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Spectra in Standard-like $\mathbf{Z}_3$ Orbifold Models

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Abstract

General features of the spectra of matter states in all 175 models found in a previous work by the author are discussed. Only twenty patterns of representations are found to occur. Accomodation of the Minimal Supersymmetric Standard Model (MSSM) spectrum is addressed. States beyond those contained in the MSSM and nonstandard hypercharge normalization are shown to be generic, though some models do allow for the usual hypercharge normalization found in $SU(5)$ embeddings of the Standard Model gauge group. The minimum value of the hypercharge normalization consistent with accommodation of the MSSM is determined for each model. In some cases, the normalization can be smaller than that corresponding to an $SU(5)$ embedding of the Standard Model gauge group, similar to what has been found in free fermionic models. Bizzare hypercharges typically occur for exotic states, allowing for matter which does not occur in the decomposition of $SU(5)$ representations—a result which has been noted many times before in four-dimensional string models. Only one of the twenty patterns of representations, comprising seven of the 175 models, is found to be without an anomalous $U(1)$. The sizes of nonvanishing vacuum expectation values induced by the anomalous $U(1)$ are studied. It is found that large radius moduli stabilization may lead to the breakdown of σ-model perturbativity. Various quantities of interest in effective supergravity model building are tabulated for the set of 175 models. In particular, it is found that string moduli masses appear to be generically quite near the gravitino mass. String scale gauge coupling unification is shown to be possible, albeit contrived, in an example model. The intermediate scales of exotic particles are estimated and the degree of fine-tuning is studied.

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Disclaimer

1 Introduction

Since its introduction, the *heterotic string* [1] has offered the possibility that it may provide a unifying description of all fundamental interactions. However, the theory as originally formulated has a ten-dimensional space-time. To construct a four-dimensional theory, one typically associates six of the spatial dimensions of the original theory with a very small compact space. One route to "compactifying" the six extra dimensions, which has been the subject of intense research for several years now, is to take the six-dimensional space to be an *orbifold* [2, 3].

Four-dimensional heterotic string theories obtained by orbifold compactification take two broad paths to the treatment of internal string degrees of freedom not associated with four-dimensional space-time. On the one hand, these degrees of freedom are associated with two-dimensional *free fermionic* fields [4]; on the other, some are associated with two-dimensional *bosonic* fields propagating in a constant background.

Remarkable progress in the construction of realistic four-dimensional *free fermionic* heterotic string models [5] has been made in the last several years: a high standard has been established recently by Cleaver, Faraggi, Nanopoulos and Walker in their construction and analysis [6] of a Minimal Superstring Standard Model based on the free fermionic model of Ref. [7]. The Minimal Superstring Standard Model has only the matter content of the Minimal Supersymmetric Model[1] (MSSM) at scales significantly below the string scale $\Lambda_H \sim 10^{17}$ GeV. Furthermore, the hypercharge normalization (discussed in detail below) is conventional.

Similarly realistic four-dimensional *bosonic* heterotic string models have not yet been engineered, though the foundations of such an effort were laid some time ago [2, 3, 9, 10]. Some of the most promising models were of the Z_3 orbifold variety, with nonvanishing *Wilson lines* (discussed below) chosen such that the matter spectrum naturally had three generations. One such model was introduced by Ibáñez, Kim, Nilles and Quevedo in Ref. [11], which we will refer to as the Bosonic Standard-Like-I (BSL-I) model. The model was subsequently studied in great detail by two groups: Ibáñez, Nilles, Quevedo et al. in Refs. [12, 13]; Casas and Muñoz in Refs. [14]. As is often the case in supersymmetric models, the vacuum in the BSL-I model is not unique; different choices lead to different low energy effective theories. A particularly encouraging vacuum was the one chosen by Font, Ibáñez, Quevedo and Sierra (FIQS) in Section 4.2 of Ref. [13]; in what follows, we will refer to this effective string-derived theory as the FIQS model. Departures from realism in the FIQS model were pointed out recently in [15] and [16]. In the latter article, we suggested that a scan over three generation constructions analogous to the BSL-I model be conducted, in the search for a more realistic model. Ultimately, we would like to attempt models comparable to the free fermionic Minimal Superstring Standard Model. Part of the purpose of this paper is to report some of our progress toward this goal.

This article is devoted to a model *dependent* study of bosonic standard-like Z_3 orbifolds. Model *independent* analyses are appealing because they paint a wide swath and highlight general predictions of a class of theories. Too often, however, one is left wondering whether the limiting assumptions made in such analyses really reflect the properties of some class of explicit, consistent underlying theories. At some point it is necessary to get dirt on oneself and investigate whether or not the broad assumptions made in model independent analyses are ever valid. This is one

[1]For a review of the MSSM, see for example Refs. [8].

of the motivations for model dependent studies such as the one contained here. Another reason to study explicit string constructions is that certain peculiarities are more readily apparent under close examination. One well-known example, which will be discussed in detail below, is the generic presence of exotic states with hypercharges which do not occur in typical Grand Unified Theories[2] (GUTs).

One objection to model dependent studies in four-dimensional string theories is that the number of possible constructions is enormous. However, in at least one respect the hugeness is not as great as it would appear. Already in the second of the two seminal papers by Dixon, Harvey, Vafa and Witten, it was realized that many "different" orbifold models are in fact equivalent [3]. Casas, Mondragon and Muñoz (CMM) have shown in detail how equivalence relations among orbifold compactifications can be used to greatly reduce the number of *embeddings* (in the present context, a set $\{V, a_1, a_3, a_5\}$ of sixteen-dimensional vectors) which must be studied in order to produce all physically distinct models within a given class of constructions [20]. In particular, they applied these techniques to a special class of bosonic standard-like heterotic string models; for convenience, we will refer to this as the *BSL_A class.* For completeness, we give its technical definition below. The meanings of the terms used here will be made clear in Section 2, as much as is required to follow the discussion in the remainder of this article. For further details, the interested reader is encouraged to consult the various reviews [21, 22], texts [23], and references therein. In simpler terms, the definition given here implies that we follow the construction outlined in [9], with three generations by the method suggested in [11], subject to additional restrictions imposed by CMM (items (iii) and (iv) below).

Definition 1 *The* $\mathbf{BSL_A}$ **class** *consists of all bosonic $E_8 \times E_8$ heterotic Z_3 orbifold models with the following properties:*

(i) symmetric treatment of left- and right-movers and a shift embedding V of the twist operator θ;

(ii) two nonvanishing Wilson lines a_1, a_3 and one vanishing Wilson line $a_5 = 0$;

(iii) observable sector gauge group

$$G_O = SU(3) \times SU(2) \times U(1)^5; \tag{1.1}$$

(iv) a quark doublet representation $(3, 2)$ in the untwisted sector.

CMM found that models satisfying (i-iv) may be described (in part) by one of just nine *observable sector* embeddings; here, "observable" refers to the first eight entries of each of the nonvanishing embedding vectors, V, a_1, a_3; it is this which determines properties (iii) and (iv) listed above. In a previous article [24], we showed that these nine observable sector embeddings are equivalent to a smaller set of six embeddings. To fully specify a model, the observable sector embedding must be completed with a *hidden sector* embedding—the last eight entries of each of the nonvanishing embedding vectors, V, a_1, a_3. In Ref. [24] we enumerated all possible ways to complete the embeddings in the hidden sector, using equivalence relations to reduce this set to a "mere" 192 models. Surprisingly, only five hidden sector gauge groups G_H were found to be possible. These possibilities are shown in Table I.

[2]For a review of non-supersymmetric GUTs see Refs. [17, 18] and for supersymmetric extensions see Refs. [19].

Case	G_H
1	$SO(10)\times U(1)^3$
2	$SU(5)\times SU(2)\times U(1)^3$
3	$SU(4)\times SU(2)^2\times U(1)^3$
4	$SU(3)\times SU(2)^2\times U(1)^4$
5	$SU(2)^2\times U(1)^6$

Table I: Allowed hidden sector gauge groups G_H.

The Z_3 orbifold models studied here have $N=1$ local supersymmetry (supergravity) at the string scale. In our analysis, we assume that this supersymmetry is broken dynamically via gaugino condensation of an asymptotically free condensing group G_C in the hidden sector. That is, the vacuum expectation value (*vev*) of the gaugino bilinear $\langle\lambda\lambda\rangle$ acquires a nonvanishing value. This operator has mass dimension three; we therefore define the dynamically generated *condensation scale* Λ_C by

$$\langle\lambda\lambda\rangle=\Lambda_C^3. \tag{1.2}$$

To estimate the value of Λ_C, consider the one loop evolution of the running gauge coupling $g_C(\mu)$ of G_C:

$$\frac{dg_C}{d\ln\mu}=\beta(g_C)=\frac{b_C g_C^3}{16\pi^2}. \tag{1.3}$$

The β function coefficient b_C is given by

$$b_C=-3C(G_C)+\sum_R X_C(R). \tag{1.4}$$

Here, $C(G_C)$ is the eigenvalue of the quadratic Casimir operator for the adjoint representation of the group G_C while $X_C(R)$ is the *Dynkin index* for the representation R, given by $\mathrm{tr}_R\,(T^a)^2=X_C(R)$ in a Cartesian basis for the generators T^a; we adhere to a normalization where $X_C=1/2$ for an $SU(N)$ fundamental representation. The sum runs over chiral supermultiplet representations. Provided b_C is negative, the coupling turns strong at low energies and the dynamical scale Λ_C is generated, in analogy to Λ_{QCD}. The running of gauge couplings from an initial unified value $g_H\sim 1$ at a unification scale, which in our case is the string scale $\Lambda_H\sim 10^{17}$ GeV, gives

$$\Lambda_C\sim\Lambda_H\exp(8\pi^2/b_C g_H^2), \tag{1.5}$$

where we have identified Λ_C with the Laundau pole of the running coupling.

Soft mass terms in the low energy effective lagrangian split the masses of supersymmetry multiplets, and thereby break supersymmetry; partners to Standard Model (SM) particles are generically heavier by the soft mass scale M_{SUSY}. The soft terms arise from nonrenormalizable interactions in the supergravity lagrangian, with masses proportional to the gaugino condensate $\langle\lambda\lambda\rangle$, suppressed by inverse powers of the (reduced) Planck mass, $m_P\equiv 1/\sqrt{8\pi G}=2.44\times10^{18}$ GeV. On dimensional grounds, one expects that the observable sector supersymmetry breaking scale M_{SUSY} is given by

$$M_{\mathrm{SUSY}}\approx\zeta\cdot\langle\lambda\lambda\rangle/m_P^2=\zeta\cdot\Lambda_C^3/m_P^2, \tag{1.6}$$

with (naively) $\zeta \sim \mathcal{O}(1)$. For supersymmetry to protect the gauge hierarchy $m_Z \ll m_P$ between the electroweak scale and the fundamental scale, one requires, say, $M_{\text{SUSY}} \lesssim 10$ TeV. Then (1.6) with $\zeta \sim \mathcal{O}(1)$ implies $\Lambda_C \lesssim 4 \times 10^{13}$ GeV. On the other hand, direct search limits [25] on charged superpartners require, say, $M_{\text{SUSY}} \gtrsim 50$ GeV, which translates into $\Lambda_C \gtrsim 7 \times 10^{12}$ GeV. More precise results may be obtained, for instance, with the detailed supersymmetry breaking models of Binétruy, Gaillard and Wu (BGW) [26] as well as subsequent ellaborations by Gaillard and Nelson [27]. These calculations confirm the naive expectation (1.6), except that

$$\mathcal{O}(10^{-2}) \lesssim \zeta \lesssim \mathcal{O}(10^{-1}), \tag{1.7}$$

which tends to increase Λ_C. For example, the lower bound implied by $M_{\text{SUSY}} \gtrsim 50$ GeV changes to $\Lambda_C \gtrsim 9 \times 10^{12}$ GeV if $\zeta \approx 0.4$, near the upper end of the range (1.7). The result is that

$$\mathcal{O}(10^{13}) \lesssim \frac{\Lambda_C}{\text{GeV}} \lesssim \mathcal{O}(10^{14}) \tag{1.8}$$

is a reasonably firm estimate.

For $G_C = SU(2)$ with no matter, one has $b_C = -6$. Substituting into (1.5), one finds $\Lambda_C \sim 10^{11}$ GeV. On the other hand, (1.5) is a crude estimate; studies of the BGW effective theory show that the naive estimate (1.5) can receive significant corrections due to a variety of effects, and deviations by an order of magnitude are certainly possible. Thus, a more reliable bound is $\Lambda_C \lesssim 10^{12}$ GeV. Since $b_C > -6$ when G_C charged matter is present, the limit $\Lambda_C \lesssim 10^{12}$ GeV is saturated by the case with no matter. In the models considered here, as will be seen below, $SU(2)$ groups always have many, many matter representations, and it is unlikely that *all* of them would acquire effective mass couplings *at the unification scale* Λ_H so that $b_C = -6$ and $\Lambda_C \sim 10^{12}$ GeV could be achieved. In any case, 10^{12} GeV is below the lower bound in (1.8), set by $M_{\text{SUSY}} \gtrsim 50$ GeV, the firmer of the soft scale requirements, so having $b_C = -6$ is marginal at best. Case 5 of Table I was therefore considered to be an unviable hidden sector gauge group. Certainly, Cases 1 to 4 appear more promising. Eliminating the models with the Case 5 gauge group, only 175 models remain. The matter spectra of these models are the topic of discussion for the present paper.

Quite commonly in the models considered here, some of the $U(1)$ factors contained in the gauge group $G = G_O \times G_H$ are apparently anomalous: tr $Q_a \neq 0$. Redefinitions of the charge generators allow one to isolate this anomaly such that only one $U(1)$ has an apparent trace anomaly. We denote this factor of G as $U(1)_X$. The associated anomaly is canceled by the Green-Schwarz mechanism [28]: tree level couplings between the $U(1)_X$ vector multiplet and the two-form field strength (dual to the universal axion) are added to the effective action in such a way that the one loop $U(1)_X$ anomaly is canceled [29]; the $U(1)_X$ only appears to be anomalous. When the cancellation is done in a supersymmetric fashion, a Fayet-Illiopoulos (FI) term ξ for $U(1)_X$ is induced; we have, for example, described this effect at the effective supergravity level in the Appendix of [16]. The result of these considerations is an effective D-term for $U(1)_X$ of the form:

$$D_X = \sum_i \frac{\partial K}{\partial \phi^i} \hat{q}_i^X \phi^i + \xi, \qquad \xi = \frac{g_H^2 \,\text{tr}\, \hat{Q}_X}{192\pi^2} m_P^2. \tag{1.9}$$

The $U(1)_X$ generator $\hat{Q}_X$ has a normalization consistent with unification (discussed further below), $\hat{q}_i^X$ is the charge of the scalar ϕ^i with respect to $\hat{Q}_X$, K is the Kähler potential and and g_H is the unified coupling mentioned above. Since the scalar potential of the effective supergravity theory at the string scale Λ_H contains the term $g_H^2 D_X^2/2$, some scalar fields generically shift to cancel the FI term (i.e., $\langle D_X \rangle = 0$ to leading order) and get vevs of order $\sqrt{|\xi|}$. Adopting the terminology of [16], we will refer to these as *Xiggs* fields, since they are associated with the breaking of $U(1)_X$ (and typically other factors of G) via the Higgs mechanism. Generally, the way in which the FI term may be canceled is not unique and continuously connected vacua result. Pseudo-Goldstone modes, *D-moduli* [15], parameterize the flat directions; dynamical supersymmetry breaking and loop effects are required to select the true vacuum and render these scalar fields massive [15, 30]. (Moduli parameterizing flat directions of the scalar potential are a generic feature of supersymmetric field theories [31]. An example of D-moduli was noted previously in the study of D-flat directions in [14], parameterized there by the quantity "λ," which interpolated between various vacuua. Such moduli have also been noted in the study of flat directions in free fermionic string models, for instance in Ref. [32].) The FI term ξ has mass dimension two and its square root therefore gives the approximate scale of $U(1)_X$ breaking, which we hereafter denote

$$\Lambda_X \equiv \sqrt{|\xi|} = \frac{\sqrt{|\operatorname{tr}\, \hat{Q}_X|}}{4\pi\sqrt{12}} \times g_H m_P. \tag{1.10}$$

In the examples below we will find by explicit calculation of $\operatorname{tr}\, \hat{Q}_X$ in each of the 175 models that $\Lambda_X \approx \Lambda_H \sim 0.2\, m_P$.

In Section 2 we discuss the determination of the spectrum of massless states from the underlying string theory. We discuss in careful detail how the gauge group G is determined. We then describe in similar detail how one determines the irreducible representations (*irreps*) and $U(1)$ charges of matter states. In Section 3 we make observations on the general features of the 175 models, as determined from the spectrum of massless states and their $U(1)$ charges. We find that only 20 patterns of irreps occur in the 175 models. In Section 4, we delve into difficulties associated with the electroweak hypercharge. We explore the most natural definition of hypercharge: to embed it into an $SU(5)$ gauge group which also contains the $SU(3) \times SU(2)$ of the observable gauge group G_O. As a further condition, we require that the $SU(5)$ is a subgroup of the observable E_8 factor of the "parent" $E_8 \times E_8$ theory. We find that none of the 175 models can accommodate the full MSSM spectrum when this is done; although adequate $SU(3) \times SU(2)$ irreps are present, the hypercharge quantum numbers are not correct for enough of the irreps. We will explain how the presence of states with unusual hypercharge values corresponds to the phenomenon of *charge fractionalization* in orbifolds. The absense of states with correct hypercharges for the $SU(5)$ embedding leads us to the less attractive alternative of engineering a hypercharge which is a general linear combination of the several $U(1)$s contained in G *and* generators of the Cartan subalgebras of nonabelian factors contained in the hidden gauge group G_H. We find that this *does* allow for the accommodation of the MSSM spectrum. At the same time, rather bizarre hypercharges for extra matter are found to be generic, as well as nonstandard hypercharge normalization. In Section 5 we illustrate these unconventional results with a detailed examination of one of the 175 models. We describe various assignments of the MSSM to the spectrum of 153 chiral multiplets of matter states present in the model, and the hypercharges and nonstandard hypercharge normalizations which occur. In spite of nonstandard hypercharge normalization, it is found that successful unification of gauge couplings at the string scale $\Lambda_H \sim 10^{17}$ GeV is possible. However, the unification scenario in this model is rather ugly, since it requires that exotic states with fractional electric charges be introduced at intermediate scales—between the electroweak scale and the string scale. We suggest how one might circumvent phenomenological difficulties with fractionally charged states having intermediate scale masses. In Section 6 we make concluding remarks and suggest directions for further research. In Appendix A we review cancellation of the *modular anomaly*. In Appendix B we present our more lengthy sets of tables.

Since each model contains $3\times\mathcal{O}(50)$ matter irreps and eight or nine independent $U(1)$ generators, it is for obvious reasons that we do not provide in full detail the spectra and charges of all 175 models. However, upon request, complete tables of the matter spectra and $U(1)$ charges are available from the author.

2 Determination of Spectra

Several textbooks discussing heterotic orbifolds are available [23]. In addition, many reviews have been written over the years [21], including the recent (and widely available) review by Bailin and Love [22]. Rather than repeat lengthy discussions given elsewhere, we have chosen to avoid many details of the underlying string theory and present a somewhat heuristic description. Our intent is to provide just enough information to allow one to determine the spectrum of gauge and matter states below the string scale, for the class of orbifold models considered here. To this end, we provide a set of "recipes" for the spectrum determination at the close of this section. These are designed as a tool for the "string novice" who merely wishes to study these models from a low energy, phenomenological point of view.

To make contact with the world of particle physics, one is interested in the effective theory produced by heterotic string theory at energy scales far below the string scale $\Lambda_H \sim 10^{17}$ GeV. The first step in constructing such a theory is to determine the string states with masses much less than Λ_H. Secondly, one must derive the interactions between these states and an appropriate description for these interactions. In the context of perturbative string theory, there exist systematic methods for the accomplishment of these tasks, subject to certain technical difficulties which we will not discuss here, since for the most part we work only at leading order in string perturbation theory. The perturbation series corresponds to string world-sheet (the two-dimensional surface swept out by the string) diagrams of increasing complexity. These are labeled by the *genus* of the diagram, starting at genus zero, often referred to as "tree level" in string theory. The next order, genus one, is often referred to as the "one loop level" in string theory, because the world-sheet diagram is a two-dimensional torus. Interactions are described by scattering amplitudes between string states. In particular, these amplitudes can be studied in the limit where external momenta are taken to be much less than the string scale, often referred to as the *zero-slope limit* [33]. One then matches the results onto a field theory; that is, one constructs a local field theory lagrangian which, when quantized, would have single particle states with the same properties (mass, spin, charge, etc.) as the low-lying string states and scattering amplitudes which match the string scattering amplitudes at low external momenta. Thus, one can talk about the "particle" states which arise from the "field theory limit" of the string.

A study of the heterotic string at tree level shows that the string states are organized into a tower of mass levels, with the lowest level of states massless. For the four-dimensional heterotic string, subject to certain qualifications which will not trouble us here (e.g., the large radius limit of the extra dimensions where massive string states can drop below Λ_H), the only string states with masses significantly below Λ_H are those which lie at the massless level of the string. However, genus one corrections can be significant if, for example, an anomalous $U(1)_X$ is present. On the effective field theory side, this correction is represented by the FI term which is induced from cancellation of the $U(1)_X$ anomaly. The tree level spectrum of masses can be dramatically altered. For this reason, we hereafter refer to the states which are massless at tree level as *pseudo-massless.* Many of the pseudo-massless states have masses near Λ_H once the one loop corrections are accounted for! This is because the Xiggses acquire $\mathcal{O}(\Lambda_X)$ vevs; explicit calculations detailed below show that $\Lambda_H/1.73 \leq \Lambda_X \leq \Lambda_H$ in the 175 models studied here, indicating that Λ_X is more or less the string scale Λ_H. The Xiggs vevs cause several chiral (matter) superfields to get effective "vector" superpotential couplings

$$W \ni \frac{1}{m_P^{n-1}} \langle \phi^1 \cdots \phi^n \rangle A A^c. \tag{2.1}$$

Here, A and A^c are conjugate with respect to the gauge group which survives after spontaneous symmetry breaking caused by the $U(1)_X$ FI term. The right-hand side of (2.1) is an effective supersymmetric mass term, which generally results in masses

$$m_{eff} \sim \mathcal{O}(\Lambda_X^n / m_P^{n-1}) \approx \mathcal{O}(\Lambda_H^n / m_P^{n-1}). \tag{2.2}$$

With $n = 1$ in (2.2), the effective masses are near the string scale. Due to the numerous gauge symmetries present in the models considered here, as well as discrete symmetries known as *orbifold selection rules* (see for example [34, 13, 22]), not all operators of the form AA^c will have couplings with $n = 1$ in (2.1). Because of this, a hierarchy of mass scales is a general prediction of models with a $U(1)_X$ factor (all but seven of the 175 models studied here). We return to this point in Section 5, where we briefly discuss gauge coupling unification.

By construction, the spectrum is that of an $N = 1$ four-dimensional locally supersymmetric theory. Furthermore, the compact space is a six-dimensional Z_3 *orbifold* (defined below). Certain parts of the spectrum are well-known to be present by virtue of these facts alone [2]. We will not discuss these states in this section except to note their existence: the supergravity multiplet, the *dilaton* supermultiplet and nine chiral multiplets T^{ij} whose scalar components correspond to the *Kähler-* or *T-moduli* of the compact space. (See for example [35] for a discussion of toric moduli.)

The remainder of the spectrum depends on the choice of *embedding,* and it is this part of the spectrum which we must calculate separately for each of the 175 models. The embedding-dependent spectrum consists of massless chiral multiplets of matter states and massless vector multiplets of gauge states. Once the vacuum shifts to cancel the FI term, some gauge symmetries are spontaneously broken and chiral matter multiplets (which are linear combinations of Xiggses) get "eaten" by some of the vector multiplets to form massive vector multiplets. Examples of the "degree of freedom balance sheet" may be found for example in [15].

2.1 The $\mathbf{Z_3}$ Orbifold

The six-dimensional Z_3 orbifold may be constructed from a six-dimensional Euclidean space $\mathbf{R}^6$. One defines basis vectors $e_1, \ldots, e_6$ satisfying

$$e_i^2 = e_{i+1}^2 = 2\,R_i^2, \qquad e_i \cdot e_{i+1} = -1\,R_i^2, \qquad i = 1, 3, 5, \tag{2.3}$$

with a vector $x \in \mathbf{R}^6$ having real-valued components:

$$x = \sum_{i=1}^{6} x^i e_i, \qquad x^i \in \mathbf{R} \quad \forall\, i = 1, \ldots, 6. \tag{2.4}$$

Each of the three pairs e_i, e_{i+1} ($i = 1, 3, 5$) define a two-dimensional subspace which is referred to below as the "ith complex plane." The ith such pair also defines a two-dimensional $SU(3)$ root lattice, obtained from the set of all linear combinations of the form $n_i e_i + n_{i+1} e_{i+1}$ with n_i, n_{i+1} both integers. Taking together all six basis vectors $e_1, \ldots, e_6$, we obtain the $SU(3)^3$ root lattice $\Lambda_{SU(3)^3}$, formed from all linear combinations of the basis vectors $e_1, \ldots, e_6$ with integer coefficients:

$$\Lambda_{SU(3)^3} = \left\{ \sum_{i=1}^{6} \ell^i e_i \;\middle|\; \ell^i \in \mathbf{Z} \right\}. \tag{2.5}$$

Note that the *radii* R_i in (2.3) are not fixed; neither are angles not appearing in (2.3), such as $e_1 \cdot e_3$. These free parameters determine the size and shape of the unit cell of the lattice $\Lambda_{SU(3)^3}$, and are encoded in the T-moduli T^{ij} mentioned above. These moduli depend on the metric $G_{ij} = e_i \cdot e_j$ ($i, j = 1, \ldots, 6$) of the six-dimensional compact space, as well as an antisymmetric two-form B_{ij}. Of particular interest are the *diagonal* T-moduli $T^i \equiv T^{ii}$. Up to normalization conventions on the T^i and B_{ij}, the diagonal T-moduli are defined by

$$T^i = \sqrt{\det G^{(i)}} + i B_{i,i+1}, \quad i = 1, 3, 5. \tag{2.6}$$

Here, $G^{(i)}$ is the metric of the ith complex plane:

$$G^{(i)} = \begin{pmatrix} e_i \cdot e_i & e_i \cdot e_{i+1} \\ e_{i+1} \cdot e_i & e_{i+1} \cdot e_{i+1} \end{pmatrix} = R_i^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.7}$$

Translations in $\mathbf{R}^6$ by elements of $\Lambda_{SU(3)^3}$,

$$x \to x + \ell, \qquad \ell \in \Lambda_{SU(3)^3}, \qquad \forall\, x \in \mathbf{R}^6, \tag{2.8}$$

form what is referred to as the *lattice group.* A rotation θ in $\mathbf{R}^6$ is defined, with action on the basis vectors:

$$\theta \cdot e_i = e_{i+1}, \qquad \theta \cdot e_{i+1} = -e_i - e_{i+1}, \qquad i = 1, 3, 5. \tag{2.9}$$

Typically, θ is referred to as the orbifold *twist operator.* It is easy to check that $\theta^3 = 1$. The twist operator θ generates the orbifold *point group,*

$$Z_3 = \{1, \theta, \theta^2\}. \tag{2.10}$$

It can be seen from (2.9) that the twist operator maps any element of $\Lambda_{SU(3)^3}$ into $\Lambda_{SU(3)^3}$. Consequently, we can define the product group generated by the combined action of the point group and the lattice group. This group is referred to as the *space group* S and a generic element is written (ω, ℓ), with $\omega \in Z_3$ and $\ell \in \Lambda_{SU(3)^3}$. Acting on any element $x \in \mathbf{R}^6$,

$$(\omega, \ell) \cdot x = \omega \cdot x + \ell = \sum_{i=1}^{6} [x^i (\omega \cdot e_i) + \ell^i e_i], \tag{2.11}$$

where $\omega \cdot e_i$ can be obtained from (2.9). It is not hard to check the multiplication rule

$$(\omega, \ell) \cdot (\omega', \ell') = (\omega\omega', \omega\ell' + \ell). \tag{2.12}$$

The space group has four generators: $(\theta, 0)$, $(1, e_1)$, $(1, e_3)$ and $(1, e_5)$. For example, using (2.9,2.12) one can write

$$(1, e_2) = (\theta, 0) \cdot (1, e_1) \cdot (\theta, 0) \cdot (\theta, 0). \tag{2.13}$$

Certain points $x_f \in \mathbf{R}^6$ are fixed under the action of space group elements with $\omega = \theta$:

$$(\theta, \ell) \cdot x_f = \theta \cdot x_f + \ell = x_f. \tag{2.14}$$

It is not hard to solve this equation; one finds that the *fixed points* are in one-to-one correspondence with elements of $\Lambda_{SU(3)^3}$:

$$x_f(\ell) = (1 - \theta)^{-1} \cdot \ell. \tag{2.15}$$

To define the orbifold, denoted $\Omega = \mathbf{R}^6/S$, one demands that points $x, x' \in \mathbf{R}^6$ be treated as equivalent if they are related to each other under the action of the space group S.

