDIX C.

The boundary integrals in Eq. (5) contain the velocity boundary conditions. The tangential velocity boundary condition is contained in the term $\hat{n} \times \underset{\sim}{u}_b$, and the normal velocity boundary condition is the term $\hat{n} \bullet \underset{\sim}{u}_b$. The quantity $\underset{\sim}{\gamma}_c$ denotes a vortex sheet which we have chosen to represent the vorticity that is created to satisfy the velocity boundary conditions. (We plan to investigate the possibility of solving for the vorticity field adjacent to the boundary rather than vortex sheets.) A vortex sheet essentially specifies a vorticity flux boundary condition as, following the work of Kinney, *et al.*, [13] and [14],

$$\underset{\sim}{\gamma}_c = \int_t^{(t+dt)} \nu(\underset{\sim}{x}, t')\,(\hat{n} \bullet \nabla)\,\underset{\sim}{\omega}(\underset{\sim}{x}, t')\,dt' . \qquad (8)$$

(see APPENDIX D) (although Wu, Wu, Ma, and Wu [34] suggest this is only an approximate solution).

The formulation presented by Wu and Thompson [30] can be obtained from Eq. (5) by setting $\underset{\sim}{\gamma}_c = 0$ and setting $c(\underset{\sim}{x})$ to the identity tensor. Similarly, Bykhovskiy and Smirnov's [5] formulation can be obtained by setting $\underset{\sim}{\gamma}_c = 0$, replacing $\hat{n} \bullet \underset{\sim}{u}_b$ with $\hat{n} \bullet \Delta\underset{\sim}{v}$, replacing $\hat{n} \times \underset{\sim}{u}_b$ with $\hat{n} \times \Delta\underset{\sim}{v}$ (where $\Delta\underset{\sim}{v}$ is an arbitrary velocity jump), and setting $\underset{\sim}{c}(\underset{\sim}{x})$ to the identity tensor.

In the following section, Eq. (5) is derived using a much simpler approach than Wu and Thompson [30] or Bykhovskiy and Smirnov [5]. The objective of the derivation is to show that for two-dimensional flows vorticity creation is properly specified using only one component of Eq. (5) (either the normal or the tangential component). This is true since as will be shown, Eq. (5) is constructed in such a way that if one component of Eq. (5) is written (on the boundary) and solved for the vortex sheet strengths, then the other component will be satisfied even though it was not considered explicitly. Thus, specification of normal and tangential velocity boundary conditions do not over-specify the creation of vorticity on the boundary.

1. Derivation of the Generalized Helmholtz Decomposition

The derivation of Eq. (5) begins with the classical Helmholtz' decomposition of a vector field. The classical formulation provides a method to recover a vector field from the curl of the field and the divergence of the field. For example, a velocity field can be recovered from the divergence of the velocity field $D(\underset{\sim}{x}) = \nabla \bullet \underset{\sim}{u}$ and the curl of the velocity field, which is the vorticity field. (Batchelor [2], Morino [19])

$$\underset{\sim}{u}(\underset{\sim}{x}) = \nabla \times \int_{R_\infty} \underset{\sim}{\omega}(\underset{\sim}{x}')\,G(\underset{\sim}{x}, \underset{\sim}{x}')\,dR(\underset{\sim}{x}') - \nabla \int_{R_\infty} D(\underset{\sim}{x}')\,G(\underset{\sim}{x}, \underset{\sim}{x}')\,dR(\underset{\sim}{x}') \qquad (9)$$

Proper use of this equation requires that the vorticity field satisfies $\nabla \bullet \underset{\sim}{\omega} = 0$, as is required of any vector which is the curl of another vector. It is noted that the integration is over the infinite domain; thus, to use the decomposition in a finite fluid domain, it is necessary that, outside the fluid, $\underset{\sim}{\omega} = 0$ and $D = 0$. This restriction is manifested in the constraint that $\underset{\sim}{\omega} \bullet \hat{n} = 0$ on the boundary of the finite domain. That is, if $\underset{\sim}{\omega} = 0$ outside the finite domain, then no vortex lines can cross the boundary, which implies that the normal component of the vorticity on the boundary is zero, $\underset{\sim}{\omega} \bullet \hat{n} = 0$. Moreover, $\underset{\sim}{\omega} \bullet \hat{n} = 0$ is a condition for which the curl of the first integral (referred to as the Biot-Savart integral) yields the vorticity. Related discussions on this topic are given by Batchelor [2], and by

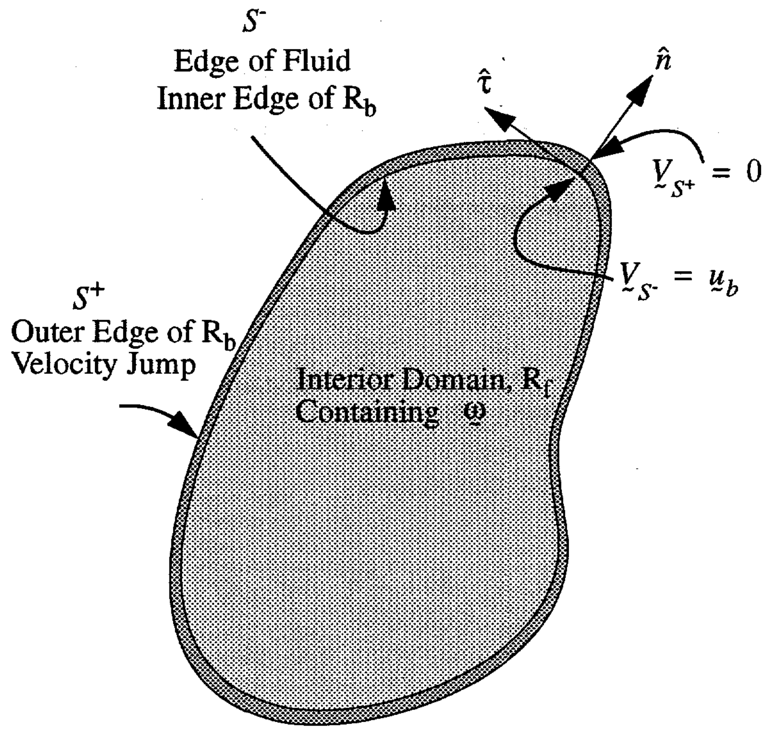

Figure 1 Configuration of fluid and boundary domains, R_f and R_b. The unit normal vector $\hat{n}$ points outward from the fluid. The side of R_b adjacent to the fluid is denoted as S^-, and the other side of R_b is denoted as S^+.

Lighthill, in Chapter 2 of Rosenhead [17].) It is also noted that Eq. (9) is arbitrary to within an irrotational, incompressible velocity field, due primarily to the fact that the velocity boundary conditions do not appear Eq. (9).

To derive a generalization of Eq. (9) that includes velocity boundary conditions, the infinite domain is considered in Figure 1 to consist of a finite fluid region R_f, a small region R_b lying on the boundary of the fluid. The velocity field in the remainder of the infinite domain is required to be incompressible and irrotational. Consider the volume of the boundary region to be defined by a thickness $\Delta \underset{\sim}{n}$ and a surface area $d\underset{\sim}{S}$,

$$R_b = d\underset{\sim}{S} \bullet \Delta \underset{\sim}{n}. \tag{10}$$

We shall consider the limit as $\Delta \underset{\sim}{n}$ approaches zero to form the boundary. The limiting process also takes into account non-zero vorticity and velocity divergence, $\underset{\sim}{\omega} \neq 0$ and $D \neq 0$ in R_b. The two sides of R_b in Figure 1 are denoted S^+ and S^-, where the "+" superscript denotes a location that is an infinitesimal distance in the positive normal direction from R_b. The "-" superscript denotes a location that is an infinitesimal distance in the negative normal direction from R_b.

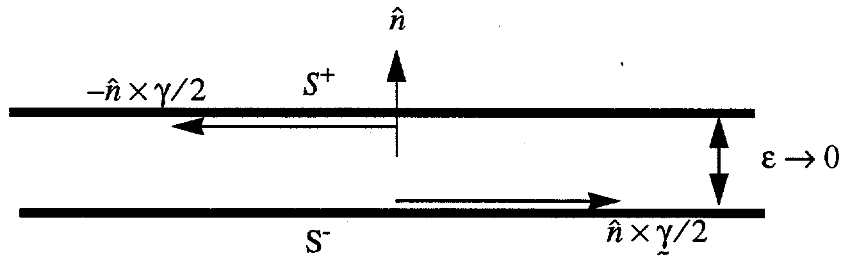

Figure 2 The velocity jump across a vortex sheet with strength $\gamma > 0$. The magnitude of the jump in tangential velocity across the sheet is $\hat{n} \times \gamma$, with velocity $-n \times \gamma/2$ on S^+, and $\hat{n} \times \gamma/2$ on S^-. S^+ and S^- denote the two sides of the vortex sheet.

1.1 Vortex Sheets

For non-zero vorticity ω_ε in the boundary region R_b, consider holding the quantity $\omega_\varepsilon \Delta n$ constant as the thickness is reduced to zero, and ω_ε approaches infinity,

$$\gamma = \lim_{\substack{\omega_\varepsilon \to \infty \\ \Delta n \to 0 \\ \hat{n} \bullet \omega \to 0}} \omega_\varepsilon \Delta n \tag{11}$$

where γ is the strength of a vortex sheet. Note that the normal component of vorticity is zero to satisfy the constraint that vortex lines cannot cross the boundary.

Applying this limiting procedure to the velocity due to ω_ε in R_b yields,

$$\lim_{\substack{\omega_\varepsilon \to \infty \\ \Delta n \to 0 \\ \hat{n} \bullet \omega \to 0}} \nabla \times \int_{R_b} \omega_\varepsilon (x') \, G(x, x') \, dR(x') \to \nabla \times \int_S \gamma(x_b') \, G(x, x'_b) \, dS(x_b') \, . \tag{12}$$

The singular contribution of the above boundary integral at a particular point is shown in Figure 2. (Kellog [11]) On S^+, the velocity is only in the tangential direction, with magnitude $-\hat{n} \times \gamma/2$; on S^-, the velocity is in the direction opposite to the tangential direction, with magnitude $\hat{n} \times \gamma/2$. Thus, there is a jump in tangential velocity from S^+ to S^-, with magnitude $-\hat{n} \times \gamma$. Accordingly, the vortex sheet strength can be defined as $\gamma = \hat{n} \times (u_{S+} - u_{S-})$. The consistency of this definition can be seen if it is noted that $-\hat{n} \times \hat{n} \times u$ is the tangential component of u ; then $-\hat{n} \times \gamma = -\hat{n} \times [\hat{n} \times (u_{S+} - u_{S-})]$ is a jump in tangential velocity from S^+ to S^-. Since the vortex sheet strength is $\gamma = n \times (u_{S+} - u_{S-})$, specification of u_{S+} and u_{S-} determines γ.

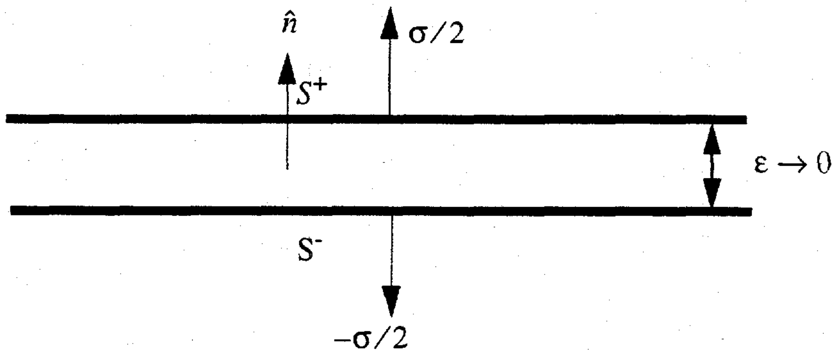

Figure 3 The velocity jump across a source sheet with strength $\sigma > 0$. The magnitude of the jump in normal velocity across the sheet is σ, with velocity $\sigma/2$ on S^+, and $-\sigma/2$ on S^-.

