

Causal inheritance in plane wave quotients

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Abstract

We investigate the appearance of closed timelike curves in quotients of plane waves along spacelike isometries. First we formulate a necessary and sufficient condition for a quotient of a general spacetime to preserve stable causality. We explicitly show that the plane waves are stably causal; in passing, we observe that some pp-waves are not even distinguishing. We then consider the classification of all quotients of the maximally supersymmetric ten-dimensional plane wave under a spacelike isometry, and show that the quotient will lead to closed timelike curves iff the isometry involves a translation along the u direction. The appearance of these closed timelike curves is thus connected to the special properties of the light cones in plane wave spacetimes. We show that all other quotients preserve stable causality.

July, 2003

1. Introduction

Plane waves are a class of metrics of great interest for string theory, for a number of reasons: Certain plane waves provide maximally supersymmetric backgrounds for string propagation [1]. Furthermore, plane waves can be obtained as the neighbourhood of a null geodesic in a generic spacetime via the Penrose limit [2], which allows us to use plane waves to explore and test the dualities between string theory in certain backgrounds and supersymmetric gauge theories [3]. These spacetimes also have vanishing curvature invariants, which makes them exact backgrounds for string propagation to all orders in l_s [4], [5].

It has recently been noted that the maximally supersymmetric plane wave of [1] is T-dual [6], [7] to a supersymmetric Gödel-like universe [8]. This might seem surprising; the plane wave is a stably causal spacetime, and is hence free of causal pathologies; the Gödel-like solution, on the other hand, has closed timelike curves (CTCs) passing through every point of the spacetime. We are used to the idea that T-duality can change the geometrical properties of the solution considerably; but is it really possible for T-duality to relate a causally well-behaved spacetime to one with CTCs¹?

In fact, the CTCs are not introduced by T-duality, but rather by quotienting. The plane wave does not have any compact circle directions. We must first quotient the spacetime by some suitable spacelike isometry to obtain a geometry with a finite-radius circle, which we can then consider the T-duality along. As we will discuss in section 2, it is this quotient that introduces the CTCs; they are present on both sides of the T-duality. That spacelike quotients can produce CTCs was first observed in [9]; for the specific case of Gödel, this was previously discussed in [10], [6], [11].

The true surprise, then, is that the quotient of a causally well-behaved spacetime along an everywhere spacelike isometry can lead to a spacetime with closed timelike curves. The main purpose of the present paper is to explore this discovery in more detail. We will follow two lines of development: we want to understand in general when such violations of causality conditions can occur, and we want to see why such a breakdown occurred for the plane wave in particular.

¹ If true, this would necessarily imply that string theory can be defined consistently on the background with CTCs, since the two geometries are given by the same worldsheet CFT. This would seem to rule out the possibility that some general mechanism in string theory forbids the existence of CTCs.

Firstly, we will consider the general question of quotients and causality. We will review the hierarchy of causality conditions that a spacetime may satisfy, and explore the conditions under which they are inherited by a quotient of the spacetime by some subgroup of the isometry group (we focus on the case of a one-parameter subgroup, *i.e.*, quotients generated by a single isometry). It is clear that for any causality condition to be inherited, it is necessary that the isometry we quotient along be everywhere spacelike. However, this is not a sufficient condition. Our main result in this general part of the discussion is to give a new necessary and sufficient condition under which a quotient of a stably causal spacetime inherits the property of stable causality².

To make these methods applicable to the case at hand, we will investigate the causality properties of plane waves and pp-waves. We will give an explicit demonstration that all plane waves are stably causal; this appears to be a known result [12], but we are unaware of any previous explicit demonstration. We will also discuss the extension to pp-waves, following [13]. We will see that they divide into two classes: either they are stably causal, or they are causal but fail to be distinguishing³.

We then consider the particular question of which quotients of plane waves preserve causality conditions, and why. We will consider the classification of the spacelike isometries of the maximally symmetric plane wave geometry in ten dimensions [1] (henceforth BFHP plane wave). This is a problem of interest in its own right: the classification of spacelike isometries of flat space led to interesting discoveries [14], and there have also been related investigations of AdS [15]. In the case of plane waves, several authors have investigated possible quotients of the BFHP plane wave [16], [6], [7], but a unified treatment has so far been lacking. We will show that the spacelike isometries of the BFHP plane wave fall into two classes: the Gödel class which leads to quotient spacetimes with CTCs, and the pp-wave class which leads to stably causal spacetimes; on Kaluza-Klein reduction to nine dimensions, quotients in the pp-wave class become pp-waves. The fact that the quotients in the Gödel class lead to CTCs turns out to be simply related to the special properties of the light cones in the plane wave spacetime.

1.1. Quotients with CTCs

To motivate the ensuing general discussion, we will begin by exhibiting two concrete

² Stable causality is the strongest of the usual causality conditions. It is equivalent to the existence of a globally-defined time function on the spacetime.

³ That is, not all distinct points have distinct pasts and futures.

examples wherein upon performing the quotient we generate CTCs (see also related examples in [9], [11]). The prototypical example involves a plane wave; for simplicity, we consider here a 4-dimensional maximally symmetric plane wave, which for convenience we write in spherical polar coordinates,

$$ds^2 = -2 du dv - \mu^2 r^2 du^2 + dr^2 + r^2 d\theta^2 . \quad (1.1)$$

To exhibit the isometry explicitly, consider the change of coordinates

$$u = t + y, \quad \theta = \phi - \mu(t + y), \quad v = \frac{1}{2}(t - y) , \quad (1.2)$$

which casts (1.1) in the form

$$ds^2 = -dt^2 + dy^2 + dr^2 + r^2 d\phi^2 - 2\mu r^2 d\phi(dt + dy) . \quad (1.3)$$

Note that ϕ is a periodic coordinate: $\phi \sim \phi + 2\pi$. Now consider a quotient of this spacetime along the orbits of the Killing vector $\left(\frac{\partial}{\partial y}\right)^a$, identifying $y \sim y + 2\pi R$ to form a compact circle of some radius R . Since $\left(\frac{\partial}{\partial y}\right)^a$ is a spacelike vector, the ‘obvious’ closed curve is not a CTC. However, since ϕ is periodic, the curve $(t, y, r, \phi) = (t_0, 2\pi n R \lambda, r_0, 2\pi \lambda)$, where $\lambda \in [0, 1]$, is also closed for every integer n (this curve only closes after winding n times around the compact y direction). The tangent to this curve,

$$\left(\frac{d}{d\lambda}\right)^a = 2\pi n R \left(\frac{\partial}{\partial y}\right)^a + 2\pi \left(\frac{\partial}{\partial \phi}\right)^a , \quad (1.4)$$

has norm

$$g_{ab} \left(\frac{d}{d\lambda}\right)^a \left(\frac{d}{d\lambda}\right)^b = 4\pi^2(n^2 R^2 + r_0^2 - 2\mu n R r_0^2) . \quad (1.5)$$

Thus, if we choose n such that $2\mu n R > 1$, then $\left(\frac{d}{d\lambda}\right)^a$ is timelike for sufficiently large r_0 , and hence the quotient spacetime contains CTCs. Thus in this prototypical example it is clear that quotienting a stably causal spacetime by an everywhere spacelike isometry can lead to CTCs. Requiring that the isometry be everywhere spacelike is necessary to avoid CTCs, but it is not always sufficient.

Another simple example involves the three-dimensional Banados-Teitelboim-Zanelli (BTZ) black hole [17]. Here the metric is

$$ds^2 = -(r^2 - r_+^2) dt^2 + \frac{dr^2}{(r^2 - r_+^2)} + r^2 d\phi^2 . \quad (1.6)$$

If we consider the Killing vector $\xi^a = \alpha \left(\frac{\partial}{\partial t}\right)^a + \left(\frac{\partial}{\partial \phi}\right)^a$, it will have norm $\xi^a \xi_a = -\alpha^2 (r^2 - r_+^2) + r^2$, so it is everywhere spacelike in the BTZ spacetime if $\alpha^2 < 1$. Consider a quotient along this Killing vector, identifying $(t, r, \phi) \sim (t + 2\pi\alpha R, r, \phi + 2\pi R)$ for some R . Again, since ϕ is periodic, $\phi \sim \phi + 2\pi$, this will also produce identifications of the form $(t, r, \phi) \sim (t + 2\pi n\alpha R, r, \phi + 2\pi(nR - m))$, for n, m integers. This produces CTCs in the region $r > r_+$, as we can choose n and m such that $-\alpha^2(r^2 - r_+^2) + r^2[1 - m/(nR)]^2 < 0$ ⁴.