Definition 2 *The points $x, x' \in \mathbf{R}^6$ are* **equivalent** *on the orbifold $\Omega = \mathbf{R}^6/S$, notated $x' \simeq x$, if and only if there exists $(\omega, \ell) \in S$ such that $x' = (\omega, \ell) \cdot x$.*

A space constructed in this way is often referred to as a *quotient space*, because we "divide out" by the action of a discrete transformation group, in this case the space group S. It is worth noting that quotient space constructions for extra dimensions were applied in a field theory context some years prior to the construction of four-dimensional strings on orbifolds, with important consequences such as chiral fermions [36].

On the orbifold, most of the fixed points (2.15) are equivalent to each other. There are only 27 inequivalent fixed points, which can be obtained from (2.15) using

$$\ell(n_1, n_3, n_5) = n_1 e_1 + n_3 e_3 + n_5 e_5, \quad n_i = 0, \pm 1. \tag{2.16}$$

Note the correspondence between this parameterization of the fixed points and the generators of the space group which are elements of the lattice group: $(1, e_1)$, $(1, e_3)$ and $(1, e_5)$.

2.2 Boundary Conditions

At the classical level, the location of the string in the six-dimensional compact space is specified by a two parameter map $X_{\rm cl}(\sigma, \tau)$ which has a component expression of the form (2.4):

$$X_{\rm cl}(\sigma, \tau) = \sum_{i=1}^{6} X_{\rm cl}^i(\sigma, \tau)\, e_i. \tag{2.17}$$

The parameter σ labels points along the string, with $\sigma \to \sigma + \pi$ as one goes once around the string; τ labels proper time in the frame of the string. The heterotic theory is a theory of closed strings, so $X_{\rm cl}(\sigma, \tau)$ and $X_{\rm cl}(\sigma + \pi, \tau)$ should be equivalent points on the orbifold. This requirement is extended to the quantized theory $X_{\rm cl}(\sigma, \tau) \to X(\sigma, \tau)$, with $X(\sigma, \tau)$ a quantum operator. As a consequence of Definition 2, $X(\sigma,\tau)$ need only be closed up to a space group element. For the "(ω,ℓ) sector,"

$$X(\sigma+\pi,\tau) = (\omega,\ell)\cdot X(\sigma,\tau). \tag{2.18}$$

If we apply some other space group element (ω',ℓ') to (2.18), we find

$$(\omega',\ell')\cdot X(\sigma+\pi,\tau) = \Big[(\omega',\ell')\cdot(\omega,\ell)\cdot(\omega',\ell')^{-1}\Big]\cdot(\omega',\ell')\cdot X(\sigma,\tau). \tag{2.19}$$

Because $(\omega',\ell')\cdot X(\sigma,\tau)$ and $X(\sigma,\tau)$ are equivalent on the orbifold, the boundary condition

$$X(\sigma+\pi,\tau) = (\omega',\ell')\cdot(\omega,\ell)\cdot(\omega',\ell')^{-1}\cdot X(\sigma,\tau) \tag{2.20}$$

must be treated as equivalent to (2.18). That is, boundary conditions in the same *conjugacy class* as (ω,ℓ),

$$\Big\{\ (\omega',\ell')\cdot(\omega,\ell)\cdot(\omega',\ell')^{-1}\quad\Big|\quad \omega'\in Z_3,\ \ell'\in\Lambda_{SU(3)^3}\ \Big\}, \tag{2.21}$$

are equivalent because they are related to each other under the action of the space group [3]. There are 27 such conjugacy classes associated with sectors twisted by θ. There exists a correspondance between each of these conjugacy classes and one of the 27 inequivalent fixed points of the Z_3 orbifold. Since these sectors do not mix with each other under the action of the space group, we regard them as 27 *different* twisted sectors.

Nontrivial boundary conditions are typically extended to internal string degrees of freedom $\Psi(\sigma,\tau)$ not associated with the location of the string in the six-dimensional compact space. For the (ω,ℓ) sector, which has (2.18), the extension may be written schematically as

$$\Psi(\sigma+\pi,\tau) = U[(\omega,\ell)]\cdot\Psi(\sigma,\tau). \tag{2.22}$$

Consistency requires this extension to be a *homomorphism* of the space group:

$$U[(\omega,\ell)]\cdot U[(\omega',\ell')] \simeq U[(\omega,\ell)\cdot(\omega',\ell')], \tag{2.23}$$

where "$\simeq$" denotes equivalence, the precise meaning of which depends on the nature of $\Psi(\sigma,\tau)$. As mentioned above, the space group has four generators; it is therefore sufficient to specify the action of U for these generators, since the homomorphism requirement then determines U for any other element of the space group.

In particular, there exist sixteen internal bosonic degrees of freedom $X^I(\sigma,\tau),\ I=1,\ldots,16$; these are employed in the construction of a current algebra which is the source of gauge symmetry in the effective field theory. In the twisted sectors, the $X^I(\sigma,\tau)$ are typically assigned nontrivial boundary conditions according to a homomorphism U. As described above, we may define U through a map of the space group generators into the internal degrees of freedom. In the construction studied here, this consists of a set of shifts:

$$\begin{aligned} U[(\theta,0)]^I_J\,X^J(\sigma,\tau) &= X^I(\sigma,\tau)+\pi V^I, \\ U[(1,e_i)]^I_J\,X^J(\sigma,\tau) &= X^I(\sigma,\tau)+\pi a_i^I, \qquad \forall\, i=1,3,5. \end{aligned} \tag{2.24}$$

The vector V is referred to as the *shift embedding* of the space group generator $(\theta,0)$; equivalently, V embeds the twist operator θ. Likewise, the vectors a_i embed the other three space group generators

$(1,e_i)$, $i=1,3,5$ respectively. They are referred to as *Wilson lines* because of their interpretation as background gauge fields in the compact space. (It is worth noting that nontrivial gauge field configurations in an extra-dimensional compact space were used by Hosotani in a field theory context to achieve gauge symmetry breaking [37]; the nontrivial a_1, a_3 in the $\mathrm{BSL_A}$ models represent a "stringy" version of the Hosotani mechanism, allowing one to obtain standard-like G.)

Taking together the embeddings (2.24), and using the space group multiplication

$$(1,e_1)^{n_1}\cdot(1,e_3)^{n_3}\cdot(1,e_5)^{n_5}\cdot(\theta,0)=(\theta,n_1e_1+n_3e_3+n_5e_5), \tag{2.25}$$

the embedding of the twisted boundary condition (2.18) for each of the 27 twisted sectors corresponding to $(\omega,\ell)=(\theta,n_1e_1+n_3e_3+n_5e_5)$ is described by a sixteen-dimensional embedding vector $E(n_1,n_3,n_5)$:

$$\begin{aligned} X^I(\sigma+\pi,\tau) &= U[(\theta,n_1e_1+n_3e_3+n_5e_5)]^I_J\,X^J(\sigma,\tau) \\ &= X^I(\sigma,\tau)+\pi E^I(n_1,n_3,n_5), \qquad (2.26) \\ E(n_1,n_3,n_5) &= V+n_1a_1+n_3a_3+n_5a_5. \qquad (2.27) \end{aligned}$$

Consistency conditions [10, 38] for $\{V,a_1,a_3,a_5\}$ following from the homomorphism condition (2.23) have been accounted for in the embeddings enumerated in [24]. For example, $(\theta,n_1e_1+n_3e_3+n_5e_5)^3=(1,0)$ implies that we must have

$$U[(\theta,n_1e_1+n_3e_3+n_5e_5)^3]^I_J\,X^J(\sigma,\tau)=X^I(\sigma,\tau)+3\pi E^I(n_1,n_3,n_5)\simeq X^I(\sigma,\tau). \tag{2.28}$$

This last step is true because the $X^I(\sigma,\tau)$ propagate on the $E_8\times E_8$ root torus where

$$X^I(\sigma,\tau)\simeq X^I(\sigma,\tau)+\pi L^I, \qquad \forall\, L\in\Lambda_{E_8\times E_8}, \tag{2.29}$$

and the embedding vectors are constrained to satisfy $3E(n_1,n_3,n_5)\in\Lambda_{E_8\times E_8}$. The results of a detailed study of these aspects of the underlying string theory [10, 38] have been built into the embeddings given in [24] and the recipes given below.

As noted above, the boundary conditions are labeled by the conjugacy classes of the space group; it is clear that in the general case, the extension U in (2.22)—and more specifically the embedding $E(n_1,n_3,n_5)$—will be different for each conjugacy class. In the description of string states, it is therefore convenient to decompose the Hilbert space into sectors, with each sector corresponding to a particular conjugacy class. For the Z_3 orbifold, one has an *untwisted* sector, 27 *twisted* sectors corresponding to fixed point (conjugacy class) labels (n_1,n_3,n_5), $n_i=0,\pm1$, and 27 *antitwisted* sectors with similar labeling. The 27 (anti)twisted sectors are often lumped together and regarded as a single (anti)twisted sector, since the (anti)twist (i.e., the point group element) is identical among them; we prefer not to do this here. The term "twisted state," when applied to a particle, must be understood to refer to the string state taken to the field theory limit, since it is not possible to go from one end of a particle to the other! The antitwisted sectors of the Z_3 orbifold merely contain the antiparticle states of the twisted sectors, so we need not discuss them below.

2.3 $\mathbf{E_8 \times E_8}$: Progenitor

Prior to FI gauge symmetry breaking, the gauge group G is a rank sixteen subgroup of $E_8 \times E_8$. The theory on the orbifold involves "twisting" the $E_8 \times E_8$ heterotic string. Even though G is a subgroup of $E_8 \times E_8$, its description on the string side reflects the $E_8 \times E_8$ symmetry of the original theory. That is, G is "embedded into $E_8 \times E_8$." To clarify what is meant by this phrase, we rehearse a well-known example.

Recall that each irrep of a Lie group[3] can be identified with a weight diagram; points on the weight diagram are labeled by weight vectors. Well-known examples are the flavor $SU(3)_F$ weight diagrams of hadrons containing only u, d, s valence quarks. In this case, the weight vectors are two-dimensional, (λ_1, λ_2), with entries corresponding to eigenvalues of two basis elements H^1, H^2 of a Cartan subalgebra of $SU(3)_F$. If we work in the limit $m_u = m_d$, $m_s \gg m_u$, then $SU(3)_F$ is not a good symmetry, but the flavor isospin subgroup $SU(2)_F$ is. In a well-chosen basis for $SU(3)_F$, the weight diagrams of $SU(2)_F$ are one-dimensional subdiagrams of the $SU(3)_F$ weight diagrams. The points of the one-dimensional $SU(2)_F$ weight diagrams are labeled by eigenvalues of the basis element I_3 of a Cartan subalgebra of $SU(2)_F$. However, we could just as well continue to label states by the $SU(3)_F$ weight vectors; the isospin quantum numbers would be determined by an appropriate linear combination

$$I_3 = \alpha^1 H^1 + \alpha^2 H^2 \tag{2.30}$$

of $SU(3)_F$ Cartan generators. The additional information contained in the two-dimensional $SU(3)_F$ weight vectors, strangeness, determines quantum numbers under a global $U(1)_S$ symmetry group which commutes with $SU(2)_F$. The generator of $U(1)_S$ is given by

$$S = s^1 H^1 + s^2 H^2. \tag{2.31}$$

Consistency of this decomposition requires that for any irrep R of $SU(3)_F$,

$$\mathrm{tr}_R\,(I_3 S) = 0 \quad \Rightarrow \sum_{i,j=1}^{2} \kappa^{ij} \alpha^i s^j = 0, \tag{2.32}$$

where κ^{ij} is defined by

$$\mathrm{tr}_R\,(H^i H^j) = X(R)\,\kappa^{ij}. \tag{2.33}$$

To summarize, the symmetry group is $G_F = SU(2)_F \times U(1)_S$; states are conveniently labeled by $SU(3)_F$ weight vectors, which allow one to determine the quantum numbers with respect to G_F; the weight diagrams of $SU(2)_F$ are best recognized as subdiagrams of $SU(3)_F$ weight diagrams. We say that G_F is embedded into $SU(3)_F$.

In complete analogy, an irrep of the gauge symmetry group G of a given orbifold model will be described by a set of basis states labeled by weight vectors of $E_8 \times E_8$. The weights with respect to nonabelian factors of G as well $U(1)$ charges of the irrep are determined by these $E_8 \times E_8$ weight vectors, just as was the case in the $SU(3)_F$ example above. The weights of the adjoint representation are referred to as *roots.* Massless states in the untwisted sector correspond to a subset of the states in the $E_8 \times E_8$ adjoint representation. For this reason, we shall often have occasion to refer to the $E_8 \times E_8$ root system. For $E_8 \times E_8$, the adjoint representation *is* the fundamental

[3] For a review of Lie algebras and groups see for example Refs. [39, 40, 18].

representation and higher dimensional representations are obtained from tensor products of the adjoint representation with itself. These higher dimensional representations appear at higher mass levels in the ten-dimensional uncompactified $E_8 \times E_8$ heterotic string. These representations are relevant to the massless spectrum in the twisted sectors of the four-dimensional theory, in a peculiar way which will be described below. Weight vectors add when the tensor products are taken to form higher dimensional representations; consequently, the weight diagrams of the higher dimensional representations fill out a weight lattice, spanned by the basis vectors of the adjoint representation weight diagram. In the case of $E_8 \times E_8$, this is the root lattice $\Lambda_{E_8 \times E_8}$, which is described in most modern string theory texts [23]; it was also reviewed in Appendix A of our previous article [24].

Briefly, the root lattice for E_8 is given by

$$\Lambda_{E_8} = \left\{ (n_1, \ldots, n_8), \; (n_1 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2}) \;\;\middle|\;\; n_1, \ldots, n_8 \in \mathbf{Z}, \; \sum_{i=1}^{8} n_i = 0 \bmod 2 \right\} \tag{2.34}$$

and $\Lambda_{E_8 \times E_8} = \Lambda_{E_8} \oplus \Lambda_{E_8}$, the direct sum of two copies of Λ_{E_8}. The sixteen entries of a root lattice vector $(n_1, \ldots, n_8; n_9, \ldots, n_{16})$ correspond to eigenvalues with respect to a basis of the $E_8 \times E_8$ Cartan subalgebra, which we write as H^I $(I = 1, \ldots, 16)$ and which is Cartesian:

$$\mathrm{tr}_R \, (H^I H^J) = X(R) \, \delta^{IJ}, \tag{2.35}$$

where the trace is taken over an $E_8 \times E_8$ irrep R. In particular, the adjoint representation (A) corresponds to the elements $\alpha \in \Lambda_{E_8 \times E_8}$ with $\alpha^2 = 2$. These are the 480 nonzero roots of $E_8 \times E_8$, which take the form $\alpha = (\beta; 0)$ or $\alpha = (0; \beta)$ with $\beta \in \Lambda_{E_8}$, $\beta^2 = 2$. It is not hard to check from (2.34) that $X(A) = 60$, which is twice the value typically used by phenomenologists. Thus, the H^I in (2.35) and the eigenvalues in (2.34) are larger by a factor of $\sqrt{2}$ than the phenomenological normalization. *Positive roots* are nonzero roots which have their first nonzero entry positive, according to an (arbitrary) ordering system. *Simple roots* are positive roots which cannot be obtained from the sum of two positive roots. The number of simple roots is equal to the rank of the Lie algebra, which for $E_8 \times E_8$ is sixteen. We label the simple roots $\alpha_1, \ldots, \alpha_{16}$. Of particular importance is the map of roots α_i into the Cartan subalgebra defined by

$$H(\alpha_i) = \sum_{I=1}^{16} \alpha_i^I H^I. \tag{2.36}$$

From this, one defines an inner product on the root space:

$$\langle \alpha_i | \alpha_j \rangle \equiv \mathrm{tr}_A \left[H(\alpha_i) \cdot H(\alpha_j) \right]. \tag{2.37}$$

Using (2.35), it is not hard to see that

$$\langle \alpha_i | \alpha_j \rangle = X(A) \; \alpha_i \cdot \alpha_j. \tag{2.38}$$

It can be seen that the Dynkin index $X(A)'$ of the basis (2.36) is related to the index of (2.35) by $X(A)' = 2X(A)$. Thus, the generators (2.36) are larger by a factor of 2 than the phenomenological normalization; we return to this point in Section 4 below. The Cartan matrix of a Lie algebra is defined by

$$A_{ij} = \frac{2 \, \langle \alpha_i | \alpha_j \rangle}{\langle \alpha_j | \alpha_j \rangle}, \tag{2.39}$$

where i, j run over the simple roots. Using (2.38) and $\alpha_i^2 = 2$, it is easy to check that (2.39) is simply expressed in terms of the sixteen-dimensional simple root vectors:

$$A_{ij} = \alpha_i \cdot \alpha_j. \tag{2.40}$$

In the orbifold constructions below, a subset of the $E_8 \times E_8$ simple roots survive, and by computing the submatrices according to (2.40), we can identify the nonabelian factors in the surviving gauge group G, using widely available tables for the Cartan matrices of Lie algebras (e.g., Ref. [39]). Finally, it is worth mentioning that by taking all linear combinations of the sixteen simple roots with integer-valued coefficients, one recovers the root lattice $\Lambda_{E_8 \times E_8}$. That is,

$$\Lambda_{E_8 \times E_8} = \left\{ \sum_{i=1}^{16} m^i \alpha_i \;\middle|\; m^i \in \mathbf{Z} \right\}. \tag{2.41}$$

2.4 Recipes

We next write down without proof recipes for the generation of the spectrum of pseudo-massless states. Where possible, we have attempted to motivate the rules in a heuristic fashion, avoiding a detailed discussion of the underlying string theory. For further details, see the reviews [21, 22], texts [23], and references therein.

Nonzero root gauge states. We write these states as $|\alpha\rangle$ where α satisfies:

$$\alpha^2 = 2, \qquad \alpha \in \Lambda_{E_8 \times E_8}, \tag{2.42}$$

$$\alpha \cdot a_i \in \mathbf{Z}, \quad \forall\, i = 1, 3, 5, \tag{2.43}$$

$$\alpha \cdot V \in \mathbf{Z}. \tag{2.44}$$

Eq. (2.42) merely states that α is an $E_8 \times E_8$ root. For nontrivial $\{V, a_1, a_3, a_5\}$, several roots of $E_8 \times E_8$ will not satisfy (2.43,2.44). Consequently, the nonzero roots of G will be a subset of the $E_8 \times E_8$ roots. The states $|\alpha\rangle$ are eigenstates of the generators H^I of the $E_8 \times E_8$ Cartan subalgebra:

$$H^I |\alpha\rangle = \alpha^I |\alpha\rangle, \qquad I = 1, \ldots, 16. \tag{2.45}$$

To determine G, one first (fully) decomposes the solutions of (2.42-2.44) into orthogonal subsets. That is, for $a \neq b$ the subset $\{\alpha_{a1}, \ldots, \alpha_{an_a}\}$ is orthogonal to the subset $\{\alpha_{b1}, \ldots, \alpha_{bn_b}\}$ provided

$$\alpha_{ai} \cdot \alpha_{bj} = 0, \qquad \forall \quad i = 1, \ldots, n_a, \quad j = 1, \ldots, n_b. \tag{2.46}$$

The ath such subset corresponds to a nonabelian simple subgroup G_a of G, and the solutions $\alpha_{a1}, \ldots, \alpha_{an_a}$ belonging to this subset are the nonzero roots of G_a. One next determines which of the $\alpha_{a1}, \ldots, \alpha_{an_a}$ are simple roots. From the simple roots one can compute the Cartan matrix for G_a using (2.40) and thereby determine the group G_a.

As an example, in all of the $\mathrm{BSL_A}$ embeddings, there are precisely eight solutions to (2.42-2.44) which *do not* have all first eight entries vanishing:

$$\alpha_{1,1}\,,\, \alpha_{1,2} = (\underline{1, -1}, 0, 0, 0, 0, 0, 0; 0, \ldots, 0), \qquad \alpha_{2,1}\,,\, \ldots\,,\, \alpha_{2,6} = (0, 0, \underline{1, -1, 0}, 0, 0, 0; 0, \ldots, 0). \tag{2.47}$$

Here (and elsewhere below), all permutations of underlined entries should be taken. These are the nonzero roots of the observable sector gauge group G_O, and should reproduce (1.1). The first set in (2.47) is orthogonal to all vectors in the second set; therefore, these two sets correspond to different simple factors, one with two nonzero roots and the other with six; the two groups must be $SU(2)$ and $SU(3)$. It is easy to check that the simple roots are

$$\alpha_{1,1} = (1, -1, 0, 0, 0, 0, 0, 0; 0, \ldots, 0), \tag{2.48}$$

$$\alpha_{2,1} = (0, 0, 1, -1, 0, 0, 0, 0; 0, \ldots, 0), \qquad \alpha_{2,2} = (0, 0, 0, 1, -1, 0, 0, 0; 0, \ldots, 0). \tag{2.49}$$

The simple roots (2.49) give the correct Cartan matrix for $SU(3)$, using (2.40).

Zero root gauge states. We write these states in an orthonormal basis $|I\rangle$, where $I = 1, \ldots, 16$. These correspond to gauge states for the Cartan subalgebra of G, in the Cartesian basis H^I discussed above. They of course have vanishing $E_8 \times E_8$ weights:

$$H^I |J\rangle = 0, \quad \forall\, I, J = 1, \ldots, 16. \tag{2.50}$$

The group G typically has a nonabelian part G_{NA} which is a product of m simple factors, and a $U(1)$ part G_{UO} which is a product of n $U(1)$s:

$$G = G_{\mathrm{NA}} \times G_{\mathrm{UO}}, \qquad G_{\mathrm{NA}} = G_1 \times G_2 \times \cdots \times G_m, \qquad G_{\mathrm{UO}} = U(1)_1 \times U(1)_2 \times \cdots \times U(1)_n. \tag{2.51}$$

For the 175 orbifold models under consideration, the simple factors G_a $(a = 1, \ldots, m)$ are either $SU(N)$ or $SO(2N)$ groups. Each G_a has its own Cartan subalgebra with a corresponding basis $H_a^1, \ldots, H_a^{r_a}$, where r_a is the rank of G_a. Each basis element H_a^i is a linear combination of the $E_8 \times E_8$ Cartan basis elements H^I:

$$H_a^i = \sum_{I=1}^{16} h_a^{iI} H^I. \tag{2.52}$$

This is the analogue of (2.30). It should not be too surprising that corresponding linear combinations of the $E_8 \times E_8$ Cartan gauge states $|I\rangle$ are taken to obtain Cartan gauge states of G_a:

$$|a; i\rangle = \sum_{I=1}^{16} h_a^{iI} |I\rangle. \tag{2.53}$$

Similarly, the generator Q_a of the factor $U(1)_a$ may be written as

$$Q_a = \sum_{I=1}^{16} q_a^I H^I \tag{2.54}$$

(this is the analogue of (2.31)) and the corresponding gauge state

$$|a\rangle = \sum_{I=1}^{16} q_a^I |I\rangle. \tag{2.55}$$

It is convenient to choose the states $|a\rangle$ to be orthogonal (we discuss normalization below):

$$\langle a|b\rangle = q_a \cdot q_b = 0 \quad \text{if} \quad a \neq b. \tag{2.56}$$

For the Cartan states $|a;i\rangle$, it is more convenient that their inner product reproduce the Cartan matrix A^a for the group G_a:

$$\langle a;i|b;j\rangle = h_a^i \cdot h_b^j = \delta_{ab} A_{ij}^a. \tag{2.57}$$

It is hopefully apparent from (2.40) that this equation is satisfied if we take h_a^i to be the sixteen-dimensional simple root vectors for G_a: $h_a^i \equiv \alpha_{ai}$. We therefore rewrite (2.52) as

$$H_a^i = H(\alpha_{ai}) = \sum_{I=1}^{16} \alpha_{ai}^I H^I, \tag{2.58}$$

where we use the notation of (2.36); as mentioned there, these generators are larger by a factor of two than the phenomenological normalization.

Naturally, we want the $G_{\rm NA}$ Cartan states orthogonal to the $G_{\rm UO}$ states:

$$\langle a|b;j\rangle = q_a \cdot \alpha_{bj} = 0, \qquad \forall\, a,b,j. \tag{2.59}$$

It can be seen from the definitions above that this gives for any irrep R of $E_8 \times E_8$

$$\mathrm{tr}_R\,(Q_a H_b^j) = 0, \tag{2.60}$$

which is the analogue of (2.32). The q_a are therefore chosen to be orthogonal to the simple roots and to each other. With n $U(1)$ factors, as in (2.51), the choice of q_a is determined only up to reparameterizations which preserve the orthogonality conditions (2.56,2.59). In practice, most choices for the $U(1)$ generators lead to several of them being anomalous. It is then useful to make redefinitions such that only one $U(1)$ is anomalous. Let

$$t_a = \mathrm{tr}\ Q_a, \quad t_b = \mathrm{tr}\ Q_b, \quad s_a = q_a^2, \quad s_b = q_b^2, \tag{2.61}$$

with t_a, t_b both nonzero. Then define generators $Q_a' = \sum_I (q_a')^I H^I$ and $Q_b' = \sum_I (q_b')^I H^I$ via

$$q_a' = t_b q_a - t_a q_b, \qquad q_b' = t_a s_b q_a + t_b s_a q_b. \tag{2.62}$$

It is easy to see that tr $Q_a' = t_b t_a - t_a t_b = 0$, so that the anomaly is isolated to Q_b'. Furthermore, orthogonality is maintained:

$$q_a' \cdot q_b' = t_a t_b (s_b q_a^2 - s_a q_b^2) = t_a t_b (s_b s_a - s_a s_b) = 0. \tag{2.63}$$

By repeating this process, one can easily isolate the anomaly to a single factor, $U(1)_X$.

