Analytical Study of Envelope Modes for a Fully Depressed Beam in Solenoidal and Quadrupole Periodic Transport Channels*

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Abstract

We present an analysis of envelope perturbations evolving in the limit of a fully space-charge depressed (zero emittance) beam in periodic, thin-lens focusing channels. Both periodic solenoidal and FODO quadrupole focusing channels are analyzed. The phase advance and growth rate of normal mode perturbations are analytically calculated as a function of the undepressed particle phase advance to characterize the evolution of envelope perturbations.

INTRODUCTION

The KV envelope equations are often employed to model the transverse evolution of the envelope of beam particles in intense beam transport channels[1]. For periodic focusing channels, there have been no fully analytical studies of perturbations in the beam envelope evolving about the matched beam envelope. Here we analytically calculate properties of small-amplitude elliptical envelope perturbations in the limit of full space-charge depression for several periodic thin-lens transport channels. Because the thin-lens model provides a reasonable approximation to the focusing effects of more realistic applied focusing elements, results derived provide a guide to the properties of envelope perturbations associated with space-charge-dominated beams.

ENVELOPE MODEL

The KV envelope equations for a fully depresed coasting beam with elliptical edge radii $r_x = 2\sqrt{\langle x^2\rangle}, r_y = 2\sqrt{\langle y^2\rangle}$ aligned along the transverse x and y axes are [2, 3]

$$r_j''(s) + \kappa_j(s) r_j(s) - \frac{2Q}{r_x(s) + r_y(s)} = 0, \qquad (1)$$

where j ranges over x and y, Q is the dimensionless beam perveance, and s is the axial coordinate. The equations (1) apply directly to a beam in a quadrupole focusing channel with $\kappa_x = -\kappa_y$, but for solenoidal focusing one has to assume zero beam canonical angular momentum with $\kappa_x = \kappa_y$ and interpret all results in a rotating Larmor frame[2, App. A]. The equations can be written in terms of scaled sum and difference coordinates $R_\pm = (r_x \pm r_y)/(2\sqrt{2Q})$ as

$$\begin{gathered} 2R_+''(s) + 2\kappa_x(s) R_+(s) - \frac{1}{R_+(s)} = 0, \\ 2R_-''(s) + 2\kappa_x(s) R_-(s) = 0 \end{gathered} \qquad (2a)$$

for solenoidal focusing, and

$$\begin{gathered} 2R_+''(s) + 2\kappa_x(s) R_-(s) - \frac{1}{R_+(s)} = 0, \\ 2R_-''(s) + 2\kappa_x(s) R_+(s) = 0 \end{gathered} \qquad (2b)$$

for quadrupole focusing. In free drift regions $\kappa_x(s) = \kappa_y(s) = 0$, and the equations can be integrated by using constancy of envelope Hamiltonian

$$R_+'^2(s) - \ln R_+(s) = \text{const} \qquad (3)$$

to yield[2]

$$\ln\frac{R_+(0)}{R_+(s)} = R_+'^2(0) - \left\{ \operatorname{erfi}^{(-1)}\left[\operatorname{erfi} R_+'(0) + \frac{e^{R_+'^2(0)} s}{\sqrt{\pi} R_+(0)} \right] \right\}^2, \qquad (4a)$$

$$R_-(s) = R_-(0) + sR_-'(0), \qquad (4b)$$

where $\operatorname{erfi}(z) = \operatorname{erf}(iz)/i$ is the imaginary error function.