This example differs from the previous one in two interesting ways. Firstly, in the plane wave case, it was observed in [11] that the identifications produce CTCs but not closed timelike geodesics. This is no longer the case in our BTZ example; it is easy to find a timelike geodesic which connects the two points (t, r, ϕ) and $(t + 2\pi n\alpha R, r, \phi + 2\pi(nR - m))$, by travelling out to larger r and reflecting off the radial potential in BTZ. Thus, we see that there is no obstruction in principle to obtaining closed timelike geodesics from such quotients. Secondly, the appearance of CTCs in the plane wave is, as we shall see later, intimately connected to the special properties of the light cone in the plane wave spacetime. In the BTZ spacetime, on the other hand, there is nothing unusual about the light cones in the region $r > r_+$, and one can easily construct similar examples where a spacelike quotient of a globally hyperbolic spacetime produces CTCs. Thus, this phenomenon seems less dependent on the special features of the causal structure of plane waves.

However, the appearance of CTCs in the BTZ geometry also depends on a special feature of this metric: that BTZ is itself a quotient. One can think of this example as the quotient of AdS_3 by a group generated by two Killing vectors, $\left(\frac{\partial}{\partial \phi}\right)^a$ and $\xi^a = \alpha \left(\frac{\partial}{\partial t}\right)^a + \left(\frac{\partial}{\partial \phi}\right)^a$. Since they differ by a timelike vector, it is perhaps not surprising that quotienting by both actions gives rise to CTCs. Put another way, the key feature of the BTZ example is that it has a spacelike isometry with non-contractible $\mathbf{S}^1$ orbits. There are clearly many other examples of spacetimes with such $\mathbf{S}^1$ s which will produce CTCs under an appropriate spacelike quotient; however, these seem somewhat artificial examples, and are less interesting than the plane wave case, where the orbits of $\left(\frac{\partial}{\partial \phi}\right)^a$ are contractible, and the appearance of CTCs comes from the causal structure.

⁴ Note that these CTCs are unrelated to the CTCs which occur if we include the region $r < 0$ in the BTZ metric; the further quotient we consider here produces CTCs which lie entirely in the ‘exterior’ region $r > r_+$.

2. Causal inheritance in general

An obvious question prompted by the above examples is, *does there exist* a sufficient condition for a given quotient of some arbitrary spacetime to be free of CTCs? In this section, we will see that there is in fact a simple and elegant condition for stable causality to be inherited under a quotient, which is both necessary and sufficient. Since stable causality is a stronger requirement than the absence of CTCs, as an immediate consequence this provides the desired sufficient condition for the absence of CTCs (albeit only in the case where the ‘parent’ spacetime is stably causal).

2.1. Causality conditions

To make the discussion comprehensible to readers unfamiliar with the intricacies of causality, we will briefly review the canonical hierarchy of causality conditions. This is a ‘pocket guide’; for a more extensive discussion of these conditions, see, e.g., [18]. These conditions are ordered by increasing strength, so each condition implies all the previous ones.

A spacetime that is free of closed timelike curves is said to satisfy the *chronology* condition. Furthermore, a spacetime is said to be *causal* if it contains no closed non-spacelike curves. The causal condition is the weakest condition that it seems desirable to impose on spacetimes. However, metrics satisfying this condition can still contain ‘almost closed’ timelike curves, which may still lead to pathologies. (What we mean formally by this is that an infinitesimal perturbation of the metric may be sufficient to produce CTCs.) It is therefore useful to consider stronger conditions on the causal structure.

We call a spacetime M *weakly distinguishing* if for $p, q \in M$, $p = q$ whenever $I^+(p) = I^+(q)$ and $I^-(p) = I^-(q)$. (Here $I^+(p)$ is the chronological future of the point p — the set of points reachable by future-directed timelike curves from p — and $I^-(p)$ is the chronological past of p). Similarly, the spacetime is *future-distinguishing* if $p = q$ whenever $I^+(p) = I^+(q)$, *past-distinguishing* if $p = q$ whenever $I^-(p) = I^-(q)$, and simply *distinguishing* if $p = q$ whenever either $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$. Clearly a spacetime satisfying any of these conditions cannot contain CTCs, since any two points p and q on a CTC γ will have $I^+(p) = I^+(q)$ and $I^-(p) = I^-(q)$.

A spacetime is called *strongly causal* if for every point p in the spacetime manifold, and every neighbourhood O of p , there exists a neighbourhood V of p contained in O which no causal curve intersects more than once. Basically, we require that causal curves

passing sufficiently near p do not come arbitrarily close to being closed curves. The strong causality condition is necessary for it to be possible to define the causal boundary of a spacetime using the technique of [19]. It also implies that the Alexandrov topology of the spacetime (the topology determined by the causal structure) agrees with the manifold topology.

A spacetime is called *stably causal* if the spacetime metric $g_{\mu\nu}$ has an open neighbourhood in the family of continuous metrics such that none of the metrics in the neighbourhood of the given metric admit closed timelike curves. Said differently, perturbing the spacetime metric by opening out light cones at every point should not lead to closed timelike curves. In a theory of quantum gravity, where the metric is subject to quantum fluctuation, the stable causality condition is likely to be the minimum required to avoid possible pathologies associated with CTCs. It is therefore usually assumed that only stably causal spacetimes are of physical relevance; this includes most of the usual standard backgrounds, such as flat space, anti-de Sitter space, plane waves (as we will see below), and p-branes. In fact, it is highly unusual to encounter a spacetime which is causal but not stably causal in practice (clearly such spacetimes are by definition a subset of measure zero in the space of continuous metrics). One of the interesting features of our discussion of the pp-waves is that they provide the first really natural example of such causally intermediate spacetimes.

Stable causality is equivalent to a much simpler requirement: it can be proved [18] that a spacetime is stably causal *iff* there is a smooth function τ on the spacetime manifold whose gradient $\nabla_a \tau$ is everywhere timelike. Such a function τ is called a *time function*, and it behaves as a good measure of time on the entire spacetime manifold, in the sense that it increases along every future directed causal curve. A time function also determines a natural foliation of the spacetime by spacelike surfaces, which are surfaces of simultaneity with respect to the time function. The definition of the time function clearly contains a lot of freedom; given a time function, we can construct another by adding to it a sufficiently small multiple of any smooth function on the spacetime. Much of our subsequent discussion will be concerned with the definition of time functions on stably causal spacetime, and on the plane waves in particular.

If the spacetime additionally has a timelike Killing vector field k^a , one can specify a class of time functions by requiring that the Lie derivative of τ along k^a $\mathcal{L}_k \tau = 1$, although this is still far from determining τ uniquely. For familiar examples, such as flat space, Anti de Sitter spacetimes, and the Einstein Static Universe (ESU), this condition is satisfied,

for example, by the canonical time function $\tau = t$ in terms of which the timelike Killing vector is $(\frac{\partial}{\partial t})^a$. However, the existence of a timelike Killing vector and stable causality are unrelated. For example, the compactification of flat space along the timelike isometry $(\frac{\partial}{\partial t})^a$ preserves the Killing vector, but clearly violates stable causality. The point is that $\tau = t$ is not a smooth function on the quotient; it must have a discontinuity somewhere.

While there is no constructive procedure for determining a time function for a given spacetime, there are various strategies at our disposal to find a candidate time function. If the given spacetime can be conformally embedded in a spacetime for which we know a time function, such as the ESU, then the same function must be a time function for the spacetime in question. This simply stems from the fact that causal properties of a spacetime are conformally invariant. Similarly, if we can find a fiducial metric $g_{\mu\nu}^{(fid)}$ such that all timelike curves in our physical metric are also timelike curves in the fiducial metric (so that the light cones of the fiducial metric ‘lie outside’ those of the physical metric), then any time function for the fiducial metric is also a time function for the physical metric. This is clear from the definition of a stably causal spacetime.

