A GENERALIZED KOLMOGOROV-VON KARMAN RELATION AND SOME FURTHER IMPLICATIONS ON THE MAGNITUDE OF THE CONSTANTS*

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The relation between the Kolmogorov and von Karman constants appropriate to the special conditions of neutrally stratified and locally dissipating flow previously given[1,2] is essentially a straightforward combination of the logarithmic wind profile, the one-dimensional spectral relation for turbulent energy density in the inertial subrange, and a reduced turbulent energy equation that balances the dissipation rate with a mechanical production term alone. This note generalizes the derivation by introducing:

a) the stability-dependent, dimensionless wind shear,

$$\phi_m \equiv \frac{kz}{u_*} \frac{d\bar{u}}{dz} \; ; \tag{1}$$

b) the diabatic wind profile (an integral of the above),

$$\bar{u}(z_2) - \bar{u}(z_1) = \frac{u_*}{k} \left[\ln z_2/z_1 - \psi_m(z_2/L) + \psi_m(z_1/L)\right] \; ; \tag{2}$$

and c) the complete energy equation which can be written in abbreviated form

$$\epsilon = \pi_T + \pi_B - D = \beta\pi_T \; . \tag{3}$$

In the last above, π_T and π_B are the mechanical and buoyant turbulent energy production rates, respectively, while the term $-D$ represents a net, apparent local production of energy effected by the (negative) divergence of the turbulent and pressure transport terms, that is

$$-D \equiv -\frac{\partial}{\partial z}\left(\overline{w'E'} + \frac{\overline{w'p'}}{\rho}\right) \; .$$

The parameter β is defined by

$$(\pi_B - D) \equiv (1 - \beta)\pi_T \; , \text{ and}$$

the remaining notation is more or less standard.

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Equations 1, 2, and 3 are combined as before with an integral of the one-dimensional, energy-density spectrum in the inertial subrange, namely

$$\int_{n}^{2n} E(n)dn \equiv \Sigma E = 0.555\alpha_1 \left[\frac{\epsilon\bar{u}}{n}\right]^{2/3} , \tag{4}$$

to obtain the general Kolmogorov-von Karman relation:

$$\alpha_1 k^{4/3} = \left[\frac{\Sigma E}{0.555}\right]\left[\frac{nz}{\beta\phi_m\bar{u}}\right]^{2/3}\left[\frac{\ln z_2/z_1 - \psi_m(z_2/L) + \psi_m(z_1/L)}{\bar{u}(z_2) - \bar{u}(z_1)}\right]^2 . \tag{5}$$

Comparison with the earlier version for the special case shows that the three stability and divergence-dependent terms (formally only two, since $\psi_m = f(\phi_m)$) introduced by the generalized derivation can be grouped in a single multiplying factor, say γ, where

$$\gamma \equiv (\beta\phi_m)^{-2/3}\left[1 - \frac{\psi_m(z_2/L) - \psi_m(z_1/L)}{\ln z_2/z_1}\right]^2 . \tag{6}$$

Substitution in Eq. 5 of the experimental measurements previously reported gives the numerical equation

$$\alpha_1 k^{4/3} = 0.1407\gamma . \tag{7}$$

In the earlier discussion, the γ factor was, in effect, assumed to be unity. The value of the K-von K product thus defined was shown to imply that $k = 0.36$, using the widely accepted value, $\alpha_1 = 0.55$. But such suggestions for a smaller value of k are in question (e.g., Garratt, 1974); therefore, this note examines the measurements summarized in Eq. 7 in the light of the possibility that $\gamma \neq 1.00$.

During the field experiment in question, the conditions were such that the atmospheric stratification might have been slightly unstable; that is, ϕ_m may have been < 1.00. On the other hand, the presence of both strong winds (and hence strong turbulent mixing) and some weak solar heating rules out the stable case. In these circumstances, some small departures from local dissipation on the side of increased local turbulence production ($\pi_B - D > 0$, hence $\beta > 1.00$) might also be expected. In this regard, it has been observed[4] that in unstable conditions, the additional production of turbulent energy by buoyancy tends to be approximately balanced by increased divergence of the turbulent transport term $(\overline{w'E'})$. However, a net imbalance in the form of an apparent positive local energy production still remains, evidently due to increased convergence of the pressure transport term, $(\overline{w'p'}/\rho)$. At any rate, as shown in the following table, the stability factor γ steadily falls off from unity over the admissible ranges of β and ϕ_m.

Table 1. The variation of the stability/divergence factor γ in conditions of weak instability ($\pi_B > 0$) and/or positive local turbulence accumulation ($-D > 0$); values of ϕ_m and ψ_m for Eq. 6 from empirical formulae after Hicks.[6]

β \ ϕ_m	1.00	0.95	0.83
1.00	1.00	0.988	0.970
1.03	0.981	0.969	0.951
1.05	0.968	0.957	0.939

It follows that, if any significant departure from neutrality and/or local dissipation did occur during these field experiments, the factor γ would have become less than unity, and the magnitude of the K-von K product would have been reduced accordingly. Consequently, the product value reported earlier, $\alpha_1 k^{4/3} = 0.141$, could not on these grounds be considered to be anything but an overestimate. As a result the inferred value of $k \simeq 0.36$ will only be made smaller by correcting for the possible effects of either weak instability or small pressure transport convergence, or both.

It further follows that to preserve the canonical value of $k = 0.4$, only two possibilities for further adjustment of the K-von K argument remain: either

1) the measured value of one or more factors in the K-von K product was indeed too small due to some as yet undetected source of experimental error, or

2) the value of the Kolmogorov constant used to estimate k, namely $\alpha_1 = 0.55$, is incorrect.

The first of these possibilities is not considered likely since an excessive experimental error of 15 percent would be required; $\alpha_1 = 0.55$ and $k = 0.40$ gives $\alpha_1 k^{4/3} = 0.162$. However, additional experiments designed to explore this possibility will be conducted during the joint field program of the International Turbulence Comparison Experiment to be conducted in Australia in October.

The second possibility may now perhaps be considered a probability. Not too many years ago, α_1 values in the range 0.46 to 0.50 were considered appropriate. Using the observed value $\alpha_1 k^{4/3} = 0.141$ (with $\gamma = 1.00$) and $k = 0.4$, we obtain $\alpha_1 = 0.48$. Moreover, two recent field measurements have again obtained values of α_1 in this range: Williams and Paulson[5] find 0.50, and Friehe (personal communication) reports $\alpha_1 = 0.51$. Again, accurate determination of the Kolmogorov constants is one of the goals of the forthcoming International Turbulence Comparison Experiment.

References

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