Untwisted matter states. We denote these states as $|K;i\rangle$, $i = 1,3,5$. Here, K is a sixteen-vector, denoting weights under the $E_8 \times E_8$ Cartan generators H^I:

$$H^I|K;i\rangle = K^I|K;i\rangle, \qquad I = 1,\ldots,16. \tag{2.64}$$

Furthermore, K must satisfy

$$K^2 = 2, \qquad K \in \Lambda_{E_8 \times E_8}, \tag{2.65}$$

$$K \cdot a_i \in \mathbf{Z}, \quad \forall\, i = 1,3,5. \tag{2.66}$$

$$K \cdot V = \frac{1}{3} \bmod 1, \tag{2.67}$$

It can be seen from comparison to (2.42-2.44) that the weights K of untwisted matter states differ from the weights of nonzero root gauge states only in the last condition, (2.44) versus (2.67): untwisted matter states correspond to a different subset of the nonzero $E_8 \times E_8$ roots which satisfy (2.43). (The remaining subset corresponds to untwisted antimatter states.) The multiplicity of three carried by the index i in $|K;i\rangle$ corresponds to a ground state degeneracy in the underlying theory [2], which we will not discuss here. It is one of the nice features of the Z_3 orbifold which aids in easily obtaining three generation constructions. However, it also means that for fixed K, the three generations $i = 1, 3, 5$ have identical $U(1)$ charges and are in identical irreps, as can easily be checked using (2.54,2.58,2.64):

$$\begin{aligned} H_a^j \, |K;i\rangle &= \alpha_{aj} \cdot K \, |K;i\rangle, & (2.68) \\ Q_a \, |K;i\rangle &= q_a \cdot K \, |K;i\rangle. & (2.69) \end{aligned}$$

That is, the weight $\lambda_{aj}^K = \alpha_{aj} \cdot K$ is independent of i and similarly for the charge $q_a^K = q_a \cdot K$.

In order to determine the matter spectrum, we need more than just the weights (2.68); we need to be able to group the basis states $|K_1;i\rangle, \ldots, |K_{d(R)};i\rangle$ which make up a given irrep R of dimension $d(R)$. Suppose an incoming matter state $|K;i\rangle$ interacts with a gauge supermultiplet state corresponding to a nonzero root α_{aj} of G_a. This interaction is described by inserting a current $J(\alpha_{aj})$, which acts like a raising or lowering operator with respect to some $SU(2)$ subgroup of G_a:

$$\langle K';i|J(\alpha_{aj})|K;i\rangle = \langle K';i|K + \alpha_{aj};i\rangle = \delta_{K',K+\alpha_{aj}}. \tag{2.70}$$

For fixed family index i, vectors K' related to K by the addition of one of the nonzero roots of G_a are in the same irrep. Collecting all vectors K' related to K in this way (and satisfying (2.65-2.67)), we fill out the vertices of a weight diagram of an irrep of G_a. Due to (2.59), K' and K give the same $U(1)$ charges (as they must):

$$q_b \cdot K' = q_b \cdot \alpha_{aj} + q_b \cdot K = q_b \cdot K. \tag{2.71}$$

Twisted non-oscillator matter states. We denote these as $|\tilde{K};n_1,n_3,n_5\rangle$, where $n_i = 0, \pm 1$ specify which of the 27 fixed points (conjugacy classes) the state corresponds to and $\tilde{K}$ is a sixteen-vector giving the weights with respect to the $E_8 \times E_8$ Cartan generators H^I, similar to Eqs. (2.45,2.64) above. However, the $\tilde{K}$ *do not* correspond to points on $\Lambda_{E_8 \times E_8}$. Rather (cf. (2.27)),

$$\tilde{K}^2 = 4/3, \qquad \tilde{K} = K + E(n_1,n_3,n_5), \qquad K \in \Lambda_{E_8 \times E_8}. \tag{2.72}$$

The condition $\tilde{K}^2 = 4/3$ guarantees $\tilde{K} \notin \Lambda_{E_8 \times E_8}$ since all elements $L \in \Lambda_{E_8 \times E_8}$ have $L^2 = 0 \bmod 2$, as can be checked by inspection of (2.41). Weights and charges under G are calculated as for the untwisted states, only now the shifted weights $\tilde{K}$ are used. In particular,

$$\begin{aligned} Q_a \, |\tilde{K};n_1,n_3,n_5\rangle &= q_a \cdot \tilde{K} \, |\tilde{K};n_1,n_3,n_5\rangle \\ &= [q_a \cdot K + q_a \cdot E(n_1,n_3,n_5)] \, |\tilde{K};n_1,n_3,n_5\rangle. & (2.73) \end{aligned}$$

Thus, the twisted matter states have charges shifted by

$$\delta_a(n_1,n_3,n_5) = q_a \cdot E(n_1,n_3,n_5) \tag{2.74}$$

from what would occur in the decomposition of $E_8 \times E_8$ representations onto a subgroup with $U(1)$ factors. The quantity $\delta_a(n_1, n_3, n_5)$ is the *Wen-Witten defect* [41], a problematic contribution which is uniform for a given twisted sector. It is precisely this feature which is responsible for difficulties accommodating the hypercharges of the MSSM spectrum and the generic appearance of states with fractional electric charge, as will be discussed below. Comparison to (2.27) shows that with $a_5 \equiv 0$, the embedding vector $E(n_1, n_3, n_5)$ is independent of n_5. It follows that states which differ only by the value of n_5 have identical $U(1)$ charges and are in identical irreps of the gauge group G. This is how three generations in twisted sectors are naturally generated in the class of models considered here. Filling out irreps of G_b is accomplished by collecting all $\tilde{K}'$ which are related to $\tilde{K}$ through $\tilde{K}' = \tilde{K} + \alpha_{bj}$, similar to what was done for untwisted states. Of course, the other quantum numbers n_1, n_3, n_5 must match.

It was stated above that higher dimensional irreps of $E_8 \times E_8$ are, in a way, relevant to massless states in the twisted sectors. We are now in a position to address this comment. In Section 5 we will discuss a model with an embedding such that

$$3E(1, 1, n_5) = (0, 0, -1, -1, -1, 5, 2, 2; 3, 1, 1, 0, 1, 0, 0, 0). \tag{2.75}$$

It is easy to check that a solution to (2.72) is obtained if

$$K = (0, 0, 0, 0, 0, -2, -1, -1; -1, -1, 0, 0, 0, 0, 0, 0). \tag{2.76}$$

However, $K^2 = 8$, so this is not a root of $E_8 \times E_8$, but the weight of a higher dimensional $E_8 \times E_8$ irrep. Of course, the weight of the state $|\tilde{K}; n_1, n_3, n_5\rangle$ is $\tilde{K}$ and not K, so it seems unimportant that $K^2 > 2$. However, $q_a \cdot K$ in (2.32) would be the "conventional" charge while $q_a \cdot E(1, 1, n_5)$ is the Wen-Witten defect; in this interpretation the charge $q_a \cdot K$ which would occur if the defect were absent is that of the decomposition a higher dimensional $E_8 \times E_8$ irrep. If nothing else, it creates the illusion that some massive states of the uncompactified $E_8 \times E_8$ heterotic string are shifted down into the massless spectrum when compactified on the six-dimensional orbifold.

Finally, we note that projections analogous to (2.66,2.67) are not required in the twisted sectors of a Z_3 orbifold [10, 38]. As a result, study of this orbifold is significantly simpler than most other orbifold constructions, where projections in the twisted sectors are rather complicated.

Twisted oscillator matter states. We denote these as $|\tilde{K}; n_1, n_3, n_5; i\rangle$, where $i = 1, 3, 5$ conveys an additional multiplicity of three, due to different ways to excite the vacuum in the underlying string theory with the analogue of harmonic oscillator raising operators; three types of oscillators—corresponding to the three complex planes of the six-dimensional compact space—excite the vacuum to generate a massless state. The $\tilde{K}$ are again shifted $E_8 \times E_8$ weights, but they have a smaller norm (to compensate for energy associated with the excited vacuum):

$$\tilde{K}^2 = 2/3, \qquad \tilde{K} = K + E(n_1, n_3, n_5), \qquad K \in \Lambda_{E_8 \times E_8}. \tag{2.77}$$

The determination of weights, irreps and charges is identical to that for the other matter states discussed above.

3 Discussion of Spectra

Automating the matter spectrum recipes given in the previous section, we have determined the spectra for all 175 models. We now make some general observations based on the results of this analysis. Ignoring the various $U(1)$ charges, only 20 patterns of irreps were found to exist in the 175 models. These are summarized in Tables VIII-XI (Appendix B). In all 175 models, twisted oscillator matter states are singlets of $G_{\rm NA}$ (cf. (2.51)). Singlets notated $(1,\ldots,1)_0$ are either untwisted matter states or twisted non-oscillator matter states while singlets notated $(1,\ldots,1)_1$ are twisted oscillator matter states. Only Patterns 2.6, 4.5, 4.7 and 4.8 have no twisted oscillator states. In Table XII (Appendix B) we show the irreps in the untwisted sector for each of the twenty patterns. Comparing to Tables VIII-XI, it can be seen that the majority of states in any given pattern are twisted non-oscillator states.

In Table XIII (Appendix B) we have cross-referenced the 175 embeddings enumerated in [24] with the twenty patterns given here. We now describe the labeling of models in Table XIII. We emphasize that the tables referenced in the following itemized list are *not* the tables contained in *this* article! Rather, table references in the following list correspond to tables in our *previous* article, Ref. [24]. Models are labeled in the format "$i.j$" where:

(a) for $i = 1, 2, 4$ or 6, i is the CMM observable sector embedding according to the labeling of Table I of Ref. [24] and j is the hidden sector embedding label as per the corresponding choice of table from the set Tables III-VI of Ref. [24];

(b) $i = 8$ corresponds to the CMM observable sector embedding 8 according to the labeling of Table I of Ref. [24] and j is the hidden sector embedding according to the labeling of Table VII of Ref. [24] ;

(c) $i = 10$ also corresponds to the CMM observable sector embedding 8 according to the labeling of Table I of Ref. [24], but now j is the hidden sector embedding according to the labeling of Table VIII of Ref. [24];

(d) $i = 9$ corresponds to the CMM observable sector embedding 9 according to the labeling of Table I of Ref. [24] and j is the hidden sector embedding according to the labeling of Table IX of Ref. [24];

(e) $i = 11$ also corresponds to the CMM observable sector embedding 9 according to the labeling of Table I of Ref. [24], but now j is the hidden sector embedding according to the labeling of Table X of Ref. [24].

We remind the reader that CMM observable sector embeddings 3, 5 and 7 do not appear because they are equivalent to 1, 4 and 6 respectively, as shown in Ref. [24].

All patterns except Pattern 1.1 have an anomalous $U(1)_X$ factor. We have determined the FI term for each of the models in the other 19 patterns. We find that all models within a particular pattern have the same FI term; the corresponding values of Λ_X, defined in (1.10) above, are displayed in Table II. As will be discussed in greater detail in Section 5.2, Kaplunovsky [42] has estimated the string scale to be

$$\Lambda_H \approx g_H \times 5.27 \times 10^{17}\,\text{GeV} = 0.216 \times g_H m_P. \tag{3.1}$$

Using the values in Table II, it is easy to check that

$$\Lambda_H/1.73 \leq \Lambda_X \leq \Lambda_H. \tag{3.2}$$

Pattern	$\Lambda_X/(g_H m_P)$	Pattern	$\Lambda_X/(g_H m_P)$
1.2	0.216	2.6, 3.3, 4.6	0.170
2.1, 4.2	0.125	3.1, 4.3	0.148
2.2, 2.3, 4.1	0.138	3.2, 4.4, 4.8	0.176
2.4	0.186	3.4	0.181
2.5	0.191	4.5, 4.7	0.157

Table II: The $U(1)_X$ symmetry breaking scale Λ_X for each of the irrep patterns.

The effective supergravity lagrangian describing the field theory limit of the string is nonrenormalizable. In principle, all superpotential and Kähler potential operators allowed by symmetries of the underlying theory should be present. As discussed in Appendix A, there exist field reparameterization invariances in the effective theory. These invariances relate different classical field configurations, or vacua. Expansion about a particular vacuum leads to a nonlinear σ model. For instance, this is reflected in the presence of superpotential operators such as (2.1) above, with ever increasing numbers n of Xiggses. For the nonlinear σ model to be perturbative, it must be possible to truncate the sequence of operators at some order $n_{\max}$ and obtain a reasonable approximation to the full theory. Since the relevant expansion parameter for nonrenormalizable operators is roughly Λ_X/m_P, which from Table II lies in the range

$$g_H/8.00 \leq \Lambda_X/m_P \leq g_H/4.63, \tag{3.3}$$

the nonlinear σ model has a reasonable chance to be perturbative, provided the unified coupling satisfies $g_H \lesssim 1$ and the number of operators contributing to an effective coupling (such as the AA^c coupling in (2.1)) is not too large. (Generically, the number of such operators increases with dimension.)

Given the importance of nonvanishing vevs to the perturbative expansion of the nonlinear σ model, we next estimate the range of Xiggs vevs. We will assume that $g_H \approx 1$ in (3.3), as suggested by analyses of the running gauge couplings; for example, see Section 5.2 below. Then from (3.3) we have

$$\Lambda_X \sim \mathcal{O}(10^{-1})\, m_P. \tag{3.4}$$

Furthermore, we assume that Xiggs fields have a nearly diagonal Kähler potential at leading order in an expansion about the vacuum:

$$K_{\text{Xiggs}} = \sum_i \left\langle \frac{\partial^2 K}{\partial\phi^i \partial\bar{\phi}^i} \right\rangle |\phi^i|^2 + \cdots, \tag{3.5}$$

with the terms represented by "$\cdots$" negligible in comparison to the explicit terms. This assumption is justified by the known form for the terms in K quadratic in matter fields for Z_3 orbifolds with nonstandard embedding [43], such as the cases considered here. In the limit of vanishing off-diagonal T-moduli (i.e., $\langle T^{ij}\rangle = 0,\ \forall\, i \neq j$),

$$K_{\text{quad.-matter}} = \sum_i \frac{|\phi^i|^2}{\prod_{j=1,3,5}(T^j + \bar{T}^j)^{q^i_j}}. \tag{3.6}$$

Here, q_j^i are the *modular weights* of the matter field ϕ^i: untwisted states $|K; i\rangle$ have modular weights $q_j^i = \delta_j^i$, while twisted non-oscillator states $|\tilde{K}; n_1, n_3, n_5\rangle$ have modular weights $q_j^i = 2/3$ and twisted oscillator states $|\tilde{K}; n_1, n_3, n_5; i\rangle$ have modular weights $q_j^i = 2/3 + \delta_j^i$. Moduli stabilization in the BGW model gives $\langle T^j \rangle = 1$ or $e^{i\pi/6}$ $\forall\, j$. Assuming the former value and applying (3.6), we find

$$\left\langle \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^i} \right\rangle_{\rm BGW} = \begin{cases} 1/2 & \text{untwisted,} \\ 1/2^2 & \text{twisted non-oscillator,} \\ 1/2^3 & \text{twisted oscillator .} \end{cases} \tag{3.7}$$

This ignores the possible contribution of terms $K \ni (c/m_P^2)\, f(T)\, |\phi^i|^2 |\phi^j|^2$, with both fields ϕ^i, ϕ^j Xiggses and $f(T)$ a function of the T-moduli. If we assume $\langle \phi^i \rangle \sim \langle \phi^j \rangle \sim \Lambda_X$, these quartic terms (which include i-j mixing) are suppressed by $\mathcal{O}(\Lambda_X^2/m_P^2)$ relative to the leading terms. However, we still have to estimate $\langle \phi^i \rangle$ and $\langle \phi^j \rangle$, so at the end of our analysis we will have to check whether or not it was consistent to neglect these quartic terms. It is also unclear what the moduli-dependent function $f(T)$ is, and whether or not the dimensionless coefficient c is $\mathcal{O}(1)$; an explicit calculation of such higher order Kähler potential terms from the underlying string theory apparently remains to be accomplished.

In large radius (LR) stabilization schemes such as in Refs. [44, 45], T-moduli vevs as large as $13 \lesssim \langle T^j \rangle \lesssim 16$ are envisioned. This greatly affects our estimates for the Xiggs vevs, since we now have (for the larger value of $\langle T^j \rangle = 16$)

$$\left\langle \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^i} \right\rangle_{\rm LR} = \begin{cases} 1/32 & \text{untwisted,} \\ 1/32^2 & \text{twisted non-oscillator,} \\ 1/32^3 & \text{twisted oscillator .} \end{cases} \tag{3.8}$$

Let N be the number of Xiggses, q^X be the average Xiggs $U(1)_X$ charge magnitude, K'' be the average value for the Xiggs metric $\langle \partial^2 K / \partial \phi^i \partial \bar{\phi}^i \rangle$ and ϕ be the average value for $|\langle \phi^i \rangle|$, where "average" is used loosely. Then from (1.9,1.10) we see that $\langle D_X \rangle = 0$ implies

$$\phi \sim \left(N q^X K'' \right)^{-1/2} \Lambda_X. \tag{3.9}$$

In Section 5 we will see in an explicit example that the (properly normalized) nonvanishing $U(1)_X$ charges vary between $1/\sqrt{84} \approx 0.11$ to $6/\sqrt{84} \approx 0.65$. We take this as an indication that $1/10 \lesssim q^X \lesssim 2/3$ is reasonable. In a typical model there are $3 \times \mathcal{O}(50)$ chiral matter multiplets. The number N which may acquire vevs to cancel the FI term varies from one flat direction to another. A reasonable range is $1 \lesssim N \lesssim 50$, given the enormous number of $G_{SM} \times G_C$ singlets in any of the models.

If a single twisted oscillator field ϕ^i of charge $1/10$ dominates the FI cancellation (i.e., ϕ^i is the only Xiggs or all of the other Xiggses have much smaller vevs so that effectively $N = 1$ in (3.9)), then with the BGW T-moduli stabilization

$$\phi \sim \sqrt{10 \times 2^3}\, \Lambda_X \sim \mathcal{O}(1)\, m_P, \tag{3.10}$$

where we have used (3.4). Such a large vev is certainly troubling. If the large radius value $\langle T^j \rangle \approx 16$ is assumed, the result is a hundred times worse:

$$\phi \sim \sqrt{10 \times 32^3}\, \Lambda_X \sim \mathcal{O}(10^2)\, m_P. \tag{3.11}$$

On the other hand, if we had, say, 50 Xiggs fields ϕ^i with more average charges of roughly 1/2 contributing equally to cancel the FI term, with the typical field a twisted nonoscillator field, and the BGW stabilization of T-moduli,

$$\phi \sim \sqrt{2 \times 2^2/50}\, \Lambda_X \sim \mathcal{O}(10^{-2})\, m_P. \tag{3.12}$$

However, for the large radius case,

$$\phi \sim \sqrt{2 \times 32^2/50}\, \Lambda_X \sim \mathcal{O}(1)\, m_P. \tag{3.13}$$

This examination of (1.9) indicates that for the BGW stabilization, Xiggs vevs are naturally $\mathcal{O}(10^{-1\pm 1})\ m_P$. At the upper end, the σ model would seem to be in trouble. The large radius case appears to be complete catastrophe, however we arrange cancellation of the FI term. To be fair, the quadratic terms $K \ni (c/m_P^2)\ f(T)\ |\phi^i|^2|\phi^j|^2$ mentioned above now need to be included in the estimation of Xiggs vevs, since they are not of sub-leading order in the large Xiggs vev limit.

It should be noted, however, that the principal motivation for the large radius assumption is to produce appreciable string scale threshold corrections to the running gauge couplings, such as was studied in [45, 46]; there, the aim was to achieve gauge coupling unification at the conventional value of approximately 2×10^{16} GeV. In a Z_3 orbifold compactification, these large T-moduli dependent threshold corrections coming from heavy string states are absent [47]. Nevertheless, it should be clear from the above analysis that orbifolds which *do* have the T-moduli dependent string threshold corrections *and* a $U(1)_X$ factor are likely to also suffer from a problem of too large Xiggs vevs in the large radius limit, because of the noncanonical Kähler potential.

Moderately large, yet perturbative, vevs such as $\phi \approx m_P/5$ would require large n in (2.1) to generate significant hierarchies. This may be a virtue: in many cases orbifold selection rules and G symmetries require that leading operators contributing to a given effective low energy superpotential term have significantly higher dimension than might be guessed from $G_{SM} \times G_C$ alone. For example, in the FIQS model (mentioned in the Introduction) the leading down-type quark masses come from dimension eleven operators. (I.e., the effective Yukawa matrix elements are sums of vevs of seventh degree monomials of Xiggs fields.)

The sum in (1.9) allows for some terms to be very small if others are $\mathcal{O}(\Lambda_X)$; we exploited this possibility in a recent study of effective quark Yukawa couplings induced by Xiggs vevs of rather different scales [16]. Such hierarchies in Xiggs vevs remain to be (dynamically) motivated from a detailed study of an explicit scalar potential which lifts the D-moduli flat directions [15] mentioned in the Introduction. The existence of these flat directions means that the upper bound estimates made here for Xiggs vevs are not at all robust. Xiggs of opposite $U(1)_X$ charge may be "turned on" along a particular flat direction (as in the FIQS model). In that case their contributions partially cancel each other; it is technically possible for the Xiggs vevs to be made arbitrarily large as a result. Of course, this would quickly spoil the nonlinear σ model expansion.

The BSL-I model mentioned in the Introduction belongs to Pattern 1.2 and is equivalent to one of the models 6.1-3 listed under that pattern in Table XIII. (CMM found that the BSL-I model observable sector embedding was equivalent to CMM 7, and in [24] we showed that CMM 7 is equivalent to CMM 6.) In [15] it was noted that the FIQS model suffers from a problem of light diagonal T-moduli masses; the conclusions made there do not depend on the choice of (hidden $SO(10)$ preserving) flat direction, and therefore hold for other vacua of the BSL-I model, such as those studied by Casas and Muñoz [14]. As will shortly be explained, the light mass problem is a consequence of having $G_C = SO(10)$ charged matter fields only in the untwisted sector. This observation extends to *all* models of Pattern 1.2, as well as to the models of Pattern 1.1. Because BGW stabilize the diagonal T-moduli with nonperturbative effects in the hidden sector (i.e., gaugino condensation), they simultaneously derive an effective (soft) mass term for these fields [26]. If the effective moduli masses are much larger than the gravitino mass, the *cosmological moduli problem* [48] can be avoided. In the BGW effective theory, one finds for the diagonal T-moduli

$$m_T \approx 2\frac{|b_{\rm GS} - b_C|}{|b_C|} m_{\tilde{G}}, \tag{3.14}$$

where b_C is the beta function coefficient for the condensing group G_C, $m_{\tilde{G}}$ is the gravitino mass and $b_{\rm GS}$ is the *Green-Schwarz counterterm coefficient,* a quantity whose origin is not important to the present discussion, but which is briefly explained in Appendix A. If $b_{\rm GS}/b_C \approx 10$, then $m_T \approx 20 m_{\tilde{G}}$; it was argued by BGW, and others [49], that this may be heavy enough to resolve the cosmological moduli problem.

However, as pointed out in Ref. [15], if G_C has only trivial irreps in the twisted sector, $b_{\rm GS} = b_C$. The T-moduli are massless to the order of the approximation made in (3.14), and the moduli problem reappears with a vengence. To see how $b_{\rm GS} = b_C$ occurs in Patterns 1.1 and 1.2, it is only necessary to note a few simple facts. In Appendix A we use well-known results to demonstrate that, for the class of models studied here, the Green-Schwarz coefficient is given by

$$b_{\rm GS} = b_a^{\rm tot} - 2 \sum_{\rho \in {\rm tw}} X_a(R^\rho), \qquad \forall\, G_a \in G_{\rm NA}, \tag{3.15}$$

where $b_a^{\rm tot}$ is the β function coefficient (given by (1.4) with $G_C \to G_a$) calculated from the entire pseudo-massless spectrum of a given model, and the index ρ runs only over twisted matter chiral supermultiplet irreps. In Table III we show $b_{\rm GS}$ for each of the twenty patterns; the value is universal to all models in a given pattern. From (3.15) it is clear that $b_{\rm GS} = b_a^{\rm tot}$ for G_a with only trivial irreps in the twisted sector. This occurs for $SO(10)$ in Patterns 1.1 and 1.2, so that one has $b_{\rm GS} = b_{10}^{\rm tot}$; we also recall $G_C = SO(10)$ in these patterns; this leads to vanishing T-moduli masses in (3.14) if $b_C = b_{10}^{\rm tot}$. One might hope to get around this by giving some of the $SO(10)$ charged matter $\mathcal{O}(\Lambda_X)$ vector mass couplings so that b_C, the effective coefficient which appears in the theory below the scale Λ_X, is different from $b_{10}^{\rm tot}$. Pattern 1.1 does not contain $SO(10)$ charged matter so this is fruitless. In Pattern 1.2, the $SO(10)$ matter is in 16s, which have as their lowest dimensional invariant $(16)^4$. To have effective vector masses for these states from superpotential terms would require breaking $SO(10)$. We leave these issues to further research. Another way resolve the light moduli problem in Patterns 1.1 and 1.2 would involve alternative inflation scenarios. For example, light moduli could be diluted via the thermal inflation of Lyth and Stewart [50]. Lastly, we note that the BGW result (3.14) is obtained in an effective theory which does not account for a $U(1)_X$; until it is understood how the BGW effective theory is modified in the presence of a $U(1)_X$ factor [30], firm conclusions about the Pattern 1.2 models cannot be drawn. (Recall that Pattern 1.1 has no $U(1)_X$ factor.)

The values for $b_{\rm GS}$ are problematic for more than just the Pattern 1.1 and 1.2 models. For example, in the $G_C = SU(5)$ Patterns 2.2-5, the Green-Schwarz coefficient is $b_{\rm GS} = -15$ and we can constrain $-15 \leq b_C \leq -6$. The bound -15 comes from a scenario of pure $SU(5)$; i.e., no

Pattern	b_{GS}	Pattern	b_{GS}
1.1	-24	1.2, 2.1	-18
2.2-5, 3.1	-15	3.2-4, 4.1-4, 4.6	-12
2.6, 4.5, 4.7-8	-9		

Table III: Green-Schwarz coefficients.

matter. Pattern 2.2 for instance allows for the possibility that the vector-like $3(5+\bar{5})$ matter acquires mass at Λ_X, so that effectively there is no $SU(5)$ charged matter in the running which dynamically generates the condensation scale. The bound -6 comes from the "marginal" case of very low Λ_C discussed in the Introduction. For this range of b_C we have from (3.14)

$$0 \leq m_T \leq 3m_{\tilde{G}}. \tag{3.16}$$

From the arguments of [26, 49], the T-moduli mass appears to be too light even in the marginal case $b_C = -6$, which gives the upper bound for m_T. Taking the $b_C = -6$ limit for each of the values of b_{GS} (except $b_{\text{GS}} = -24$ which corresponds to Pattern 1.1 discussed above—where it seems $m_T \approx 0$ is unavoidable), we find upper bounds of $m_T^{\max}/m_{\tilde{G}} \approx 4, 3, 2, 1$ for $b_{\text{GS}} = -18, -15, -12, -9$ respectively. Thus, the light T-moduli mass problem is a general feature of the BSL_A models.

Most of the 20 patterns contain $(3+\bar{3}, 1)$ representations under $SU(3)_C \times SU(2)_L$. It is necessary to find a vacuum solution which gives these fields vector mass couplings at a high enough scale. The greater the number of such pairs, the more difficult this is to achieve, since one must simultaneously avoid high scale supersymmetry breaking; more and more fields must be identified as Xiggses in order to give all of the required effective supersymmetric mass couplings. As each new Xiggs is introduced, it is harder to avoid nonzero F-terms at the scale Λ_X. Similarly, large vector masses are generally required for the many additional $(1,2)$ and $(1,1)$ representations present in all of the models. The electroweak hypercharges of these representations depend on how the several $U(1)$s are broken in choosing a D-flat direction. States with exotic electric charge (i.e., leptons with fractional charges and quarks which may form fractionally charged color singlet bound states) typically occur. We will address constraints on the presence of such matter in Section 5 below.