Finally, there is one stronger causality condition, which we include for completeness: a spacetime is *globally hyperbolic* if it has a global Cauchy surface. That is, there must be some spacelike surface Σ such that for any point p in the manifold, every causal curve through p meets Σ exactly once. This condition implies stable causality. To see this, note that we can define a time function τ by setting $\tau(p)$ equal to the volume of the intersection of $I^-(p)$ with Σ minus the volume of the intersection of $I^+(p)$ with Σ . While global hyperbolicity simplifies various constructions on the spacetime, imposing global hyperbolicity would be too restrictive: important examples, such as anti-de Sitter space or plane waves, are not globally hyperbolic. Moreover, as mentioned in the Introduction, assuming the spacetime is globally hyperbolic will not in general prevent spacelike quotients from leading to CTCs. We will therefore focus on the weaker condition of stable causality.

2.2. Quotients and causal inheritance

Generally, it would be interesting to know when a quotient of a spacetime satisfying one of these conditions continues to satisfy that condition. Clearly, the quotient of a causal spacetime along an isometry which is timelike or null somewhere will fail to be causal, as we are explicitly introducing closed timelike or null curves. However, as the example in the previous section illustrated, choosing an isometry which is everywhere spacelike is only a

necessary condition to preserve the causal condition, and not a sufficient one. So we would like to ask what more we can do.

For a given spacetime, we can always adopt the brute force method. Take a given isometry and perform the quotient. In the resulting geometry, analyze the causal curves to check explicitly whether or not there are CTCs. While this at least in principle provides a general solution of the problem of determining which quotients satisfy the causal condition, it does not provide us with much intuition. It would be preferable to have a more geometric criterion. At the same time it would be more useful to have a condition for the preservation of stable causality, since we have argued that this is the condition we should generically impose on physical spacetimes.

Fortunately, there is a simple criterion. *Let M/G be the quotient of a stably causal spacetime M by a subgroup G of its isometry group generated by some Killing vector ξ^a . Then M/G is itself stably causal iff there exists a time function τ on M such that $\mathcal{L}_\xi \tau = 0$.* That is, we require that the quotient only relates points lying in the same level surface of some time function τ ; this time function is invariant under the isometry we wish to quotient along.

Proof: First, imagine that M admits such a time function. Then τ will be single-valued on the quotient M/G , and hence defines a time function on M/G . Thus, M/G is stably causal. Now suppose M/G is stably causal. Then take any time function τ on M/G . We can pull this back to a function on M , which will again be a time function, and trivially satisfies the condition $\mathcal{L}_\xi \tau = 0$. QED.

One attractive feature of this condition is that it has the same form as the conditions for the preservation of other structures: for example, a quotient will preserve the symmetry associated with some other Killing vector ζ^a iff $\mathcal{L}_\xi \zeta = [\xi, \zeta] = 0$, and it will preserve the supersymmetry associated with a Killing spinor ϵ iff the spinorial Lie derivative $\mathcal{L}_\xi \epsilon = 0$ [20]. It is perhaps slightly surprising that we can formulate a local condition expressing the invariance under an isometry of a causality condition, since the causality condition expresses a global property of the geometry, like the absence of CTCs. The beauty of the formulation of stable causality in terms of time functions is that it hides all the global aspects in the time function.

Note that we only require that there exist at least one time function such that $\mathcal{L}_\xi \tau = 0$; this need not be true for any arbitrary time function we happen to consider. This makes this condition difficult to disprove, as there seems to be too much freedom in the choice of

time function on M to allow us to check it systematically. Hence, in practice we will only use this condition to show that stable causality is in fact preserved in some examples; where stable causality fails to be preserved, we will demonstrate the failure by the brute-force method.

3. Time functions for plane waves

The specific example to which we wish to apply these ideas is the plane wave spacetime. We will now show that plane wave spacetimes are stably causal by explicitly writing down time functions for them. This shows that we can apply our criterion to their quotients; it will also develop important intuition on the construction of time functions.

Plane wave spacetimes are characterised by a covariantly constant null Killing vector, along with full planar symmetry in the transverse directions. One can write the general metric as

$$ds^2 = -2 du dv - A_{ij}(u) x^i x^j du^2 + dx^i dx^i \quad (3.1)$$

where A_{ij} are arbitrary functions of u (although the equations of motion will impose some constraint on the trace $A_{ii}(u)$), and $i, j = 1, \dots, d$. We will proceed by first giving a time function for the BFHP plane wave, where there is an obvious choice. We will then generalise this to the general plane wave.

3.1. Time functions for the BFHP plane wave

The BFHP plane wave is the ten-dimensional metric (so $d = 8$) with $A_{ij}(u) = \mu^2 \delta_{ij}$. For convenience, we perform the coordinate transformation to set $\mu = 1$; the metric is then

$$ds^2 = -2 du dv - x^i x^i du^2 + dx^i dx^i. \quad (3.2)$$

The only obvious guess for a time function is the coordinate u , which is indeed treated as the ‘light-cone time’ in studies of string theory on this background. However, since the gradient $\nabla_a u$ is null, this does not provide a good time function. Since it is null, one might guess that there will be good time functions which only differ from u by ‘a little bit’. We will see that this is in fact true, but we will arrive at our time function by a somewhat different route. We want to exploit the fact that the metric (3.2) is conformally flat, and thus can be mapped to the ESU. The ESU is stably causal, and has an obvious time function, namely the global time in the usual coordinate system. We can pull this function back to obtain a good time function on the entire BFHP plane wave spacetime.

By making a sequence of coordinate transformations as in [21], starting with

$$U = \tan u , \quad V = -v - \frac{x^i x^i \tan u}{2} , \quad X^i = \frac{x^i}{\cos u} , \quad (3.3)$$

followed by

$$\begin{aligned} 2U &= \left(\tan \frac{\psi + \zeta}{2} - \tan \frac{\psi - \zeta}{2} \right) \cos \theta + \tan \frac{\psi + \zeta}{2} + \tan \frac{\psi - \zeta}{2} \\ 4V &= \left(\tan \frac{\psi + \zeta}{2} - \tan \frac{\psi - \zeta}{2} \right) \cos \theta - \left(\tan \frac{\psi + \zeta}{2} + \tan \frac{\psi - \zeta}{2} \right) \\ 2\sqrt{X^i X^i} &= \left(\tan \frac{\psi + \zeta}{2} - \tan \frac{\psi - \zeta}{2} \right) \sin \theta , \end{aligned} \quad (3.4)$$

and finally,

$$\begin{aligned} \cos \zeta &= -\cos \alpha \cos \beta \\ \cos \theta \sin \zeta &= \cos \alpha \sin \beta , \end{aligned} \quad (3.5)$$

we can rewrite the BFHP plane wave (3.2) in a form conducive to comparison to the ESU:

$$ds^2 = \frac{1}{4|e^{i\psi} - \cos \alpha e^{i\beta}|^2} (-d\psi^2 + d\alpha^2 + \cos^2 \alpha d\beta^2 + \sin^2 \alpha d\Omega_7^2) . \quad (3.6)$$

The part of the metric inside the parenthesis is the metric of the ten dimensional ESU, where we have decomposed the metric of the $\mathbf{S}^9$ into an $\mathbf{S}^7$ and the $\mathbf{S}^1$ parameterised by β . The conformal factor $\frac{1}{4|e^{i\psi} - \cos \alpha e^{i\beta}|^2}$ blows up when $\alpha = 0$ and $\psi = \beta + 2\pi n$ for some integer n — this corresponds to the one-dimensional conformal boundary of the plane wave.