1.2 Source Sheets

Now consider a region of non-zero velocity divergence D_ε in the boundary region R_b. Following a similar procedure as above, a surface distribution of a source σ is defined as

$$\sigma = \lim_{\substack{D_\varepsilon \to \infty \\ \Delta n \to 0}} D_\varepsilon \Delta n . \tag{13}$$

Applying this limiting procedure to the velocity due to D_ε in R_b yields,

$$\lim_{\substack{D_\varepsilon \to \infty \\ \Delta n \to 0}} \nabla \int_R D_\varepsilon (\underset{\sim}{x}') \, G(\underset{\sim}{x}, \underset{\sim}{x}') \, dR(\underset{\sim}{x}') \to \nabla \int_S \sigma(\underset{\sim}{x}') \, G(\underset{\sim}{x}, \underset{\sim}{x}') \, dS(\underset{\sim}{x}_b') \ . \tag{14}$$

The singular contribution from the above boundary integral is shown in Figure 3. (Kellog [11]) On S^+, the velocity is only in the normal direction, with magnitude $\sigma/2$; on S^-, the velocity is $-\sigma/2$, in the direction opposite to the normal direction. Thus, there is a jump in normal velocity from S^+ to S^-, with magnitude σ. Accordingly, the vortex sheet strength can be defined as $\sigma = \hat{n} \bullet (\underset{\sim}{u}_{S^+} - \underset{\sim}{u}_{S^+})$. Therefore, specification of $\underset{\sim}{u}_{S^+}$ and $\underset{\sim}{u}_{S^-}$ determines σ.

1.3 Generalized Decomposition

Adding the boundary integrals from the previous section to Helmholtz' decomposition Eq. (9) yields a generalized Helmholtz decomposition (in the domain),

$$\underset{\sim}{u}(\underset{\sim}{x}) = \tag{15}$$

$$\nabla\times\int_R \underset{\sim}{\omega}(\underset{\sim}{x}')\,G(\underset{\sim}{x},\underset{\sim}{x}')\,dR(\underset{\sim}{x}') + \nabla\times\int_S \underset{\sim}{\gamma}(\underset{\sim}{x}_b')\,G(\underset{\sim}{x},\underset{\sim}{x}_b')\,dS(\underset{\sim}{x}_b')$$

$$-\nabla\int_R D(\underset{\sim}{x}')\,\nabla G(\underset{\sim}{x},\underset{\sim}{x}')\,dR(\underset{\sim}{x}') - \nabla\int_S \sigma(\underset{\sim}{x}_b')\,G(\underset{\sim}{x},\underset{\sim}{x}_b')\,dS(\underset{\sim}{x}_b')$$

1.4 Evaluation of Generalized Decomposition on the Boundary

The generalized decomposition is evaluated on the boundary to define the strengths of the vortex sheets and source sheets in the boundary integrals. Topics regarding the well-posedness of vorticity creation and the need for additional constraints are also discussed. Hereafter, the flow is assumed to be incompressible; *i.e.*, $D(\underset{\sim}{x}) = 0$.

The decomposition is evaluated at a point on the boundary denoted by $\underset{\sim}{x}_b$. For notational simplicity, the velocity due to all vorticity in the domain, and all vortex sheets and sources, except those at $\underset{\sim}{x}_b$, is denoted as $\underset{\sim}{u}_1(\underset{\sim}{x}_b)$.

$$\underset{\sim}{u}_1(\underset{\sim}{x}_b) = \tag{16}$$

$$\nabla\times\int_R \underset{\sim}{\omega}(\underset{\sim}{x}')\,G(\underset{\sim}{x}_b,\underset{\sim}{x}')\,dR(\underset{\sim}{x}')$$

$$+\nabla\times\int_{S(\underset{\sim}{x}_b'\neq\underset{\sim}{x}_b)} \underset{\sim}{\gamma}(\underset{\sim}{x}_b')\,G(\underset{\sim}{x}_b,\underset{\sim}{x}_b')\,dS(\underset{\sim}{x}_b')$$

$$-\nabla\int_{S(\underset{\sim}{x}_b'\neq\underset{\sim}{x}_b)} \sigma(\underset{\sim}{x}_b')\,G(\underset{\sim}{x}_b,\underset{\sim}{x}_b')\,dS(\underset{\sim}{x}_b')$$

The restriction $\underset{\sim}{x}_b \neq \underset{\sim}{x}_b'$ on the limits of the boundary integrals indicates that $\underset{\sim}{u}_1(\underset{\sim}{x}_b)$ has the same value at $S^+(\underset{\sim}{x}_b)$ and $S^-(\underset{\sim}{x}_b)$. That is, the velocity jump at $\underset{\sim}{x}_b$ is due to $\underset{\sim}{\gamma}(\underset{\sim}{x}_b)$ and $\sigma(\underset{\sim}{x}_b)$, and is not included in $\underset{\sim}{u}_1(\underset{\sim}{x}_b)$.

The decomposition is evaluated on both sides of the boundary, S^+ and S^- to yield two equations. The equations differ in the sign (+ or -) of the singular contribution from the boundary integrals. (See Kellog [11], or consider the velocity induced by the sheets, as shown in Figure 2 and Figure 3.)

For $\underset{\sim}{x}_b$ on S^+,

$$\underset{\sim}{u}_{S^+}(\underset{\sim}{x}_b) = -\frac{1}{2}\hat{n}(\underset{\sim}{x}_b) \times \underset{\sim}{\gamma}(\underset{\sim}{x}_b) + \frac{1}{2}\hat{n}(\underset{\sim}{x}_b)\sigma(\underset{\sim}{x}_b) + \underset{\sim}{u}_1(\underset{\sim}{x}_b) \tag{17}$$

For $\underset{\sim}{x}_b$ on S^-,

$$\underset{\sim}{u}_{S^-}(\underset{\sim}{x}_b) = \frac{1}{2}\hat{n}(\underset{\sim}{x}_b) \times \underset{\sim}{\gamma}(\underset{\sim}{x}_b) - \frac{1}{2}\hat{n}(\underset{\sim}{x}_b)\sigma(\underset{\sim}{x}_b) + \underset{\sim}{u}_1(\underset{\sim}{x}_b) \tag{18}$$

The first two terms in these two equations are the singular contributions to the velocity by the vortex and source sheets at the point. The boundary is assumed to be smooth for this discussion, hence the coefficients are 1/2.

At this point, values must be chosen for $\underset{\sim}{u}_{S^+}$ and $\underset{\sim}{u}_{S^-}$. Since $\underset{\sim}{u}_{S^-}$ is the velocity at the edge of the fluid, $\underset{\sim}{u}_{S^-}$ should be the velocity boundary condition, $\underset{\sim}{u}_{S^-} = \underset{\sim}{u}_b$. To choose $\underset{\sim}{u}_{S^+}$ consider that $\underset{\sim}{u}_{S^+}$ is essentially a reference velocity for the fluid. The most general reference velocity is $\underset{\sim}{u}_{S^+} = 0$. This choice has other clear advantages, as discussed below.

For $\underset{\sim}{u}_{S^-} = \underset{\sim}{u}_b$ and $\underset{\sim}{u}_{S^+} = 0$:

1) Values for $\underset{\sim}{\gamma}$, and σ are given by

$$\underset{\sim}{u}_b(\underset{\sim}{x}_b) = \hat{n}(\underset{\sim}{x}_b) \times \underset{\sim}{\gamma}(\underset{\sim}{x}_b) - \hat{n}(\underset{\sim}{x}_b)\sigma(\underset{\sim}{x}_b) . \tag{19}$$

That is, $\sigma = -\hat{n} \bullet \underset{\sim}{u}_b$, and $\underset{\sim}{\gamma} = -\hat{n} \times \underset{\sim}{u}_b$. This result can be obtained by subtracting Eq. (18) from Eq. (17), substituting $\underset{\sim}{u}_{S^+} - \underset{\sim}{u}_{S^-} = 0 - \underset{\sim}{u}_b$, and noting that $-\hat{n}(\underset{\sim}{x}_b) \times \hat{n}(\underset{\sim}{x}_b) \times \underset{\sim}{u}_b(\underset{\sim}{x}_b)$ is the tangential component of $\underset{\sim}{u}_b$.

2) The generalized decomposition yields the same equation on both S^+ and S^-

$$-\frac{1}{2}\hat{n}(\underset{\sim}{x}_b) \times \hat{n}(\underset{\sim}{x}_b) \times \underset{\sim}{u}_b(\underset{\sim}{x}_b) + \frac{1}{2}\hat{n}(\underset{\sim}{x}_b)[\hat{n} \bullet \underset{\sim}{u}_b] = \underset{\sim}{u}_1(\underset{\sim}{x}_b) . \tag{20}$$

Recall that $-\hat{n}(\underset{\sim}{x}_b) \times \hat{n}(\underset{\sim}{x}_b) \times \underset{\sim}{u}_b(\underset{\sim}{x}_b)$ is the tangential component of $\underset{\sim}{u}_b$. Eq. (20) can be obtained by substituting $\underset{\sim}{u}_{S^+} = 0$ into Eq. (17), with $\sigma = -\hat{n} \bullet \underset{\sim}{u}_b$ and $\underset{\sim}{\gamma} = -\hat{n} \times \underset{\sim}{u}_b$ or, substituting $\underset{\sim}{u}_{S^-} = \underset{\sim}{u}_b$ into Eq. (18) with $\sigma = -\hat{n} \bullet \underset{\sim}{u}_b$ and $\underset{\sim}{\gamma} = -\hat{n} \times \underset{\sim}{u}_b$. Accordingly, Eq. (20) is an unambiguous definition of what it means to evaluate the decomposition on boundary; *i.e.*, it does not matter whether the evaluation is considered to be on S^+ or S^-.

3) $\underset{\sim}{u}_{S^+} = 0$ has special implications regarding the issue of over-specification of vorticity creation by velocity boundary conditions. That is, the value of $\underset{\sim}{u}_{S^+}$ can be chosen arbi-

trarily, but it will be shown that the choice $\underset{\sim}{u}_{S+} = 0$ allows both components of the velocity boundary condition to be satisfied simultaneously. To see this, first consider the potential velocity field $\underset{\sim}{u} = \nabla\phi$ outside the fluid domain, as determined from the Laplace equation $\nabla^2\phi = 0$ and appropriate boundary conditions. The boundary conditions must satisfy

$$\int_{S^+} \nabla\phi \bullet \hat{n}\, dS = 0 \text{ and } \int_{S^+} \nabla\phi \bullet \hat{\tau} dS = 0\,. \tag{21}$$

The boundary of the entire domain is S^+, and for $\nabla\phi_{S+} = 0$, these constraints are obviously satisfied, so a Laplace solution can be considered.

It will be shown that solutions to the Laplace equation have the property that if one of the boundary conditions ($\hat{n} \bullet \nabla\phi$ or $\hat{\tau} \bullet \nabla\phi$) is zero everywhere on the boundary, then the other un-specified boundary is determined to be zero everywhere on the boundary.

To show this, consider that if the normal velocity boundary condition, $\nabla\phi \bullet \hat{n} = 0$ everywhere on the boundary, and is also specified at infinity, it is well-known that $\nabla\phi = 0$ everywhere, including on S^+, so that $\nabla\phi \bullet \hat{\tau} = 0$ on S^+. (Batchelor [2])

Similarly, the solution obtained by specifying the tangential velocity boundary condition $\nabla\phi \bullet \hat{\tau} = 0$ everywhere on the boundary is $\nabla\phi = 0$, including $\nabla\phi \bullet \hat{n} = 0$ on S^+. This can be seen by considering that $\nabla\phi \bullet \hat{\tau} = 0$ implies that ϕ = constant on the boundary. Then, from the maximum-minimum modulus theorem (which states that harmonic functions can have maxima and minima only on boundaries[1]), the solution to the Laplace equation is ϕ = constant in the non-fluid domain. Thus, $\nabla\phi = 0$, including $\nabla\phi \bullet \hat{n} = 0$ on S^+.

Thus, it is proved that for the potential flow outside the fluid domain, satisfaction of one component (normal or tangent) of the boundary condition as being zero implies that the other component of the boundary condition is zero.