Without loss of generality[2, Sec. II E], we assume that the length of the free drift interval between the two adjacent thin lenses is 2 as in Fig. 1. By symmetry we need only to consider the envelope evolution of the beam between two neighboring lenses only. We take the first lens to be at axial location $s = -1$ and the second one to be at $s = 1$. We also assume that in alternating gradient channel the second lens (at $s = 1$) is focusing in x. Then for both thin lens solenoids and quadrupoles we take near $s = 1$

$$\kappa_x(s) = \tfrac{1}{f}\delta(s-1), \qquad (5)$$

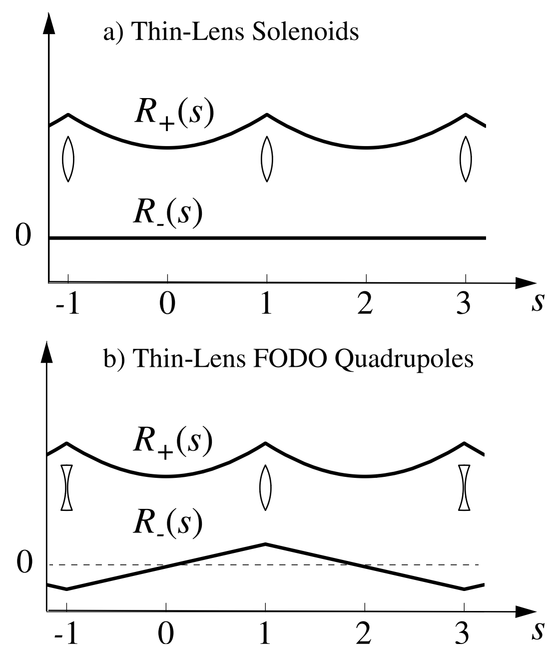

FIG. 1: Matched beam envelopes $R_\pm(s)$ and transport lattice for (a) solenoid, and (b) FODO quadrupole thin-lens channels.

*This research was performed at LBNL and LLNL under US DOE contact Nos. DE-AC03-76SF0098 and W-7405-Eng-48.

where f = const is the thin lens focal length and $\delta(s)$ is the Dirac delta-function. The focal length f can be related to the undepressed particle phase advance over one lattice period σ_0 as [2, Sec. II D]

$$\frac{1}{f} = \begin{cases} 2\sin^2\frac{\sigma_0}{2}, & \text{solenoidal focusing,} \\ \sin\frac{\sigma_0}{2}, & \text{quadrupole focusing.} \end{cases} \tag{6}$$

We analyze the perturbations of the envelope coordinate vector $\mathbf{R}(s) = (R_+(s), R'_+(s), \zeta(s)R_-(s), \zeta(s)R'_-(s))$ from the mid-drift at $s = 0$ to the next mid-drift at $s = 2$. Here, $\zeta(s) = 1$ when the next lens to be traversed is focusing, and $\zeta(s) = -1$ when the next lens is defocusing.

PERTURBATIVE ANALYSIS

To analyze the first-order perturbations in the coordinate vector $\mathbf{R}(s)$ we compute the Jacobian matrix $\mathbf{M}(0,2)$ where $\mathbf{M}(s_1|s_2) = \partial\mathbf{R}(s_2)/\partial\mathbf{R}(s_1)$ and derivatives are evaluated for a matched envelope. Since $\mathbf{M}(0|2)$ is simplectic, then the first-order perturbations are stable if and only if all eigenvalues of $\mathbf{M}$ lie on the unit circle $|z| = 1$.

In calculating $\mathbf{M}(0|2)$, we henceforth denote $\mathcal{F}(s\pm0) \equiv \lim_{\delta\to\pm0}\mathcal{F}(s+\delta)$ to represent the discontinuous action of the thin lenses on the beam envelope functions. To exploit lattice symmetries, we split the interval $(0,2)$ into three parts $(0, 1-0)$, $(1-0, 1+0)$ and $(1+0, 2)$, and calculate $\mathbf{M}(0,2)$ as $\mathbf{M}(0|2) = \mathbf{M}(1+0|2)\mathbf{M}(1-0|1+0)\mathbf{M}(0|1-0)$. By symmetry, $\mathbf{M}(1+0|2) = \mathbf{M}(0|-1+0)^{-1}$. Thus,

$$\mathbf{M}(0|2) = \mathbf{M}_f(-1+0)^{-1}\mathbf{M}_s\mathbf{M}_f(1-0), \tag{7}$$

where $\mathbf{M}_s = \mathbf{M}(1-0|1+0)$ is the "singular Jacobian" associated with the thin lens focusing kick, and $\mathbf{M}_f(s) = \mathbf{M}(0|s)$ for $|s| < 1$ is the "free drift Jacobian" associated with the half-drift.