The global time ψ is clearly a good time function for the ESU. Our claim is that $\tau = \psi$ is a good time function for the geometry (3.2). To see this, note that

$$g^{\psi\psi} = -4|e^{i\psi} - \cos \alpha e^{i\beta}|^2 , \quad (3.7)$$

so the one-form $d\psi$ is timelike everywhere, except where the conformal factor diverges *i.e.*, at the conformal boundary. Since the conformal boundary is not part of the spacetime manifold, the function ψ provides the good time function that we sought. This time function can be written in terms of the coordinate chart used in (3.2) as

$$\begin{aligned} \tau(u, v, x^i) &= \tan^{-1} \left[\frac{(1 + x^i x^i)}{2} \tan u + v + \sqrt{\left[\frac{(1 - x^i x^i)}{2} \tan u - v \right]^2 + \frac{x^i x^i}{\cos^2 u}} \right] \\ &\quad + \tan^{-1} \left[\frac{(1 + x^i x^i)}{2} \tan u + v - \sqrt{\left[\frac{(1 - x^i x^i)}{2} \tan u - v \right]^2 + \frac{x^i x^i}{\cos^2 u}} \right] \\ &= u + \tan^{-1} \left(\frac{2v}{1 + x^i x^i} \right) . \end{aligned} \quad (3.8)$$

We can also check that this is a time function directly in the Brinkmann coordinates used in (3.2): a short calculation gives

$$\nabla_a \tau(u, v, x^i) \nabla^a \tau(u, v, x^i) = -\frac{4}{(1 + x^i x^i)^2 + 4v^2} , \quad (3.9)$$

which implies that $\nabla_a \tau$ is timelike for all finite values of the coordinates. $\tau(u, v, x^i)$ is also a continuous function of its arguments, so it is a global time function⁵.

Since the second term in (3.8) is a $\tan^{-1}$, we see that this time function is indeed of the form $\tau = u +$ ‘a little bit’; in particular, the level surfaces of τ approach the surfaces $u = \text{constant}$ as v or the x^i go to infinity. Note also that this time function satisfies $\partial_u \tau(u, v, x^i) = 1$; that is, relative to the timelike Killing vector ∂_u , this time function satisfies the condition we argued in the previous section could be used to pick out some ‘natural’ time functions for us. It is possible to find other time functions by a simple deformation of (3.8). Consider functions of the form $\tau(u, v, x^i) = u + \tilde{\tau}(v, x^i)$. One would expect that any function $\tilde{\tau}(y)$ with $y = \frac{2v}{1+x^i x^i}$ that “looks like” $\tan^{-1} y$, such as $\tanh y$ or $\frac{y}{1+|y|}$, ought to work; and indeed, explicit checks confirm this. For instance, $\tau = u + \frac{av}{1+x^i x^i + b|v|}$ is also a good time function provided $0 < a \leq 2$ and $b^2 \geq 2a$ with $b > 0$.

3.2. Time functions for general plane waves

Having constructed an explicit time function for the BFHP plane wave (3.2), we can now try to generalise this construction to other plane waves. In general, it is still true that u has a null gradient, so it is reasonable to look for a time function by using the ansatz $\tau(u, v, x^i) = u + \tilde{\tau}(v, x^i)$. For a general plane wave, we have the metric (3.1) where $A_{ij}(u)$ is an arbitrary function of u . For functions $A_{ij}(u)$ that are bounded above, it is very simple to generalise the time function (3.8) found for the BFHP plane wave. Consider for example the candidate time function

$$\tau(u, v, x^i) = u + \frac{1}{\alpha} \tan^{-1} \left(\frac{2\alpha v}{1 + B_{ij} x^i x^j} \right) , \quad (3.10)$$

where B_{ij} is a constant matrix. In order for $\nabla_a \tau$ to be timelike, *i.e.*, $\nabla_a \tau \nabla^a \tau < 0$, it suffices to pick B_{ij} such that

$$\begin{aligned} (A_{ij}(u) - B_{ij}) x^i x^j &\leq 0, \\ (B_{ik} B_{kj} - \alpha^2 B_{ij}) x^i x^j &\leq 0 . \end{aligned} \quad (3.11)$$

⁵ In passing, we note that this calculation also shows that the conformal factor relating the plane wave to the ESU can be rewritten as $\frac{4}{(1+x^i x^i)^2 + 4v^2}$.

i.e., $B_{ij} - A_{ij}(u)$ and $B_{ij} - \frac{B_{ik}B_{kj}}{\alpha^2}$ are positive definite metrics on the transverse $\mathbf{R}^d$. Now for any *bounded above* functions $A_{ij}(u)$, we can always find a matrix B_{ij} and a constant α satisfying this requirement.

There is a simple physical reason why it is easy to find a time function for plane waves with $A_{ij}(u)$ bounded above: we can bound the quadratic form $A_{ij}(u) x^i x^j$ by $\mu^2 \delta_{ij} x^i x^j$ for some μ^2 . That is, there is some maximally symmetric plane wave which provides a fiducial metric for our physical plane wave; if $A_{ij}(u) x^i x^j$ is bounded by $\mu^2 \delta_{ij} x^i x^j$, the light cones of the metric (3.1) will be bounded by the light cones of this fiducial plane wave, and the time function constructed previously for this maximally symmetric plane wave gives a time function on the general plane wave.

The case where $A_{ij}(u)$ can diverge as a function of u would appear to be more difficult, and indeed in this case we will have to give up on the ansatz $\tau(u, v, x^i) = u + \tilde{\tau}(v, x^i)$ used in the previous case, and allow some more general dependence on u . Let us isolate the divergence by writing $A_{ij}(u) x^i x^j = f(u) \bar{A}_{ij}(u) x^i x^j$, where $f(u)$ carries any divergence to $+\infty$ and $\bar{A}_{ij}(u)$ is bounded above as a function of u . We will assume without loss of generality that $f(u) \geq 0 \forall u$. One might expect that a sufficiently severe divergence in the metric will be somehow reflected in the time function. This leads us to adopt an ansatz which satisfies $\partial_u \tau = 1 + f(u)$.

Having isolated the divergent piece, we assume the remaining part of the time function behaves as before, adopting the following ansatz:

$$\tau(u, v, x^i) = u + \int^u d\bar{u} f(\bar{u}) + \frac{1}{\alpha} \tan^{-1} \left(\frac{2\alpha v}{1 + B_{ij} x^i x^j} \right) . \quad (3.12)$$

For such a time function,

$$\nabla_a \tau = (1 + f(u)) (du)_a + \frac{2 (1 + B_{ij} x^i x^j)}{(1 + B_{ij} x^i x^j)^2 + 4\alpha^2 v^2} (dv)_a - \frac{4 v B_{ij} x^j}{(1 + B_{ij} x^i x^j)^2 + 4\alpha^2 v^2} (dx^i)_a . \quad (3.13)$$

Evaluating the norm of $\nabla_a \tau$ we obtain

$$\begin{aligned} \nabla_a \tau \nabla^a \tau = & - \frac{4 (1 + f(u))}{(1 + B_{ij} x^i x^j)^2 + 4\alpha^2 v^2} \\ & - \frac{4 (1 + B_{ij} x^i x^j)^2}{\left[(1 + B_{ij} x^i x^j)^2 + 4\alpha^2 v^2 \right]^2} (B_{ij} x^i x^j (1 + f(u)) - f(u) \bar{A}(u, x^i)) \\ & - \frac{16 v^2}{\left[(1 + B_{ij} x^i x^j)^2 + 4\alpha^2 v^2 \right]^2} (4\alpha^2 (1 + f(u)) B_{ij} x^i x^j - B_{ik} B_{kj} x^i x^j) . \end{aligned} \quad (3.14)$$

We can therefore make $\nabla_a \tau$ timelike everywhere by satisfying

$$\begin{aligned} (B_{ij} - \bar{A}_{ij}(u)) x^i x^j &\geq 0, \\ (\alpha^2 B_{ij} - B_{ik} B_{kj}) x^i x^j &\geq 0, \end{aligned} \tag{3.15}$$

as in (3.11). As before, it is easy to find B_{ij} and α satisfying the above requirements so long as $\bar{A}_{ij}(u)$ is bounded above. Thus, we have obtained time functions for all plane waves, explicitly demonstrating that these spacetimes are stably causal⁶.