The distinction between observable and hidden sectors is blurred by twisted states in nontrivial representations of both G_O and G_H. Gauge interactions communicating with both sectors are a well-known effect in orbifold models. Communication via $U(1)$s was for example noted in Refs. [10, 51, 52], while the occurence of states in nontrivial representations of both observable and hidden nonabelian factors has been noted in other orbifold constructions, for example in a $Z_3 \times Z_3$ model in Ref. [46]. Cases 2 through 4 (cf. Table I) have at least one hidden $SU(2)$ factor (which we denote $SU(2)'$), and $(1,2,2)$ representations under $SU(3)_C \times SU(2)_L \times SU(2)'$ occur in several of the patterns. No $(\bar{3},1,2)$ representations occur, so it is not possible to use $SU(2)'$ to construct a left-right symmetric model in any of the 175 models studied here. (Left-right symmetric models would place the u^c- and d^c-type quarks in $(\bar{3},1,2)$ representations.) All 175 models contain twisted states in nontrivial irreps of $SU(3)_C \times SU(2)_L$ charged under $U(1)$s contained in G_H.

It is an interesting question to what degree these features might communicate supersymmetry breaking to the observable sector. A similar scenario has been considered by Antoniadis and Benakli

[53]. Specifically, they examined hidden sector matter with supersymmetric masses M and a soft mass δM splitting the matter scalars from fermions, gauginos from vector gauge bosons, with the assumption $\delta M \ll M$; this "hidden" matter was also assumed to be in nontrivial irreps of $G_{\rm SM}$. They found significant contributions to the soft terms which break supersymmetry in the MSSM. To evaluate the implications of such gauge mediation of supersymmetry breaking in the 175 models at hand requires a significant extension of their results, given the strong dynamics of the hidden sector in a gaugino condensation scenario; much of the hidden sector matter now consists of bound states of G_C which are $G_{\rm SM}$ neutral (certainly the case for those condensates which acquire the supersymmetry breaking nonvanishing vevs) yet contain particles in nontrivial $G_{\rm SM}$ irreps. We leave these matters to future research.

The generic presence of an anomalous $U(1)_X$ has implications for low energy supersymmetric models which aim to be "string-inspired" or "string-derived." The effective theory in the low energy limit is obtained by integrating out states which get large masses due to the $U(1)_X$ FI term. The surviving spectrum of states will generally contain superpositions of the original states, mixing the various sectors. Thus, assigning each state in the MSSM to a *definite* sector (i.e., the untwisted sector or one of the 27 (n_1, n_3, n_5) twisted sectors) is in many cases inconsistent with the mixing which occurs in the presence of a $U(1)_X$, as was for instance remarked recently in Ref. [54]. Mixings of sectors was considered for quarks, for example, in the FIQS model and in the toy model of Ref. [16]. In addition to modified properties for the spectrum, integrating out the massive states will modify the interactions of the light fields and create threshold effects for running couplings. These threshold effects can be large due to the large number of extra states, and need to be considered in any analysis of gauge coupling unification, for example.

4 Hypercharge

4.1 Normalization in GUTs

An important feature of GUTs is that the $U(1)$ generator corresponding to electroweak hypercharge does not have arbitrary normalization. This is because the hypercharge generator is embedded into the Lie algebra of the GUT group. That is, $G_{\rm GUT} \supset SU(3)_C \times SU(2)_L \times U(1)_Y$. The *unified normalization* is most clear when one identifies a Cartesian basis for the GUT group generators T^a for a given representation R:

$$\mathrm{tr}_R \; T^a T^b = X(R)\; \delta^{ab}. \tag{4.1}$$

The normalization prevalent in phenomenology has $X(F) = 1/2$ for an $SU(N)$ fundamental representation F. Because of the GUT symmetry, the interaction strength of a gauge particle with matter is given by

$$g_U(\mu)\; T^a, \quad \forall\; a, \tag{4.2}$$

where $g_U(\mu)$ is the running coupling for the GUT gauge group at the scale $\mu \geq \Lambda_U$, with Λ_U the unification scale. One of the T^a, say T^1, is then identified with the electroweak hypercharge generator. However, to obtain the usual eigenvalues for MSSM particles (e.g., $Y = 1$ for e^c) we generally must rescale the generator:

$$Y \equiv \sqrt{k_Y}\; T^1. \tag{4.3}$$

The reason for writing the rescaling constant in this way will become clear below. Because of (4.2,4.3), the hypercharge coupling $g_Y(\mu)$ will be related to $g_U(\mu)$ at the boundary scale Λ_U. More precisely,

$$g_U(\Lambda_U)\, T^1 = g_Y(\Lambda_U)\, Y = \sqrt{k_Y}\, g_Y(\Lambda_U)\, T^1, \tag{4.4}$$

since the interaction strength should not depend on normalization conventions for the generators. We maintain the GUT normalization for the generators T^a which correspond to the unbroken $SU(2)$ and $SU(3)$ groups, so that there are no rescalings analogous to (4.3) for these two groups; their running couplings are denoted by $g_2(\mu)$ and $g_3(\mu)$ respectively. Because of (4.2), they too must be matched to the boundary value $g_U(\Lambda_U)$ when $\mu = \Lambda_U$; thus, we obtain the well-known GUT boundary conditions

$$g_3(\Lambda_U) = g_2(\Lambda_U) = \sqrt{k_Y}\, g_Y(\Lambda_U) = g_U(\Lambda_U). \tag{4.5}$$

For example, consider an SU(5) GUT [55]. The $SU(5)$ embedding of hypercharge, which we write as T^1, can be determined from the requirement that $\mathrm{tr}\,(T^1)^2 = 1/2$ for a fundamental or antifundamental irrep. For example,

$$T^1 = \frac{1}{\sqrt{60}}\,\mathrm{diag}\,(-3,-3,2,2,2), \qquad \text{for} \qquad \bar{5} = \begin{pmatrix} L \\ d^c \end{pmatrix}. \tag{4.6}$$

Here, L is a $(1,2)$ lepton, and d^c is a $(\bar{3},1)$ down-type quark, where we denote $SU(3)_C \times SU(2)_L$ quantum numbers. On the other hand, the electroweak normalization has by convention

$$Y = \frac{1}{6}\,\mathrm{diag}\,(-3,-3,2,2,2) \tag{4.7}$$

for the same set of states. Since $Y = \sqrt{5/3}\, T^1$, we see from (4.3) that

$$k_Y = 5/3. \tag{4.8}$$

It is this value which, when assumed in (4.5), yields the amazingly successful gauge coupling unification in the MSSM, detailed for example in Refs. [56, 57].

4.2 Normalization in String Theory

As in GUTs, the normalization of $U(1)$ generators in string-derived field theories requires care. Above, we have alluded to the fact that gauge coupling unification at the heterotic string scale Λ_H is a prediction of the underlying theory [58]. Just as in GUTs, unification of the hypercharge coupling with the couplings of other factors of the gauge symmetry group G corresponds to a particular normalization. However, the unified normalization of hypercharge is often different than the one which appears in $SU(5)$ or $SO(10)$ GUTs; in fact it is often difficult or impossible to obtain (4.8). Examples of this hypercharge normalization "difficulty" will be examined below. We will show how the unified normalization can be identified from very simple arguments. In the process we will make it very clear why, in the class of orbifold models considered here, nonstandard hypercharge normalization is generic and fractionally charged exotic matter is abundant.

It was noted in Section 2 that the basis (2.58) is larger by a factor of two than the phenomenological normalization. Thus, $\mathrm{tr}\,(T^a)^2 = 2$ for an $SU(N)$ fundamental representation. For instance, consider an untwisted $SU(2)_L$ doublet with respect to $\alpha_{1,1}$ in (2.48) above, for CMM 2 observable sector embeddings. (The embedding label here corresponds to Table I of Ref. [24].) The lowest and highest weight states are respectively

$$\begin{aligned} K_1 &= (0,1,0,0,0,1,0,0;0,\ldots,0), \\ K_2 &= K_1 + \alpha_{1,1} = (1,0,0,0,0,1,0,0;0,\ldots,0). \end{aligned} \tag{4.9}$$

Using Eqs. (2.48,2.68), the corresponding weights are ∓ 1; this gives $\mathrm{tr}\,(H_1^1)^2 = 2$, where $H_1^1 = T^3$, the isospin operator of $SU(2)_L$. To get to the phenomenological normalization, we should rescale generators by $1/2$. Thus, instead of (2.58), we define our properly normalized Cartan generators $\hat{H}_a^i$ according to

$$\hat{H}_a^i = \sum_{I=1}^{16} \hat{h}_a^{iI} H^I \equiv \sum_{I=1}^{16} \frac{1}{2} \alpha_{ai}^I H^I. \tag{4.10}$$

In this case, the sixteen-vectors $\hat{h}_a^i$ satisfy

$$(\hat{h}_a^i)^2 = 1/2. \tag{4.11}$$

It is hardly surprising that the properly normalized generator $\hat{Q}_a$ of $U(1)_a$ must also satisfy $(\hat{q}_a)^2 = 1/2$, where $\hat{q}_a$ is the sixteen-vector appearing in (2.54), but now with a special normalization. After all, the generator of $U(1)_a$ just corresponds to a different linear combination of the $E_8 \times E_8$ Cartan generators H^I, and taking a linear combination of the same norm is the logical choice. If, on the other hand, we work with a generator $Q_a = \sqrt{k_a}\hat{Q}_a$, then it follows that $q_a^2 = k_a/2$. This is one way of motivating the "affine level" of a $U(1)$ factor:

$$k_a = 2\sum_{I=1}^{16} (q_a^I)^2. \tag{4.12}$$

(This relation also follows from a consideration of the double-pole Schwinger term which occurs in the operator product of $U(1)$ currents in the underlying conformal field theory [13, 59, 60, 61], details which we have purposely avoided here.) The unified normalization, where nonabelian Cartan generators $\hat{H}_b^i$ and $U(1)$ generators $\hat{Q}_a$ have in common $(\hat{h}_b^i)^2 = \hat{q}_a^2 = 1/2$, corresponds to $k_a = 1$.

4.3 SU(5) Hypercharge Embeddings

Note that the generator

$$Y_1 = \sum_{I=1}^{16} y_1^I H^I, \qquad y_1 = \frac{1}{6}(-3,-3,2,2,2,0,0,0;0,\ldots,0), \tag{4.13}$$

satisfies $k_{Y_1} = 5/3$, is orthogonal to the $SU(3)_C \times SU(2)_L$ roots in (2.47), and has nonzero entries only in the subspace where the $SU(3)_C \times SU(2)_L$ roots have nonzero entries. Furthermore, it gives $Y_1 = y_1 \cdot K_{1,2} = -1/2$ to the doublet in (4.9), corresponding to the lepton doublets L or the H_d Higgs doublet of the MSSM. The $\mathrm{BSL_A}$ models with observable sector embedding CMM 2 also include $(\bar{3},1)$ states in the untwisted sector with weights

$$K_{3,4,5} = (0,0,\underline{1,0,0},-1,0,0;0,\ldots,0). \tag{4.14}$$

These have $Y_1 = y_1 \cdot K_{3,4,5} = 1/3$, corresponding to the d^c states. Finally, the untwisted sector contains $(3,2)$ states with weights

$$K_{6,\ldots,11} = (\underline{-1,0},\underline{-1,0,0},0,0,0;0,\ldots,0) \tag{4.15}$$

which have $Y_1 = y_1 \cdot K_{6,\ldots,11} = 1/6$, corresponding to the quark doublets Q. Thus, the untwisted sector contains a $\bar{5}$ and an incomplete 10 under the "would-be" $SU(5)$ into which we wish to embed $SU(3)_C \times SU(2)_L \times U(1)_{Y_1}$, taking (4.13) to be the hypercharge generator. The fact that the e^c and u^c representations needed to fill out the 10 irrep are not present in the untwisted sector is a troubling feature which is generic to the 175 models studied here.

In Table XII (Appendix B) we display the irreps present in the untwisted sector for each of the twenty patterns. In no case do we have the required irreps to build a 10 of $SU(5)$. In those cases where one finds $(3,2)+(\bar{3},1)$, the states which are singlets of the observable $SU(3)\times SU(2)$ are in nontrivial irreps of the hidden sector group. One could imagine breaking the hidden sector group and using a singlet of the surviving group to give the necessary $(1,1)$ irrep to fill out a 10. For instance, in Pattern 2.2, the $(1,1,1,2)$ irrep, a 2 of the hidden $SU(2)'$, would give two singlets if we break $SU(2)'$ with nonvanishing vevs for a pair of twisted sector $(1,1,1,2)$ irreps along a D-flat direction. (A pair is required to have vanishing D-terms for $SU(2)'$.) We would thereby obtain three generations of two $(1,1,1)$ irreps with respect to the surviving nonabelian gauge symmetry $SU(3)\times SU(2)\times SU(5)$, where the $SU(5)$ shown here is the hidden condensing gauge group. However, the untwisted $(1,1,1,2)$ irrep which gives these states has an $E_8\times E_8$ weight vector K of the form $K=(0;\beta)$, $\beta \in \Lambda_{E_8}$, since it is an untwisted state charged under the hidden sector gauge group. Then it has vanishing charge with respect to the generator Y_1 according to (4.13), rather than the required $Y_1 = 1$. We could overcome this by modifying y_1 to have nonzero entries in the hidden sector portion, represented by $0,\ldots,0$ in (4.13). However, according to (4.12), this would increase k_Y over the value of $5/3$ which y_1 gives. Moreover, it can be seen that one never has enough untwisted $(\bar{3},1)$ irreps to give three generations of both u^c- and d^c-type quarks, and that untwisted $(1,2)$ irreps always occur when an untwisted $(\bar{3},1)$ is present. Thus, even if we break the hidden gauge group, use a singlet to complete the 10, are willing to consider $k_Y > 5/3$, and find the $(\bar{3},1)$ has $Y_1 = -2/3$ so that it fits into a 10, the $(1,2)$ would stand for an incomplete $\bar{5}$. It is inevitable that we use states from the twisted sectors to fill out the MSSM; as we have already alluded to in Section 2, twisted states have unusual $U(1)$ charges (partly) because the $E_8\times E_8$ weights are shifted by the embedding vectors $E(n_1,n_3,n_5)$.

Let us now examine the relationship of (4.13) to $SU(5)$. To begin with we relabel the $SU(3)\times SU(2)$ simple roots in (2.48,2.49) as

$$\alpha_1 \equiv \alpha_{1,1}, \qquad \alpha_2 \equiv \alpha_{2,1}, \qquad \alpha_3 \equiv \alpha_{2,2}. \tag{4.16}$$

These may be supplemented by a fourth $E_8\times E_8$ root

$$\alpha_4 = (0,1,-1,0,0,0,0,0;0,\ldots,0) \tag{4.17}$$

to give the correct Cartan matrix for $SU(5)$, according to (2.40). In this way we embed $SU(3)\times SU(2)$ into a would-be $SU(5)$ subgroup of the observable E_8 factor of $E_8\times E_8$. A (properly normalized) basis $\hat{H}_1,\ldots,\hat{H}_4$ for the Cartan subalgebra of the would-be $SU(5)$ is given in terms of the $E_8\times E_8$ Cartan generators H^I according to the methods described in Section 2, supplemented

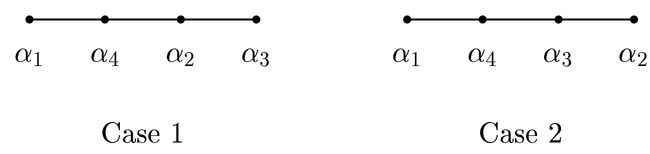

Figure I: Would-be $SU(5)$ Dynkin diagrams.

by the normalization considerations which led to (4.10). That is, we take linear combinations described by sixteen-vectors $\hat{h}^i = \alpha_i/2$, so that

$$\hat{H}^i = \sum_{I=1}^{16} \frac{1}{2} \alpha_i^I H^I. \tag{4.18}$$

However, when we decompose $SU(5) \supset SU(3) \times SU(2) \times U(1)$ we want to take the $U(1)$ generator to be orthogonal to the generators $\hat{H}^{1,2,3}$ associated with the simple roots (4.16), unlike $\hat{H}^4$. (This is the analogue of (2.32).) We thus make a change of basis, keeping $\hat{h}^i = \alpha_i/2$ for $i = 1, 2, 3$ while taking the fourth vector to be an orthogonal linear combination of the four simple roots:

$$y = \sum_{i=1}^{4} r_i \alpha_i, \qquad \text{where} \qquad y \cdot \alpha_i = 0, \quad i = 1, 2, 3. \tag{4.19}$$

The orthogonality constraint in (4.19) and the fact that $\alpha_i^I = 0$ for $I = 6, \ldots, 16$ requires

$$y = (a, a, b, b, b, 0, 0, 0; 0, \ldots, 0), \tag{4.20}$$

while $\sum_I \alpha_i^I = 0$ requires $2a = -3b$. From here it is easy to check that with normalization $k_Y = 5/3$, we have $y = y_1$, Eq. (4.13). Thus we see that y_1 corresponds to a natural completion of the $SU(3) \times SU(2)$ roots (4.16) into a would-be $SU(5)$ subgroup of the observable E_8. We note that (4.20) has the form of a *minimal embedding* of hypercharge, in the spirit of the analysis carried out in [59].

Now we come to the origin of the subscript in (4.13). It turns out that (4.17) is not the unique E_8 root which may be appended to $\alpha_1, \alpha_2, \alpha_3$ to obtain the simple roots of an $SU(5)$ subalgebra of the observable E_8. The two ways that a supposed α_4 could be related to the roots $\alpha_1, \alpha_2, \alpha_3$ are shown in the Dynkin diagrams of Figure I. A line connecting α_i to α_j indicates $\alpha_i \cdot \alpha_j = -1$; if not connected by a line, $\alpha_i \cdot \alpha_j = 0$.

We define y as in (4.19), except that now we allow α_4 to be any observable E_8 root (i.e., $\alpha_4 = (\beta; 0)$, $\beta \in \Lambda_{E_8}$, $\beta^2 = 2$) consistent with Figure I. We simultaneously demand $2y^2 = 5/3$, corresponding to $k_Y = 5/3$ from (4.12). This gives solutions:

$$y = \pm\frac{1}{6}(3\alpha_1 + 4\alpha_2 + 2\alpha_3 + 6\alpha_4) \qquad \text{Case 1}, \tag{4.21}$$

$$y = \pm\frac{1}{6}(3\alpha_1 + 2\alpha_2 + 4\alpha_3 + 6\alpha_4) \qquad \text{Case 2}. \tag{4.22}$$

In each of the 175 models we consider here, the only $(3,2)$ representations under the observable $SU(3) \times SU(2)$ are contained in the untwisted sector, and they all take the form (4.15). To accommodate the MSSM we require that this representation have $Y = y \cdot K_{6,\ldots,11} = 1/6$. It suffices to demand this for any of the six K_i since by (4.19)

$$(K_i + \alpha_j) \cdot y = K_i \cdot y, \qquad \forall \quad i = 6, \ldots, 11, \quad j = 1, 2, 3. \tag{4.23}$$

(Recall from the discussion in Section 2 that the weights $K_{6,\ldots,11}$ are related to each other by the addition of $SU(3) \times SU(2)$ roots.) We choose to employ

$$K_6 = (-1, 0, -1, 0, 0, 0, 0, 0; 0, \ldots, 0). \tag{4.24}$$

It is easy to check that for Eq. (4.21), $K_6 \cdot y = 1/6$ imposes

$$K_6 \cdot \alpha_4 = \begin{cases} 4/3 & (+), \\ 1 & (-). \end{cases} \tag{4.25}$$

Since α_4 can only have integral or half-integral entries, we must take the negative sign in (4.21) and $K_6 \cdot \alpha_4 = 1$. For Eq. (4.22), $K_6 \cdot y = 1/6$ imposes

$$K_6 \cdot \alpha_4 = \begin{cases} 1 & (+), \\ 2/3 & (-). \end{cases} \tag{4.26}$$

Now we must take the positive sign in (4.22). To summarize, imposing that the quark doublet have $Y = 1/6$ constrains α_4 to satisfy the additional constraint

$$K_6 \cdot \alpha_4 = 1 \tag{4.27}$$

and determines the signs in (4.21,4.22):

$$y \;=\; -\frac{1}{6}(3\alpha_1 + 4\alpha_2 + 2\alpha_3 + 6\alpha_4) \qquad \text{Case 1}, \tag{4.28}$$

$$y \;=\; \frac{1}{6}(3\alpha_1 + 2\alpha_2 + 4\alpha_3 + 6\alpha_4) \qquad \text{Case 2}. \tag{4.29}$$

As noted briefly in Section 2, the ordering by which nonzero $E_8 \times E_8$ roots are determined to be positive is arbitrary. A particular *lexicographic ordering* for the first E_8 can be specified by an eight-tuple $(n_1, n_2, \ldots, n_8)$. Here, n_1 tells us which entry should be checked first, n_2 tells us which entry should be checked second, etc. For example, $(8,7,6,5,4,3,2,1)$ would instruct us to determine positivity by reading the entries of a given E_8 root vector backwards, right to left. It is easy to see that several *lexicographic orderings* are consistent with $\alpha_1, \alpha_2, \alpha_3$ being regarded as positive; in fact, the number of such orderings is 3360. Our final restriction on α_4 is that it for one of these 3360 orderings, α_4 is also positive. This is necessary if it is to be regarded as a simple root of a would-be $SU(5)$.

When all of the conditions described above are taken into account, the complete list of observable E_8 roots α_4 and the corresponding vectors y which result can be determined by straightforward analysis of the 240 nonzero E_8 roots. The results are given in Table IV. We label the four additional y solutions according to:

$$\begin{aligned} y_2 &= \frac{1}{6}(0,0,-1,-1,-1,-3,-3,-3;0,\ldots,0), \\ y_{3,4,5} &= \frac{1}{6}(0,0,-1,-1,-1,\underline{-3,3,3};0,\ldots,0). \end{aligned} \tag{4.30}$$

In what follows, we refer to Y_i, $i=1,\ldots,5$, as the five possible $SU(5)$ *embeddings* of the hypercharge in the $\mathrm{BSL_A}$ models.

α_4	y
$(0,1,-1,0,0,0,0,0;0,\ldots,0)$	$\frac{1}{6}(-3,-3,2,2,2,0,0,0;0,\ldots,0)$
$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};0,\ldots,0)$	$\frac{1}{6}(0,0,-1,-1,-1,-3,-3,-3;0,\ldots,0)$
$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\underline{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}};0,\ldots,0)$	$\frac{1}{6}(0,0,-1,-1,-1,\underline{-3,3,3};0,\ldots,0)$
$(-1,0,0,0,1,0,0,0;0,\ldots,0)$	$\frac{1}{6}(-3,-3,2,2,2,0,0,0;0,\ldots,0)$
$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\underline{-\frac{1}{2},\frac{1}{2},\frac{1}{2}};0,\ldots,0)$	$\frac{1}{6}(0,0,-1,-1,-1,\underline{-3,3,3};0,\ldots,0)$

Table IV: Observable E_8 roots which embed $SU(3)_C \times SU(2)_L$ into a would-be $SU(5)$.

A model must also have Y non-anomalous for it to survive unmixed with other $U(1)$ factors below Λ_X. Many models have a trace anomaly for one or more of the five Y_i. This would not occur if complete $SU(5)$ irreps were present. We have already seen that the untwisted sector does not contain complete would-be $SU(5)$ irreps for any of the 175 models (cf. Table XII). Of course, whether or not Y_1 is anomalous in those models also depends on the matter content of the twisted sectors. This in turn depends on the hidden sector embedding through (2.72); consequently, each of the 175 models must be studied separately.

We have determined the charges of all matter irreps with respect to Y_i ($i=1,\ldots,5$) for all of models. In those models where a given Y_i is not anomalous, the MSSM particle spectrum is never accommodated. That is not to say that we do not have enough $(3,2)$s, $(\bar{3},1)$s, $(1,2)$s and $(1,1)$s; in fact, we typically have too many of the latter three types, as can be seen from Tables VIII-XI. The difficulty comes in their hypercharge assignments when we take Y to be one of the five Y_i. Although there are always a few irreps with the right hypercharges, there are never enough.

As suggested by the discussion in Section 2, the origin of bizarre hypercharges with respect to the $SU(5)$ embeddings Y_i is due to the fact that twisted states generically have $E_8 \times E_8$ weights on a shifted lattice, as is apparent in (2.72). To further understand these matters, we now discuss the decomposition of the two lowest lying E_8 representations, of dimension 248 and 3875 respectively. The decomposition of these irreps under $E_8 \supset SU(5)$ is tabulated, for instance, in the review by Slansky [18]. We identify this $SU(5)$ as the subgroup of E_8 in which irreps of G_{SM} are embedded. The decompositions are (numbers in parentheses denote $SU(5)$ irreps)

$$\begin{aligned} 248 &= 24(1)+(24)+10(5+\overline{5})+5(10+\overline{10}), \\ 3875 &= 100(1)+65(5+\overline{5})+50(10+\overline{10})+5(15+\overline{15})+25(24)+5(40+\overline{40}) \\ &\quad +10(45+\overline{45})+(75). \end{aligned} \tag{4.31}$$

Although these are real representations, a chiral four-dimensional theory is obtained by compactification on a quotient manifold (i.e., the Z_3 orbifold), a mechanism pointed out some time ago [36]. Also from Slansky, we take the decomposition of the $SU(5)$ irreps shown in (4.31) with respect to $SU(5) \supset SU(3) \times SU(2) \times U(1)$, with the standard electroweak normalization for the $U(1)$ charge given in the last entry:

$$\begin{aligned}
1 &= (1,1,0) \\
5 &= (1,2,1/2) + (3,1,-1/3) \\
10 &= (1,1,1) + (\bar{3},1,-2/3) + (3,2,1/6) \\
15 &= (1,3,1) + (3,2,1/6) + (6,1,-2/3) \\
24 &= (1,1,0) + (1,3,0) + (3,2,-5/6) + (\bar{3},2,5/6) + (8,1,0) \\
40 &= (1,2,-3/2) + (3,2,1/6) + (\bar{3},1,-2/3) + (\bar{3},3,-2/3) + (8,1,1) + (\bar{6},2,1/6) \\
45 &= (1,2,1/2) + (3,1,-1/3) + (3,3,-1/3) + (\bar{3},1,4/3) + (\bar{3},2,-7/6) \\
&\quad + (\bar{6},1,-1/3) + (8,2,1/2) \\
75 &= (1,1,0) + (3,1,5/3) + (\bar{3},1,-5/3) + (3,2,-5/6) + (\bar{3},2,5/6) \\
&\quad + (6,2,5/6) + (\bar{6},2,-5/6) + (8,1,0) + (8,3,0)
\end{aligned} \tag{4.32}$$

While the higher dimensional $SU(5)$ irreps certainly contain states with unusual hypercharge (e.g., the $(1,2)$ irrep in the 40 of $SU(5)$ with $Y=-3/2$), given the number of 5, $\bar{5}$ and 10 representations present in (4.31) it is perhaps surprising that we do not obtain the $SU(3) \times SU(2) \times U(1)$ irreps to fill out the MSSM for any of the 175 models.