Now consider the separate components of the generalized decomposition, Eq. (5), evaluated on the boundary. If the normal or tangential component of Eq. (5) satisfies the normal or tangential component of $\underset{\sim}{u}_{S+} = 0$, the previous discussion indicates that the unspecified component also satisfies $\underset{\sim}{u}_{S+} = 0$. That is, specifying one component of $\underset{\sim}{u}_{S+} = 0$ for Eq. (5) fully determines the velocity field. As a result, there is no over-specification of vorticity creation even though there are more components of velocity boundary conditions than unknown components of vorticity.

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1. The maximum-minimum modulus theorem applies to closed bounded regions; *i.e.*, the theorem is not generally stated as applying to unbounded domains. However, as described by Wu [31] and Morino [19], velocity boundary conditions at infinity can be properly represented by the boundary integrals in Eq. (5) when the integrals are applied to a boundary whose location approaches infinity. In this sense, all domains can be considered as bounded.

6. The generalized decomposition implicitly satisfies the integral relationships that have been used as constraint equations in previous analyses. These constraints are typically indicate that the integral of the vorticity is zero over the infinite domain, as given by

$$\int_{R_\infty} \underset{\sim}{\omega} dR = 0, \tag{22}$$

This constraint can be deduced from the identity $\nabla \bullet \underset{\sim}{\omega} = 0$, which implies that vortex lines cannot end in space, and therefore must form closed loops. Solenoidality of vorticity also shows that $\int \underset{\sim}{\omega} \bullet dA$ is the same at any cross section A of a vortex tube, where a vortex tube can be defined in terms of the surface which none of vortex lines cross. Since the vortex lines must be closed, vortex tubes must be closed, and since $\int \underset{\sim}{\omega} \bullet dA$ is constant at every cross section of the tube, the integral of the vorticity of a closed tube is zero. Hence, since all vorticity must consist of closed loops, and the integral of the vorticity in any closed loop is zero, the integral of vorticity over the infinite domain is zero. For two-dimensions, the integral of the vorticity in the infinite plane must also be zero. The vortex lines also form closed loops, but with non-zero curvature occurring only at +/- infinity in the direction perpendicular to the plane. In the plane, the loops are straight and perpendicular to the plane.

This constraint is implicit in the generalized decomposition, as can be shown by considering the identity [15]

$$\int_R (\nabla \times \underset{\sim}{u})\, dR = \int_S (\hat{n} \times \underset{\sim}{u})\, dS \tag{23}$$

on the region enclosed by S^+, and the region consisting of R_f the fluid domain, and R_b is the infinitesimal boundary region containing the vortex and source sheets. Since $\underset{\sim}{u}_{S^+} = 0$, the boundary integral in Eq. (23) is zero.

Now consider the volume integrals in Eq. (23). Let the velocity in the domain be given as $\underset{\sim}{u} = \underset{\sim}{u}_\omega + \underset{\sim}{u}_\sigma + \underset{\sim}{u}_\gamma$, representing the three terms in the generalized decomposition, the Biot-Savart Law, the normal velocity boundary integral, and the tangential velocity boundary integral. In R_f, $\nabla \times \underset{\sim}{u}_\omega = \underset{\sim}{\omega}$, $\nabla \times \underset{\sim}{u}_\sigma = 0$, and $\nabla \times \underset{\sim}{u}_\gamma = 0$ so that

$$\int_{R_f} (\nabla \times \underset{\sim}{u})\, dR = \int_{R_f} \underset{\sim}{\omega} dR\,. \tag{24}$$

In R_b, the theorem in Eq. (23) is invoked again so that

$$\int_{R_b} (\nabla \times \underset{\sim}{u})\, dR = \int_{(S^+,\, S^-)} (\hat{n} \times \underset{\sim}{u})\, dS \tag{25}$$

where the boundary integral is performed around the entire periphery of R_b; *i.e.*, on both S^+ and S^-. On $S^+, \underset{\sim}{u}_{S^+} = 0$, so the boundary integral is zero. On S^-, $\underset{\sim}{u}_{S^-} = \underset{\sim}{u}_{b^-}$, and the direction of integration is opposite that on S^+, so that

$$\int_{(S^+,\, S^-)} (\hat{n} \times \underset{\sim}{u})\, dS = \int (-\hat{n} \times \underset{\sim}{u}_b)\, dS \tag{26}$$

Using Eqns. (24), (25), and (26), Eq. (23) becomes

$$\int_{R_f} \underset{\sim}{\omega} dR + \int (-\hat{n} \times \underset{\sim}{u}_b)\, dS = 0\ . \tag{27}$$

Using $\underset{\sim}{\gamma} = -\hat{n} \times \underset{\sim}{u}_b$ Eq. (27) becomes

$$\int_{R_f} \underset{\sim}{\omega} dR + \int \underset{\sim}{\gamma} dS = 0 \tag{28}$$

or,

$$\int_{R_f} \underset{\sim}{\omega} dR + \lim_{\substack{\underset{\sim}{\omega} \to \infty \\ \Delta n \to 0 \\ \hat{n} \bullet \underset{\sim}{\omega} \to 0}} \int_{R_b} \underset{\sim}{\omega} dR = 0\ . \tag{29}$$

Eq. (29) shows how the generalized decomposition implicitly satisfies the kinematic requirement that the total circulation be zero in the infinite domain. Accordingly, the need to explicitly specify Eq. (29) as a constraint must arise from discretization of the generalized decomposition. For example, discretizations based on point collocation and piecewise constant boundary elements do not generally satisfy Eq. (29). Failure to satisfy Eq. (29) results in erroneous solutions. (The vorticity rapidly approaches numbers larger than allowed on computers.) To satisfy Eq. (29), it is specified as an additional constraint, resulting in too many equations, for which only a least squares solution can be obtained. The typical result is that the velocity boundary conditions are not satisfied as well as might be desired. These observations are presently motivating us to construct numerical methods that satisfy Eq. (29) implicitly.

2. Vorticity Creation

The generalized decomposition provides a mathematical prescription for the vorticity created to satisfy velocity boundary conditions. To begin, consider that velocity boundary conditions and vorticity fields cannot be specified arbitrarily. For an arbitrary vorticity field, and an arbitrary normal velocity boundary condition, $\hat{n} \bullet \underset{\sim}{u}_b$, the generalized decomposition determines the tangential velocity in terms of vortex sheet strengths $\underset{\sim}{\gamma}$ on the boundary, as denoted in Eq. (30),

$$\underset{\sim}{u}(\underset{\sim}{x}) = \tag{30}$$

$$\nabla \times \int_R \underset{\sim}{\omega}(\underset{\sim}{x}')\, G(\underset{\sim}{x}, \underset{\sim}{x}')\, dR(\underset{\sim}{x}') + \nabla \times \int_S \underset{\sim}{\gamma}(\underset{\sim}{x}_b')\, G(\underset{\sim}{x}, \underset{\sim}{x}_b')\, dS(\underset{\sim}{x}_b')$$

$$-\nabla \int_S [-\hat{n}(\underset{\sim}{x}_b') \bullet \underset{\sim}{u}_b(\underset{\sim}{x}_b')]\, G(\underset{\sim}{x}, \underset{\sim}{x}_b')\, dS(\underset{\sim}{x}_b')$$

The unknown $\underset{\sim}{\gamma}$ can be determined by writing the tangential component of Eq. (30) at discrete locations on the boundary and solving the resulting set of linear equations. The tangential velocity on S^- is $\hat{n} \times \underset{\sim}{\gamma}$ which is not generally the specified tangential velocity boundary condition.[2] The vortex sheet $\underset{\sim}{\gamma}$, however, contains the "boundary condition vortex sheet" $\underset{\sim}{\gamma}_{bc} = -\hat{n} \times \underset{\sim}{u}_b$, with the excess vortex sheet strength $\underset{\sim}{\gamma}_c = \underset{\sim}{\gamma} - \underset{\sim}{\gamma}_{bc}$ representing the vorticity that is created in the fluid.

From another point of view, the tangential velocity boundary condition is used to partition $\underset{\sim}{\gamma}$ into two vortex sheets: one which remains outside the fluid, $\underset{\sim}{\gamma}_{bc}$, and the other which enters the fluid (representing vorticity created in the fluid $\underset{\sim}{\gamma}_c = \underset{\sim}{\gamma} - \underset{\sim}{\gamma}_{bc}$). Figure 4 shows the configuration of the two sheets. The fact that the tangential velocity boundary condition is satisfied at the interface between the two sheets can be seen by considering that for the velocity $\underset{\sim}{u}_1$ as defined in Eq. (16), the tangential component of Eq. (30) on the boundary S^+ yields

$$\hat{n} \times \underset{\sim}{\gamma}/2 = -\hat{n} \times \hat{n} \times \underset{\sim}{u}_1 \tag{31}$$

For the configuration shown in Figure 4b, the tangential velocity on the interface between the two vortex sheets is

$$u_\tau = -\hat{n} \times \underset{\sim}{\gamma}_c/2 + \hat{n} \times \underset{\sim}{\gamma}_{bc}/2 - \hat{n} \times \hat{n} \times \underset{\sim}{u}_1 \tag{32}$$

Substituting $\underset{\sim}{\gamma}_c = \underset{\sim}{\gamma} - \underset{\sim}{\gamma}_{bc}$ yields

2. Although only the tangential component of was Eq. (30) solved, the normal velocity boundary condition is enforced since $\underset{\sim}{u}_{S^+} = 0$, as discussed in item 5 of the previous section.

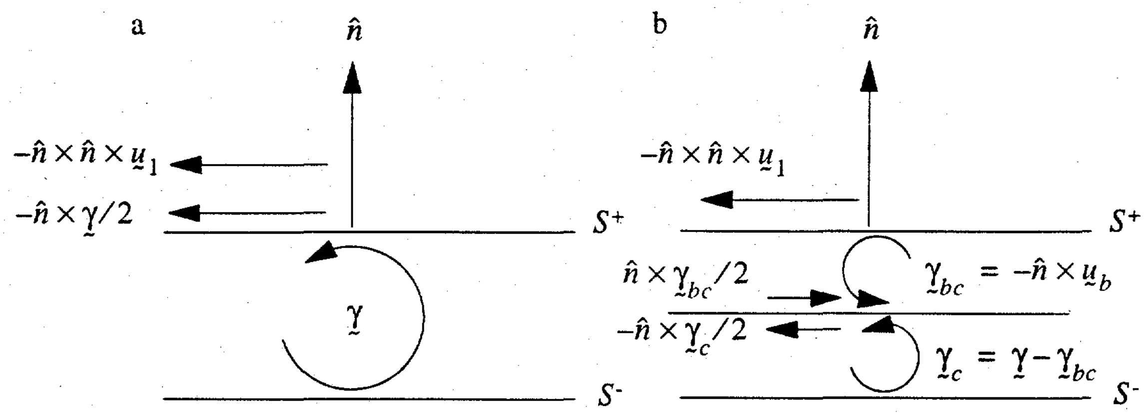

Figure 4 Partitioning of a vortex sheet γ which satisfies zero tangential velocity on S^+ into a boundary condition vortex sheet $\gamma_{bc} = -\hat{n}\times u_b$, and a sheet representing the creation of vorticity in the fluid, $\gamma_c = \gamma-\gamma_{bc}$. Tangential velocities are shown, with the velocity u_1 is defined in Eq. (16).

$$u_\tau = -\hat{n}\times(\gamma-\gamma_{bc})/2+\hat{n}\times\gamma_{bc}/2-\hat{n}\times\hat{n}\times u_1 \tag{33}$$

Substituting Eq. (31) into Eq. (33) yields

$$u_\tau = \hat{n}\times\gamma_{bc}\Big|_{\gamma_{bc}=-\hat{n}\times u_b} = \hat{n}\times(-\hat{n}\times u_b)\ \text{or,} \tag{34}$$

$$u_\tau = -\hat{n}\times\hat{n}\times u_b \tag{35}$$

where the operator $-\hat{n}\times\hat{n}\times(\ )$ extracts the tangential component of any vector. This exercise shows that the tangential velocity boundary condition is satisfied on the interface between the two sheets. Substituting $\gamma = \gamma_{bc}+\gamma_c$ and $\gamma_{bc} = -\hat{n}\times u_b$ into Eq. (30) Eq. (5).