Note that if the divergence in $A_{ij}(u)$ is slower than $\frac{1}{u-u_0}$, τ will remain bounded. In this case it seems to make sense to continue the time function through the singularity at $u = u_0$. On the other hand, if $A_{ij}(u)$ diverges faster than $\frac{1}{u-u_0}$, τ will diverge as $u \rightarrow u_0$, and we only have a good time function on the whole spacetime if we treat the singularity as the ‘end of the universe’ and don’t try to evolve through it. The singularity is weak in the sense of Tipler [23],[24] if $A_{ij}(u)$ diverges slower than $(u - u_0)^{-2}$, so it might appear that the time functions we constructed are not good enough to account for all weak singularities. It is possible that there might exist better time functions for these cases. However, this weak singularity is unstable in the sense that a generic stress tensor will diverge as one approaches it unless $A_{ij}(u)$ diverges slower than $(u - u_0)^{-1}$ [25]. Hence we have constructed time functions which cover what we regard as the full physical spacetime region in all cases.

4. Causal properties of pp-waves

All plane waves are hence stably causal. What about pp-waves? Consider the general pp-wave metric,

$$ds^2 = -2 du dv - F(u, x^i) du^2 + dx^i dx^i. \tag{4.1}$$

The key new possibility that allowing a general function $F(u, x^i)$ permits, compared to the previous plane wave case, is that this function can diverge to $+\infty$ as a function of the x^i faster than a quadratic at large x^i , or at some finite x^i . This additional ‘opening out’ of the light cones can make the metric less well-behaved causally than the plane waves.

⁶ It was shown in [22] that plane waves are, on the other hand, not globally hyperbolic. We remark in passing that Penrose’s argument only applies to the case where A_{ij} has at least one positive eigenvalue. The case where A_{ij} is negative semidefinite is globally hyperbolic—however, the nontrivial plane waves in this class violate the energy conditions.

Spacetimes of the form (4.1) will be stably causal *iff* the behaviour of the function $F(u, x^i)$ is “subquadratic” as a function of x^i [13]. Subquadratic means $F(u, x^i) \leq A_{ij}(u) x^i x^j \forall x^i$ for some $A_{ij}(u)$. If this is true, we see that the light cones of (4.1) are bounded by the light cones of some fiducial plane wave, and hence it is clear that there is a good time function on the pp-wave. This is thus a sufficient condition for stable causality; remarkably, it is also necessary. As was shown in [13], if the pp-wave is not subquadratic, it will remain causal but fail to be even weakly distinguishing.

We will give a simple general proof of this statement. If the pp-wave is not subquadratic, either $F(u, x^i)$ diverges to $+\infty$ faster than a quadratic term $x^i x^i$ in some direction at large x^i , or $F(u, x^i)$ diverges to $+\infty$ at some finite x^i , which for convenience, we take to be the origin. In [13] it was shown that the spacetimes of the form (4.1) are not distinguishing if $F(u, x^i)$ grows faster than $x^i x^i$ at large x^i . We will review this argument and show that it can easily be extended to address divergences at finite x^i .

The essential point in both cases is that for pp-waves of this kind, the future of any point (u_0, v_0, x_0^i) will be the entire region $u > u_0$. In particular, this implies that any two points in the plane $u = u_0$ have the same future. By a similar argument, they will have the same pasts. Hence, the spacetime is not weakly distinguishing. To demonstrate that the future of any point (u_0, v_0, x_0^i) is the entire region $u > u_0$, we need to find a timelike curve γ connecting (u_0, v_0, x_0^i) to (u_1, v_1, x_1^i) for any $u_1 > u_0$ and arbitrary v_1, x_1^i . We will show this is possible for infinitesimal separation, $u_1 - u_0 = \epsilon$. It then follows immediately that it is true for finite separation.

For the case where $F(u, x^i)$ grows faster than $x^i x^i$ at large x^i , we will use a slight simplification of the argument in [13]. We consider a path composed of three pieces; first, we go in a straight line from (u_0, v_0, x_0^i) to a point $(u_0 + \epsilon/4, V, X^i)$. We then travel in a straight line from $(u_0 + \epsilon/4, V, X^i)$ to $(u_0 + 3\epsilon/4, -V', X^i)$. Finally, we travel in a straight line from $(u_0 + 3\epsilon/4, -V', X^i)$ to $(u_0 + \epsilon, v_1, x_1^i)$. We assume that $V, V', |X|^2$ are large positive numbers; in particular, assume $V, V' \gg |v_0|, |v_1|$, and $|X| \gg |x_0|, |x_1|$. Then the conditions for these three segments to be timelike reduce to

$$-\frac{1}{2} \epsilon V + |X|^2 < 0, \quad (4.2)$$

$$\epsilon (V + V') - \frac{1}{4} F(u_0, X^i) \epsilon^2 < 0, \quad (4.3)$$

$$-\frac{1}{2} \epsilon V' + |X|^2 < 0. \quad (4.4)$$

We can use the first and third conditions to eliminate V and V' in the second condition, which then relates ϵ and X^i :

$$F(u_0, X^i) \epsilon^2 > 16 |X|^2. \quad (4.5)$$

If $F(u_0, x^i)$ grows faster than $|x|^2$ at large x^i , this condition can be satisfied for arbitrary ϵ simply by choosing X^i large enough. This shows that the indicated path is timelike in this case⁷.

This argument can be trivially extended to the case where $F(u_0, x^i)$ is singular at some finite position, which we take W.L.O.G. to be at $x^i = 0$. We simply take X^i in the above argument to be a value near the origin, rather than a large value. This modifies (4.2)-(4.4) to read

$$-\frac{1}{2} \epsilon V + |x_0|^2 < 0, \quad (4.6)$$

$$\epsilon (V + V') - \frac{1}{4} F(u_0, X^i) \epsilon^2 < 0, \quad (4.7)$$

$$-\frac{1}{2} \epsilon V' + |x_1|^2 < 0. \quad (4.8)$$

Again, eliminating V and V' , the main condition we need to satisfy is

$$F(u_0, X^i) \epsilon^2 > 8 (|x_0|^2 + |x_1|^2), \quad (4.9)$$

which we can clearly satisfy for arbitrary ϵ by choosing X^i sufficiently close to the origin, so that $F(u_0, X^i)$ is big enough. Thus, for all “superquadratic” pp-waves, we can construct a timelike path linking any two points (u_0, v_0, x_0^i) and (u_2, v_1, x_1^i) with $u_1 > u_0$. As a result, the future $I^+(u_0, v_0, x_0^i) = \{(u, v, x^i) : u > u_0\}$ for any v_0, x_0^i . This completes the demonstration that all “superquadratic” pp-waves are not weakly distinguishing.

In the other case, where $F(u, x^i)$ is a subquadratic function of x^i , we have already argued that the light cones of the pp-wave are bounded by the light cones of a fiducial plane wave, so it must be stably causal. It is straightforward to construct an explicit time function in this case, as we will do now. As in the plane wave discussion, let us isolate from $F(u, x^i)$ any divergence to $+\infty$ as a function of u , writing $F(u, x^i) = f(u) \bar{F}(u, x^i)$, where $\bar{F}(u, x^i)$ is a bounded function of u .

⁷ If $F(u, x^i)$ is quadratic at large x^i we see that this condition can still be satisfied, but it now requires a finite separation $\epsilon = u_1 - u_0$, in agreement with previous discussions of light cones for plane waves [26].

A suitable ansatz for a time function is then

$$\tau = u + \int^u d\bar{u} f(\bar{u}) + \frac{1}{\alpha} \tan^{-1} \left(\frac{2\alpha v}{1 + G(x^i)} \right) . \quad (4.10)$$

It is easy to check that is a good time function for the geometry (4.1) so long as

$$\begin{aligned} G(x^i) - \bar{F}(u, x^i) &\geq 0 , \\ 4\alpha^2 G(x^i) - \partial_i G(x^i) \partial_i G(x^i) &\geq 0 . \end{aligned} \quad (4.11)$$

Given that $\bar{F}(u, x^i)$ is bounded above as a function of u , and a subquadratic function of x^i , we can satisfy the first condition by choosing some suitably large, smooth function $G(x^i)$, which grows at most as $|x|^2$ at large x^i . We will assume that $G(x^i)$ is everywhere positive; we can then satisfy the second condition by choosing α large enough (here the fact that G does not grow faster than $|x|^2$ is crucial).