Beside the projections (2.66,2.67) in the untwisted sector—which lead to incomplete would-be $SU(5)$ irreps as discussed in detail above—the problem, of course, is that in the twisted sectors the $E_8 \times E_8$ weights do not correspond to the decomposition of E_8 representations described by (4.31,4.32). The weights are of the form $\tilde{K} = K + E(n_1,n_3,n_5)$; whereas $K \in \Lambda_{E_8\times E_8}$, for any twisted sector with solutions to (2.72) the embedding vector is a strict fraction of a lattice vector:

$$3E(n_1,n_3,n_5) \in \Lambda_{E_8\times E_8}, \qquad E(n_1,n_3,n_5) \notin \Lambda_{E_8\times E_8}. \tag{4.33}$$

Specializing (2.73), the hypercharge for any of the Y_i is given by

$$Y_i(\tilde{K}; n_1,n_3,n_5) = y_i \cdot K + \delta_{y_i}(n_1,n_3,n_5), \qquad \delta_{y_i}(n_1,n_3,n_5) = y_i \cdot E(n_1,n_3,n_5). \tag{4.34}$$

For a massless state, the value of $y_i \cdot K$ will take values corresponding to the decompositions (4.32); $y_i \cdot K$ values from the 3875 of E_8 occur because $K^2 > 2$ is possible, as discussed in Section 2. The second term on the right-hand side is the Wen-Witten defect, briefly discussed above in Section 2. Since each y_i is nonzero only in the first eight entries, the Wen-Witten defect only depends on the observable sector embeddings enumerated by CMM. It is easy to check that for each of the y_i the defect in each twisted sector is a multiple of $1/3$. This is consistent with general arguments [62, 63] which show that fractionally charged color singlet (bound) states in Z_N orbifolds have electric charges which are quantized in units of $1/N$.

4.4 Extended Hypercharge Embeddings

Having failed to accommodate the MSSM with any of the five Y_i, we envision the most general hypercharge consistent with leaving at least a hidden $SU(3)'$ unbroken to serve as the condensing group G_C. (Such a Y is of the *extended* hypercharge embedding variety, studied for example in Ref. [64].) That is, we include the possibility that Cartan generators of the nonabelian hidden sector group mix into Y under a Higgs effect, perhaps induced by the FI term. (A well-known example of the mixing of a nonabelian Cartan generator into a surviving $U(1)$ is the electroweak symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_E$.) Thus, we assume a hypercharge generator of the form

$$6Y = \sum_{a \neq X} c_a Q_a + \sum_{a,i} c_a^i H_a^i. \tag{4.35}$$

A factor of six has been included for later convenience. The Cartan generators written here are *not* those of (2.58) or (4.18). Rather, we choose a basis where the H_a^i are mutually orthogonal (i.e., $\mathrm{tr}_R\ H_a^i H_a^j = 0$ for $i \neq j$, any irrep R of G_a).

Nontrivial irreps of the hidden sector gauge group G_H may decompose under the partial breaking of G_H implied by (4.35) to give some of the $(1,2)$ and $(1,1)$ irreps of the MSSM. For instance, if the pattern of gauge symmetry breaking in an irrep Pattern 2.5 model is

$$SU(3)_C \times SU(2)_L \times SU(5) \times SU(2)' \times U(1)^8 \to SU(3)_C \times SU(2)_L \times SU(3)' \times U(1)_Y, \tag{4.36}$$

then we have the following decompositions of nontrivial irreps of G_H onto the surviving gauge symmetry group:

$$\begin{aligned} (1,1,5,1) &\to (1,1,3) + 2(1,1,1), \\ (1,1,10,1) &\to 2(1,1,3) + (1,1,\bar{3}) + (1,1,1), \\ (1,2,1,2) &\to 2(1,2,1). \end{aligned} \tag{4.37}$$

Thus, we get many candidates for e^c as well as candidates for L, H_d or H_u. The Cartan generator of $SU(2)'$ is allowed to mix into Y; this is also true of the two Cartan generators of $SU(5)$ which commute with all of the generators of the surviving $G_C = SU(3)'$. The weights of the $(1,2,1)$ and $(1,1,1)$ states in (4.37) with respect to these generators then contribute to the hypercharges of these states.

Corresponding to (4.35) is an assumption for the sixteen-vector y which describes the linear combination of $E_8 \times E_8$ Cartan generators H^I which give Y:

$$6y = \sum_{a \neq X} c_a q_a + \sum_{a,i} c_a^i h_a^i. \tag{4.38}$$

To calculate k_Y, we use Eq. (4.12) and the orthogonality of the sixteen-vectors appearing in (4.38):

$$k_Y = \frac{1}{36}\left(\sum_{a \neq X} c_a^2 k_a + \sum_{a,i} 2(c_a^i h_a^i)^2 \right). \tag{4.39}$$

We define, as above, $\hat{H}_a^i$ to be the generator H_a^i rescaled to the unified normalization (e.g., $\mathrm{tr}\,(\hat{H}_a^i)^2 = 1/2$ for an $SU(N)$ fundamental irrep). We express the rescaling by $H_a^i = \sqrt{k_a^i}\hat{H}_a^i$. Then in terms of the sixteen-vectors associated with these generators, using Eq. (4.11),

$$2(h_a^i)^2 = 2k_a^i(\hat{h}_a^i)^2 = k_a^i. \tag{4.40}$$

Thus, the hypercharge normalization may be expressed as

$$k_Y = \frac{1}{36}\left(\sum_{a\neq X} c_a^2 k_a + \sum_{a,i}(c_a^i)^2 k_a^i\right). \tag{4.41}$$

Eq. (4.41) gives k_Y as a quadratic form of the real coefficients c_a and c_a^i, a function which is easy to minimize subject to the linear constraints imposed by demanding that the seven types of chiral supermultiplets in the MSSM $(Q, u^c, d^c, L, H_d, H_u, e^c)$ be accommodated, including hypercharges. (For instance, we used standard routines available on the math package Maple.) We have performed an automated analysis to determine the minimum $\delta k_Y \equiv k_Y - 5/3$ values allowed by each model, for each possible assignment of the MSSM to the full pseudo-massless spectrum. Our results are shown in Table V.

Pattern	$\delta k_Y^{\rm min}$	Pattern	$\delta k_Y^{\rm min}$	Pattern	$\delta k_Y^{\rm min}$	Pattern	$\delta k_Y^{\rm min}$
1.1	0	2.4	8/29	3.3	-4/61	4.4	16/61
1.2	1/5	2.5	11/73	3.4	16/59	4.5	-1/31
2.1	4/29	2.6	4/11	4.1	-8/113	4.6	11/73
2.2	-8/167	3.1	1/7	4.2	-8/113	4.7	-1/31
2.3	0	3.2	-8/119	4.3	8/81	4.8	14/5

Table V: Minimum values of $\delta k_Y = k_Y - 5/3$.

It can be seen from the table that $k_Y = 5/3$ is possible in some patterns. We remark, however, that this value has lost most of its motivation in the present context. Whereas in a GUT the normalization $k_Y = 5/3$ came out naturally, we now obtain this value by artifice, choosing a "just so" linear combination of observable and hidden sector generators. Perhaps this is to be expected, since $SU(3)_C \times SU(2)_L$ was obtained from the start at the string scale, without ever being—properly speaking—embedded into a GUT.

For some of the assignments of $Q, u^c, d^c, L, H_d, H_u, e^c$ to the pseudo-massless spectrum, other states in the spectrum may have the right charges with respect to $SU(3)_C \times SU(2)_L \times U(1)_Y$ to also be candidates for some of these MSSM states. In this case, the MSSM states will generally be a mixture of all the candidate states from the pseudo-massless spectrum, as described above in Section 3. An example of this will be seen in the following section. This, however, does not alter our conclusions for the coefficients c_a and c_a^i, as well as the hypercharge normalization k_Y.

5 Example: $\mathbf{BSL_A}$ 6.5

The model labeling here is the same as described in Section 3: the observable embedding is CMM 6 from Table I of Ref. [24] and the hidden sector embedding is No. 5 from Table VI of Ref. [24]. Thus, the model has embedding

$$\begin{aligned} 3V &= (-1,-1,0,0,0,2,0,0;2,1,1,0,0,0,0,0), \\ 3a_1 &= (1,1,-1,-1,-1,2,1,0;-1,0,0,1,0,0,0,0), \end{aligned}$$

$$3a_3 \;=\; (0,0,0,0,0,1,1,2;2,0,0,-1,1,0,0,0). \tag{5.1}$$

(Recall that $a_5 \equiv 0$ in the class of models studied here.) Using the recipes of Section 2, it is easy to determine the simple roots and to check that the unbroken gauge group is

$$G = SU(3)_C \times SU(2)_L \times SU(5) \times SU(2)' \times U(1)^8. \tag{5.2}$$

The untwisted sector pseudo-massless matter states are also obtained by simple calculations; the twisted sectors are somewhat tedious because of the large number of states involved. The full spectrum of pseudo-massless matter states is given in Table XIV (Appendix B). Each entry corresponds to a *species* of chiral matter multiplets, with three families to each species. We have assigned labels 1 through 51 to the species for convenience of reference in the discussion which follows. The irrep of each species with respect to the nonabelian factors of G is given in the second column of Table XIV, with the order of entries corresponding to the order of the nonabelian factors in (5.2). It is not hard to check that the model falls into Pattern 2.6 of Table IX. This pattern has the attractive feature that it contains only three extra $(3+\bar{3},1)$ representations. Thus, we can expect less finagling with flat directions to arrange masses for these exotic isosinglet quarks. The subscript on the Irrep column data denotes the sector to which a species belongs: "U" is for untwisted, while for the twisted species, n_1, n_3 pairs of fixed point labels are given. The n_5 fixed point label now serves as a family index, so that for each twisted species, it takes on all three values $n_5 = 0, \pm 1$. Twisted oscillator matter states do not occur in the pseudo-massless spectrum of this model. The remainder of the columns in Table XIV provide information about $U(1)$ charges.

As discussed in Section 2, the eight $U(1)$ generators correspond to sixteen-dimensional vectors q_a which are orthogonal to the simple roots and to each other. It is not hard to determine a set of eight q_as. However, once the pseudo-massless spectrum of matter states has been calculated using the recipes of Section 2, one finds that a naive choice of the q_as does not isolate the trace anomaly to a single $U(1)$. Using the redefinition technique described in Section 2, we have isolated the anomaly to the eighth generator, which we denote Q_X. Unfortunately, the redefinitions required to do this, while maintaining orthogonality of the q_as, lead to large entries for many of the q_as when the charges of states are kept integral. We display our choice of q_as in Table VI, along with k_a (determined by Eq. (4.12)) and tr Q_a (determined from the pseudo-massless spectrum). We note that $q_1/6 = y_1$ of (4.13). States 27 and 42 would be electrically neutral exotic isoscalar quarks if we took $Q_1/6$ as hypercharge. This provides an explicit example of the effects of charge fractionalization; in the low energy theory these states would bind with ordinary quarks to form fractionally charged color singlet composite states.

For fields which are not Q_X neutral, we see from Table XIV that $|Q_X|$ has minimum value 1 and maximum value 6. On the other hand, from Table VI we see that $k_X = 84$. Then the generator with unified normalization is $\hat{Q}_X = Q_X/\sqrt{84}$ and for fields which are not $\hat{Q}_X$ neutral, $|\hat{Q}_X|$ has minimum value $1/\sqrt{84} \approx 0.11$ and maximum value $6/\sqrt{84} \approx 0.65$. We appealed to this range in Section 3 above.

Finally, we note that the $SU(5)$ charged states in the model consist of

$$3\,[\,(1,1,5,1) + 3(1,1,\bar{5},1) + (1,1,10,2)\,]. \tag{5.3}$$

Using $C(SU(5)) = 5$, $X(5) = X(\bar{5}) = 1/2$, and $X(10) = 3/2$ (apparent from (4.32) taking tr T^aT^a

a	q_a	tr Q_a	$k_a/4$
1	$(-3,-3,2,2,2,0,0,0;0,0,0,0,0,0,0,0)$	0	15
2	$3(-1,-1,-1,-1,-1,15,0,0;0,0,0,0,0,0,0,0)$	0	1035
3	$3(3,3,3,3,3,1,-46,0;0,0,0,0,0,0,0,0)$	0	9729
4	$\frac{3}{2}(-3,-3,-3,-3,-3,-1,-1,-47;0,0,0,0,0,0,0,0)$	0	2538
5	$\frac{3}{2}(-15,-15,-15,-15,-15,-5,-5,5;12,-12,-12,-48,-12,0,0,0)$	0	4590
6	$\frac{1}{2}(-15,-15,-15,-15,-15,-5,-5,5;-22,-12,-12,20,22,0,0,0)$	0	357
7	$3(0,0,0,0,0,0,0,0;1,0,0,0,1,0,0,0)$	0	9
X	$\frac{1}{2}(-3,-3,-3,-3,-3,-1,-1,1;4,6,6,4,-4,0,0,0)$	504	21

Table VI: Charge generators of $\mathrm{BSL_A}$ 6.5 (cf. (2.54)).

with respect to a generator of an $SU(3)$ subgroup of $SU(5)$), we find that

$$b_5^{\mathrm{tot}} = -3 \cdot 5 + 3(4 \cdot 1/2 + 2 \cdot 3/2) = 0. \tag{5.4}$$

Thus, in order to have supersymmetry broken by gaugino condensation in the hidden sector, it is necessary that vector masses be given to some of the states in (5.3). If we can arrange to give large masses to the $3(5+\bar{5})$ vector pairs, then the effective β function coefficient is only $b_5 = -3$. This gives a lower Λ_C than the pure $G_C = SU(2)$ case ($b_C = -6$) which was regarded as "marginal" in the Introduction. Consequently, the hidden $SU(5)$ must be broken to a subgroup so that vevs can be given to components of the $(\bar{5} \cdot \bar{5} \cdot 10)$ and $(5 \cdot 10 \cdot 10)$ invariants, allowing more states to get large masses. (For the $SU(5)$ invariant $(\bar{5} \cdot \bar{5} \cdot 10)$ to generate an effective mass term, the hidden $SU(2)'$ would also have to be broken since the 10s belong to doublet representations of $SU(2)'$, as is evident from Eq. (5.3).)

As an example, consider breaking $SU(5) \to SU(4)$. For many choices of the hypercharge generator, some (but generally not all) of the 5 and $\bar{5}$ irreps are hypercharge neutral. Decomposing these onto $SU(4)$ irreps, we have $5 = 4 + 1$ and $\bar{5} = \bar{4} + 1$. The breaking can be achieved by giving vevs to the $SU(4)$ singlets in these decompositions, though one should be careful to avoid generating non-vanishing F- or D-terms in the process. The 10 of $SU(5)$ decomposes according to $10 = 4 + 6$. The invariants mentioned above may generate masses for many of the nontrivial $SU(4)$ irreps, since under the $SU(5) \supset SU(4)$ decomposition

$$(5 \cdot \bar{5}) \ni (4 \cdot \bar{4}), \qquad (\bar{5} \cdot \bar{5} \cdot 10) \ni (1 \cdot \bar{4} \cdot 4), \qquad (5 \cdot 10 \cdot 10) \ni (1 \cdot 6 \cdot 6). \tag{5.5}$$

It is conceivable that *all* of the $SU(4)$ charged matter may be given $\mathcal{O}(\Lambda_X)$ masses in this way, yielding $b_4 = -12$. If some matter remains light and $SU(4)$ is identified as the condensing group G_C, values in the range $-12 < b_C \leq -6$ could be obtained. To say whether or not these arrangements can actually be made requires an analysis of D- and F-flat directions which is beyond the scope of the present work.

5.1 Accomodating the MSSM

Inspection of Table XIV shows that while appropriate $SU(3)_C \times SU(2)_L$ charged multiplets are present to accommodate the MSSM spectrum, the "obvious" choice for hypercharge, $Y_1 = Q_1/6$, does not provide for the three e^c supermultiplets nor does it provide enough $(1, 2)$ representations with hypercharge $-1/2$ to accommodate three Ls and an H_d. As discussed above, one problem is that most of the twisted states have bizarre Y_1 charges due to the Wen-Witten defect. We also have the problem that $\tilde{K}^2 = 4/3$ for twisted (non-oscillator) states (versus $K^2 = 2$ for untwisted), so that the $E_8 \times E_8$ weights are "smaller" and it is harder to obtain the "large" e^c hypercharge; this explains why $k_Y > 5/3$ is generically required. Note that the Y_1 charges are ordinary in the untwisted sector: the hidden irreps $(1, 1, 10, 2)$ and $(1, 1, 5, 1)$ are Y_1 neutral while the observable irreps $(3, 2, 1, 1)$, $(1, 2, 1, 1)$ and $(\bar{3}, 1, 1, 1)$ have Y_1 charges $1/6$, $1/2$ and $-2/3$ respectively. Furthermore, if we subtract off the Wen-Witten defect, we expect Y_1 charges which would appear in the decompositions (4.32) for twisted states. With this in mind, we define *Z charge* to be $Z = Y_1$ for untwisted states while for twisted states

$$Z(n_1, n_3, n_5) \equiv Y_1 - y_1 \cdot E(n_1, n_3, n_5) = \frac{Q_1}{6} - \frac{1}{3} + n_1 \frac{2}{3}, \tag{5.6}$$

where the last equality is easy to check using the embedding vectors (5.1). The Z charges are given in Table XIV. To see that these charges are ordinary, one should compare to the decompositions (4.31,4.32). Checking the Z charges and $SU(3) \times SU(2)$ irrep labels from Table XIV, it can be seen that all are in correspondence to some irrep contained in a decomposition of the 248 and 3875 irreps of E_8. An example of the role of the 3875 irrep can be seen in state 11 of Table XIV, which is a $(1, 2)$ irrep of $SU(3)_C \times SU(2)_L$ with Z charge $-3/2$; from (4.32) we see that this occurs in the 40 of $SU(5)$, which itself occurs in the 3875 but not the 248 of E_8. This shows how it is precisely the peculiar role of higher dimensional $E_8 \times E_8$ irreps and the shift $E(n_1, n_3, n_5)$ that is responsible for the bizarre Y_1 charges in the twisted sectors.

Thus, we are forced to assume hypercharge of the more general form (4.35), which in the present case we write as

$$6Y = c_1 Q_1 + \cdots + c_7 Q_7 + c_8 H_{(2')} + c_9 H^1_{(5)} + c_{10} H^2_{(5)}. \tag{5.7}$$

The generator $H_{(2')}$ is the Cartan element for the hidden $SU(2)'$, which we take to be

$$H_{(2')} = \mathrm{diag}\,(1, -1) \tag{5.8}$$

in the fundamental irrep. The generators $H^1_{(5)}, H^2_{(5)}$ are the two Cartan elements for the hidden $SU(5)$ which could combine into hypercharge while still leaving unbroken a hidden $SU(3)'$ for the condensing group G_C, as explained in Section 4. We take them to be given by

$$H^1_{(5)} = \mathrm{diag}\,(4, -1, -1, -1, -1), \qquad H^2_{(5)} = \mathrm{diag}\,(0, 3, -1, -1, -1), \tag{5.9}$$

for the fundamental representation. We seek solutions $c_1, \ldots, c_{10}$ which allow for the accommodation of the MSSM. As mentioned in Section 4, assigning the MSSM amounts to the imposition of seven linear constraints on the coefficients c_i, one for each of the species $Q, u^c, d^c, L, H_d, H_u, e^c$. Because of the enormous number of species to which L, H_d, H_u and e^c could be assigned, a very large number of assignments accommodate the MSSM. However, it is also important to consider the hypercharge normalization k_Y. From the discussion given in Section 4, we know that

$$k_Y = \frac{1}{36}(c_1^2 k_1 + \cdots + c_{10}^2 k_{10}), \tag{5.10}$$

with $k_1, \ldots, k_7$ given in Table VI, and where k_8, k_9, k_{10} depend on the normalization of the hidden $SU(2)' \times SU(5)$ Cartan generators (5.8,5.9). It is easy to see that the generators (5.8,5.9) have been rescaled from the unified normalization according to

$$\begin{gathered} H_{(2')} = \sqrt{k_8}\hat{H}_{(2')}, \qquad H^1_{(5)} = \sqrt{k_9}\hat{H}^1_{(5)}, \qquad H^2_{(5)} = \sqrt{k_{10}}\hat{H}^2_{(5)}, \\ k_8 = 4, \qquad k_9 = 40, \qquad k_{10} = 24. \end{gathered} \tag{5.11}$$

We have investigated the range of k_Y that is allowed in $\mathrm{BSL_A}$ 6.5, consistent with assignment of the MSSM spectrum to the model. This is not a difficult exercise. We first obtain seven linear constraint equations on the c_is from a given assignment of the seven types of fields in the MSSM. We use these constraint equations to rewrite (5.10) in terms of a set of independent c_is. The result is a quadratic form k_Y depending on the independent c_is. We minimize this quadratic form subject to the constraint of real c_i using a standard algorithm provided with the math package Maple. We have verified the automated results by hand in a few sample cases and find agreement. An exhaustive analysis of all possible assignments of the MSSM to the $\mathrm{BSL_A}$ 6.5 spectrum shows that in every case $k_Y > 5/3$, consistent with Table V (Pattern 2.6). As above, it is convenient to define $\delta k_Y = k_Y - 5/3$. We find that constraining $\delta k_Y \leq 2$ still gives 274 possible assignments. A manageable set is obtained if we impose the limit $\delta k_Y \leq 1$. The only possible assignments in this case are given in Table VII. We also give the minimum value $\delta k_Y^{\min}$ for each of the assignments. For the cases where $\delta k_Y^{\min} = 4/11$ or $\delta k_Y^{\min} = 1/2$, some of the MSSM states have been assigned to $(1, 2, 1, 2)$ irreps, which are each effectively two $(1, 2, 1)$ irreps when the hidden $SU(2)'$ is broken to give an effective nonabelian gauge symmetry group $SU(3) \times SU(2) \times SU(5)$. None of the assignments in Table VII require breaking the hidden $SU(5)$ to provide the e^c species or $SU(5)$ Cartan generators contributing to Y; that is, each of these assignments has $c_9 = c_{10} = 0$ for the minimum value $\delta k_Y^{\min}$. These two coefficients are independent parameters for any of the assignments in Table VII and *could* be made nonzero without affecting the Y values of the MSSM spectrum; however, this would alter the Y charges of $SU(5)$ charged states and would increase δk_Y above the minimum value $\delta k_Y^{\min}$. In principle, k_Y could be made arbitrarily large! Subscripts on species labels in Table VII denote which of the two $H_{(2')}$ eigenstates the MSSM state has been assigned to. For instance, in the $\delta k_Y^{\min} = 1/2$ assignments, 30_1 and 30_2 are states of opposite $SU(2)'$ isospin.

With these assignments and δk_Y set to its minimum value $\delta k_Y^{\min}$, the coefficients c_i in (5.7) are uniquely determined for each case; examples are:

$$\begin{aligned} \text{Assign. 1}: \quad (c_1, \ldots, c_{10}) &= (1, 3/253, 1/11891, -4/517, 0, 0, 2/11, -18/11, 0, 0), \\ \text{Assign. 9}: \quad (c_1, \ldots, c_{10}) &= (2/5, 1/10, 0, 0, 1/68, -3/68, 3/4, 0, 0, 0), \\ \text{Assign. 11}: \quad (c_1, \ldots, c_{10}) &= (1, -6/115, -2/5405, 8/235, 0, 0, 2/5, 0, 0, 0). \end{aligned} \tag{5.12}$$

From these one can calculate the hypercharges of the pseudo-massless spectrum, using the Q_a values and $SU(2)'$ irrep data provided in Table XIV. As an example, we have calculated the hypercharges of the spectrum for Assignment 11. These are tabulated in the last column of Table XIV.

No.	$Q, u^c, d^c, \underline{L, H_d}, H_u, e^c$	$\delta k_Y^{\rm min}$	No.	$Q, u^c, d^c, \underline{L, H_d}, H_u, e^c$	$\delta k_Y^{\rm min}$
1	$1, 3, 10, 11, 30_1, 2, 48_2$	$4/11$	10	$1, 3, 42, 30_1, 30_2, 2, 29$	$1/2$
2	$1, 3, 10, 25, 30_1, 2, 33_2$	$4/11$	11	$1, 3, 10, 11, 25, 2, 43$	$4/5$
3	$1, 3, 10, 30_1, 31, 2, 28_2$	$4/11$	12	$1, 3, 10, 11, 31, 2, 49$	$4/5$
4	$1, 3, 10, 30_1, 44, 2, 16_2$	$4/11$	13	$1, 3, 10, 25, 44, 2, 34$	$4/5$
5	$1, 3, 42, 11, 30_1, 2, 48_2$	$4/11$	14	$1, 3, 10, 31, 44, 2, 23$	$4/5$
6	$1, 3, 42, 25, 30_1, 2, 33_2$	$4/11$	15	$1, 3, 42, 11, 25, 2, 35$	$4/5$
7	$1, 3, 42, 30_1, 31, 2, 28_2$	$4/11$	16	$1, 3, 42, 11, 31, 2, 24$	$4/5$
8	$1, 3, 42, 30_1, 44, 2, 16_2$	$4/11$	17	$1, 3, 42, 25, 44, 2, 9$	$4/5$
9	$1, 3, 10, 30_1, 30_2, 2, 29$	$1/2$	18	$1, 3, 42, 31, 44, 2, 17$	$4/5$

Table VII: Assignments satisfying $\delta k_Y \leq 1$ in BSL$_{\rm A}$ 6.5. Underlining on H_d and L indicates that either permutation may be assigned to the fourth and fifth entries. Where applicable, the subscript on a state label denotes which of the two $H_{(2')}$ eigenstates of a $(1,2,1,2)$ irrep is used in an assignment.

For all of the $\delta k_Y^{\rm min} = 4/5$ cases, the $SU(3)_C \times SU(2)_L$ charged exotic matter is

$$3\ [\ (3,1,1/15) + (\bar{3},1,-1/15) + 2(1,2,1/10) + 2(1,2,-1/10)\ ] + 2\ [\ (1,2,1/2) + (1,2,-1/2)\ ]. \quad (5.13)$$

The last number in each term gives the hypercharge of the corresponding state. We refer to the $SU(3)_C$ charged states as *exoquarks* and to the $SU(2)_L$ charged states as *exoleptons.* The last four exolepton states correspond to the two extra families of H_u-like and H_d-like states which are an artifact of the three generation construction. However, the other exoleptons have $Y = \pm 1/10$, a rather bizarre value, and certainly not one that appears in GUT scenarios, as can be seen by comparison to (4.32). Here again we see the effect of charge fractionalization. Similar comments apply to the exoquarks which have $Y = \pm 1/15$.

For all of the $\delta k_Y^{\rm min} = 1/2$ assignments, the $SU(3)_C \times SU(2)_L$ charged exotic matter is

$$3\ [\ (3,1,-1/3) + (\bar{3},1,1/3) + 4(1,2,0)\ ] + 2\ [\ (1,2,1/2) + (1,2,-1/2)\ ]. \quad (5.14)$$

The exoquarks in these assignments have SM charges of the colored Higgs fields in an $SU(5)$ GUT. Whether or not their masses are similarly constrained by proton decay depends on a detailed study of the allowed effective superpotential couplings along a given flat direction, since we do not have the $SU(5)$ symmetry to relate Yukawa couplings. Since altogether we have six $(\bar{3},1,1/3)$ representations, each of the three d^c-type quarks and their three exoquark relatives will generally be a mixture of States 10 and 42, corresponding to a cross between Assignments 9 and 10. Such mixing was discussed above in Section 3.