This formulation provides a mathematical basis for the classical approach suggested by Lighthill [17]. In Lighthill's approach, a potential velocity field is added to the velocity field induced by vorticity in order to satisfy the normal velocity boundary condition, but without satisfying the tangential velocity boundary condition, in general. The deviation from the tangential velocity boundary condition, or "slip" velocity is then eliminated by adding a vortex sheet to cancel the slip velocity. Although this approach can yield correct results, to some observers, it appears to be essentially *ad hoc*. Moreover, it is not generally recognized that both components of the velocity boundary condition are satisfied simultaneously. For example, in Chorin's algorithm, vortex sheets are added to satisfy the tangential velocity boundary condition, and image sheets are used to satisfy the normal velocity boundary condition.

The generalized decomposition shows that in order to satisfy the normal velocity boundary condition, a vortex sheet $\underset{\sim}{\gamma}$ must exist on the boundary. In Lighthill's approach, the slip velocity u_{slip} which occurs after the normal velocity boundary condition is satisfied is $u_{slip}\hat{\tau} = \hat{n} \times \underset{\sim}{\gamma}$, or $\underset{\sim}{\gamma} = -\hat{n} \times u_{slip}\hat{\tau}$. For zero tangential velocity, this result agrees with the results of Chorin [6] and Anderson [1]; *i.e.*, the slip velocity u_{slip} is eliminated by adding vortex sheets of strength $\underset{\sim}{\gamma} = -\hat{n} \times u_{slip}\hat{\tau}$. Note that the vortex sheet at a boundary location does not by itself eliminate the slip velocity; the vortex sheet at that location eliminates only half of the slip, and the other half is eliminated by the motion induced by the other vortex sheets and the previously existing vorticity.

An additional feature deduced from the generalized decomposition allows non-zero tangential velocity boundary conditions to be specified, and shows that non-zero tangential velocity boundary conditions are satisfied on an internal interface of the vortex sheet, in which case, only a portion of the vortex sheet is vorticity created within the fluid. The tangential velocity boundary condition therefore specifies how $\underset{\sim}{\gamma}$ is to be partitioned, thus specifying how much vorticity is created in the fluid.

Finally, it is noted that the boundary integrals in the generalized decomposition are not merely a representation of the traditional potential velocity field $\nabla\phi$. Consider that $\nabla\phi$ is constrained by $\int \nabla\phi \bullet \hat{\tau} dS = 0$, whereas u_{slip} is constrained more generally by $\int \underset{\sim}{\omega} dV = \int \hat{n} \times u_{slip}\hat{\tau} dS$. If $\underset{\sim}{\omega} = 0$, this constraint simplifies to the traditional constraint for potential flows $\int \hat{n} \times u_{slip}\hat{\tau} dS = 0$.

3. Example of Non-Uniqueness If the Vorticity Constraint is Violated

Consider a two-dimensional vorticity field that does not satisfy the constraint that the area integral of the vorticity must be zero. It will be shown that if this constraint is not satisfied, the velocity field outside the vorticity field has a different solution depending on whether it is calculated using the Biot-Savart law or a Laplace solution. This inconsistency is eliminated by including the boundary integrals in the generalized decomposition.

Consider a two-dimensional, circular region of constant vorticity, ω_o, contained within a domain with radius R_c. This vorticity field does not satisfy Eq. (22),

$$\int_A \underset{\sim}{\omega} \bullet \hat{n}_A \, dA = \omega_o \pi R_c^2 \neq 0 \, . \tag{36}$$

The velocity field specified by the Biot-Savart law includes an azimuthal velocity component u_θ,

$$u_\theta = \frac{\omega_o}{2} r \quad \text{for } r \leq R_c \tag{37}$$

and

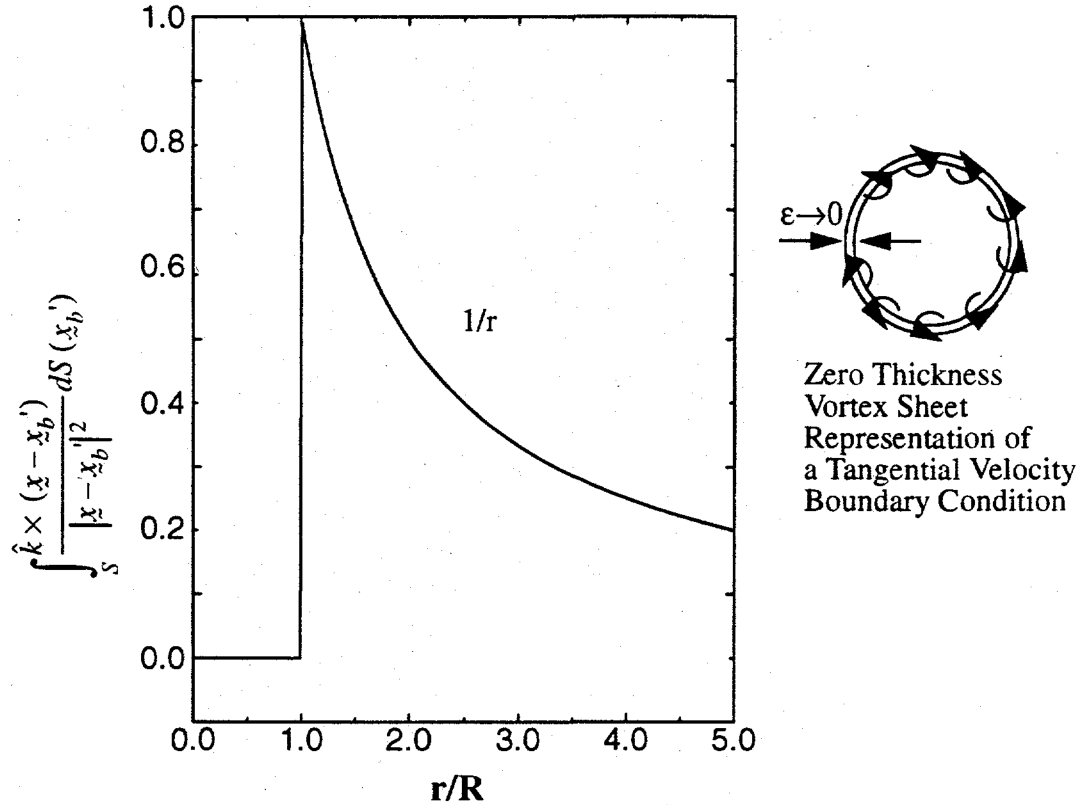

Figure 5 Radial variation of the azimuthal velocity field induced by a circular vortex sheet (radius R) with a uniform strength of unity. The velocity induced inside the sheet is 0. Outside the sheet, the velocity varies as 1/r, the same variation as the velocity induced by a point vortex at r = 0.

$$u_\theta = \frac{\omega_o}{2}\frac{R_c^2}{r} \text{ for } r \geq R_c \tag{38}$$

The Biot-Savart law specifies that the radial velocity is zero everywhere.

Now consider the boundary of the domain to be $r = R_c$, and calculate the potential velocity field in $R_c < r < \infty$, by solving $\nabla^2\phi = 0$. On $r = R_c$, the radial velocity is zero, so $\nabla\phi \bullet \hat{n} = 0$. Similarly, $\nabla\phi \bullet \hat{n} = 0$ is assumed on $r = \infty$. The unique solution is $\nabla\phi = 0$, which includes the zero tangential velocity on $r = R_c$, which differs with the solution obtained from the decomposition.

In reality, the Biot-Savart solution is incomplete since the effects of boundary conditions are not included. In particular, the boundary integral containing the tangential velocity must be included. The motion induced by the boundary integral is shown in Figure 5 for unit vortex sheet strength. In $r \geq R_c$, the outward pointing normal is toward the center of the circle, so the tangent direction is counterclockwise. Since the tangential velocity on the boundary is $\omega_o R_c/2$ (clockwise, from Eq. (38)), the result in Figure 5 must be multiplied by $-\omega_o R_c/2$. Adding that velocity field to the field given by Eq. (38) yields the correct velocity field; *i.e.*, the velocity outside $r \geq R_c$ is zero since the motion induced by the boundary integral is equal and opposite the motion induced by the vorticity field. For this problem, the solution in the domain $r < R_o$ is unaffected by the boundary integral.

Lastly, consider the integral constraint Eq. (29). $\underset{\sim}{\gamma} = -\omega_o R_c/2$ in the line integral, so that

$$\oint \underset{\sim}{\gamma} ds = (-\omega_o R_c/2) \bullet (2\pi R_c) = -\omega_o \pi R_c^2 .$$

The circulation of the domain vorticity is,

$$\int_A \underset{\sim}{\omega} dA = 2\pi \int_0^{R_c} \omega_o r dr = \omega_o \pi R_c^2 .$$

Thus,

$$\int_A \underset{\sim}{\omega} dA + \oint \underset{\sim}{\gamma} ds = 0$$

as is necessary to satisfy the constraint on vorticity, Eq. (29).

Fractional Step Numerical Formulation

1. General Solution Procedure:

As mentioned earlier, the transport of vorticity in a constant density and constant viscosity fluid is described by the vorticity form of the Navier-Stokes equations,

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + (\underset{\sim}{u} \bullet \nabla)\, \underset{\sim}{\omega} = (\underset{\sim}{\omega} \bullet \nabla)\, \underset{\sim}{u} + \nu \nabla^2 \underset{\sim}{\omega} \qquad \text{in the domain } R. \qquad (39)$$

Vorticity creation on the boundaries and outflow velocities are determined by evaluating the generalized decomposition on the boundary, Eq. (5). The flux of vorticity on boundaries is specified using Eq. (8). In the domain, the velocity field in the domain is obtained from the vorticity field and the velocity boundary conditions using Eq. (5) with $D = 0$.

Eq. (39) describes simultaneous inviscid transport and viscous transport, and could be solved using finite element or finite difference methods. However, inviscid transport can be greatly simplified for vorticity transport so, inviscid transport and viscous transport are considered separately. Inviscid transport is described by

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + (\underset{\sim}{u} \bullet \nabla)\, \underset{\sim}{\omega} = (\underset{\sim}{\omega} \bullet \nabla)\, \underset{\sim}{u} \qquad (40)$$

which, according to Helmholtz' theorem, is also described by moving vortex lines at the local fluid velocity, which is described in a Lagrangian reference frame by,

$$\frac{d\underset{\sim}{x}(\underset{\sim}{\alpha}, t)}{dt} = \underset{\sim}{u}(\underset{\sim}{x}(\underset{\sim}{\alpha}, t)) \qquad (41)$$

where $\underset{\sim}{x}(\underset{\sim}{\alpha}, t{=}0) = \underset{\sim}{\alpha}$ is the starting point of a point on a vortex line.

This is the basis for Lagrangian vortex blob methods, and is closely related to the basis by which momentum transport is described in many shock wave physics or "hydrocodes" used to model high pressure (mega-bars), high velocity (km/s) phenomena. See Benson [3]. Hydrocodes describe momentum transport by solving,

$$\frac{d^2\underset{\sim}{x}}{dt^2} = \underset{\sim}{a} = \frac{\sum \underset{\sim}{F}}{m} \qquad (42)$$

wherein Eq. (42) describes the motion of points on discrete volumes, which contain quantities to be transported, such as mass and energy.

The algorithm used in hydrocodes has been adapted to solve Eq. (41) to describe vorticity transport. Two other investigations are also in progress regarding this approach, and have had encouraging results. See Bless and Chacon [4], and Russo and Strain [26].

Diffusion of vorticity into the domain and diffusion of vorticity within the domain are described by solving

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} = \nu \nabla^2 \underset{\sim}{\omega} \tag{43}$$

with the boundary condition

$$\nu (\hat{n} \bullet \nabla \omega_\tau) = \frac{\gamma_c}{\Delta t} .$$

To briefly summarize the algorithm, first, the vortex sheet is determined which satisfies the normal velocity boundary condition. Second, the vortex sheet is moved to lie inside the fluid, such that the tangential velocity boundary condition is also satisfied. (This step requires no effort--it's purely conceptual.) Third, the vorticity field is transported inviscidly. Fourth, viscous diffusion of the vorticity field is described, including a flux of vorticity from the boundaries. At this point, the velocity boundary conditions are no longer satisfied, so that new vortex sheets must be found on the boundary, which begins the repetition of the algorithm.