We have thus seen that pp-waves are either stably causal or nondistinguishing. Stable causality does not rule out singularities in $F(u, x^i)$; it only requires that any divergence in $F(u, x^i)$ is to $-\infty$. Curiously enough, $F(x^i) \rightarrow -\infty$ guarantees a genuine curvature singularity, while $F(x^i) \rightarrow +\infty$ need not in fact correspond to a singular spacetime [27]. For example, the stably causal pp-wave with $F(u, x^i) = -1/|x|^{d-2}$ is singular (recall d is the number of x^i). On the other hand, the opposite sign, $F(u, x^i) = 1/|x|^{d-2}$, which was shown to be geodesically complete in [27], is nondistinguishing. Similarly, the pp-wave spacetime [28],

$$ds^2 = -2 du dv - (\cosh x - \cos y) du^2 + dx^2 + dy^2 + dz^i dz^i , \quad (4.12)$$

leads to a geodesically complete but nondistinguishing spacetime. Note also that the fact that certain pp-waves are nondistinguishing implies that the technique for constructing causal completions described in [19] cannot be applied to them. In particular, the conclusions of [27], wherein it was claimed that the spacetime (4.12) has a one-dimensional causal boundary are thereby rendered invalid; since (4.12) is not distinguishing, it does not make sense to talk about its causal boundary.

5. Quotients of the BFHP plane wave

We have demonstrated that general plane waves are stably causal; we would now like to ask what happens to the causal properties of these plane waves when we quotient them

by some spacelike isometry. We will focus on the classification of the quotients for the BFHP plane wave; this is the case of greatest interest, and will also provide the richest structure, since it has the largest isometry group. We will comment briefly on the lessons of this detailed analysis for quotients of more general plane waves in the conclusions.

5.1. Classification of spacelike isometries

The BFHP plane wave in ten dimensions has metric

$$ds^2 = -2 du dv - \mu^2 x^i x^i du^2 + dx^i dx^i, \quad (5.1)$$

which is supported by a null five-form flux

$$F^{(5)} = \mu du \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8). \quad (5.2)$$

The solution has 30 Killing vectors. Some of these, such as translations along u and v and rotations in the transverse space, are manifest in the coordinate chart used to write the metric (5.1). There are also translation symmetries in the transverse directions that are not manifest in these coordinates. Note that while the metric admits an $SO(8)$ rotation symmetry, some of this is broken by the fluxes that are necessary to support the metric. If we support the metric by 5-form Ramond-Ramond flux, as above, then we break the $SO(8)$ down to $SO(4) \times SO(4)$. However, if we choose to consider the 28 supercharge solution supported by 3-form flux as in [29],[30], then we have only a $U(4)$ rotation symmetry. We will write the isometry algebra for the maximally supersymmetric solution above.

The Killing vectors for the geometry (5.1) are given as:

$$\begin{aligned} \xi_{e_u} &= -\partial_u \\ \xi_{e_v} &= \partial_v \\ \xi_{e_i} &= -\cos(\mu u) \partial_{x^i} + \mu \sin(\mu u) x^i \partial_v, \quad i = 1, \dots, 8 \\ \xi_{e_i^*} &= -\mu \sin(\mu u) \partial_{x^i} - \mu^2 \cos(\mu u) x^i \partial_v, \quad i = 1, \dots, 8 \\ \xi_{M_{ij}} &= x^i \partial_j - x^j \partial_i, \quad \text{both } i, j = 1, \dots, 4, \text{ or both } i, j = 5, \dots, 8. \end{aligned} \quad (5.3)$$

The algebra generated by the Killing vectors is

$$\begin{aligned} [\xi_{e_v}, \xi_{e_u}] &= 0, \quad [\xi_{e_v}, \xi_{e_i}] = 0, \quad [\xi_{e_v}, \xi_{e_i^*}] = 0, \quad [\xi_{e_v}, \xi_{M_{ij}}] = 0, \\ [\xi_{e_u}, \xi_{e_i}] &= \xi_{e_i^*}, \quad [\xi_{e_u}, \xi_{e_i^*}] = -\mu^2 \xi_{e_i}, \quad [\xi_{e_u}, \xi_{M_{ij}}] = 0, \\ [\xi_{e_i}, \xi_{e_j^*}] &= -\mu^2 \delta_{ij} \xi_{e_v}, \\ [\xi_{M_{ij}}, \xi_{e_k}] &= \delta_{jk} \xi_{e_i} - \delta_{ik} \xi_{e_j}, \quad [\xi_{M_{ij}}, \xi_{e_k^*}] = \delta_{jk} \xi_{e_i^*} - \delta_{ik} \xi_{e_j^*} \\ [\xi_{M_{ij}}, \xi_{M_{kl}}] &= \delta_{jk} \xi_{M_{il}} - \delta_{ik} \xi_{M_{jl}} - \delta_{jl} \xi_{M_{ik}} + \delta_{il} \xi_{M_{jk}}. \end{aligned} \quad (5.4)$$

Let us now consider a general linear combination of the Killing vectors,

$$\zeta = a \xi_{e_u} + b \xi_{e_v} + c_i \xi_{e_i} + d_i \xi_{e_i^*} + \omega_{ij} \xi_{M_{ij}} . \quad (5.5)$$

The norm of the Killing vector can be calculated from the metric (5.1) to be

$$\begin{aligned} |\zeta|^2 = & -a^2 \mu^2 x^i x^i + 2a [b + \mu \sin(\mu u) c_i x^i - \mu^2 \cos(\mu u) d_i x^i] \\ & + [\cos(\mu u) c_i + \mu \sin(\mu u) d_i + 2\omega_{ij} x^j]^2 . \end{aligned} \quad (5.6)$$

The first line vanishes when $a = 0$ and the second line being a perfect square is non-negative. This motivates a natural distinction between the situation when a is vanishing and when a is non-vanishing. As we will see, this choice is also motivated by the distinct causal properties of these two cases. So we divide our isometries into two classes; the Gödel class where $a \neq 0$ and the pp-wave class where $a = 0$.

5.2. Quotients by the Gödel class of isometries

The Gödel class of spacelike isometries includes the quotient involved in the construction of the supersymmetric Gödel-like spacetime from T-duality of the plane wave [6]. Thus, we expect to find that at least some of these quotients contain CTCs. In fact, it turns out that all the quotients in the Gödel class contain CTCs.

First, let us see that there are spacelike isometries in this class. For $a \neq 0$, the first term in (5.6) is negative. To achieve a spacelike norm, we must balance this negative term against a positive contribution coming from the part involving the rotation matrices ω_{ij} . We must require $[-a^2 \mu^2 \delta_{ij} + 4\omega_{ik} \omega_{jk}] x^i x^j \geq 0$; this implies that the rotation matrix ω_{ij} must be of maximal rank. Also, to make the norm non-vanishing at the origin of the transverse space $x^i = 0$, we must require either $b \neq 0$ or both c_i and d_j non-vanishing for some i, j . Once these conditions are satisfied, the Killing vector will be everywhere spacelike.

The argument for the presence of CTCs in these quotients is simple. Consider a point on the $u = 0$ plane, say $p = (u_0 = 0, v_0, x_0^i)$. The future of p contains all points with $u \geq \frac{\pi}{\mu}$ [26]. The Killing vector has a ξ_{e_u} component, which is just translation along u . When we quotient by this isometry we identify some points on its orbits. In particular, we will identify some point $q = (u_1 \geq \frac{\pi}{\mu}, v_1, x_1^i)$ with the point p . But q is in the future of p , so these two points are connected by a future directed timelike curve, thus giving us a CTC. So whenever we have a ξ_{e_u} component to the isometry we have a CTC. The example

mentioned in Section 1.1 was a prototype of this phenomenon in the simple setting of the four-dimensional maximally symmetric plane wave.

We now see that the presence of CTCs in such quotients is intimately connected to the special features of the light cones in plane wave spacetimes—that the future of the origin contains *all* points with $u \geq \pi/\mu$. This provides some understanding of why CTCs can appear in quotients involving spacelike isometries in the plane waves but not, for example, in flat space ⁸.

5.3. Quotients by the *pp*-wave class of isometries

Now consider the case when we have $a = 0$. This set of isometries has no ξ_{e_u} component, and we will show that all the quotients by spacelike Killing vectors in this class inherit the property of stable causality from the parent spacetime.