For all of the $\delta k_Y^{\rm min} = 4/11$ assignments, the $SU(3)_C \times SU(2)_L$ charged exotic matter is

$$\begin{aligned} 3\ [\ &(3,1,-2/33) + (\bar{3},1,2/33) + (1,2,1/22) + 2(1,2,-3/22) + (1,2,5/22)\ ] \\ &+2\ [\ (1,2,1/2) + (1,2,-1/2)\ ]. \end{aligned} \quad (5.15)$$

Note that a portion of the exolepton spectrum is chiral and would lead to a massless states if the usual electroweak symmetry breaking is assumed. For this reason the Assignments 1-8 are not viable.

5.2 Gauge Coupling Unification

Gauge coupling unification in semi-realistic four-dimensional string models has been a topic of intense research for several years. The situation in the heterotic theory has been reviewed by Dienes in Ref. [61], which contains a thorough discussion and extensive references to the original articles. We will only present a brief overview; the interested reader is recommended to Dienes' review for further details.

It has been known since the earliest attempts [65] to use closed string theories as unified "theories of everything" that

$$g^2 \sim \kappa^2/\alpha', \tag{5.16}$$

where g is the gauge coupling, κ is the gravitational coupling and α' is the *Regge slope,* related to the string scale by $\Lambda_{\text{string}} \approx 1/\sqrt{\alpha'}$. In particular, this relation holds for the heterotic string [1]. However, g and κ in (5.16) are the ten-dimensional couplings. By dimensional reduction of the ten-dimensional effective field theory obtained from the ten-dimensional heterotic string in the zero slope limit, the relation (5.16) may be translated into a constraint relating the heterotic string scale Λ_H to the four-dimensional Planck mass m_P. One finds, as expected on dimensional grounds, $m_P \sim 1/\sqrt{\kappa}$, where the coefficients which have been supressed depend on the size of the six compact dimensions; similarly, the four-dimensional gauge coupling satisfies $g_H \sim g$; for details see Ref. [66]. Then (5.16) gives

$$\Lambda_H \sim g_H m_P. \tag{5.17}$$

Kaplunovsky has made this relation more precise, including one loop effects from heavy string states [42]. Subject to various conventions described in [42], including a choice of the $\overline{\text{DR}}$ renormalization scheme in the effective field theory, the result is:

$$\Lambda_H \approx 0.216 \times g_H m_P = g_H \times 5.27 \times 10^{17}\,\text{GeV}\,. \tag{5.18}$$

In (5.18), a single gauge coupling, g_H, is shown. However, in the heterotic orbifolds under consideration the gauge group G has several factors, each of which will have its own running gauge coupling. One may ask how these running couplings are related to g_H. This question was studied by Ginsparg [58], with the result that the running couplings unify to a common value g_H at the string scale Λ_H, up to string threshold effects and affine levels (discussed below). (In the case of $U(1)$s, normalization conventions must be accounted for, as we have described in detail in Section 4.) Specifically, unification in four-dimensional string models makes the following requirements on the running gauge couplings $g_a(\mu)$:

$$k_a g_a^2(\Lambda_H) = g_H^2, \quad \forall\, a. \tag{5.19}$$

Here, k_a for a nonabelian factor G_a is the *affine* or *Kac-Moody level* of the current algebra in the underlying theory which is responsible for the gauge symmetry in the effective field theory. It is unnecessary for us to trouble ourselves with a detailed explanation of this quantity or its string theoretic origins, since $k_a = 1$ for any nonabelian factor in the heterotic orbifolds we are considering.

For this reason, these heterotic orbifolds are referred to as affine level one constructions. In the case of G_a a $U(1)$ factor, k_a carries information about the normalization of the corresponding current in the underlying theory, and hence the normalization of the charge generator in the effective field theory. We saw explicit examples of this in the previous section.

The important point, which has been emphasized many times before, is that a gauge coupling unification prediction is made by the underlying string theory. The SM gauge couplings are known (to varying levels of accuracy), say, at the Z scale (approximately 91 GeV). Given the particle content and mass spectrum of the theory between the Z scale and the string scale, one can easily check at the one loop level whether or not the unification prediction is approximately consistent with the Z scale boundary values. To go beyond one loop requires some knowledge of the other couplings in the theory, and the analysis becomes much more complicated. However, the one loop success is not typically spoiled by two loop corrections, but rather requires a slight adjustment of flexible parameters (such as superpartner masses) which enter the one loop analysis.

In what follows we briefly discuss the one loop running of SM gauge couplings in $\mathrm{BSL_A}$ 6.5, Assignment 11 of Table VII, estimating two loop effects using previous studies of the MSSM. Due to the presence of exotic matter, we are able to achieve string scale unification. This sort of unification scenario has been studied many times before, for example in Refs. [67, 68, 69, 70]. However, in contrast to the Refs. [67, 69, 70], we have states which would not appear in decompositions of standard GUT groups, such as (4.32). Indeed, it was found by Gaillard and Xiu in Ref. [67] that $(3+\bar{3},2)$ representations with hypercharge $Y=\pm 1/6$ were necessary to string scale unification, while Faraggi achieved string unification in Ref. [68] in a model where the only colored exotics were $(3+\bar{3},1)$ states. The resolution of this apparent conflict is that the unification scenario of Faraggi contains $(1,2)$ exoleptons with vanishing hypercharge and $(3+\bar{3},1)$ exoquarks with hypercharge $Y=\pm 1/6$; such states have exotic electric charge and *do not* appear in (4.32). The appearance of these states is due to the Wen-Witten defect in the free fermionic construction used in the model of Ref. [68], which has a $Z_2 \times Z_2$ orbifold underlying it, leading to shifts in hypercharge values by integer multiples of 1/2. Because exotics with small hypercharge values, much like the $(3+\bar{3},2)$ representations used by Gaillard and Xiu, appear in the model employed by Faraggi, the $SU(3)\times SU(2)$ running can be altered to unify at the string scale without having an overwhelming modification on the running of the $U(1)_Y$ coupling.

Similar to the unification scenario of Faraggi, in the model studied here exotic representations with small hypercharges are present and allow us to unify at the string scale without the presence of exotic quark doublets. However, we also have nonstandard hypercharge normalization: for Assignment 11 the minimum value is $k_Y = 37/15 > 5/3$. Nonstandard hypercharge normalization has been studied previously, for example in Refs. [71, 59]. In these analyses, it was found that *lower* values $k_Y < 5/3$ were preferred if only the MSSM spectrum is present up to the unification scale; the preferred values were between 1.4 to 1.5. Unfortunately, we are faced with the opposite effect—a larger than normal $k_Y = 37/15$. This larger value requires a larger correction to the running from the exotic states, and has the effect of pushing down the required mass scale of the exotics from what was found in Faraggi's analysis—particularly in the case of the exoquarks.

Standard evolution of the gauge couplings from the Z scale (i.e., the solution to (1.3) for groups other than G_C), together with the unification prediction (5.19), leads to three constraint equations:

$$4\pi\alpha_H^{-1} = \frac{1}{k_Y}\left[4\pi\alpha_Y^{-1}(m_Z) - b_Y \ln\frac{\Lambda_H^2}{m_Z^2} - \Delta_Y\right], \tag{5.20}$$

$$4\pi\alpha_H^{-1} = 4\pi\alpha_a^{-1}(m_Z) - b_a \ln\frac{\Lambda_H^2}{m_Z^2} - \Delta_a, \quad a = 2, 3. \tag{5.21}$$

The notation is conventional, with $\alpha_a = g_a^2/4\pi$ $(a = H, Y, 2, 3)$. Corrections are captured by the quantities Δ_a, and will be discussed below. The quantities b_a, $a = Y, 2, 3$ are the β function coefficients

$$b_a = -3C(G_a) + \sum_R X_a(R) \tag{5.22}$$

evaluated for the MSSM spectrum. Here, $C(SU(N)) = N$ while $C(U(1)) = 0$. For a fundamental or antifundamental representation of $SU(N)$ we have $X_a = 1/2$ while for hypercharge $X_Y(R) = Y^2(R)$. This gives

$$b_Y = 11, \qquad b_2 = 1, \qquad b_3 = -3. \tag{5.23}$$

Throughout, we use Z scale boundary values from the Particle Data Group 2000 review [25], which are given in the $\overline{\mathrm{MS}}$ scheme. For a supersymmetric running, these boundary values should be converted to the $\overline{\mathrm{DR}}$ scheme, so that the supersymmetry algebra is kept four-dimensional [72, 73]. These scheme conversion effects are included in the corrections Δ_a. Due to very small errors (relative to other uncertainties in the analysis), we take as precise

$$m_Z = 91.19\,\mathrm{GeV}, \qquad \alpha_e^{-1}(m_Z) = 127.9. \tag{5.24}$$

For the other couplings we utilize global fits to experimental data [25]:

$$\sin^2\theta_W(m_Z) = 0.23117 \pm 0.00016, \qquad \alpha_3(m_Z) = 0.1192 \pm 0.0028. \tag{5.25}$$

Using

$$\alpha_2^{-1} = \alpha_e^{-1}\sin^2\theta_W, \qquad \alpha_Y^{-1} = \alpha_e^{-1}\cos^2\theta_W, \tag{5.26}$$

we obtain the boundary values

$$\alpha_Y^{-1}(m_Z) = 98.333 \pm 0.020, \qquad \alpha_2^{-1}(m_Z) = 29.567 \pm 0.020, \qquad \alpha_3^{-1}(m_Z) = 8.39 \pm 0.20. \tag{5.27}$$

We now discuss the various corrections contributing to Δ_a $(a = Y, 2, 3)$. Each may be written as the sum of six terms:

$$\Delta_a = \Delta_a^{\mathrm{conv}} + \Delta_a^{\mathrm{HL}} + \Delta_a^{\mathrm{string}} + \Delta_a^{\mathrm{light}} + \Delta_a^{\mathrm{exotic}} + \Delta_a^{\mathrm{heavy}}. \tag{5.28}$$

The quantities Δ_a^{conv} convert the $\overline{MS}$ renormalization scheme input values (5.27) to the $\overline{DR}$ scheme [73, 57]. They are given by:

$$\Delta_a^{\mathrm{conv}} = \frac{1}{3}C(G_a) \quad \Rightarrow \quad \Delta_Y^{\mathrm{conv}} = 0, \quad \Delta_2^{\mathrm{conv}} = 2/3, \quad \Delta_3^{\mathrm{conv}} = 1. \tag{5.29}$$

As will be seen below, these corrections are negligible in comparison to the other terms in Δ_a, and we could ignore them without changing our results in a meaningful way.

The quantities $\Delta_a^{\rm HL}$ represent corrections from higher loop orders, which are sensitive to Yukawa couplings for the MSSM spectrum and the exotic states. If either the top or bottom Yukawa coupling evolves to nonperturbative values somewhere between Z scale and the string scale (as can happen for small or very large values of the ratio of MSSM Higgs vevs, $\tan\beta$), the $\Delta_a^{\rm HL}$ correction is out of control. However, if the Yukawa couplings arise from a weakly coupled heterotic string theory, as we assume, then this does not occur; $\Delta_a^{\rm HL}$ will take more reasonable values. For example, Dienes, Faraggi and March-Russell [59] have studied the range of MSSM two loop corrections with the Yukawa couplings taking values $\lambda_t(m_Z) \approx 1.1$ and $\lambda_b(m_Z) \approx 0.175$. (Using $m_b(m_Z) \approx 3.0$ GeV from Ref. [74] and $m_t(m_Z) \approx 174$ GeV from [25], these Yukawa couplings correspond to $\tan\beta \approx 9.2$.) These authors found that the two loop (TL) correction terms took approximate values

$$\Delta_Y^{\rm TL} \approx 11.6, \qquad \Delta_2^{\rm TL} \approx 12.3, \qquad \Delta_3^{\rm TL} \approx 6.0. \tag{5.30}$$

These should dominate $\Delta_a^{\rm HL}$, so we assume that to the same level of approximation $\Delta_a^{\rm HL} \approx \Delta_a^{\rm TL}$, $\forall\ a = Y, 2, 3$. Relative to the boundary values for $4\pi\alpha_a^{-1}$, these are 0.9%, 3.3% and 5.7% corrections, respectively. By comparison, the largest experimental error is 2.4% for α_3^{-1}.

The third type of correction is peculiar to unified theories with large numbers of gauge-charged states above or near the unification scale. These effects have been extensively studied [75] in GUTs. In attempts to bring unification predictions into good agreement with precision data these corrections play an important role [57]. When very large GUT group representations are introduced near the unification scale, these corrections can be considerable [76]. With the standard-like string constructions which we study here, a GUT symmetry group and heavy states which complete GUT multiplets are not restored at the unification scale. Rather, the chief concern is with threshold effects due to the enormous towers of massive string states. These may be computed from one loop diagrams in the underlying string theory, using background field methods quite similar to those exploited in ordinary field theory [42]. As noted above, in some four-dimensional heterotic theories, string threshold corrections exist which grow in size as the T-moduli vevs increase [47]. This corresponds to the large volume limit for the compact dimensions; the potentially large contribution in this limit can also be understood from the fact that the compactification scale drops below the string scale and entire excited mass levels of the string enter the running below the string scale. In any event, such T-moduli dependent string threshold effects are irrelevant for the 175 models studied here, as they do not occur in Z_3 orbifold compactifications of the heterotic string [47].

However, threshold corrections which do not increase with the vevs of T-moduli must also be considered. These threshold effects have been calculated by Mayr, Nilles and Stieberger [77] for an example model which is equivalent to one of the 175 studied here. They find that the string threshold effects are given by

$$\Delta_a^{\rm string} = 0.079\ b_a^{\rm tot} + 4.41\ k_a. \tag{5.31}$$

(Actually, Ref. [77] states that (5.31) is valid with $k_a = 1$. However, starting with the hypercharge coupling in the unified normalization $\alpha_1^{-1} = \alpha_Y^{-1}/k_Y$, it can be seen from (5.20) that by our conventions $b_1^{\rm tot} = b_Y^{\rm tot}/k_Y$ and $\Delta_1 = \Delta_Y/k_Y$. Substituting these expressions into (5.31) for $a = 1$, and then solving for $\Delta_Y^{\rm string}$, one finds that the formula is also valid for $a = Y$ where $k_Y \neq 1$.) It is important to keep in mind that $b_a^{\rm tot}$ is the β function coefficient for G_a with the full spectrum of pseudo-massless states. This includes those states which get $\Lambda_X \approx \Lambda_H$ scale masses when the vacuum shifts to cancel the FI term. Because of the large number of states with charge under a given $U(1)$ factor, the hypercharge correction $\Delta_Y^{\rm string}$ is usually much larger than $\Delta_2^{\rm string}$ or $\Delta_3^{\rm string}$. The precise values of the coefficients in (5.31) will vary from model to model; these must be worked out by the numerical evaluation of a rather complicated integral, as explained in [77]. However, Mayr, Nilles and Stieberger analyzed a few other Z_3 orbifold models, which do not fall into the class of models considered here, and found that the threshold corrections differed only slightly from (5.31). This was found to be due to the fact that the leading term in the integrand did not depend on the embedding. From this we conclude that Eq. (5.31) gives a fair estimate of the string threshold corrections in all 175 models which we study here.

The hypercharge values of the 51 species must be calculated in order to compute $b_Y^{\rm tot}$ for the example model. This of course depends on what linear combination (5.7) of generators we take to be the hypercharge generator Y. As an example we take Assignment 11 from Table VII, which has (for $\delta k_Y = \delta k_Y^{\rm min}$) hypercharge normalization $k_Y = 37/15$ and hypercharges Y given in Table XIV. It is easy to check that

$$b_Y^{\rm tot} = {\rm tr}\ Y^2 = 171/5, \qquad b_2^{\rm tot} = 9, \qquad b_3^{\rm tot} = 0. \tag{5.32}$$

Applying (5.31), one finds

$$\Delta_Y^{\rm string} \approx 13.6, \qquad \Delta_2^{\rm string} \approx 5.1, \qquad \Delta_3^{\rm string} \approx 4.4, \tag{5.33}$$

which are comparable to the two loop corrections in (5.30).

Next we discuss one loop threshold corrections for pseudo-massless states which have masses greater than the Z mass but less than the string scale Λ_H. Heuristically, these corrections may be understood as follows. At a running scale μ, only states with masses less than this scale contribute significantly to the running of the gauge couplings. Then the more accurate one loop β function coefficients in this regime are calculated using the spectrum of states with masses less than μ. If some of the superpartner states are more massive than μ, the β function coefficients will not take the MSSM values given in (5.23). Non-MSSM values for the coefficients will also be obtained if exotic states with masses less than μ are present. In (5.20,5.21) we assumed the MSSM values for the β function coefficients. The threshold corrections we now discuss account for the non-MSSM β function coefficients which "should" have been used over regimes where the MSSM was not the spectrum of states with masses less than μ. This simple picture is valid in the $\overline{DR}$ renormalization scheme; in other schemes there are modifications to the one loop threshold corrections presented below, as has been recounted for example in [57].

The first correction is due to MSSM superpartners to the SM. In the coefficients (5.23), we have implicitly included these particles in the running all the way from the Z scale; however, if they are more massive than the Z scale, this is not quite right. We introduce "light" threshold corrections which subtract out the running which should never have been there in the first place:

$$\Delta_a^{\rm light} = - \sum_{m_i > m_Z} b_{a,i} \ln \frac{m_i^2}{m_Z^2}, \tag{5.34}$$

where $b_{a,i}$ is the contribution to the MSSM b_a coming from the state i of mass m_i. Properly speaking, the top quark and the light scalar Higgs doublet threshold corrections should also be included in $\Delta_a^{\rm light}$. The top mass is near enough to the Z mass that the correction is negligibly small for our purposes; we assume that this is likewise true for the light scalar Higgs doublet.

Following Langacker and Polonsky [57], one often defines effective thresholds T_a ($a = Y, 2, 3$) which give the same $\Delta_a^{\rm light}$ as (5.34):

$$\Delta_a^{\rm light} \equiv -(b_a - b_a^{\rm SM}) \ln \frac{T_a^2}{m_Z^2}. \qquad (5.35)$$

Here, $b_a^{\rm SM}$ are the β function coefficients in the SM (which we take to include a light Higgs doublet and the top quark):

$$b_Y^{\rm SM} = 7, \quad b_2^{\rm SM} = -3, \quad b_3^{\rm SM} = -7, \qquad \Rightarrow \qquad b_a - b_a^{\rm SM} = 4, \quad a = Y, 2, 3, \qquad (5.36)$$

where we make use of (5.23). Eq. (5.35) has the interpretation that it gives the equivalent threshold correction to α_a^{-1} if all superpartners contributing to b_a had a uniform mass scale T_a. One may study how the prediction for $\alpha_3(m_Z)$ in terms of $\sin^2\theta_W(m_Z)$ depends on T_a and determine a combination of the three effective thresholds which would give the same effect as a uniform superpartner mass threshold $\Lambda_{\rm SUSY}$ [57]:

$$\begin{aligned}&(b_Y - b_3 k_Y)(b_2 - b_2^{\rm SM}) \ln\frac{T_2}{m_Z} - (b_2 - b_3)(b_Y - b_Y^{\rm SM}) \ln\frac{T_Y}{m_Z} - (b_Y - b_2 k_Y)(b_3 - b_3^{\rm SM}) \ln\frac{T_3}{m_Z} \\ \equiv\ & \Big[(b_Y - b_3 k_Y)(b_2 - b_2^{\rm SM}) - (b_2 - b_3)(b_Y - b_Y^{\rm SM}) - (b_Y - b_2 k_Y)(b_3 - b_3^{\rm SM})\Big] \ln\frac{\Lambda_{\rm SUSY}}{m_Z}. \qquad (5.37)\end{aligned}$$

From this one can define the single effective threshold $\Lambda_{\rm SUSY}$ in terms of a geometric average of superpartner masses [78]. Because of terms of opposite sign in (5.37), it should be clear that $\Lambda_{\rm SUSY}$ can be much lower than the typical superpartner mass, which we denoted $M_{\rm SUSY}$ in the Introduction; $\Lambda_{\rm SUSY} \lesssim m_Z$ is not at all unreasonable, even with the typical superpartner mass $M_{\rm SUSY}$ several hundred GeV. Furthermore, it should be noted that the formulae for $\Lambda_{\rm SUSY}$ given in Refs. [57, 78] are modified in the present context due to the nonstandard hypercharge normalization, as has been accounted for in (5.37), which holds generally. (Our $b_a^{\rm SM}$, as given in (5.36), also differ slightly due to the inclusion of a light scalar Higgs doublet; however, Eq. (5.37) has been written such that it is valid in either case.) Lastly, the effective threshold $\Lambda_{\rm SUSY}$ completely encodes the effects of split thresholds on the $\alpha_3(m_Z)$ versus $\sin^2\theta_W(m_Z)$ prediction, but for other unification predictions, such as the unified coupling and scale of unification, a fixed value of $\Lambda_{\rm SUSY}$ corresponds to many different outcomes [78]; this is because other unification predictions depend on combinations of the T_a other than (5.37). In the present context, simply using $\Lambda_{\rm SUSY}$ would not cover the full range of g_H, Λ_H and the predictions for intermediate scales where exotic matter thresholds alter the running. An exhaustive analysis would require scanning over the parameters T_a ($a = Y, 2, 3$) independently, or subject to model constraints on the generation of soft masses by supersymmetry breaking. Our purpose here is simply to demonstrate the possibility of string scale unification with nonstandard hypercharge normalization and to estimate the order of magnitude required for the exotic scales. For these purposes it is therefore sufficient to take $\Lambda_{\rm SUSY} \approx T_a$ ($a = Y, 2, 3$). Within this universal scale $\Lambda_{\rm SUSY}$ approximation,

$$\Delta_a^{\rm light} = -4 \ln \frac{\Lambda_{\rm SUSY}^2}{m_Z^2}, \quad a = Y, 2, 3. \qquad (5.38)$$

If we limit $m_Z \lesssim \Lambda_{\rm SUSY} \lesssim 1$ TeV, then

$$0 \gtrsim \Delta_a^{\rm light} \gtrsim -19.2, \quad a = Y, 2, 3. \qquad (5.39)$$

The second set of mass threshold corrections comes from exotic matter at intermediate scales. For the sake of simplicity, we assume that exoleptons with mass much less than the string scale enter the running at a *single* scale Λ_2. We assume that *all* the exoquarks enter at a single scale Λ_3. (Introducing only *some* of the exoquarks forces Λ_3 to even lower values than we will find below, which are already a bit of a problem given the exotic hypercharges that these exoquarks have.) The exotic matter threshold corrections can be thought of as due to shifts in the total β function coefficients between $\Lambda_{2,3}$ and the string scale. Since we introduce $3(3+\bar{3},1)$ chiral multiplets q_i and q_i^c at Λ_3, we have

$$\Delta_3^{\rm exotic} = 3 \ln \frac{\Lambda_H^2}{\Lambda_3^2}. \tag{5.40}$$

The shift in the β function coefficient for $SU(2)_L$ due to extra $(1,2)$ representations—the exolepton chiral multiplets ℓ_i and ℓ_i^c introduced at Λ_2—is given by

$$\delta b_2 = \sum_{\ell_i,\ell_i^c} \frac{1}{2}. \tag{5.41}$$

That is, δb_2 is just the number of exolepton pairs $\ell_i + \ell_i^c$. The threshold corrections are

$$\Delta_2^{\rm exotic} = \delta b_2 \ln \frac{\Lambda_H^2}{\Lambda_2^2}. \tag{5.42}$$

The exoquark and exolepton chiral multiplets also carry hypercharge. We denote the shifts in the β function coefficient for $U(1)_Y$ by

$$\delta b_Y = \sum_{q_i,q_i^c} (Y_i)^2, \qquad \delta b_Y' = \sum_{\ell_i,\ell_i^c} (Y_i)^2. \tag{5.43}$$

In this notation the threshold corrections are

$$\Delta_Y^{\rm exotic} = \delta b_Y \ln \frac{\Lambda_H^2}{\Lambda_3^2} + \delta b_Y' \ln \frac{\Lambda_H^2}{\Lambda_2^2}. \tag{5.44}$$

Let m, n denote the numbers of exolepton pairs entering the running at Λ_2, where m is the number of $Y = \pm 1/2$ exolepton pairs and n is the number of $Y = \pm 1/10$ exolepton pairs. We then have

$$\delta b_Y = \frac{2}{25}, \qquad \delta b_Y' = m + \frac{n}{25}, \qquad \delta b_2 = m + n. \tag{5.45}$$

For purposes of illustration below, we will study only the case $(m,n) = (0,6)$, for which

$$\delta b_Y = \frac{2}{25}, \qquad \delta b_Y' = \frac{6}{25}, \qquad \delta b_2 = 6. \tag{5.46}$$

It is not difficult to generalize our results to other (m,n) values.

Finally, there is the spectrum of particles which get masses of order Λ_X when the vacuum shifts to cancel the FI term. Since $\Lambda_X < \Lambda_H$ in BSL$_{\rm A}$ 6.5 (cf. Table II, Pattern 2.6), these can give an appreciable heavy threshold correction. Corrections of this type have been noted previously; for example, in Ref. [46]. We assume that all pseudo-massless states other than the MSSM spectrum plus exotics associates with $\Lambda_{2,3}$ enter the running at Λ_X, which is convenient because the ratio

$$\ln \frac{\Lambda_H^2}{\Lambda_X^2} = 2 \ln \frac{0.216 \times g_H m_P}{0.170 \times g_H m_P} = 0.479 \tag{5.47}$$

is independent of g_H (both Λ_X and Λ_H are proportional to g_H); here we use the value for Pattern 2.6 from Table II. Taking into account the exotic matter assumed at intermediate scales $\Lambda_{2,3}$ and the total β function coefficients mentioned above, we have

$$\Delta_Y^{\text{heavy}} = (b_Y^{\text{tot}} - b_Y - \delta b_Y - \delta b'_Y) \ \ln \frac{\Lambda_H^2}{\Lambda_X^2} = 10.3, \tag{5.48}$$

$$\Delta_2^{\text{heavy}} = (b_2^{\text{tot}} - b_2 - \delta b_2) \ \ln \frac{\Lambda_H^2}{\Lambda_X^2} = 1.0, \qquad \Delta_3^{\text{heavy}} = 0. \tag{5.49}$$

The hypercharge threshold correction is comparable to the larger corrections discussed above. On the other hand, we could just as well ignore $\Delta_{2,3}^{\text{heavy}}$ at the level of approximation made here.