Example of the Algorithm: Impulsively Started, Driven Lid Cavity

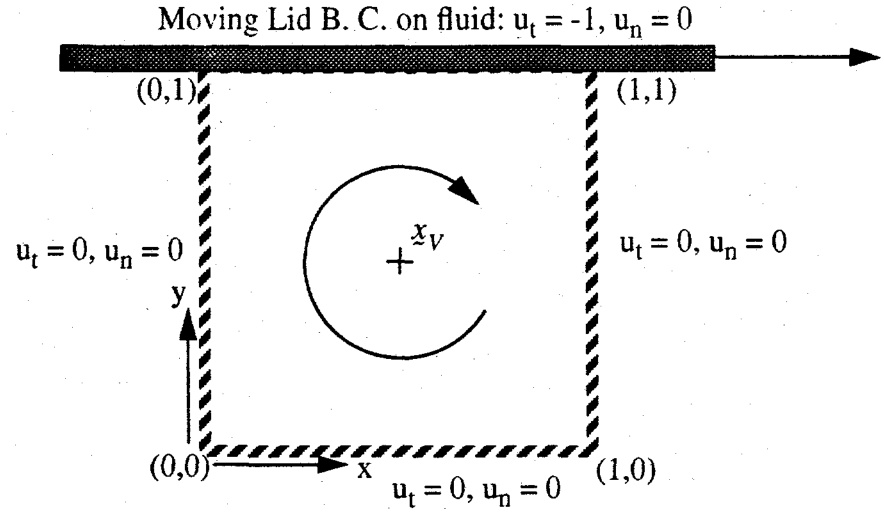

Figure 6 Schematic of a driven-lid cavity. The top of the cavity moves from left to right, imparting motion to the fluid via the no-slip boundary condition. The tangential velocity on all other boundaries is zero. In addition, the normal velocity is zero on all boundaries.

The incompressible flowfield in a two-dimensional cavity with a moving lid is simulated to demonstrate the introduction of vorticity into fluid using the generalized decomposition. This is intended to be a demonstration, rather than a validation, of the proposed algorithm, although the results presented fall within the ranges of previously reported solutions for a Reynolds number of unity. A validation of the algorithm would require comparisons of solutions over a wide range of Reynolds numbers, which has not been completed at this time. This demonstration merely shows the algorithmic process by which vorticity is introduced into the fluid. The Lagrangian transport algorithm in the hydrocode ALEGRA was modified to perform this simulation. Steady-state results were obtained by solving the transient equations until steady-state was reached.

For this preliminary calculation, constant, discontinuous boundary elements were used to represent the boundary integrals, with a single point collocation scheme for evaluating the vortex sheet strengths. For this simple representation, the velocity boundary conditions are satisfied only on average over an element, and the integrals constraints are satisfied only to within a few percent. As discussed below, however, this simple scheme yields results that agree with calculations from previous analyses. Thus, we view this simple and not very accurate scheme as a preliminary numerical validation of the generalized decomposition, and are presently developing more accurate schemes, including a Galerkin weighted residual method.

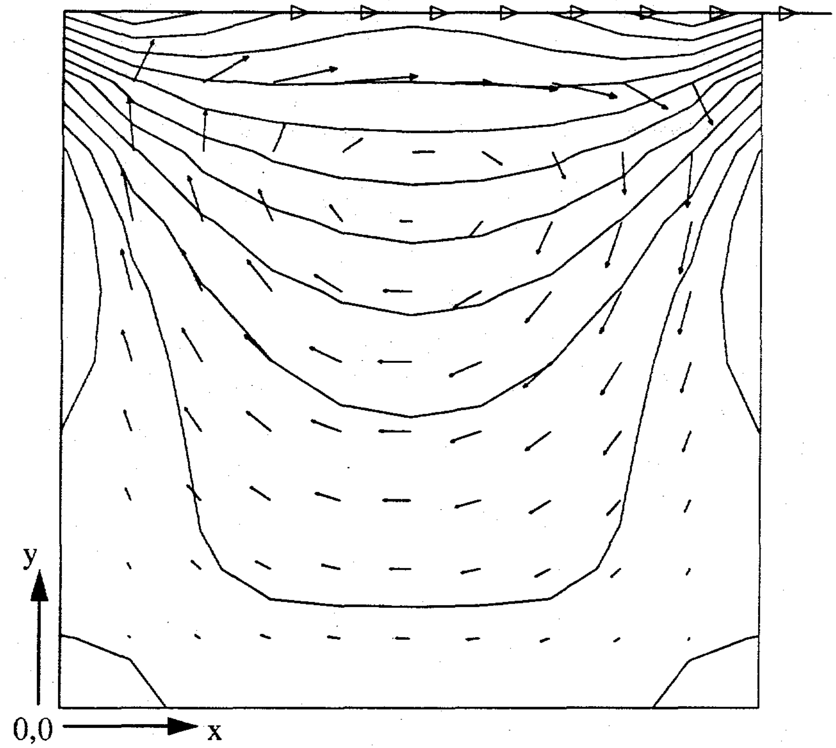

Figure 7 Velocity vectors and vorticity contours for a lid-driven cavity, fora Reynolds number of unity. The cavity is square, each side has unit length, the lid motion is left to right, the grid is 10 X 10, and the center of the primary vortex is $\underset{\sim}{x}_V = (0.5, 0.754)$, which is within the range of previously reported results

The major feature of the flow field in a lid-driven cavity is the recirculation motion shown in Figure 9. (The flow field is generally non-symmetric, except at low Reynolds numbers, and smaller recirculation regions, or Moffatt eddies [18] occur in the corners.) This problem has been used as a test of the ability of Navier-Stokes codes to resolve recirculating motion (see Olson [22],Winters and Cliffe [29], and Ghia, *et al.* [8]. A principal quantitative diagnostic is the location of the center of the largest recirculation region $\underset{\sim}{x}_V = (x_V, y_V)$. As summarized by Olson [22], for a Reynolds number of unity, in a square of unit width and height, the center of the recirculation region lies at $x_V = 0.5$, and $0.794 \geq y_V \geq 0.75$ for discretizations ranging from 10 X 10 to 101 X 101, with no apparent dependence on discretization, for a number of analyses. Using the proposed vorticity formulation with a 10 X 10 grid, the vortex center obtained is $\underset{\sim}{x}_V = (0.5, 0.754)$, which is in the range of previously reported results. Velocity vectors and lines of constant vorticity are shown in Figure 7.

As mentioned earlier, the objective of this discussion is to describe the process of vorticity generation at boundaries. The results shown in Figure 7 were obtained by solving the transient equations until steady-state was reached. The initial condition is

$$\underset{\sim}{u}(\underset{\sim}{x}, 0) = 0 \text{ in the domain (and, implicitly } \underset{\sim}{\omega}(\underset{\sim}{x}, 0) = 0\text{)}$$

and the boundary conditions are

$$\hat{n} \bullet \underset{\sim}{u}_b = u_n = 0 \text{ on all boundaries (impermeable boundaries)}$$

and

$$\hat{\tau} \bullet \underset{\sim}{u}_b = u_\tau = -1 \quad \text{on the lid}$$

and

$$\hat{\tau} \bullet \underset{\sim}{u}_b = 0 \text{ on all other surfaces.}$$

This problem is mathematically ill-posed, as are all impulsively-started problems. To see this, consider Stokes' theorem,

$$\int_A \underset{\sim}{\omega} \bullet \hat{n}_A dA = \oint u_\tau ds \ . \tag{44}$$

The line integral has value -1 since the tangential velocity is -1 over the lid length which has unit length (the tangential velocity is zero elsewhere). In order for the equality in Eq. (44) to hold, vorticity must be added to the fluid. If this problem were being numerically simulated using a velocity-pressure formulation of the Navier-Stokes equations, a vortex sheet would occur on the boundaries of each wall after the first time-step, which would render the problem well-posed. (See Gresho [9] for a description of the vortex sheet that forms in impulsively-started flows.) In a velocity-vorticity formulation, we must explicitly determine the nature of the vortex sheet associated with the initial condition, and in doing so, explain (kinematically) its formation.

As discussed previously, the tangential velocity boundary condition can be viewed as a vortex sheet with strength $\underset{\sim}{\gamma}_{bc}$,

$$\underset{\sim}{\gamma}_{bc} = -\hat{n} \times \underset{\sim}{u}_b = -\hat{n} \times (u_n \hat{n} + u_\tau \hat{\tau}) = -u_\tau \hat{k}\Big|_{u_\tau = -1} = \hat{k} \, .$$

At t = 0, $\underset{\sim}{\gamma}_{bc}$ induces non-zero normal velocities on the side walls of the cavity, thus violating the normal velocity boundary condition, as shown in Figure 8. The addition of $\underset{\sim}{\gamma}_c = -\hat{k}$ on the driven lid, as obtained by solving Eq. (5), induces equal and opposite normal velocities on the cavity side walls. (see APPENDIX B for analysis of a 1 X 1 discretization.) Thus, the normal velocity boundary conditions are satisfied, and the total circulation of the flow is zero, $\underset{\sim}{\gamma}_{bc} + \underset{\sim}{\gamma}_c = 0$, as required. Additionally, with $\underset{\sim}{\gamma}_{bc}$ and $\underset{\sim}{\gamma}_c$ superposed, the velocity in the cavity is zero, which also satisfies the initial condition for velocity.

If not for viscous diffusion, there would be no further evolution of the flow. Viscous diffusion, however, injects circulation from $\underset{\sim}{\gamma}_c$ into the fluid. With the spatial separation of cir-

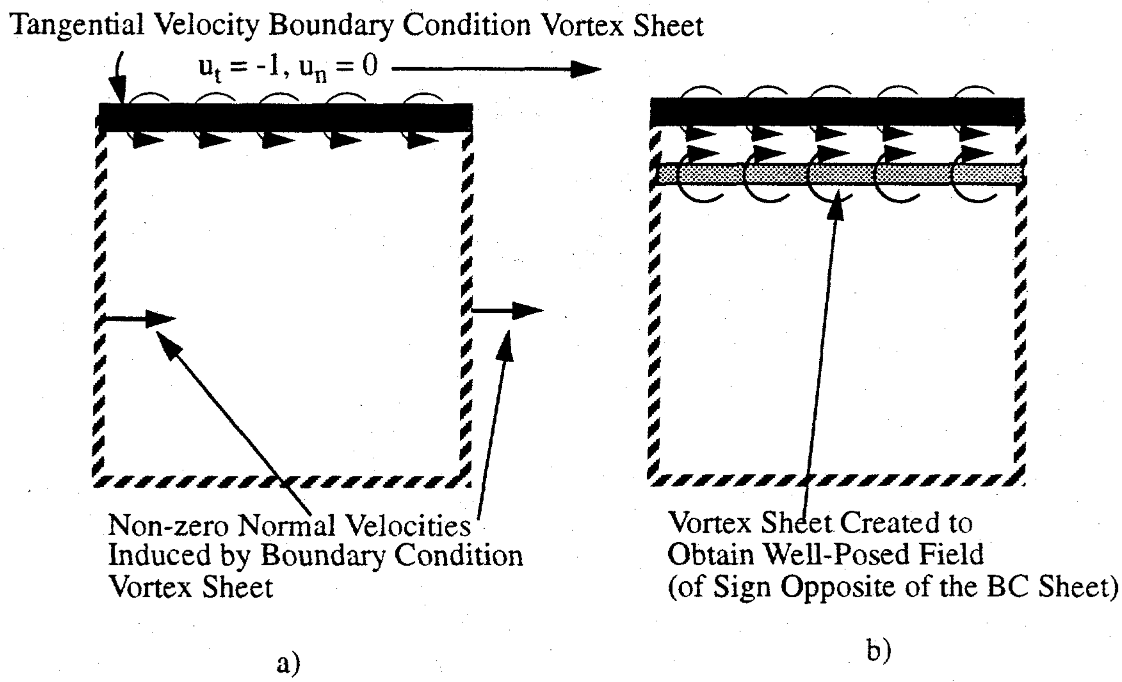

Figure 8 a) Motion induced by the vortex sheet which represents the tangential velocity boundary condition for the lid-driven cavity. The normal velocity induced on the side-walls and lid violates the normal velocity boundary condition, thus making the initial condition ill-posed. b) The additional vortex sheet obtained using the generalized decomposition cancels the motion induced by the boundary condition to obtain a well-posed problem.

culation associated with γ_c from the circulation associated with γ_{bc}, non-zero velocities are generated in the fluid. As a result, the velocity boundary conditions are no longer satisfied (on any of the boundaries). In order to satisfy the boundary conditions, new vortex sheets are solved for on all the boundaries, and are diffused into the domain, as before. In this way, the evolution of the flow is described numerically.