To evaluate $\mathbf{M}_s$, we consider the action of the thin lens according to Eqs. (2) and (5). We obtain

$$\mathbf{M}_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 1 \end{bmatrix}, \quad \mathbf{M}_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{f} & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{f} & 0 & 0 & -1 \end{bmatrix} \tag{8}$$

for solenoidal and quadrupole channels respectively.

To evaluate $\mathbf{M}_f(s)$, the free expansion solutions in Eqs. (4) and the matched beam symmetry condition $R'_+(0) = 0$ are employed to evaluate Jacobian elements:

$$\mathbf{M}_f(s) = \begin{bmatrix} \frac{R_+(s)-sR'_+(s)}{R_+(0)} & 2R_+(0)R'_+(s) & 0 & 0 \\ -\frac{s}{2R_+(0)R_+(s)} & \frac{R_+(0)}{R_+(s)} & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

To complete the evaluation of $\mathbf{M}_f(1-0)$, we find relations of the elements to σ_0 by deriving equations connecting $R_+(1-0) \equiv R_+(1)$, $R'_+(1-0)$, and $R_+(0)$ to these quantities for the matched beam envelope. By symmetry, for a periodic, matched envelope

$$R'_\pm(1-0) = -R'_\pm(1+0), \tag{10}$$

For solenoids, Eqs. (2a) and (5) can be integrated once about $s = 1$ to obtain

$$R'_\pm(1+0) = R'_\pm(1-0) - \frac{1}{f}R_\pm(1).$$

Combining these constraints with the matching conditions (10), we get

$$R'_\pm(1-0) = \frac{1}{2f}R_\pm(1). \tag{11}$$

Similarly, using Eqs. (2b) and (5) for alternating gradient focusing and matched beam symmetries (10), we obtain

$$R'_\pm(1-0) = \frac{1}{2f}R_\mp(1). \tag{12}$$

The solenoidal and quadrupole matching conditions in Eq. (12) for R_+ can be expressed as

$$\hat{k}R_+(1) = 2R'_+(1-0), \tag{13}$$

$$\text{where } \hat{k} = \begin{cases} \frac{1}{f} = 1-\cos\sigma_0, & \text{solenoidal focusing,} \\ \frac{1}{2f^2} = \frac{1}{4}(1-\cos\sigma_0), & \text{quadrupole focusing.} \end{cases}$$

Applying Eqs.(3) between $s = 0$ and $s = 1-0$ with the matched beam condition $R'_+(0) = 0$ leads to

$$R_+(1) = R_+(0)e^{R'^2_+(1-0)}. \tag{14}$$

Using Eqs. (13) and (14) in Eq. (4) then yields

$$\hat{k} = 2\sqrt{\pi}e^{-R'^2_+(1-0)}R'_+(1-0)\,\mathrm{erfi}\,R'_+(1-0). \tag{15}$$