As we tune $\Lambda_{2,3}$ to satisfy the unification constraints, it is convenient to define the sum of all the corrections *except* Δ_a^{exotic}:

$$\Delta_a^0 \equiv \Delta_a - \Delta_a^{\text{exotic}} = \Delta_a^{\text{conv}} + \Delta_a^{\text{HL}} + \Delta_a^{\text{string}} + \Delta_a^{\text{light}} + \Delta_a^{\text{heavy}}. \tag{5.50}$$

Using the above estimates for each of the terms, we find for the case of $\Lambda_{\text{SUSY}} = m_Z$

$$\Delta_Y^0 \approx 35.5, \qquad \Delta_2^0 \approx 19.1, \qquad \Delta_3^0 \approx 11.4. \tag{5.51}$$

For the case of $\Lambda_{\text{SUSY}} = 1$ TeV, the estimate is

$$\Delta_Y^0 \approx 16.3, \qquad \Delta_2^0 \approx -0.1, \qquad \Delta_3^0 \approx -7.8. \tag{5.52}$$

We now proceed to study the unification constraint in BSL_A 6.5, Assignment 11, subject to the assumptions described above. For convenience, we define

$$a_H = 4\pi\alpha_H^{-1}; \qquad d_a = 4\pi\alpha_a^{-1}(m_Z) - \Delta_a^0, \quad a = Y, 2, 3; \tag{5.53}$$

$$t_2 = \ln \frac{\Lambda_2^2}{m_Z^2}, \qquad t_3 = \ln \frac{\Lambda_3^2}{m_Z^2}. \tag{5.54}$$

Because the string scale Λ_H contains a dependence on g_H through (5.18), it will prove convenient to write

$$\ln \left(\frac{\Lambda_H}{m_Z}\right)^2 = t_P - \ln(4\pi\alpha_H^{-1}), \tag{5.55}$$

$$t_P \equiv 2 \ln \left(\frac{4\pi\Lambda_H}{g_H m_Z}\right) = 2 \ln \left(\frac{4\pi \times \zeta \times 5.27 \times 10^{17}}{91.19}\right) = 77.6 + 2 \ln \zeta. \tag{5.56}$$

Here we introduce a coefficient ζ which expresses uncertainty in (5.18) described in [42]; we study 10% deviations by taking $0.9 \leq \zeta \leq 1.1$, leading to $t_P = 77.6 \pm 0.2$. Eqs. (5.20,5.21) give the following equations which must be simultaneously satisfied:

$$\begin{aligned} a_H &= d_3 + 3t_3 && (5.57) \\ a_H &= d_2 + \delta b_2\, t_2 - (1 + \delta b_2)(t_P - \ln a_H) && (5.58) \\ k_Y a_H &= d_Y + \delta b_Y\, t_3 + \delta b'_Y\, t_2 - (11 + \delta b_Y + \delta b'_Y)(t_P - \ln a_H) && (5.59) \end{aligned}$$

The first equation shows the nice feature that since the $SU(3)_C$ coupling becomes conformal above Λ_3, the $\ln a_H$ dependence is gone and we can solve for a_H explicitly. Since this equation does not depend at all on t_2, we obtain $a_H = a_H(d_3, t_3)$. Substituting this into the second equation allows us to solve for t_2 explicitly, yielding $t_2 = t_2(d_2, d_3, t_3)$. Thus, the last equation becomes the only nontrivial constraint, which is transcendental and must be solved numerically. Through it we can determine $t_3 = t_3(d_Y, d_2, d_3)$ after having substituted the expressions for a_H and t_2 from the first two equations. Taking the values (5.46) for the $(m, n) = (0, 6)$ example, the implicit equation for t_3 is

$$t_3 = \frac{1}{540}\left[75 d_Y - 182 d_3 - 3 d_2\right] - \frac{23}{15}\left[t_P - \ln(d_3 + 3t_3)\right], \qquad (5.60)$$

which can easily be solved iteratively. Once t_3 is determined, a_H is easily obtained from (5.57) and

$$t_2 = \frac{1}{6}\left[a_H - d_2 + 7(t_P - \ln a_H)\right]. \qquad (5.61)$$

Note that if g_H and Λ_H were independent, as in the GUT case, we would have one more degree of freedom and we could not uniquely determine t_2, t_3, g_H, Λ_H in terms of d_Y, d_2, d_3. Related to this is an alternative, but equivalent, method of solution to that employed above. We could treat Λ_H and g_H as independent and solve (5.20,5.21) keeping t_3 as the extra free parameter. Then solutions to (5.20,5.21) would have $\Lambda_H = \Lambda_H(t_3)$ and $g_H = g_H(t_3)$. We could then determine the range of t_3 which allow the fourth constraint (5.18) to be satisfied to within, say, 10%. Instead we impose (5.18) from the start and address uncertainty of $\pm 10\%$ with the parameter ξ. The results are of course the same by either method.

In the case of $\Lambda_{\text{SUSY}} = m_Z$, we find

$$\begin{aligned} \Lambda_2 &= (2.25 \mp 0.07 \mp 0.006 \pm 0.09) \times 10^{13}\ \text{GeV}, \\ \Lambda_3 &= (5 \mp 0.1 \mp 3 \mp 1) \times 10^{6}\ \text{GeV}, \\ g_H &= 0.995 \pm 0.0004 \pm 0.0001 \pm 0.003, \\ \Lambda_H &= (5.1 \pm 0.002 \pm 0.0005 \pm 0.6) \times 10^{17}\ \text{GeV}. && (5.62) \end{aligned}$$

The first two uncertainties for each quantity give the modified estimates if $\sin^2\theta_W(m_Z)$ and $\alpha_3^{-1}(m_Z)$ are taken at the ends of the 1σ ranges given in (5.25) and (5.27) respectively. Upper signs in (5.62) correspond to the upper limits of the 1σ ranges; asymmetric uncertainties (due to logarithms) have been rounded up to the larger of the two. The third uncertainty gives the modified estimates if the "fudge parameter" ζ in (5.56) is taken at the ends of the range $0.9 \leq \zeta \leq 1.1$. Again, the upper signs in (5.62) correspond to the upper limit of the range for ζ. Sensitivities are logical: the exoquark scale Λ_3 is most sensitive to $\alpha_3^{-1}(m_Z)$, while the sensitivity to $\sin^2\theta_W(m_Z)$

is below significance. Only the exolepton scale Λ_2 is has significant sensitivity to $\sin^2\theta_W(m_Z)$; Λ_2, Λ_H and g_H, quantities more closely related to the high scale physics, are sensitive the high scale uncertainty ζ. For the case of $\Lambda_{\rm SUSY} = 1$ TeV, we find

$$\begin{aligned}
\Lambda_2 &= (8.4 \mp 0.3 \mp 0.02 \pm 0.4) \times 10^{12} \text{ GeV}, \\
\Lambda_3 &= (7 \mp 0.1 \mp 4 \mp 1) \times 10^{5} \text{ GeV}, \\
g_H &= 0.972 \pm 0.0003 \pm 0.0001 \pm 0.003, \\
\Lambda_H &= (5.0 \pm 0.002 \pm 0.0004 \pm 0.5) \times 10^{17} \text{ GeV}.
\end{aligned} \tag{5.63}$$

We next address concerns over fine-tuning in the unification scenario considered here. Ghilencea and Ross have recently argued that a realistic string model should not disturb the "significance of the prediction for the gauge couplings" which occurs in the MSSM [79]. They note that for reasonable values of $\Lambda_{\rm SUSY}$, the portion of the $\alpha_3(m_Z)$ versus $\sin^2\theta_W(m_Z)$ plane allowed by conventional MSSM unification is a very small strip. We can rewrite Eq. (5.60) as an implicit equation $d_3 = d_3(d_Y, d_2, t_3)$, so that for fixed value of the exoquark scale, and thereby t_3, we can predict $\alpha_3(m_Z)$ as a function of $\sin^2\theta_W(m_Z)$. In Figure II we show our results for values of Λ_3 which step by a factor of four; we assume $\Lambda_{\rm SUSY} = 1$ TeV for these (solid) curves. For comparison we also show the MSSM unification predictions (dashed) with $\Lambda_{\rm SUSY}$ stepping by factors of four; in the MSSM case we take $k_Y = 5/3$ and assume threshold corrections

$$\Delta_a^{\rm MSSM} \approx \Delta_a^{\rm conv} + \Delta_a^{\rm HL} + \Delta_a^{\rm light}, \qquad a = Y, 2, 3, \tag{5.64}$$

where each of the quantities on the right-hand side are assumed as above. We also show with error bars the experimental values (5.25). The experimental error bars define the major and minor axes of an "error ellipse." In any give direction, the distance from the center of this ellipse to its edge gives a measure which is independent of how we scale the axes of the graph. We compare the widths of strips to those of the MSSM in these units. It can be seen that the sensitivity to Λ_3 is only a factor of approximately three greater than the sensitivity to $\Lambda_{\rm SUSY}$ in the MSSM. Roughly speaking, the tuning is not much worse than in the MSSM. Another way to see that the tuning is not "fine" is that deviations of up to roughly 60% in Λ_3 from the central value given in (5.63) can be accommodated by the uncertainty in $\alpha_3^{-1}(m_Z)$. It is also interesting to note that setting the scale Λ_3 is equivalent to predicting $\alpha_3^{-1}(m_Z)$, since the (solid) curves in Figure II are nearly horizontal; this is reflected in that fact that uncertainty in $\sin^2\theta_W(m_Z)$ had no appreciable effects on the estimates of Λ_3 in Eqs. (5.62,5.63).

In Figure III we present a similar analysis for Λ_2, the exolepton scale. We fix t_2 and solve Eqs. (5.57-5.59) numerically eliminating t_3 and a_H to obtain $d_3 = d_3(d_Y, d_2, t_2)$. For a given value of t_2 we obtain a curve for $\alpha_3(m_Z)$ as a function of $\sin^2\theta_W(m_Z)$; we take $\Lambda_{\rm SUSY} = 1$ TeV. The sensitivity to the exolepton scale is *much* higher, so we only step by $\pm 10\%$ from $\Lambda_2 = 8.4 \times 10^{12}$ GeV, the approximate central value of (5.63). We compare the widths of the strips to those of the MSSM unification as describe above. It can be seen that they are roughly three times wider, implying that a 10% variation of Λ_3 in the string unification scenario studied here is on a par with a 1200% variation of $\Lambda_{\rm SUSY}$ in the MSSM unification scenario. That is, sensitivity to the exolepton scale is roughly 120 times worse than the $\Lambda_{\rm SUSY}$ sensitivity of the MSSM. From (5.63) we note that deviations of up to 3.5% for Λ_2 from the central value can be accommodated by the uncertainty

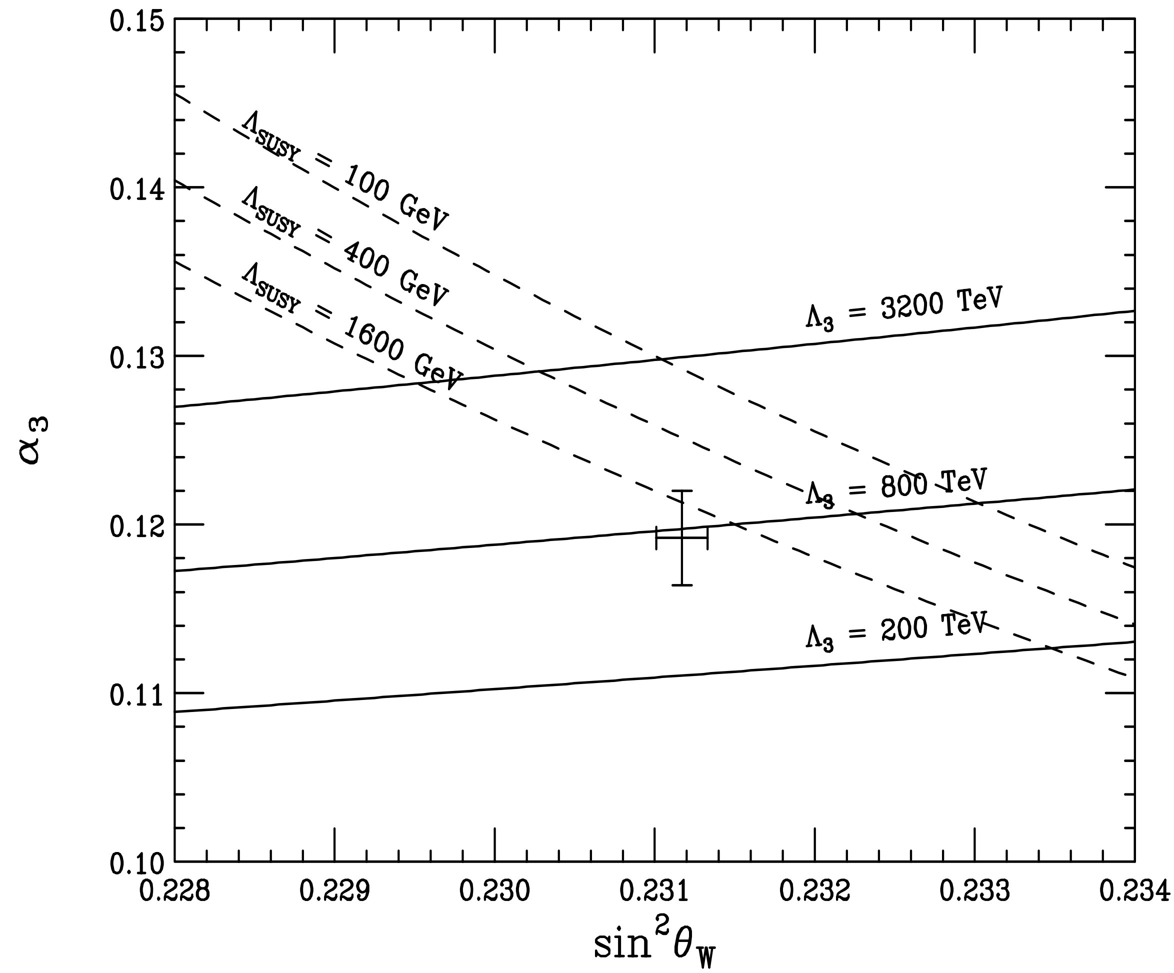

Figure II: Predicted Z scale values per the string unification scenario (solid), for values of Λ_3 stepping by factors of four, with $\Lambda_{\text{SUSY}} = 1$ TeV. For comparison, the MSSM unification prediction is shown (dashed), with Λ_{SUSY} stepping by factors of four. Experimental values are show with error bars.

in $\sin^2\theta_W(m_Z)$. Although this tuning is "fine," it is not horrendous. The vertical (solid) curves in Figure III demonstrate that choosing Λ_2 is essentially equivalent to predicting $\sin^2\theta_W(m_Z)$; this is reflected in (5.63) by the fact that Λ_2 has no significant sensitivity to the uncertainty in $\alpha_3^{-1}(m_Z)$.

To summarize, relative to the tuning of superpartner thresholds in the MSSM unification scenario, the the exoquark scale is *not* finely-tuned, but the exolepton scale *is* finely-tuned; however, the fine-tuning of the exolepton scale is not astronomical and is perhaps acceptable. If one is prepared to accept a tuning 120 times worse than the tuning of $\Lambda_{\rm SUSY}$ in the MSSM, then one still must explain *why* the exotic scales have the order of magnitudes that they do. Presumably, this would be determined by a detailed study of the flat directions which produce Xiggs vevs and the selection rules which restrict couplings in the effective theory. If the leading couplings giving exoquarks mass were of high enough dimension, a natural explanation of the low exoquark scale may be possible; the exolepton scale may be easier to explain because it is near the condensation scale.

Using our results for the scales $\Lambda_{2,3}$, we can extract the range of exotic thresholds corrections $\Delta_a^{\rm exotic}$ which are required:

$$9 \lesssim \Delta_Y^{\rm exotic} \lesssim 10, \qquad 120 \lesssim \Delta_2^{\rm exotic} \lesssim 130, \qquad 150 \lesssim \Delta_3^{\rm exotic} \lesssim 160. \tag{5.65}$$

Comparing to (5.51,5.52), it can be seen that the exotic threshold corrections for α_2^{-1} and α_3^{-1} are quite large compared to other effects; they represent roughly 35% and 150% corrections to $4\pi\alpha_2^{-1}$ and $4\pi\alpha_3^{-1}$ respectively! However, the hypercharge correction is fairly modest (0.8%). (To a good approximation, we could have neglected the Δ_a^0 of Eq. (5.50) and solved for the order of magnitude of the exotic threshold corrections.) This can be traced to the fact that the exoquarks and exoleptons which we have introduced at Λ_3 and Λ_2 have very small hypercharges. This is precisely what is needed to overcome the nonstandard hypercharge normalization. It can be seen from (5.20) that as k_Y is increased above its standard value, the prediction for α_H^{-1} will tend to decrease, all other quantities being held constant and ignoring the constraints (5.18,5.21). We can correct for this tendancy by making Δ_2 and Δ_3 significantly larger than what is typical in the MSSM, so long as we do not greatly change Δ_Y. This is possible because we have exoquarks and exoleptons with very small hypercharge.

The bizarre hypercharges of the exotic particles lead to fractionally charged particles; the most problematic are the exoquarks, given the rather low value of Λ_3. Thermal production of exoquarks or exoleptons at an early stage of the universe would violate relic abundance bounds on fractionally charged particles (FECs) by several orders of magnitude, as discussed for example in Refs. [62, 80, 81]. Thus, viabilty of this unification scenario requires inflation, to dilute the abundances of FECs, with a reheating temperature T_R which is sufficiently low that the FECs will not be appreciably produced following inflation; such scenarios have been examined for example in free fermionic models [80]. Chung, Kolb and Riotto [82] have recently pointed out that the dilution of heavy particle abundances by inflation imposes a much stronger limit than was initially imagined: to avoid thermal production of heavy particles with G_{SM} gauge quantum numbers, the masses of these heavy particles must be greater than T_R by a factor of roughly 10^3. Then to escape conflict with the relic density data for fractionally charged particles, we require inflation with

$$T_R \lesssim 10^{-3}\Lambda_3 \lesssim 5\,\mathrm{TeV}\,. \tag{5.66}$$

While inflationary scenarios with such low reheating temperatures have certainly been proposed

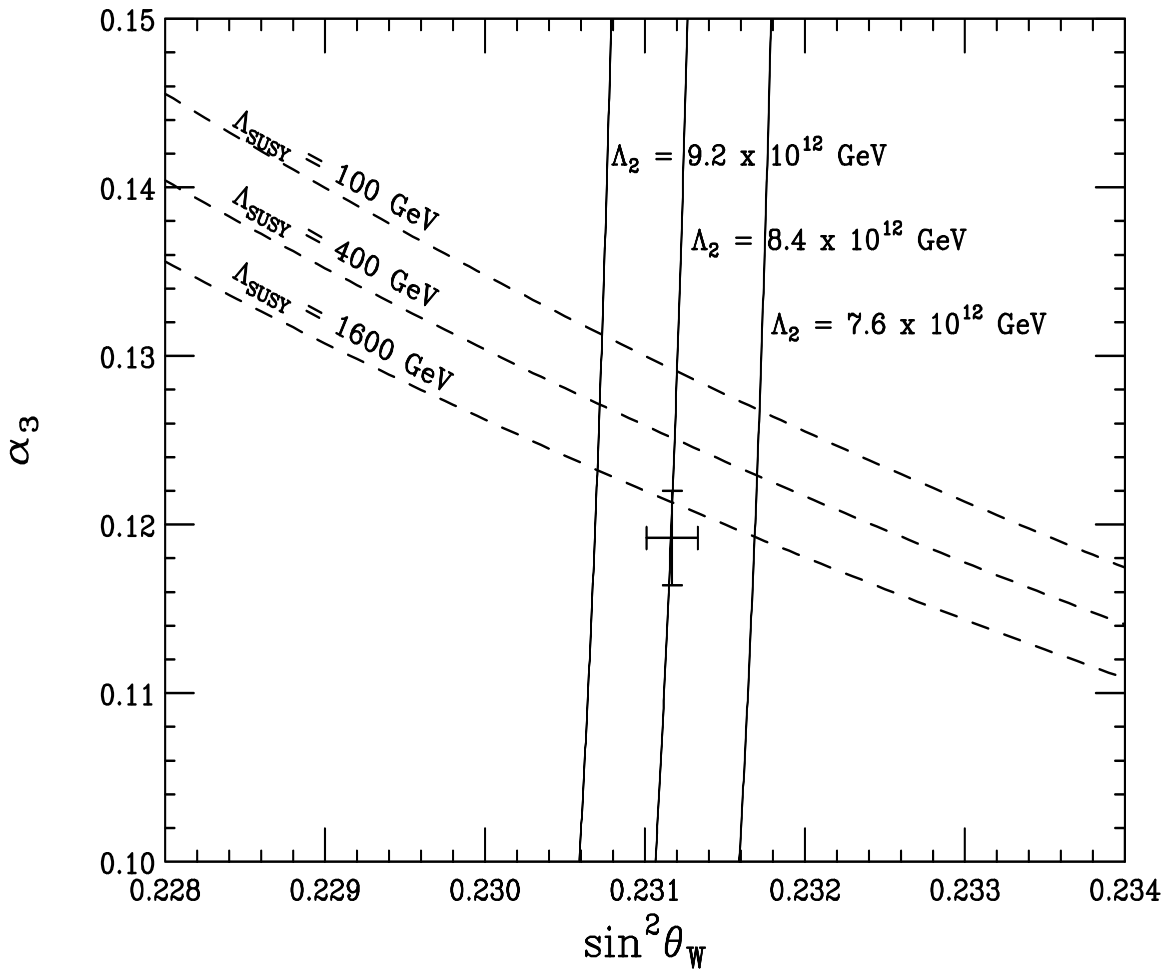

Figure III: Predicted Z scale values per the string unification scenario (solid), for values of Λ_2 stepping by $\pm 10\%$ from the best fit value, with $\Lambda_{\rm SUSY} = 1$ TeV. For comparison, the MSSM unification prediction is shown (dashed), with $\Lambda_{\rm SUSY}$ stepping by factors of four. Experimental values are show with error bars.

(see for example Ref. [83]), it is not at all clear that such scenarios can be achieved in the present context. We will not address this question here, leaving it to further investigation.

6 Conclusions

In this work we have made a systematic tabulation of detailed properties of *all* orbifold models falling within the $\mathrm{BSL_A}$ class defined in the Introduction; by so doing, we can legitimately say what is "typical" in this class of models. We have determined the hidden sector gauge group G_H and matter representations charged under the nonabelian part of G_H. These details are key to predicting the low energy phenomenology which arises from supersymmetry breaking in a hidden sector, such as in the effective theory of BGW. We have listed all of the patterns of irreps under the nonabelian factors of G. Using these results, one can easily select a model from the $\mathrm{BSL_A}$ class which has the desired exotic matter. The tables of irreps also suggest topics for further study, such as gauge mediation of hidden sector supersymmetry breaking from mixed representations of the observable and hidden sector gauge groups. While such communications may be suppressed by large masses, they are likely competitive with gravity mediation, which is suppressed by inverse powers of the Planck mass.

For each model, we have given a number of quantities which are useful for phenomenological studies. The FI terms in Table II allow one to determine the scale of initial gauge symmetry breaking. An understanding of the details of how this occurs is important to the construction of the low energy effective theory. Because many of the low energy effective operators have coefficients which at leading order depend on large powers of the Xiggs vevs, $\mathcal{O}(1)$ variations in the FI term can be greatly enhanced. For this reason, an accurate determination of the FI term is of practical interest. Table III gives the Green-Schwarz coefficient b_{GS} for each model, which plays a prominent role in formulae in the effective theory of BGW—for example, the T-moduli mass formula (3.14). In particular, we found that this implies a problem of too light T-moduli masses in the $\mathrm{BSL_A}$ class.

The minimum hypercharge normalization k_Y (consistent with accommodation of the MSSM and at least $SU(3)'$ surviving in the hidden sector to provide for gaugino condensation) was determined for each model. If one is determined to obtain the standard normalization $k_Y = 5/3$, Table V spares effort on fruitless models where this is not possible—over half of the 175 studied here. We are able to conclude that "extended" hypercharge embeddings allow for $k_Y < 5/3$ in some of the models, similar to what was found for free fermionic models in Ref. [64]. However, it is not possible to obtain small enough k_Y, in the range of 1.4 to 1.5, to achieve string scale unification with only the MSSM field content—a string unification scenario studied in Refs. [71, 59] and reviewed in [61].

All of the quantities tabulated here are necessary to detailed model-building in the effective supergravity theory and have implications for soft terms in the MSSM and the unification of running gauge couplings. To our knowledge, this is the first complete and systematic survey of three generation standard-like bosonic heterotic orbifold models performed *at this level of detail.* By organizing the models into twenty patterns of irreps and enumerating various other properties which are universal to models within a given pattern, we allow the phenomenologist to quickly select a subset of the models within the $\mathrm{BSL_A}$ class which have the desired properties. It is an interesting result that so many of the features of the various models within an irrep pattern are universal. Cross-referencing with the embeddings enumerated in [16] using Table XIII, one can employ the recipes provided in Section 2 to quickly generate the matter spectra for a given model, without a detailed understanding of the underlying theory; alternatively, full tables of all 175 models are available from the author upon request. It is hoped that through these efforts the $\mathrm{BSL_A}$ class of string-derived models has been rendered more readily accessible for further study to a wider audience.

The unusual features of string-derived models, charge fractionalization and nonstandard hypercharge normalization, have been discussed in the simplest of terms. We have endeavored to make clear as is possible how it is that states occur which would not be discovered through straightforward dimensional reduction and irrep decompositions of the original ten-dimensional $E_8 \times E_8$ theory. We have discussed at length the problems which these features present for the construction of a phenomenologically viable model. We have described the size of Xiggs vevs in general terms, and have found that large T-moduli vevs would seem to spoil perturbativity of the σ model expansion of the effective theory.

In an example model where nonstandard hypercharge normalization cannot be avoided, we have described the lengths to which one must go in order to achieve unification at the string scale. Exotic matter states with very small hypercharges were introduced at intermediate scales to obtain agreement with Z scale data for the gauge couplings. The exoquark scale was found to be rather low. The exotic hypercharges of the exotic matter in turn implied a low reheating temperature to avoid problems with FEC relic abundance constraints. Fine-tuning of the intermediate scales was examined and was shown to be, in our opinion, rather mild. However, we did not study flat directions and superpotential couplings in the example model, and for this reason the intermediate scales and intermediate field content remain to be justified.

To defend the unification scenario presented in Section 5.2, one must be willing to take the position that the apparent unification at roughly 2×10^{16} GeV in the MSSM with $k_Y = 5/3$ is purely accidental; we find this point of view difficult to accept. On the other hand, the unification scenario we have studied serves as an illustration of how ugly things really are when one attempts to refine many of the models into a realistic theory. Though we have studied only one example, it can be seen from Table V that a good fraction of the $\mathrm{BSL_A}$ class models have $k_Y > 5/3$ and the unification constraint in these models leads inevitably to the contortions exhibited in our example.

In conclusion, the more promising models will be those with $k_Y \leq 5/3$. One might invoke M-theory [84] to explain unification at 2×10^{16}, as was done in Refs. [6]; or, one might introduce *many* exotics at a intermediate scale with a "just so" arrangement of irreps and charges in the hope that with enough exotics the intermediate scale would quite near the unification scale of the MSSM and the apparent approximate unification at 2×10^{16} would not be an accident. In either case, the classification performed here, together with the identification of equivalences performed in Ref. [20, 24], has moved the effort further along for the $\mathrm{BSL_A}$ class and has narrowed down the number of "attractive models."

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Appendices

A Cancellation of the Modular Anomaly

For the Z_3 orbifold, $SU(3,3,\mathbf{Z})$ reparameterizations of the nine T-moduli T^{ij} are symmetries [85] of the underlying perturbative string theory, at least to one loop in string perturbation theory [86, 47]. These are referred to as *target space modular transformations* or *duality transformations* of the T-moduli. Most commonly, projective $SL(2,\mathbf{Z})$ subgroups acting on the diagonal moduli are studied:

$$T^i \to \frac{a^i T^i - ib^i}{ic^i T^i + d^i}, \qquad a^i d^i - b^i c^i = 1, \qquad \forall\, i = 1,3,5, \tag{A.1}$$

with a^i, b^i, c^i, d^i all integers. The indices on these integers indicate that each of the three T^i may transform with its own set. In addition to transformations on the T-moduli, accompanying T-dependent reparameterizations of chiral matter superfields must be made:

$$\Phi^A \to \frac{\sum_B M^A{}_B \Phi^B}{\prod_{i=1,3,5}(ic^i T^i + d^i)^{q_i^A}}. \tag{A.2}$$

Here, q_i^B is the modular weight of the field Φ^B; these quantities were given in Section 3. The matrix $M^A{}_B$ is identity for untwisted fields while it mixes subsets of twisted fields with the same modular weight [87] in a way which depends on the parameters a^i, b^i, c^i, d^i.