In the course of obtaining the solutions, Stoke's theorem was not satisfied exactly after the new vortex sheets were determined. This is in spite of the fact that Stokes' theorem is satisfied analytically by the generalized decomposition. The non-zero "residual" for Stokes' theorem essentially is an indication of discretization error. In particular, the generalized decomposition is written at the mid-point of each boundary element. As a result, the boundary condition is satisfied at the mid-point of each element, but not necessarily at other points on the element. For example, consider a different problem in a unit square in which the initial condition is that the vorticity field has a value of unity throughout the domain, and all the boundaries are motionless and impermeable. The tangential velocity induced by the vorticity at the midpoint of each side is approximately 0.55, and decreases to

0.37 at the corners. If a 1 X 1 grid is used (the coarsest grid possible), and the evaluation point is the midpoint, then the vortex sheet created on each side satisfies the boundary condition at the evaluation point to within machine precision at the midpoint, but the boundary condition is satisfied to only 1 - 0.37/0.55 = 30% at the corners. With errors this large, the numerical solution overflows at some short time. Of course this error decreases rapidly with finer discretization.

Two approaches are being considered to reduce the discretization error. First, the residual to Stokes' theorem is subtracted on an area-weighted basis from the vortex sheets created at each time step. This approach was used in the results presented here, and appears to yield good results with little computational effort.

Another approach is to require that the boundary condition on each element be satisfied in an average sense. That is, instead of writing one component of

$$\underset{\sim}{u}_b = \underset{\sim}{f}(\underset{\sim}{\omega}, \underset{\sim}{u}_b, \gamma_c)$$

at a point on each boundary element and solving for γ_c on the element, as done in this work, the equation for each element might be one component of a vector function

$$\int_{element} [\underset{\sim}{u}_b - \underset{\sim}{f}(\underset{\sim}{\omega}, \underset{\sim}{u}_b, \gamma_c)]\, ds = 0 .$$

To test this approach, we are writing a Galerkin, weighted residual boundary element formulation. At this early stage, a Galerkin approach appears to have the advantage that the near-singular behavior of the generalized decomposition on boundaries is mollified by the additional integration.

Summary

A generalization of Helmholtz' decomposition is used to formulate velocity boundary conditions for vorticity forms of the incompressible Navier-Stokes equations. The generalized decomposition is obtained by requiring the velocity field outside the fluid domain to be zero. This solution matches a solution to a Laplace equation for the velocity potential. This matching procedure requires the use of singular forms of vorticity and velocity divergence just outside the fluid; *i.e.*, on the boundary of the fluid. The strength of these singularities is specified in terms of the velocity jumps across them. It was shown that it is consistent to specify the velocity on the fluid side of the singular forms as the velocity boundary condition, and the velocity on the non-fluid side of the singular forms as zero. The resulting formulation allows all components of the velocity boundary condition to be specified simultaneously. Moreover, it was shown that a single (normal or tangential) component of the boundary velocity vector is sufficient to determine the vorticity generated in the fluid that satisfies the velocity boundary conditions.

The generalized decomposition also satisfies implicitly the integral constraint that the total circulation in the domain must be zero. Thus, the need to specify the constraint explicitly arises from discretization of the governing equations. This suggests that discrete formulations should be developed to satisfy the integral constraint implicitly, in much the same way the many numerical methods to solve transport equation have been developed to satisfy conservation principles.

The generalized decomposition provides the basis for a no-slip boundary condition in which velocity boundary conditions are satisfied by the creation of vorticity in fluid, adjacent to the boundary. The unknown vorticity is determined from in the form of a vortex sheet, from which a diffusive flux of vorticity into the fluid is determined.

The same formulation can be used to determine inviscid flow fields, wherein the vorticity on the boundary (in the form of vortex sheets) does not diffuse into the fluid, but instead represents a slip velocity.

The use of ALEGRA's Lagrangian step and remap capability to solve the inviscid transport equation for vorticity provides an accurate formulation to describe incompressible transient flows. The high accuracy and ALEGRA's interface reconstruction algorithm is expected to be particularly useful in describing transient instabilities in coating flows.

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APPENDIX A: Analysis of the Generalized Decomposition

To gain a better understanding of how to use the generalized decomposition, consider the divergence and the curl of the decomposition. If the velocity jump on the boundary is $\Delta \mathbf{v}$, then

$$\mathbf{u}(\mathbf{x}) = \nabla \times \int_R \boldsymbol{\omega}(\mathbf{x}')\, G(\mathbf{x}, \mathbf{x}')\, dR(\mathbf{x}')$$

$$\nabla \times \int_S [\hat{n}(\mathbf{x}_b') \times \Delta \mathbf{v}(\mathbf{x}_b')]\, G(\mathbf{x}, \mathbf{x}')\, dS(\mathbf{x}_b') \quad . \tag{45}$$

$$-\nabla \int_S G(\mathbf{x}, \mathbf{x}')\, [\hat{n}(\mathbf{x}_b') \bullet \Delta \mathbf{v}(\mathbf{x}_b')]\, dS(\mathbf{x}_b')$$

Upon taking the divergence (where ∇ operates only on functions of $\mathbf{x}$, not functions of $\mathbf{x}'$), the first two integrals are zero identically, yielding

$$\nabla \bullet \ \mathbf{u}(\mathbf{x}) = \int_S \nabla^2 G(\mathbf{x}, \mathbf{x}')\, [\hat{n}(\mathbf{x}_b') \bullet \Delta \mathbf{v}(\mathbf{x}_b')]\, dS(\mathbf{x}_b') \tag{46}$$

The Green's function is defined so that its Laplacian is,

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \tag{47}$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function. For $\mathbf{x}$ restricted to the fluid domain (not including the boundary), the delta function is zero, so

$$\nabla \bullet \mathbf{u} = 0 . \tag{48}$$

To satisfy the divergence theorem,

$$\int_R \nabla \bullet \mathbf{u}\, dR = \int_S \hat{n} \bullet \mathbf{u}\, dS \ . \tag{49}$$

for incompressible flows, the condition

$$\int_S [\hat{n} \bullet \Delta \underset{\sim}{v}]\, dS = 0 , \tag{50}$$

must be satisfied.

Now, consider the curl of Eq. (45) for the purpose of ensuring that $\nabla \times \underset{\sim}{u} = \underset{\sim}{\omega}$, which is necessary for consistency. Using the identity (for any vector $\underset{\sim}{A}$)

$$\nabla \times (\nabla \times \underset{\sim}{A}) = -\nabla^2 \underset{\sim}{A} + \nabla (\nabla \bullet \underset{\sim}{A}) \tag{51}$$

the curl of Eq. (45) can be expressed as,

$$\begin{aligned}
&\nabla \times \underset{\sim}{u} = \\
&-\int_R \underset{\sim}{\omega}(\underset{\sim}{x}')\, \nabla^2 G(\underset{\sim}{x}, \underset{\sim}{x}')\, dR(\underset{\sim}{x}') + \nabla \int_R \nabla \bullet \{ \underset{\sim}{\omega}(\underset{\sim}{x}')\, G(\underset{\sim}{x}, \underset{\sim}{x}') \}\, dR(\underset{\sim}{x}') \\
&-\int_S [\hat{n}(\underset{\sim}{x}_b') \times \Delta \underset{\sim}{v}(\underset{\sim}{x}_b')]\, \nabla^2 G(\underset{\sim}{x}, \underset{\sim}{x}'_b)\, dS(\underset{\sim}{x}_b') \\
&+\nabla \int_S \nabla \bullet \{ [\hat{n}(\underset{\sim}{x}_b') \times \Delta \underset{\sim}{v}(\underset{\sim}{x}_b')]\, G(\underset{\sim}{x}, \underset{\sim}{x}'_b) \}\, dS(\underset{\sim}{x}_b')
\end{aligned} \tag{52}$$

where the curl of the source (normal velocity) integral is zero identically. Using Eq. (47), in the first and third integrals, and the divergence theorem on the second integral,

$$\begin{aligned}
&\nabla \times \underset{\sim}{u} = \\
&\underset{\sim}{\omega}(\underset{\sim}{x}) - \nabla \int_S [\underset{\sim}{\omega}(\underset{\sim}{x}') \bullet \hat{n}]\, G(\underset{\sim}{x}, \underset{\sim}{x}')\; dS(\underset{\sim}{x}') \\
&-\nabla \int_S \nabla \bullet ([\hat{n}(\underset{\sim}{x}_b') \times \Delta \underset{\sim}{v}(\underset{\sim}{x}_b')])\, G(\underset{\sim}{x}, \underset{\sim}{x}'_b)\, dS(\underset{\sim}{x}_b')
\end{aligned} \tag{53}$$

The integrand in the last integral is the divergence of a vortex sheet of strength, $\underset{\sim}{\gamma} = \hat{n} \times \Delta \underset{\sim}{v}$, which is zero identically. The next to last term is zero since $\underset{\sim}{\omega} = 0$ outside the fluid domain, which implies that no vorticity crosses the boundary, or equivalently, $\underset{\sim}{\omega} \bullet \hat{n} = 0$ on the boundary.

Thus, the result of this analysis is that

$$\nabla \times \underset{\sim}{u} = \underset{\sim}{\omega} . \tag{54}$$

Boundary conditions that are compatible with the vorticity field must satisfy the integral identity,

$$\int_V \nabla \times \underset{\sim}{u} dV = \int_S \hat{n} \times \underset{\sim}{u} dS , \tag{55}$$

or, for system of interest,

$$\int_V \underset{\sim}{\omega} dV = \int_S \hat{n} \times \Delta \underset{\sim}{v} dS . \tag{56}$$

The conclusion to be drawn from this analysis is that the decomposition Eq. (45) satisfies $\nabla \bullet \underset{\sim}{u} = 0$ and $\nabla \times \underset{\sim}{u} = \underset{\sim}{\omega}$ independent of the velocity boundary conditions. As a result, the velocity boundary conditions must satisfy certain constraints, as given by equations (50) and (56).

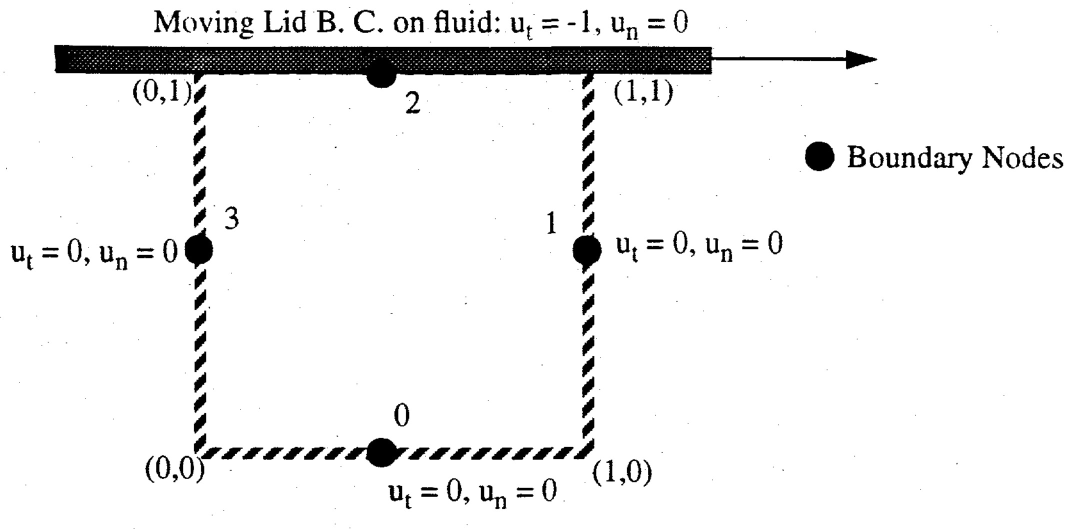

Figure 9 Schematic of a driven-lid cavity. The top of the cavity moves from left to right, imparting motion to the fluid via the no-slip boundary condition. The tangential velocity on all other boundaries is zero. In addition, the normal velocity is zero on all boundaries.