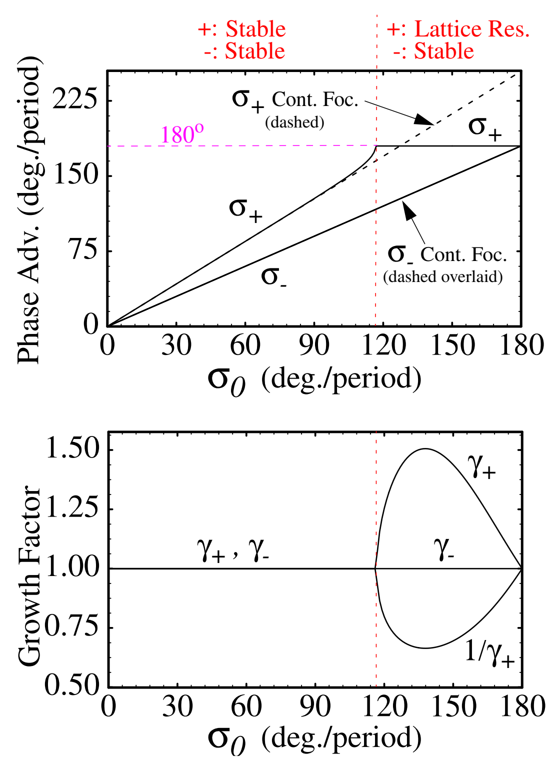

FIG. 2: Phase advances ($\sigma_\pm$) and growth factors ($\gamma_\pm$) for the breathing and quadrupole modes for a thin-lens solenoidal focusing channel and a fully depressed beam. Continuous focusing model predictions for $\sigma_\pm$ are superimposed (dashed curves).

Equations (13)–(15) provide the needed constraints to relate the elements of $\mathbf{M}_f(1-0)$ to σ_0. Elements of $\mathbf{M}_f(-1+0)$ can be calculated from these constraints using the matched beam symmetries

$$R_+(-1) = R_+(1), \qquad R'_+(-1+0) = -R'_+(1-0). \quad (16)$$

For solenoidal focusing $R_\pm$ are uncoupled, and $\mathbf{M}(0|2)$ is of block diagonal form with $\mathbf{M}(0|2) = \begin{bmatrix} \mathbf{M}_+(0|2) & 0 \\ 0 & \mathbf{M}_-(0|2) \end{bmatrix}$, where $\mathbf{M}_\pm(0|2)$ are 2×2 symplectic matrices that can be independently analyzed for the stability of perturbations. We compute $\mathbf{M}_\pm(0|2)$ from Eq. (7):

$$\begin{aligned}
\mathbf{M}_+(0|2) &= \begin{bmatrix} \frac{R_+(-1)+R'_+(-1+0)}{R_+(0)} & 2R_+(0)R'_+(-1+0) \\ \frac{1}{2R_+(0)R_+(-1)} & \frac{R_+(0)}{R_+(-1)} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} \frac{R_+(1)-R'_+(1-0)}{R_+(0)} & 2R_+(0)R'_+(1-0) \\ -\frac{1}{2R_+(0)R_+(1)} & \frac{R_+(0)}{R_+(1)} \end{bmatrix} \\
&= \begin{bmatrix} \cos\sigma_0 - 4R'^2_+(1-0)\cos^2(\frac{\sigma_0}{2}) & 2\frac{R_+^2(0)}{f}[1-2R'^2_+(1-0)] \\ \frac{-f}{R_+^2(0)}\cos^2(\frac{\sigma_0}{2})[1-\cos\sigma_0+4R'^2_+(1-0)\cos^2(\frac{\sigma_0}{2})] & \cos\sigma_0 - 4\cos^2(\frac{\sigma_0}{2})R'^2_+(1-0) \end{bmatrix}, \qquad (17) \\
\mathbf{M}_-(0|2) &= \begin{bmatrix} \cos\sigma_0 & 1+\cos\sigma_0 \\ -1-\cos\sigma_0 & \cos\sigma_0 \end{bmatrix}.
\end{aligned}$$

Eigenvalues $\lambda_\pm$ of the matrices $\mathbf{M}_\pm(0|2)$ are

$$\begin{aligned}
\lambda_+ &= \cos\sigma_0 - 4R'^2_+(1-0)\cos^2(\tfrac{\sigma_0}{2}) \pm 2i\cos(\tfrac{\sigma_0}{2}), \\
&\cdot\sqrt{\left[1-2R'^2_+(1-0)\right]\left[\sin^2(\tfrac{\sigma_0}{2})+2R'^2_+(1-0)\cos^2(\tfrac{\sigma_0}{2})\right]} \\
\lambda_- &= \cos\sigma_0 \pm i\sin\sigma_0.
\end{aligned} \qquad (18)$$