Let us first classify the physically distinct spacelike Killing vectors with $a = 0$. A general isometry of this form is

$$\zeta_P = c_i \xi_{e_i} + d_i \xi_{e_i^*} + \omega_{ij} \xi_{M_{ij}} + b \xi_{e_v} , \quad (5.7)$$

with norm

$$|\zeta_P|^2 = [\cos(\mu u) c_i + \mu \sin(\mu u) d_i + 2 \omega_{ij} x^j]^2 . \quad (5.8)$$

Now ζ_P is guaranteed to be non-timelike, but for it to be spacelike at the origin of transverse space, we need c_i or d_i to be non-vanishing. Moreover, since the trigonometric functions have zeros at multiples of $\frac{\pi}{2\mu}$, we need to have at least one c_i and one d_i non-vanishing simultaneously.

Some of these isometries are related by conjugation by the isometry group, and should not be counted as physically distinct. Since our main interest is in the causal structure, we will consider conjugation by the full $SO(8)$ symmetry of the metric; thus, we are treating as equivalent isometries which only differ by their action on the fluxes. Let us use this freedom to bring the isometry to a canonical form. By an $SO(8)$ rotation we can set c_i for

⁸ If we regard this plane wave as the Penrose limit of the supersymmetric $\text{AdS}_5 \times \mathbf{S}^5$ solution, the $\text{AdS}_5 \times \mathbf{S}^5$ isometry which gives ξ_{e_u} is $\partial_t - \partial_\psi$, while ξ_{e_v} is obtained from $(\partial_t + \partial_\psi)/R^2$, where the Penrose limit is $R \rightarrow \infty$. Thus, the Killing vector ζ on the plane wave corresponds to a Killing vector $\xi_{AdS} = a(\partial_t - \partial_\psi) + \omega_{ij}\xi_{M_{ij}}$ in the Penrose limit. When $a \neq 0$, the quotient along this Killing vector leads to CTCs in the $\text{AdS}_5 \times \mathbf{S}^5$ space—and in particular in its boundary—for precisely the same reasons as above. We thank Vijay Balasubramanian for raising this issue.

$i \neq 1$ to zero. Since the ξ_{e_i} and $\xi_{e_i^*}$ differ only by a translation in u and scale, we may translate u so as to set d_1 to zero. A further rotation which leaves the x^1 direction fixed may be used to set d_i for $i \neq 2$ to zero. We can then use the freedom to choose the overall scale to set $c_1 = 1$, and finally set $b = 0$ by conjugation with respect to $\xi_{e_1^*}$. Hence, the canonical form for the spacelike isometries in the pp-wave class is

$$\zeta_P = \xi_{e_1} + \alpha \xi_{e_2^*} + \omega_{ij} \xi_{M_{ij}} , \quad \alpha \neq 0 \quad (5.9)$$

where we have redefined $d_2/c_1 = \alpha$ for notational convenience, and absorbed c_1 into ω_{ij} .

Since the CTCs in the Gödel class clearly had their origin in the translation in u , one might guess that the pp-wave class of isometries will lead to stably causal spacetimes. This is indeed true. An elegant way to show this is to observe that the (string frame) geometry obtained under Kaluza-Klein reduction to nine dimensions along the general quotient in the pp-wave class is a pp-wave. This follows from the basic property of the pp-wave class, that the Killing vector (5.9) does not involve ∂_u . As a consequence, ∂_v remains a covariantly constant null vector in the Kaluza-Klein reduced spacetime.

In more detail, since the Killing vector (5.9) does not involve ∂_u , one can make a coordinate transformation to bring this Killing vector to the form $\zeta_P^a = \left(\frac{\partial}{\partial z}\right)^a$ without redefining u . In fact, the coordinate transformation can be taken to have the form

$$\begin{aligned} x^i &= b_a^i(U, z) X^a + c^i(U, z) , \\ v &= V + f(U, z, X^a) , \\ u &= U \end{aligned} \quad (5.10)$$

where $a = 2, \dots, d$. In terms of these adapted coordinates, the ten-dimensional metric can then be rewritten as

$$\begin{aligned} ds^2 &= -2 dU dV - k(U, X^c) dU^2 + l_a(U, X^c) dU dX^a + h_{ab}(U, X^c) dX^a dX^b \\ &\quad + g(U, X^c) (dz + j(U, X^c) dU + m_a(U, X^c) dX^a)^2 . \end{aligned} \quad (5.11)$$

The first line of this gives the string-frame metric in nine dimensions obtained if we consider the Kaluza-Klein reduction along ζ_P . The essential point is that, because $u = U$ and $v = V + \dots$, the only non-zero $g_{\mu V}$ metric component in the nine-dimensional metric is still $g_{UV} = -1$. This implies that in this metric, the vector $\left(\frac{\partial}{\partial V}\right)^a$ is covariantly constant and null. Thus, the metric is a pp-wave⁹.

⁹ Since the dilaton depends on $g(U, X^a)$, in the Einstein frame metric $\left(\frac{\partial}{\partial V}\right)^a$ is no longer covariantly constant, but continues to be a null isometry.

Now we saw in the previous section that pp-waves can be divided into two classes: the subquadratic ones, which are stably causal, and the superquadratic ones, which are causal but not distinguishing. Which of these can arise under this Kaluza-Klein reduction? In any superquadratic pp-wave, the function $F(u, x^i)$ must diverge to $+\infty$ either at finite x^i or faster than a quadratic at large x^i . This implies that the curvature $R_{+i+j} = \partial_i \partial_j F(u, x^k)$ must also grow unboundedly. Now we are considering the quotient of the smooth, constant curvature BFHP plane wave by an everywhere spacelike isometry, whose proper length is bounded below for any given α, ω_{ij} . This cannot lead to a Kaluza-Klein reduced geometry with unbounded curvatures. Hence Kaluza-Klein reduction along ζ_P must give a subquadratic pp-wave.

The subquadratic pp-waves are all stably causal, and hence have global time functions defined on them. We can therefore construct a time function satisfying $\mathcal{L}_{\zeta_P} \tau = 0$ on the BFHP plane wave for any Killing vector ζ_P in the pp-wave class by constructing the Kaluza-Klein reduction, identifying a time function on the resulting nine-dimensional pp-wave geometry, and pulling it back to the ten-dimensional geometry. This shows that all Killing vectors of the form (5.9) satisfy the condition in section 2.2, and hence that all quotients in the pp-wave class are stably causal.

This discussion gives a procedure for obtaining suitable time functions, but it may seem somewhat abstract. To illustrate the foregoing general discussion, we will now explicitly construct time functions satisfying the condition of section 2.2 for a subclass of the Killing vectors (5.9). We will start by considering the isometry ζ_P in (5.9) with $\omega_{ij} = 0$. We will subsequently show that the time function τ we obtain for this case will continue to satisfy $\mathcal{L}_{\zeta_P} \tau = 0$ with $\omega_{ij} \neq 0$ for $i, j \neq 1, 2$, as the time functions will be rotationally invariant in the other directions. We thus obtain explicit time functions for a large subclass of the pp-wave class. Explicit time functions could also be constructed for the remaining cases, but they would be considerably more messy.

Let us consider the spacelike isometry

$$\zeta_P = \xi_{e_1} + \alpha \xi_{e_2^*} , \quad (5.12)$$

with $\alpha \neq 0$. The case $\alpha = \frac{1}{\mu}$ was discussed in [16]. For general non-zero α we wish to find the geometry resulting from quotienting (5.1) by (5.12). To this end it is useful to find a coordinate chart such that $\zeta_P^a = \left(\frac{\partial}{\partial z}\right)^a$. This is achieved by the following change of

coordinates:

$$\begin{aligned}
x^1 &= -\cos(\mu u) z - \alpha \mu \sin(\mu u) \frac{w}{\sqrt{g(u)}} \\
x^2 &= -\alpha \mu \sin(\mu u) z + \cos(\mu u) \frac{w}{\sqrt{g(u)}} \\
-v &= \frac{1}{2} (1 - \alpha^2 \mu^2) \mu \sin(\mu u) \cos(\mu u) z^2 + \alpha \mu^2 z \frac{w}{\sqrt{g(u)}} - V,
\end{aligned} \tag{5.13}$$

where

$$g(u) \equiv \cos^2(\mu u) + \alpha^2 \mu^2 \sin^2(\mu u). \tag{5.14}$$

In terms of the new coordinates we can rewrite the metric (5.1) as

$$\begin{aligned}
ds^2 &= -2 du dV - \mu^2 [w^2 + \sum_{i \neq 1,2} x^i x^i] du^2 + dw^2 + \sum_{i \neq 1,2} dx^i dx^i \\
&+ g(u) \left(dz + \frac{\alpha \mu^2}{g(u)^{3/2}} w du \right)^2.
\end{aligned} \tag{5.15}$$

The metric (5.15) can in the Kaluza-Klein sense be thought of as a nine-dimensional metric along with a dilaton and gauge field. The nine dimensional metric in string frame is just the first line of (5.15). This is clearly a plane wave metric written in standard Brinkmann coordinates¹⁰; we can therefore use the analysis of section 3.2 to determine an appropriate time function for the nine-dimensional part of the geometry. Since z is a spacelike isometry and $\mathcal{L}_z \tau = 0$, this will then immediately lift to a good time function for the ten-dimensional geometry (5.15).