APPENDIX B: Example of Vorticity Creation: Lid-Driven Cavity

As an example of how Eq. is used to generate vorticity on boundaries, consider the flow in a lid-driven cavity, as shown in Figure 9. The lid of the cavity is impulsively started, and is the only source of motion in the fluid. The initial condition for the fluid is

$$\underset{\sim}{u}(\underset{\sim}{x}, 0) = 0 \text{ in the domain (and, implicitly } \underset{\sim}{\omega}(\underset{\sim}{x}, 0) = 0)$$

and the boundary conditions are

$$\hat{n} \bullet \underset{\sim}{u}_b = u_n = 0 \text{ on all boundaries (impermeable boundaries)}$$

and

$$\hat{\tau} \bullet \underset{\sim}{u}_b = u_\tau = -1 \quad \text{on the lid}$$

and

$$\hat{\tau} \bullet \underset{\sim}{u}_b = 0 \text{ on all other surfaces.}$$

As stated, this problem is mathematically ill-posed. To see this, consider Stokes' theorem,

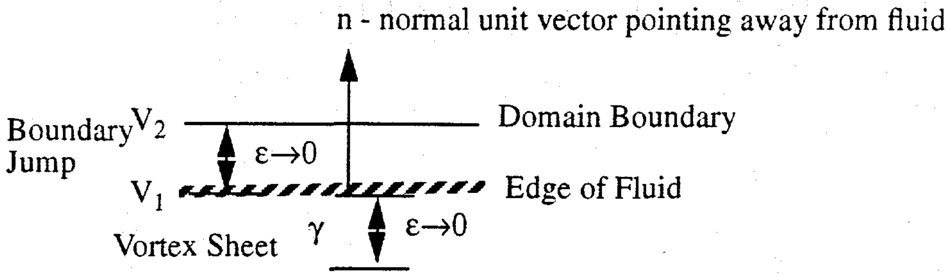

Figure 10 Configuration of boundary velocity jump and vortex sheet near the boundary of the cavity lid. The vortex sheet lies in the fluid, whereas the boundary jump occurs in the zero thickness region between the edge of the fluid and the boundary.

$$\int_A \underset{\sim}{\omega} \bullet \hat{n}_A dA = \oint u_\tau ds \ . \tag{57}$$

The line integral has value -1 since the tangential velocity is -1 over the lid length which has unit length. In order for the equality in Eq. (57) to hold, vorticity must be added to the fluid. If this problem were being numerically simulated using a velocity-pressure formulation of the Navier-Stokes equations, a vortex sheet would occur on the boundaries of each wall after the first time-step, which would render the problem well-posed. In a velocity-vorticity formulation, we must explicitly determine the nature of the vortex sheet associated with the initial condition.

Eq. is evaluated for $\underset{\sim}{V}_{S+} = 0$, so that $\Delta \underset{\sim}{v} = 0 - \underset{\sim}{V}_{S-} = -\underset{\sim}{V}_{S-}$. The objective of this analysis is to determine the strength of the sheet on each surface of the cavity. The configuration of the velocity jump $\Delta \underset{\sim}{v}$ and the vortex sheet is shown in Figure 10. Performing the curl and gradient operations, setting $\hat{n} \times (-\underset{\sim}{V}_{S-}) = -(\hat{\tau} \bullet \underset{\sim}{V}_{S-})\hat{k} = -u_\tau \hat{k}$ where u_τ is the tangential velocity boundary condition, and setting $\underset{\sim}{\omega} = 0$ (the initial condition in the domain), yields the operational equation to determine the strength of vortex sheets,

$$
\begin{aligned}
\frac{1}{2}\underset{\sim}{u}_b(\underset{\sim}{x}_b) &= -\hat{n}(\underset{\sim}{x}_b)\times\frac{1}{2}\underset{\sim}{\gamma}_{cb}(\underset{\sim}{x}_b) \\
&+\frac{1}{2\pi}\int_S \frac{\gamma_{cb}(\underset{\sim}{x}_b')\left[(x_b-x_b')\hat{j}-(y_b-y_b')\hat{i}\right]}{\left|\underset{\sim}{x}_b-\underset{\sim}{x}_b'\right|^2}dS(\underset{\sim}{x}_b') \\
&+\frac{1}{2\pi}\int_S \frac{-u_\tau(\underset{\sim}{x}_b')\left[(x_b-x_b')\hat{j}-(y_b-y_b')\hat{i}\right]}{\left|\underset{\sim}{x}_b-\underset{\sim}{x}_b'\right|^2}dS(\underset{\sim}{x}_b') \\
&+\frac{1}{2\pi}\int_S \frac{-u_n(\underset{\sim}{x}_b')\left[(x_b-x_b')\hat{i}+(y_b-y_b')\hat{j}\right]}{\left|\underset{\sim}{x}_b-\underset{\sim}{x}_b'\right|^2}dS(\underset{\sim}{x}_b')
\end{aligned}
\qquad (58)
$$

A crude discretization is assumed to allow hand calculation of the vortex sheet strengths. Each vortex sheet is assumed to be uniform over each surface of the cavity, and represented by a constant boundary element whose node is at the mid point of each cavity surface, as shown in Figure 9. For this coarse discretization and a single point quadrature, the discrete terms in the integrand of Eq. (58) have the values

$$
\frac{(x-x')}{\left|\underset{\sim}{x}_b-\underset{\sim}{x}_b'\right|^2}\Delta S = \pm 1 \text{ and } \frac{(y-y')}{\left|\underset{\sim}{x}_b-\underset{\sim}{x}_b'\right|^2}\Delta S = \pm 1,
$$

with the + or - depending on the two points being considered.

First, the normal component of Eq. (58) is considered at each of the node points, 0 - 3 (see Figure 9).

$$
\begin{aligned}
\frac{1}{2}u_{n,0}(\underset{\sim}{x}_b) &= \hat{n}(\underset{\sim}{x}_b)\bullet\frac{1}{2\pi}\int_S \frac{\gamma_{cb}(\underset{\sim}{x}_b')\left[(x_b-x_b')\hat{j}-(y_b-y_b')\hat{i}\right]}{\left|\underset{\sim}{x}_b-\underset{\sim}{x}_b'\right|^2}dS(\underset{\sim}{x}_b') \\
&+\hat{n}(\underset{\sim}{x}_b)\bullet\frac{1}{2\pi}\int_S \frac{-u_\tau(\underset{\sim}{x}_b')\left[(x_b-x_b')\hat{j}-(y_b-y_b')\hat{i}\right]}{\left|\underset{\sim}{x}_b-\underset{\sim}{x}_b'\right|^2}dS(\underset{\sim}{x}_b') \\
&+\hat{n}(\underset{\sim}{x}_b)\bullet\frac{1}{2\pi}\int_S \frac{-u_n(\underset{\sim}{x}_b')\left[(x_b-x_b')\hat{i}+(y_b-y_b')\hat{j}\right]}{\left|\underset{\sim}{x}_b-\underset{\sim}{x}_b'\right|^2}dS(\underset{\sim}{x}_b') \\
&+\hat{n}(\underset{\sim}{x}_b)\bullet\left[-\hat{n}(\underset{\sim}{x}_b)\times\frac{1}{2}\underset{\sim}{\gamma}_{cb}(\underset{\sim}{x}_b)\right]
\end{aligned}
\qquad (59)
$$

Denoting a quantity at node *i* with the subscript *i*, Eq. (59) evaluated at node 0 (the bottom of the cavity, x = 1/2, y = 0), is

$$\frac{1}{2}u_{n,0} = \gamma_0(0) + \gamma_1(1) + \gamma_2(0) + \gamma_3(-1)$$
$$+u_{\tau,0}(0) + u_{\tau,1}(1) + u_{\tau,2}(0) + u_{\tau,3}(-1)$$
$$+u_{n,1}(-1) + u_{n,2}(0) + u_{n,3}(-1)$$

Note that although this is the normal component of Eq. (58), both normal and tangential velocity boundary conditions are included. This aspect is an advantage over other approaches in which normal and tangential boundary conditions are specified serially, resulting in the so-called over-specification problem wherein (in two-dimensions) there are two boundary conditions, but only one component of vorticity to satisfy them. In the present formulation, normal and tangential velocity boundary conditions are specified simultaneously, thus overcoming the over-specification problem.

Returning to the problem at hand, the only non-zero velocity on the boundary is the lid velocity $u_{\tau,2}$, thus the above equation simplifies to

$$\text{Node 0: } 0 = \gamma_1 - \gamma_3 . \tag{60}$$

Evaluating Eq. (58) at the other three node points results in the following equations.

$$\text{Node 1: } \quad u_\tau = -\gamma_0 + \gamma_2$$

$$\text{Node 2: } 0 = -\gamma_1 + \gamma_3$$

$$\text{Node 3: } \quad -u_\tau = \gamma_0 - \gamma_2$$

Notice that the equations for nodes 0 and 2 are linearly dependent, as are the equations for nodes 1 and 3. Thus, γ_i cannot be determined from these equations. If this set of equations is supplemented with a discrete form of Stokes' theorem Eq. (57)

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = u_\tau \tag{61}$$

the rank of the system increases to 3, but a rank of 4 is needed to determine γ_i. The normal velocity constraint Eq. (50) is satisfied implicitly, thus it offers no help.

Having failed at finding γ_i from the normal component of Eq. on the boundary, a similar procedure is attempted with the tangential component of Eq. on the boundary,

$$\frac{1}{2}u_\tau(\underset{\sim}{x}_b) = \hat{\tau}(\underset{\sim}{x}_b) \bullet \frac{1}{2\pi}\int_S \frac{\gamma(\underset{\sim}{x}_b')\left[(x_b - x_b')\hat{j} - (y_b - y_b')\hat{i}\right]}{\left|\underset{\sim}{x}_b - \underset{\sim}{x}_b'\right|} dS(\underset{\sim}{x}_b')$$

$$+\hat{\tau}(\underset{\sim}{x}_b) \bullet \frac{1}{2\pi}\int_S \frac{-u_\tau(\underset{\sim}{x}_b')\left[(x_b - x_b')\hat{j} - (y_b - y_b')\hat{i}\right]}{\left|\underset{\sim}{x}_b - \underset{\sim}{x}_b'\right|} dS(\underset{\sim}{x}_b') \tag{62}$$

$$+\frac{1}{2}\underset{\sim}{\gamma}_c(\underset{\sim}{x}_b)$$

Multiplying both sides of the equation by 2π, the equations at the nodes are:

Node 0: $u_\tau = \pi\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$

Node 1: $u_\tau = \gamma_0 + \pi\gamma_1 + \gamma_2 + \gamma_3$

Node 2: $\pi u_\tau = \gamma_0 + \gamma_1 + \pi\gamma_2 + \gamma_3$

Node 3: $u_\tau = \gamma_0 + \gamma_1 + \gamma_2 + \pi\gamma_3$

Note that if the normal velocity boundary conditions were non-zero, they would be included in this equation set, so that the solution would satisfy both the normal and the tangential velocity boundary conditions.

In matrix form, the set of linear equations is,

$$\begin{bmatrix} \pi & 1 & 1 & 1 \\ 1 & \pi & 1 & 1 \\ 1 & 1 & \pi & 1 \\ 1 & 1 & 1 & \pi \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} u_\tau \\ u_\tau \\ \pi u_\tau \\ u_\tau \end{bmatrix} \tag{63}$$

The solution to Eq. (63) is

$$\gamma_0 = \gamma_1 = \gamma_3 = 0, \gamma_2 = u_\tau \tag{64}$$

This solution specifies that a vortex sheet lies just inside the boundary of the fluid adjacent to the lid of the cavity. The velocity induced by the vortex sheet cancels the motion imposed the motion of the lid, so as to obtain the initial condition that the fluid is initially motionless.

This solution also satisfies the normal component equations, Eqns. (60), and Stokes' theorem, Eq. (61), thus providing confidence that the overall formulation is correct. Note that if Stokes' theorem is used to replace any of the equations, incorrect answers are obtained.