Real-valued mode phase advances $\sigma_\pm$ and growth factors $\gamma_\pm$ per lattice period satisfy $\lambda_\pm = \gamma_\pm e^{i\sigma_\pm}$. With proper branch selection[2] we get

$$\begin{aligned}
\sigma_+ &= \arg\lambda_+ \quad \text{with + sign in Eq. (18)}, \\
\sigma_- &= \sigma_0,
\end{aligned} \qquad (19)$$

and growth factors as

$$\begin{aligned}
\gamma_+ &= \begin{cases} 1, & \text{stable}, \\ \sqrt{2\left[\cos\sigma_0 - 4R'^2_+(1-0)\cos^2(\frac{\sigma_0}{2})\right]^2 - 1}, & \text{unstable}, \end{cases} \\
\gamma_- &= 1.
\end{aligned}$$

These solutions are plotted in Fig. 2 as a function of σ_0. The extent of the band of instability ($\gamma_+ \neq 1$) in σ_0 can be calculated from γ_+ directly as

$$\sigma_0 \in \left[\arccos\left(1-\sqrt{\frac{2\pi}{e}}\,\mathrm{erfi}\,\frac{1}{\sqrt{2}}\right), \pi\right] \approx [116.715^\circ, 180^\circ].$$

The stability of quadrupole focusing can be investigated analogously except that we must work with the full 4×4 Jacobian matrix $\mathbf{M}(0|2)$. After multiplying out the matrices in Eq. (7) and calculating the eigenvalues using the constraints in Eqs. (12)–(15) yields

$$\lambda = w - \tfrac{1}{2}\hat{k} \pm i\sqrt{w\hat{k} + \left[1-\tfrac{1}{2}\hat{k}\right]\left[\hat{k}+8R'^2_+(1-0)\right]}, \quad (20)$$

where $w = \pm\sqrt{\left[1-\frac{1}{2}\hat{k}\right]\left[1-\frac{1}{2}\hat{k}-8R'^2_+(1-0)\right]}$ and $\hat{k}$ is given by Eq. (13). These eigenvalues can be employed to calculate phase advances (σ_B and σ_Q) and growth factors (γ_B and γ_Q) of the breathing and quadrupole modes as $\sigma_{B,Q} = 2\arg\lambda$ and $\gamma_{B,Q} = \left|\lambda^2\right|$ (see Fig. 3). Using Eqs. (15) and Eq. (20) we find numerically that the instability band is located on the interval $\sigma_0 \in (121.055^\circ, 180^\circ)$.

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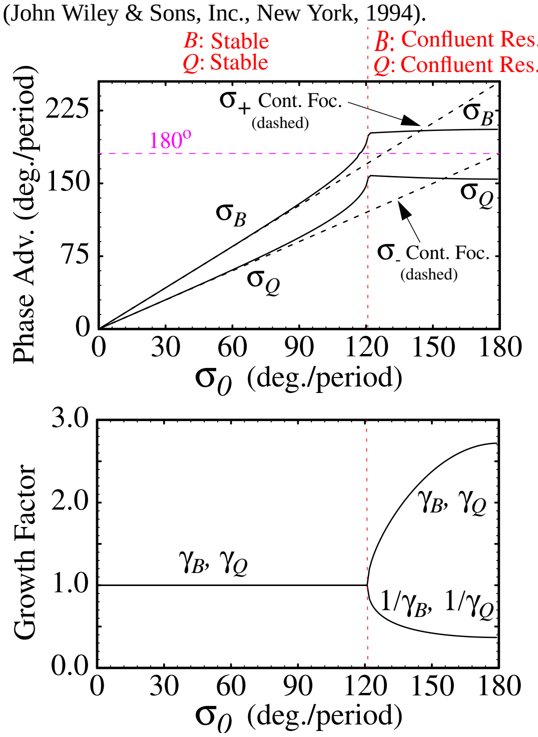

FIG. 3: Phase advance (σ_Q and σ_B) and growth factors (γ_Q and γ_B) for the breathing and quadrupole modes for a thin-lens FODO quadrupole focusing channel and a fully depressed beam.Continuous focusing model predictions for $\sigma_\pm$ are superimposed (dashed curves).