Transformations (A.1,A.2) are symmetries of the effective supergravity action at the classical level—isometries of the nonlinear σ model. However, at the quantum level there is a σ model anomaly [88] associated with the duality tranformations, as originally pointed in Refs. [89, 90]. To study this *modular anomaly*, one calculates the quantum corrections to the supergravity lagrangian, in particular triangle diagrams involving the composite σ model connections of T-moduli to other fields at one vertex and gauge boson currents at the other two vertices. Various calculations of the modular anomaly have been performed. Most often, supergravity interactions have been studied at the component level and then the anomaly written as a globally supersymmetric superspace integral, which is an approximation to the true supergravity anomaly [89, 90, 91, 92]. The supergravity one loop effective lagrangian and its transformation properties has been studied in great detail by Gaillard and collaborators, using Pauli-Villars regularization techniques [93]. These calculations were recently used to infer a locally supersymmetric superspace expression for the anomaly at one loop [94]. Equivalent expressions have also been obtained in Ref. [95]. Keeping only the leading term important to the present analysis, the quantum part of the one loop effective supergravity lagrangian transforms under (A.1) as

$$\delta\mathcal{L}_Q = \sum_{a,j} \frac{\alpha_a^j}{64\pi^2} \int d^4\theta \frac{E}{R} \ln(ic^j T^j + d^j) \sum_i (\mathcal{W}^\alpha \mathcal{W}_\alpha)_a^i + \text{h.c.} \tag{A.3}$$

The expression on the right-hand side is a superspace integral in the Kähler $U(1)$ formulation of supergravity [96]. The quantity E is the superdeterminant of the *vielbein;* it generalizes the tensor density $e = \sqrt{g}$ which appears in the Einstein-Hilbert action to a superfield. The superfield R is chiral and has as its lowest component the scalar auxilliary field of supergravity. The chiral spinor superfield $\mathcal{W}^i_{\alpha,a}$ is the superfield-strength corresponding to the generator T^i_a of the factor G_a of the gauge group G and has as its lowest component the gaugino $\lambda^i_{\alpha,a}$. The coefficient α^j_a reflects particles in the triangle loop which contribute to the anomalous transformation, and is given by [92]

$$\alpha^j_a = -C(G_a) + \sum_A (1 - 2q^A_j) X_a(R^A), \tag{A.4}$$

where the sum runs over matter irreps R^A of G_a and q^A_j is the modular weight appearing in (A.2).

Since the transformations (A.1,A.2) are known to be anomaly free in the underlying four-dimensional string theory, we must add effective terms to cancel the anomaly. One possible cancellation is from the shift in the T-moduli dependent threshold corrections alluded to in Sections 3 and 5.2. As mentioned there, however, such threshold corrections are absent in Z_3 orbifold compactifications [47]. Thus, the entire modular anomaly given by (A.3) must be canceled by the Green-Schwarz mechanism. That is, we include in the effective supergravity lagrangian a term which will have an anomalous transformation under (A.1,A.2), just such as to cancel (A.3). The overall coefficient $b_{\rm GS}$ of the Green-Schwarz term is determined by this matching.

We now describe this term in the BGW effective theory. However, we note that in expressions below, we use a slightly different normalization for the Green-Schwarz coefficient $b_{\rm GS}$ than BGW; rather, we adopt the more common convention of Refs. [97, 98]. In the BGW notation, the Green-Schwarz coefficient is written as b, which is related to $b_{\rm GS}$ by the equation $b = -b_{\rm GS}/24\pi^2$. In addition, in our formulae we do not use the BGW conventions for the β function coefficients of the gauge groups. The two conventions are related by $b_a^{\rm BGW} = -b_a^{\rm here}/24\pi^2$.

In addition to the supergravity multiplet, gauge multiplets, and matter multiplets, string theory predicts the existence of other supermultiplets of dynamic states. One particularly important set of fields is the following: a real scalar field ℓ called the *dilaton*, an antisymmetric tensor B_{mn} whose field strength is dual to the universal axion, and a Majorana spinor φ which is referred to as the *dilatino*. This is on-shell content of the superfield L, which is a *linear* multiplet. It satisfies the modified linearity condition [96]

$$(\bar{\mathcal{D}}^2 + 8R)L = -\sum_{a,i} (\mathcal{W}^\alpha \mathcal{W}_\alpha)^i_a. \tag{A.5}$$

Following BGW, we write the Green-Schwarz counterterm for the modular anomaly as

$$\mathcal{L}_{GS} = \frac{b_{\rm GS}}{24\pi^2} \int d^4\theta \; EL \sum_j \ln(T^j + \bar{T}^j). \tag{A.6}$$

Using (A.1), integration by parts in superspace [99], chirality of T^j and the modified linearity condition (A.5),

$$\delta \mathcal{L}_{GS} \;=\; \frac{-b_{\rm GS}}{24\pi^2} \int d^4\theta E L \sum_j \ln(ic^j T^j + d^j) + \text{h.c.}$$

$$
\begin{aligned}
&= \frac{b_{\rm GS}}{8\cdot 24\pi^2}\int d^4\theta \frac{E}{R}(\bar{\mathcal{D}}^2+8R)[L\sum_j \ln(ic^jT^j+d^j)] + \text{h.c.} \\
&= \frac{-b_{\rm GS}}{192\pi^2}\sum_{j,a}\int d^4\theta \frac{E}{R}\ln(ic^jT^j+d^j)\sum_i(\mathcal{W}^\alpha\mathcal{W}_\alpha)^i_a + \text{h.c.} \qquad \text{(A.7)}
\end{aligned}
$$

Comparing to (A.3), it is easy to see that in the present context (i.e., in the absence of T-moduli dependent string threshold corrections),

$$
b_{\rm GS} = 3\alpha_a^j \qquad \forall\ a, j. \tag{A.8}
$$

A generic spectrum of massless states which is free of chiral gauge anomalies will not satisfy (A.8), since it requires that we get the same result, $b_{\rm GS}$, for each factor G_a in the gauge group G. Thus, (A.8) is a highly nontrivial constraint on the matter spectrum. This was exploited by Ibáñez and Lüst to draw a number of phenomenological conclusions for Z_3 orbifold models [97].

As discussed in Section 2, untwisted states come in families of three; we make explicit the family index $i = 1,3,5$ by taking $A \to (\alpha, i)$ for untwisted fields, so that α denotes the species of untwisted field. For the twisted fields we take $A \to \rho$ to distinguish them, but do not separate out the family label. For nonabelian factors G_a in the models considered here, a nice simplification can be made. As mentioned in Section 3, none of the pseudo-massless twisted fields which are in nontrivial representations of G_a are oscillator states. Consequently, it follows from the discussion of Section 3 that they all have modular weights $q_j^\rho = 2/3$. Also from Section 3, we have for the untwisted states $q_j^{(\alpha,i)} = \delta_j^i$. With these facts, it is easy to show that Eqs. (A.4,A.8) can be rewritten

$$
b_{\rm GS} = -3C_a + \sum_{(\alpha,i)\in{\rm untw}} X_a(R^{(\alpha,i)}) - \sum_{\rho\in{\rm tw}} X_a(R^\rho) = b_a^{\rm tot} - 2\sum_{\rho\in{\rm tw}} X_a(R^\rho), \tag{A.9}
$$

where the last equality follows from (5.22), only now it is the total β function coefficient which appears, since all pseudo-massless states contribute. In the absence of twisted states in nontrivial irreps of G_a, the last term on the right-hand side vanishes. This occurs for $SO(10)$ in Patterns 1.1 and 1.2. But then for a G_a with only trivial irreps in the twisted sector $b_{\rm GS} = b_a^{\rm tot}$. This is the source of the (approximately vanishing) T-moduli mass problem discussed in Section 3 and Ref. [15].

As an example of the surprising matching of (A.9) for different G_a, we examine Pattern 1.1. The $SO(10)$ factor of G has no nontrivial matter representations, as can be seen from Table VIII, which gives

$$
b_{\rm GS} = b_{10}^{\rm tot} = -3C(SO(10)) = -24. \tag{A.10}
$$

For the $SU(3)$ factor, we have $15(3+\bar{3},1,1)$ beyond the MSSM which gives $\delta b_3 = b_3^{\rm tot} - b_3 = 15$, and consequently $b_3^{\rm tot} = 12$. Comparison of Table VIII to Table XII shows that the twisted sector irreps are $15(3,1,1) + 21(\bar{3},1,1)$, which gives

$$
b_{\rm GS} = b_3^{\rm tot} - 2\sum_{\rho\in{\rm tw}} X_3(R^\rho) = 12 - 36 = -24. \tag{A.11}
$$

Finally, the $SU(2)$ factor has $40(1,2,1)$ beyond the MSSM, so that $\delta b_2 = b_2^{\rm tot} - b_2 = 20$ and $b_2^{\rm tot} = 21$. Again comparing Table VIII to Table XII, we find that the $SU(2)$ charged twisted matter is $45(1,2,1)$ and so

$$b_{\rm GS} = b_2^{\rm tot} - 2\sum_{\rho\in{\rm tw}} X_2(R^\rho) = 21 - 45 = -24. \tag{A.12}$$

It is reassuring that each group $SO(10), SU(3)$ and $SU(2)$ gives the same answer for $b_{\rm GS}$, as they must for universal cancellation of the modular anomaly [97]. As a nontrivial check on our results, we have verified that this matching holds among the nonabelian factors in each of the twenty patterns.

B Tables

Pattern	$SU(3)\times SU(2)\times SO(10)$ Irreps
1.1	$3[(3,2,1)+5(3,1,1)+7(\bar{3},1,1)+15(1,2,1)+48(1,1,1)_0+15(1,1,1)_1]$
1.2	$3[(3,2,1)+4(3,1,1)+6(\bar{3},1,1)+13(1,2,1)+(1,1,16)+48(1,1,1)_0$ $+9(1,1,1)_1]$

Table VIII: Patterns of irreps in Case 1 models.

Pattern	$SU(3)\times SU(2)\times SU(5)\times SU(2)$ Irreps
2.1	$3[(3,2,1,1)+3(3,1,1,1)+5(\bar{3},1,1,1)+9(1,2,1,1)+(1,1,5,1)$ $+(1,1,\bar{5},1)+6(1,1,1,2)+(1,2,1,2)+34(1,1,1,1)_0+9(1,1,1,1)_1]$
2.2	$3[(3,2,1,1)+3(3,1,1,1)+5(\bar{3},1,1,1)+9(1,2,1,1)+(1,1,5,1)$ $+(1,1,\bar{5},1)+6(1,1,1,2)+(1,2,1,2)+37(1,1,1,1)_0+6(1,1,1,1)_1]$
2.3	$3[(3,2,1,1)+3(3,1,1,1)+5(\bar{3},1,1,1)+11(1,2,1,1)+(1,1,5,1)$ $+(1,1,\bar{5},1)+8(1,1,1,2)+33(1,1,1,1)_0+6(1,1,1,1)_1]$
2.4	$3[(3,2,1,1)+2(3,1,1,1)+4(\bar{3},1,1,1)+9(1,2,1,1)+(1,1,5,1)$ $+2(1,1,\bar{5},1)+(1,1,10,1)+6(1,1,1,2)+32(1,1,1,1)_0+6(1,1,1,1)_1]$
2.5	$3[(3,2,1,1)+2(3,1,1,1)+4(\bar{3},1,1,1)+7(1,2,1,1)+(1,1,5,1)$ $+2(1,1,\bar{5},1)+(1,1,10,1)+4(1,1,1,2)+(1,2,1,2)+36(1,1,1,1)_0$ $+6(1,1,1,1)_1]$
2.6	$3[(3,2,1,1)+(3,1,1,1)+3(\bar{3},1,1,1)+5(1,2,1,1)+(1,1,5,1)$ $+3(1,1,\bar{5},1)+(1,1,10,2)+10(1,1,1,2)+(1,2,1,2)+25(1,1,1,1)_0]$

Table IX: Patterns of irreps in Case 2 models.

Pattern	$SU(3)\times SU(2)\times SU(4)\times SU(2)^2$ Irreps
3.1	$3[(3,2,1,1,1)+2(3,1,1,1,1)+4(\bar{3},1,1,1,1)+7(1,2,1,1,1)+2(1,1,4,1,1)$ $+2(1,1,\bar{4},1,1)+6(1,1,1,2,1)+4(1,1,1,1,2)+(1,2,1,1,2)+27(1,1,1,1,1)_0$ $+6(1,1,1,1,1)_1]$
3.2	$3[(3,2,1,1,1)+2(3,1,1,1,1)+4(\bar{3},1,1,1,1)+7(1,2,1,1,1)+2(1,1,\bar{4},1,1)$ $+8(1,1,1,2,1)+4(1,1,1,1,2)+(1,1,4,2,1)+(1,2,1,1,2)+26(1,1,1,1,1)_0$ $+3(1,1,1,1,1)_1]$
3.3	$3[(3,2,1,1,1)+2(3,1,1,1,1)+4(\bar{3},1,1,1,1)+7(1,2,1,1,1)+2(1,1,\bar{4},1,1)$ $+6(1,1,1,2,1)+6(1,1,1,1,2)+(1,1,4,2,1)+(1,2,1,2,1)+26(1,1,1,1,1)_0$ $+3(1,1,1,1,1)_1]$
3.4	$3[(3,2,1,1,1)+(3,1,1,1,1)+3(\bar{3},1,1,1,1)+5(1,2,1,1,1)+2(1,1,4,1,1)$ $+2(1,1,\bar{4},1,1)+8(1,1,1,2,1)+4(1,1,1,1,2)+(1,1,6,2,1)+(1,2,1,2,1)$ $+24(1,1,1,1,1)_0+3(1,1,1,1,1)_1]$

Table X: Patterns of irreps in Case 3 models.

Pattern	$SU(3)\times SU(2)\times SU(3)\times SU(2)^2$ Irreps
4.1	$3[(3,2,1,1,1)+2(3,1,1,1,1)+4(\bar{3},1,1,1,1)+9(1,2,1,1,1)+(1,1,3,1,1)$ $+(1,1,\bar{3},1,1)+6(1,1,1,2,1)+6(1,1,1,1,2)+30(1,1,1,1,1)_0+3(1,1,1,1,1)_1]$
4.2	$3[(3,2,1,1,1)+2(3,1,1,1,1)+4(\bar{3},1,1,1,1)+7(1,2,1,1,1)+(1,1,3,1,1)$ $+(1,1,\bar{3},1,1)+4(1,1,1,2,1)+6(1,1,1,1,2)+(1,2,1,2,1)+34(1,1,1,1,1)_0$ $+3(1,1,1,1,1)_1]$
4.3	$3[(3,2,1,1,1)+(3,1,1,1,1)+3(\bar{3},1,1,1,1)+7(1,2,1,1,1)+3(1,1,3,1,1)$ $+3(1,1,\bar{3},1,1)+4(1,1,1,2,1)+4(1,1,1,1,2)+36(1,1,1,1,1)_0$ $+3(1,1,1,1,1)_1]$
4.4	$3[(3,2,1,1,1)+(3,1,1,1,1)+3(\bar{3},1,1,1,1)+7(1,2,1,1,1)+(1,1,3,1,1)$ $+3(1,1,\bar{3},1,1)+4(1,1,1,2,1)+7(1,1,1,1,2)+(1,1,3,1,2)+30(1,1,1,1,1)_0$ $+3(1,1,1,1,1)_1]$
4.5	$3[(3,2,1,1,1)+(3,1,1,1,1)+3(\bar{3},1,1,1,1)+7(1,2,1,1,1)+(1,1,3,1,1)$ $+3(1,1,\bar{3},1,1)+4(1,1,1,2,1)+7(1,1,1,1,2)+(1,1,3,1,2)+33(1,1,1,1,1)_0]$
4.6	$3[(3,2,1,1,1)+(3,1,1,1,1)+3(\bar{3},1,1,1,1)+5(1,2,1,1,1)+(1,1,3,1,1)$ $+3(1,1,\bar{3},1,1)+4(1,1,1,2,1)+5(1,1,1,1,2)+(1,1,3,1,2)+(1,2,1,1,2)$ $+34(1,1,1,1,1)_0+3(1,1,1,1,1)_1]$
4.7	$3[(3,2,1,1,1)+(3,1,1,1,1)+3(\bar{3},1,1,1,1)+5(1,2,1,1,1)+3(1,1,3,1,1)$ $+(1,1,\bar{3},1,1)+4(1,1,1,2,1)+5(1,1,1,1,2)+(1,2,1,1,2)+(1,1,\bar{3},1,2)$ $+37(1,1,1,1,1)_0]$
4.8	$3[(3,2,1,1,1)+2(\bar{3},1,1,1,1)+3(1,2,1,1,1)+(1,1,3,1,1)+5(1,1,\bar{3},1,1)$ $+8(1,1,1,2,1)+6(1,1,1,1,2)+(1,2,1,1,2)+(1,1,3,2,2)+25(1,1,1,1,1)_0]$

Table XI: Patterns of irreps in Case 4 models.

Patterns	Untwisted Irreps
1.1	$3[(3,2,1)+3(1,1,1)_0]$
1.2	$3[(3,2,1)+(\bar{3},1,1)+(1,2,1)+(1,1,16)]$
2.1	$3[(3,2,1,1)+3(1,1,1,1)_0]$
2.2, 2.3	$3[(3,2,1,1)+(\bar{3},1,1,1)+(1,2,1,1)+(1,1,5,1)+(1,1,1,2)]$
2.4, 2.5	$3[(3,2,1,1)+(1,1,10,1)+2(1,1,1,1)_0]$
2.6	$3[(3,2,1,1)+(\bar{3},1,1,1)+(1,2,1,1)+(1,1,5,1)+(1,1,10,2)]$
3.1	$3[(3,2,1,1,1)+(1,1,4,1,1)+2(1,1,1,1,1)_0]$
3.2, 3.3	$3[(3,2,1,1,1)+(\bar{3},1,1,1,1)+(1,2,1,1,1)+(1,1,1,1,2)+(1,1,4,2,1)]$
3.4	$3[(3,2,1,1,1)+(1,1,6,2,1)+3(1,1,1,1,1)_0]$
4.1, 4.2	$3[(3,2,1,1,1)+(\bar{3},1,1,1,1)+(1,2,1,1,1)+(1,1,1,2,1)+(1,1,1,1,2)]$
4.3	$3[(3,2,1,1,1)+(1,1,3,1,1)+(1,1,\bar{3},1,1)+3(1,1,1,1,1)_0]$
4.4, 4.6	$3[(3,2,1,1,1)+(1,1,3,1,2)+3(1,1,1,1,1)_0]$
4.5, 4.7	$3[(3,2,1,1,1)+(\bar{3},1,1,1,1)+(1,2,1,1,1)+(1,1,\bar{3},1,1)+(1,1,1,2,1)$
	$+(1,1,1,1,2)+(1,1,3,1,2)]$
4.8	$3[(3,2,1,1,1)+(1,1,3,1,1)+(1,1,3,2,2)+3(1,1,1,1,1)_0]$

Table XII: Irreps of the untwisted sectors for each pattern of total irreps.

Pattern	Models
1.1	1.1, 1.2, 1.3, 4.1, 4.2, 4.3, 8.1
1.2	2.1, 2.2, 2.3, 6.1, 6.2, 6.3, 9.1
2.1	1.4, 1.5, 1.11, 1.12, 4.4, 4.6, 4.9, 4.11, 8.2, 8.3
2.2	2.4, 2.5, 2.6, 2.7, 6.4, 6.6, 6.9, 6.11, 9.2, 11.3
2.3	2.9, 2.10, 2.12, 6.8, 6.10, 6.12, 9.3
2.4	1.6, 1.8, 1.10, 4.5, 4.8, 4.10, 10.2
2.5	1.7, 1.9, 4.7, 4.12, 10.1, 10.3
2.6	2.8, 2.11, 6.5, 6.7, 11.1, 11.2
3.1	1.14, 1.15, 1.16, 1.17, 4.13, 4.15, 4.16, 4.18, 10.4, 10.5, 10.6, 10.7
3.2	2.13, 2.14, 6.15, 6.17, 11.5
3.3	2.15, 2.16, 2.17, 2.18, 6.13, 6.14, 6.16, 6.18, 9.4, 11.4
3.4	1.13, 1.18, 4.14, 4.17, 8.4
4.1	2.19, 2.20, 2.21, 6.22, 6.23, 6.29, 11.16
4.2	2.22, 2.23, 2.27, 2.32, 6.24, 6.26, 6.30, 6.31, 9.5, 11.9, 11.11, 11.14
4.3	1.19, 1.32, 1.33, 4.20, 4.27, 4.31, 8.5
4.4	1.21, 1.22, 1.23, 1.25, 1.28, 1.29, 4.21, 4.24, 4.25, 4.28, 4.30, 4.33, 10.8, 10.11, 10.14
4.5	2.24, 2.25, 2.28, 2.29, 2.30, 2.31, 6.19, 6.20, 6.21, 6.27, 6.28, 6.32, 11.6, 11.7, 11.12, 11.13, 11.15
4.6	1.20, 1.24, 1.27, 1.30, 4.19, 4.22, 4.26, 4.29, 10.10, 10.12, 10.15, 10.16, 10.17
4.7	2.26, 2.33, 6.25, 6.33, 11.8, 11.10
4.8	1.26, 1.31, 4.23, 4.32, 10.9, 10.13

Table XIII: Irrep patterns versus the models enumerated in [24]. See explanation of model labeling in Section 3.

Table XIV: $\mathrm{BSL_A}$ 6.5 Pseudo-Massless Spectrum

No.	Irrep	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_X	Z	Y
1	$(3,2,1,1)_U$	1	6	−18	9	45	15	0	3	1/6	1/6
2	$(1,2,1,1)_U$	3	18	−54	27	−45	−15	0	−3	1/2	1/2
3	$(\bar{3},1,1,1)_U$	−4	−24	72	−36	0	0	0	0	−2/3	−2/3
4	$(1,1,10,2)_U$	0	0	0	0	−18	−6	0	3	0	0
5	$(1,1,5,1)_U$	0	0	0	0	36	12	0	−6	0	0
6	$(1,1,1,1)_{-1,-1}$	0	−20	−32	−31	−35	−23	0	1	−1	0
7	$(1,1,1,1)_{-1,-1}$	0	−35	13	17	25	−3	0	5	−1	2/5
8	$(1,1,1,1)_{-1,-1}$	0	10	16	−55	25	−3	0	5	−1	−2/5
9	$(1,1,1,1)_{-1,-1}$	0	10	−122	14	10	−8	0	4	−1	0
10	$(\bar{3},1,1,1)_{-1,-1}$	2	7	25	11	−5	−13	0	3	−2/3	1/3
11	$(1,2,1,1)_{-1,-1}$	−3	7	25	11	−5	−13	0	3	−3/2	−1/2
12	$(1,1,1,2)_{-1,0}$	0	−5	61	−7	−5	−13	0	−4	−1	0
13	$(1,1,1,1)_{-1,0}$	0	−5	61	−7	−95	−9	0	1	−1	0
14	$(1,1,1,2)_{-1,0}$	0	−20	−32	−31	55	7	0	0	−1	0
15	$(1,1,1,1)_{-1,0}$	0	−20	−32	−31	−35	11	0	5	−1	0
16	$(1,1,1,2)_{-1,0}$	0	25	−29	38	40	2	0	−1	−1	0
17	$(1,1,1,1)_{-1,0}$	0	25	−29	38	−50	6	0	4	−1	0
18	$(1,1,1,2)_{-1,1}$	0	−5	61	−7	−5	21	0	0	−1	0
19	$(1,1,\bar{5},1)_{-1,1}$	0	−5	61	−7	49	5	0	1	−1	0
20	$(1,1,1,1)_{-1,1}$	0	−5	61	−7	−5	4	−3	5	−1	−1/5
21	$(1,1,1,1)_{-1,1}$	0	−5	61	−7	−5	4	3	5	−1	1/5
22	$(1,1,1,1)_{0,-1}$	2	−28	38	−19	−5	4	−1	5	0	2/5
23	$(1,1,1,1)_{0,-1}$	2	17	41	50	−20	−1	−1	4	0	2/5
24	$(1,1,1,1)_{0,-1}$	2	17	−97	−22	−20	−1	−1	4	0	0
25	$(1,2,1,1)_{0,-1}$	−1	14	50	−25	−35	−6	−1	3	−1/2	−1/2
26	$(1,1,1,1)_{0,-1}$	−4	−4	−34	17	−5	4	−1	5	−1	−3/5
27	$(3,1,1,1)_{0,-1}$	0	−10	−16	8	−50	−11	−1	2	−1/3	1/15
28	$(1,1,1,2)_{0,0}$	2	32	−4	2	10	9	−1	−1	0	0
29	$(1,1,1,1)_{0,0}$	2	32	−4	2	10	−8	2	4	0	1/5
30	$(1,2,1,2)_{0,0}$	−1	−16	2	−1	−5	4	−1	−2	−1/2	−1/10
31	$(1,2,1,1)_{0,0}$	−1	−16	2	−1	−5	−13	2	3	−1/2	1/10
32	$(1,1,\bar{5},1)_{0,1}$	2	2	−52	26	4	7	−1	0	0	2/5
33	$(1,1,1,2)_{0,1}$	2	2	−52	26	40	2	2	−1	0	3/5
34	$(1,1,1,1)_{0,1}$	2	2	−52	26	40	−15	−1	4	0	2/5
35	$(1,1,1,1)_{0,1}$	2	2	−52	26	−50	6	2	4	0	3/5

Table XIV: $\mathrm{BSL_A}$ 6.5 Pseudo-Massless Spectrum (Cont.)

No.	Irrep	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_X	Z	Y
36	$(1,1,1,2)_{1,-1}$	-2	3	-9	-19	55	7	-2	0	0	$-3/5$
37	$(1,1,\bar{5},1)_{1,-1}$	-2	3	-9	-19	19	12	1	1	0	$-2/5$
38	$(1,1,1,1)_{1,-1}$	-2	3	-9	-19	-35	11	-2	5	0	$-3/5$
39	$(1,1,1,1)_{1,-1}$	-2	3	-9	-19	55	-10	1	5	0	$-2/5$
40	$(1,1,1,1)_{1,0}$	-2	-27	-57	5	-5	4	1	5	0	0
41	$(1,1,1,1)_{1,0}$	-2	18	84	5	-5	4	1	5	0	$-2/5$
42	$(\bar{3},1,1,1)_{1,0}$	0	15	-45	-1	-35	-6	1	3	$1/3$	$-1/15$
43	$(1,1,1,1)_{1,0}$	4	-6	18	38	-20	-1	1	4	1	1
44	$(1,2,1,1)_{1,0}$	1	-9	27	-37	-35	-6	1	3	$1/2$	$1/10$
45	$(1,1,1,1)_{1,0}$	-2	-12	36	29	-65	-16	1	1	0	0
46	$(1,1,1,2)_{1,1}$	-2	-12	36	29	25	14	1	0	0	0
47	$(1,1,1,1)_{1,1}$	-2	-12	36	29	25	-3	-2	5	0	$-1/5$
48	$(1,1,1,2)_{1,1}$	4	9	-27	-10	10	9	1	-1	1	$3/5$
49	$(1,1,1,1)_{1,1}$	4	9	-27	-10	10	-8	-2	4	1	$2/5$
50	$(1,1,1,2)_{1,1}$	-2	3	-9	-19	-35	-6	1	-4	0	$-2/5$
51	$(1,1,1,1)_{1,1}$	-2	3	-9	-19	-35	-23	-2	1	0	$-3/5$

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