The solution Eq. (64) also represents the situation if the flow was inviscid. Our interest is in viscous flows, however, in which the vortex sheet diffuses (as a result of viscosity) into the domain. Essentially, vorticity diffuses into the domain and then must be considered in the area integral of vorticity in the domain. New vortex sheet strengths can then be calculated as responses to the vorticity in the domain and the velocity boundary conditions. These new vortex sheets will generally have non-zero strengths, and as they diffuse into the domain, they correspond directly to the aforementioned vortex sheets observed in numerical simulations of the velocity-pressure form of the Navier-Stokes equations.

To continue the simulation, the vorticity in the domain would be transported according the vorticity form of the Navier-Stokes equations, which we consider in fractional steps of inviscid transport (which is specified by solving $d\underset{\sim}{x}/dt = \underset{\sim}{u}$), and viscous transport, including the diffusion of the new vortex sheets into the domain. In this way, the time evolution of the vorticity field, and the velocity field in the cavity is simulated.

APPENDIX C: Calculation of Coefficients on the Boundary

The general expression for a velocity field in terms of its vorticity field and velocity boundary conditions is,

$$c(\underset{\sim}{x})\,\underset{\sim}{u}(\underset{\sim}{x}) = \int_A \frac{\underset{\sim}{\omega}(\underset{\sim}{x}') \times (\underset{\sim}{x} - \underset{\sim}{x}')}{2\pi|\underset{\sim}{x} - \underset{\sim}{x}'|^2} dA(\underset{\sim}{x}') \tag{65}$$

$$+ \int_S \frac{-[\underset{\sim}{u}_b(\underset{\sim}{x}_b') \bullet \hat{n}(\underset{\sim}{x}_b')](\underset{\sim}{x} - \underset{\sim}{x}_b') + [\underset{\sim}{u}_b(\underset{\sim}{x}_b') \times \hat{n}(\underset{\sim}{x}_b')] \times (\underset{\sim}{x} - \underset{\sim}{x}_b')}{2\pi|\underset{\sim}{x} - \underset{\sim}{x}_b'|^2} dS(\underset{\sim}{x}_b') \quad .$$

In general, c is a tensor. In the domain, $c_{ij} = 1$. Outside the domain, $c_{ij} = 0$. On the boundary, c_{ij} depends only on the geometry of the boundary. In x-y coordinates,

$$c\underset{\sim}{u} = \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} . \tag{66}$$

The values of c_{ij} in Eq. (66) are determined in the section that follows. In normal-tangential coordinates,

$$c\underset{\sim}{u} = \begin{bmatrix} c_{nn} & c_{n\tau} \\ c_{\tau n} & c_{\tau\tau} \end{bmatrix} \begin{bmatrix} u_n \\ u_\tau \end{bmatrix} , \tag{67}$$

In section 2, the values of each component in Eq. (67) are determined in terms of the coefficients in Eq. (66).

It is shown that

$$c_{xx} = c_{yy} = c_{nn} = c_{\tau\tau} = -\int_S \frac{\hat{n}(\underset{\sim}{x}_b') \bullet (\underset{\sim}{x}_b - \underset{\sim}{x}_b')}{2\pi|\underset{\sim}{x}_b - \underset{\sim}{x}_b'|^2} dS(\underset{\sim}{x}_b') \quad \text{(1/2 on smooth surfaces)}$$

and

$$c_{yx} = -c_{xy} = c_{\tau n} = -c_{n\tau} = -\int_S \frac{\hat{k} \bullet [\hat{n}(\underset{\sim}{x}_b') \times (\underset{\sim}{x}_b - \underset{\sim}{x}_b')]}{2\pi|\underset{\sim}{x}_b - \underset{\sim}{x}_b'|^2} dS(\underset{\sim}{x}_b')$$

(0 on smooth surfaces)

1. Determination of c_{ij} in x-y Coordinates

c_{ij} are independent of the velocity field. Thus, to determine c_{ij}, any flow field can be considered, but a uniform flow is particularly useful. In a uniform flow, there is zero vorticity,

$$\underset{\sim}{\omega} = 0. \tag{68}$$

The x-y components of velocity and the normal unit vector are,

$$\underset{\sim}{u}_b = \hat{i}u_x + \hat{j}u_y \tag{69}$$

$$\hat{n} = \hat{i}n_x + \hat{j}n_y \tag{70}$$

Substituting (68), (69), and (70) into Eq. (65) yields

$$\begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \int_S \frac{-u_x\eta + u_y\zeta}{2\pi|\underset{\sim}{x}_b - \underset{\sim}{x}_b'|^2} dS\,(\underset{\sim}{x}_b') \\ \int_S \frac{-u_x\zeta - u_y\eta}{2\pi|\underset{\sim}{x}_b - \underset{\sim}{x}_b'|^2} dS\,(\underset{\sim}{x}_b') \end{bmatrix} \tag{71}$$

where

$$\eta = n_x(x - x') + n_y(y - y') = \hat{n} \bullet (\underset{\sim}{x} - \underset{\sim}{x}') \tag{72}$$

$$\zeta = n_x(y - y') - n_y(x - x') = \hat{k} \bullet [\hat{n} \times (\underset{\sim}{x} - \underset{\sim}{x}')] \tag{73}$$

For $u_x = 1$ and $u_y = 0$, the x-component of Eq. (71) reduces to

$$c_{xx} = -\int_S \frac{\hat{n}(\underset{\sim}{x}_b') \bullet (\underset{\sim}{x}_b - \underset{\sim}{x}_b')}{2\pi|\underset{\sim}{x}_b - \underset{\sim}{x}_b'|^2} dS\,(\underset{\sim}{x}_b') \tag{74}$$

and the y-component of Eq. (71) reduces to

$$c_{yx} = -\int_S \frac{\hat{k} \bullet [\hat{n}(\underset{\sim}{x}_b') \times (\underset{\sim}{x}_b - \underset{\sim}{x}_b')]}{2\pi|\underset{\sim}{x}_b - \underset{\sim}{x}_b'|^2} dS\,(\underset{\sim}{x}_b') \tag{75}$$

For $u_x = 0$ and $u_y = 1$, the x-component of Eq. (71) reduces to

$$c_{yy} = c_{xx} \tag{76}$$

and the y-component of Eq. (71) reduces to

$$c_{xy} = -c_{yx}. \tag{77}$$

2. Determination of c_{ij} in Normal-Tangential Coordinates

The results of section 1 will be transformed into normal-tangential coordinates. First, the x-y components of velocity in Eq. (71) are represented in terms of normal and tangential velocity components,

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} \begin{bmatrix} u_n \\ u_\tau \end{bmatrix}. \tag{78}$$

to obtain the vector,

$$\begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \left\{ \begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} \begin{bmatrix} u_n \\ u_\tau \end{bmatrix} \right\}. \tag{79}$$

This vector is still in x-y coordinates; *e.g.*, the x-component of Eq. (79) is $c_{xx}[n_x u_n - n_y u_\tau] + c_{xy}[n_y u_n + n_x u_\tau]$. Accordingly, the normal and tangential components of Eq. (79) are obtained by operating on it with the tensor

$$\begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix} \tag{80}$$

to obtain

$$\begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix} \left\{ \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \left\{ \begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} \begin{bmatrix} u_n \\ u_\tau \end{bmatrix} \right\} \right\} . \tag{81}$$

The normal component of this vector is

$$\begin{aligned} & n_x [c_{xx}(n_x u_n - n_y u_\tau) + c_{xy}(n_y u_n + n_x u_\tau)] \\ & + n_y [c_{yx}(n_x u_n - n_y u_\tau) + c_{xy}(n_y u_n + n_x u_\tau)] \end{aligned} \tag{82}$$

According to Eq. (67), the coefficients of u_n and u_τ are c_{nn} and $c_{n\tau}$. Rearranging Eq. (82) yields,

$$u_n\,[n_x\,(n_x c_{xx} + n_y c_{xy}) + n_y\,(n_x c_{yx} + n_y c_{yy})\,] \tag{83}$$

$$u_\tau\,[n_x\,(-n_y c_{xx} + n_x c_{xy}) + n_y\,(-n_y c_{yx} + n_x c_{yy})$$

Using $c_{xx} = c_{yy}$ and $c_{xy} = -c_{yx}$, Eq. (83) becomes

$$u_n\left[\,(n_x^2 + n_y^2)\,c_{xx} + (n_x n_y - n_x n_y)\,c_{xy}\right] \tag{84}$$

$$+\;u_\tau\left[n_x n_y\,(c_{xx} - c_{xx}) + (n_x^2 + n_y^2)\,c_{xy}\right]$$

Noting that $n_x^2 + n_y^2 = 1$, the normal component of the velocity vector is

$$u_n c_{xx} + u_\tau c_{xy}$$

which indicates that $c_{nn} = c_{xx}$ and $c_{n\tau} = c_{xy}$. Similar analysis of the tangential component of the vector yields $c_{\tau\tau} = c_{yy}\ \ (= c_{xx})$ and $c_{\tau n} = c_{yx}\ \ (= -c_{xy})$.

APPENDIX D: Viscous Diffusion of Vortex Sheets into the Fluid

In a viscous fluid, the (zero-thickness) vortex sheet $\underset{\sim}{\gamma}_c$ can be used to introduce new vorticity into the fluid. The approach described below is also used by Koumoutsakos, *et al.* [16]. Recall the definition of a vortex sheet,

$$\underset{\sim}{\gamma} = \lim_{\substack{\omega_\tau \to \infty \\ dn \to 0}} \underset{\sim}{\omega}_\tau dn.$$

Essentially, the amount of vorticity $\Delta\underset{\sim}{\omega}_c$ created at a boundary will be represented as a vortex sheet that is expanded from zero thickness to a finite thickness Δn over a time interval Δt. The equation

$$\Delta\underset{\sim}{\omega}_c = \underset{\sim}{\gamma}/\Delta n \tag{85}$$

relates the vortex sheet strength to the vorticity introduced within the fluid. One choice for Δn is the approximate distance a quantity will diffuse over a time interval Δt, which is $\Delta n \approx \sqrt{\nu \Delta t}$, where ν is the kinematic viscosity. Another choice could be the height of the discrete elements or cells adjacent to the boundary into which vorticity will enter. As will be shown, one need not choose Δn, since the final result is independent of it.

The addition of vorticity to the fluid can be specified as a flux of vorticity from the boundary into a volume

$$V = A\Delta n \tag{86}$$

where A is the surface over which the flux is applied.

This balance of vorticity is given by (the normal is assumed to point outward),

$$\int_t^{(t+\Delta t)} [\nu(\hat{n} \bullet \nabla)\omega_\tau A]\, dt = \Delta\underset{\sim}{\omega}_{cb} V. \tag{87}$$

In discrete form,

$$\nu(\hat{n} \bullet \nabla)\omega_\tau A\Delta t = \Delta\underset{\sim}{\omega}_{cb} V \tag{88}$$

Using Eq. (85), Eq. (86), and Eq. (87), the normal gradient of vorticity at the boundary is

$$(\hat{n} \bullet \nabla)\omega_\tau = \frac{1}{\nu}\frac{\underset{\sim}{\gamma}_c}{\Delta t} \tag{89}$$

To interpret this, consider that $\underset{\sim}{\gamma}_c$ represents a deviation from the desired tangential velocity on the boundary, $\underset{\sim}{\gamma}_c = \hat{n} \times \delta \underset{\sim}{u}_{slip}(t)$, at the time t. The tangential velocity boundary condition will be satisfied, however, by diffusing the vorticity into the domain over the time interval Δt, so that $\delta \underset{\sim}{u}_{slip}(t + \Delta t) = 0$. Or,

$$\frac{\delta \underset{\sim}{u}_{slip}(t+\Delta t) - \delta \underset{\sim}{u}_{slip}(t)}{\Delta t} \approx \frac{0 - \underset{\sim}{\gamma}_c}{\Delta t} \approx \frac{d(\delta \underset{\sim}{u}_{slip})}{dt} \tag{90}$$

That is, $\underset{\sim}{\gamma}_c / \Delta t$ in Eq. (89) is an approximation for the time-rate of change of the deviation from the tangential velocity boundary condition. To summarize, the flux of vorticity into the domain can be specified using the vortex sheet strength, as given by Eq. (89).

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