Explicitly, a suitable time function in the coordinates of (5.15) is

$$\tau(u, V, w, x^i) = u + \frac{1}{\mu} \tan^{-1} \left(\frac{2 \mu V}{1 + \mu^2 w^2 + \sum_{i \neq 1,2} \mu^2 (x^i x^i)} \right). \tag{5.16}$$

Expressing this time function in the original coordinate system of (5.1), we see that it takes the form

$$\tau(u, v, x^i) = u + \frac{1}{\mu} \tan^{-1} \left(\frac{2 \mu [v + A_1(u, x^1, x^2)]}{1 + \sum_{i \neq 1,2} \mu^2 (x^i x^i) + A_2(u, x^1, x^2)} \right), \tag{5.17}$$

¹⁰ Note that the dilaton here is just a function of u and hence the Einstein frame metric in nine dimensions is also a plane wave. Also, since the nine-dimensional metric is a plane wave, this quotient is generically preserving 16 supersymmetries; it can preserve additional supersymmetries at special values of α , as discussed in [16]. Note that once we consider $\omega_{ij} \neq 0$, we will generically break all the supersymmetry.

where

$$\begin{aligned}
A_1(u, x^1, x^2) &= \frac{\mu}{2g(u)^2} [\cos(\mu u) x^1 + \alpha \mu \sin(\mu u) x^2] \times \\
&\quad [(g(u) + \alpha^2 \mu^2) \sin(\mu u) x^1 - \alpha \mu (g(u) + 1) \cos(\mu u) x^2] \quad (5.18) \\
A_2(u, x^1, x^2) &= \frac{\mu^2}{g(u)} (\alpha \mu \sin(\mu u) x^1 - \cos(\mu u) x^2)^2.
\end{aligned}$$

This is somewhat more complicated than the time functions considered previously, but it is straightforward to check that it is a good time function on the plane wave geometry, and furthermore that it satisfies the condition $\mathcal{L}_{\zeta_P} \tau = 0$ for ζ_P given by (5.9) even once we allow $\omega_{ij} \neq 0$ for $i, j \neq 1, 2$. This thus provides explicit time functions for a substantial subclass of the Killing vectors (5.9).

We have argued that all quotients in the pp-wave class lead to subquadratic pp-wave metrics on Kaluza-Klein reduction to nine dimensions. Hence all quotients in this class are stably causal. We will not discuss the construction of time functions for the more general case, as the expressions become considerably more complicated.

6. Discussion

The main aim of this paper has been to explore the recent discovery [9], [10], [6], [11], that there are quotients of the plane wave spacetime by everywhere spacelike isometries which lead to CTCs. First, we note that this implies that requiring the isometry to be spacelike is a necessary but, disappointingly, not a sufficient condition for the quotient to inherit the causal property (*i.e.*, the absence of CTCs) from the parent spacetime. Our first result was to find a necessary and sufficient condition to preserve the property of stable causality under a quotient. The condition is simply that there exist a time function τ on the parent spacetime M which is invariant under our isometry: $\mathcal{L}_{\xi} \tau = 0$, where ξ^a is the Killing vector we wish to quotient along. Since stable causality implies causality, this provides a sufficient condition for the absence of CTCs in the quotient.

Our other main result was to classify the quotients of the BFHP plane wave by everywhere spacelike isometries. We showed that the quotient by a spacelike isometry will introduce CTCs if and only if the isometry includes some translation in u . This is simply because any two points which are separated in u by more than π/μ are timelike separated, so if we identify points with images that are translated in u , eventually we must identify timelike separated points. For the cases where the isometry does not involve translation

in u , we applied the general condition we derived to show that the quotients in this class all preserve stable causality.

We have seen that CTCs in spacelike quotients can appear in two circumstances. They can appear in plane wave spacetimes, where their occurrence is connected to the special features of the light cones in these solutions. They can also appear when we consider a spacetime M which is itself the quotient of some parent spacetime N along a spacelike isometry, $M = N/G$, so that M has a spacelike isometry with non-contractible S^1 orbits. A similar construction is possible for any spacetime with a noncontractible odd-dimensional sphere factor. We gave the construction for BTZ, which is an example of this type, in section 1.1, but there are clearly many more examples in this class which might be discussed. It is not clear if these are the only circumstances in which such CTCs can occur; it would be interesting to understand the possibilities more systematically. In particular, it is tempting to conjecture: *For any globally hyperbolic space-time M with trivial $\pi_n(M)$ for n odd, any quotient M/G by a subgroup G of the isometry group generated by an everywhere spacelike Killing vector ξ^a will be causal.* That is, we conjecture that spacelikeness of the Killing vector is a sufficient condition for the absence of CTCs so long as the spacetime does not have any noncontractible cycles of odd dimension. An interesting direction for further exploration of this family of ideas is to investigate whether or not the null brane class of quotients of flat space are stably causal. Note that the examples discussed in [9], are not globally hyperbolic (as is the case with many p-brane geometries) and hence are not a counter-example to our conjecture.

It is interesting to compare our classification of quotients for the BFHP plane wave to the classification for flat space [14] and the investigation of AdS [15]. Unlike in those cases, we have not found any qualitatively new features in the general quotients; there is no analog of the null brane quotient for the plane wave. This is simply because the canonical form for the isometry (5.9) involves just translation-like and rotation generators. The quotients which preserve stable causality are simple generalisations of the quotients studied in [16]. In particular, as in [16], Kaluza-Klein reduction along the isometries which do not involve rotations gives rise to plane wave metrics in nine dimensions. More general quotients in this class lead to subquadratic pp-waves.

It should be easy to extend this analysis to quotients of more general plane waves. It seems natural to conjecture that there will be quotients along spacelike isometries which

lead to CTCs only when the $A_{ij}(u)$ are actually constants¹¹, so that translation in u is an isometry, and the isometry we quotient along involves a translation in u . A similar discussion should also be possible for the stably causal pp-waves where $F(u, x^i)$ is independent of u , in cases where they have such spacelike isometries.

In passing, we also noted that, as first demonstrated in [13], there are pp-waves which are not stably causal; in fact, which are not even weakly distinguishing. This is interesting for two reasons: There have been efforts to study string theory on such pp-waves [28], [31], [32]. Although the analysis works in the usual way in light-cone gauge, the fact that the spacetime is not weakly distinguishing should encourage us to treat such analyses with some care. On the other hand, it is interesting to have a natural example of a spacetime which fails to be distinguishing; the textbook examples of such partial failures of the causality conditions tend to be quite artificial. Perhaps further study of these spacetimes will enable us to learn more about the consequences of violating some of the causality conditions.

Acknowledgements

We would like to thank Liat Maoz and Jeremy Michelson for discussions. We are grateful to the organisers of the workshop on String theory and Quantum Gravity in Amsterdam for their hospitality during the concluding stages of this project. VH is supported by NSF Grant PHY-9870115, while MR acknowledges support from the Berkeley Center for Theoretical Physics and also partial support from the DOE grant DE-AC03-76SF00098 and the NSF grant PHY-0098840. SFR is supported by an EPSRC Advanced Fellowship.

¹¹ The matrix A_{ij} must also have at least one positive eigenvalue, so that the light cones have the same feature as in the BFHP case [26].

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