Technical Note:

On Analytical Solutions of Transient Flow into Unsaturated Rock Matrix

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Abstract

This paper presents a new class of analytical solutions for transient flow into unsaturated rock matrix. These analytical solutions are derived using specially correlated but physically meaningful relative permeability and capillary functions, while the transient flow process in unsaturated rock matrix blocks is generally described by the Richards' equation. The analytical solutions describe the full transient behavior of flow into unsaturated matrix blocks with the special relative permeability and capillary functions, and are proven (through various examples) to be useful to verify numerical model results.

1. Introduction

Fluid flow through variably saturated fractured porous media occurs in many subsurface systems related to petroleum-reservoir engineering, vadose zone hydrology, and soil sciences. For quantitative analysis of such flow in unsaturated soil or rock, Richards' equation has been used as a fundamental basis. However, because of its nonlinear nature, Richards' equation solutions for general unsaturated flow may be obtained only by a numerical approach. Even though significant progress has been made in numerical modeling of unsaturated flow and infiltration since the late 1950s, analytical approaches still prove to be irreplaceable. This is because analytical solutions, if available, provide more direct insight into the physics of unsaturated flow phenomena than numerical or laboratory studies, especially when dealing with effects of various parameters. Moreover, even in numerical studies, analytical solutions are often needed to examine and verify numerical schemes or results.

In the past few decades, a considerable amount of effort has been devoted in groundwater hydrology and soil science to mathematical modeling of steady-state and transient Richards' flow through unsaturated porous media (see various articles *Milly* [1988], *Pullan* [1990], and *Bodvarsson et al.* [2000]). As a result, many exact and approximate analytical solutions have been developed. In general, the analytical solutions derived for Richards' equation are dependent upon the level of the applied linearizations or approximations. These existing, closed-form analytical solutions may be divided into the following classes: (1) steady-state solutions using the exponential hydraulic conductivity model [*Gardner*, 1958] and quasi-linear approximations

[*Pullan*, 1990]; (2) transient infiltration solutions using special forms of soil retention curves [e.g., *Warrick et al.,* 1990 and 1991; *Hills and Warrick*, 1993; *Warrick and Parkin*, 1995; *Chen et al*., 2001] or using linearization and the Kirchhoff transformation [e.g., *Basha,* 1999]; (3) approximate and asymptotic solutions [e.g., *Philip,* 1969; *Zimmerman* and *Bodvarsson*, 1989; *Zimmerman et al*., 1990; *Zimmerman and Bodvarsson*, 1995]. Most of these analytical solutions, however, are limited to one-dimensional (1-D) flow or a constant soil-water-diffusivity approximation with semi-infinite flow domains.

Despite the advances made so far, exact forms of analytical solutions to Richards' equation remain intractable under general flow conditions, because of the known nonlinearity of Richards' equation. This explains why continual research efforts have been made to find new solutions over the past half century. The present work is motivated by our modeling studies of characterizing fluid flow and tracer transport in the unsaturated zone of Yucca Mountain, Nevada, a potential repository site for storing high-level radioactive waste. The dual-continuum numerical-modeling method has been used in those studies to handle fracture-matrix flow and interaction in unsaturated fractured tuffs at the site. The effort of verifying the accuracy of numerical schemes and calculations for fracture-matrix interactions motivated us to resort to analytical solutions to examine numerical-model results.

The objective of this work is to present a new class of analytical solutions for unsaturated flow within a matrix block, which can be used to examine numerical solutions and the accuracy of different modeling approaches for handling fracture-matrix interactions. These analytical solutions are derived from a linearized Richards' equation, which requires a specially correlated relationship between relative permeability and capillary-pressure functions.

2. Linearization of Richards' Equation

Consider the flow of an incompressible liquid in a homogeneous, isothermal, incompressible, and isotropic porous medium, such as an unsaturated rock matrix. Ignoring air dynamics and gravity, the flow is commonly described by Richards' equation:

$$\nabla \bullet \left(\frac{k\, k_{rw}}{\mu_w} \nabla P_w \right) = \phi \frac{\partial}{\partial t}\left(S_w\right) \tag{2.1}$$

where k is the absolute permeability, k_{rw} is the relative permeability to the water phase, μ_w is the viscosity of the water phase, P_w is the pressure in the water phase, ϕ is the effective porosity of the formation, and S_w is the water saturation.

To find analytical solutions for Equation (2.1) but keep P_c and k_{rw} as nonlinear functions of S_w, we select a relative permeability in the form:

$$k_{rw}(S_w) = C_k \left(S_w^*\right)^{\alpha} \tag{2.2}$$

and capillary pressure in the form:

$$P_c(S_w) \equiv P_g - P_w = C_p \left(S_w^*\right)^{-\beta} \tag{2.3}$$

where P_g(Pa) is a constant air (or gas) pressure, C_k and C_p(Pa) are coefficients, α and β are exponential constants, respectively, of relative permeability and capillary-pressure functions, and S_w^* is the effective water saturation,

$$S_w^* = \frac{S_w - S_{wr}}{1 - S_{wr}} \tag{2.4}$$

with S_{wr} being the residual water saturation. Note that if Brooks and Corey's capillary function is used [*Brooks* and *Corey*, 1964], the coefficient C_p in Equation (2.3) becomes the air entry pressure P_b (i.e., $C_p = P_b$) and $\beta = 1/\lambda$, with λ being an index of pore size distribution [*Honarpour et. al*., 1986].

If the following condition

$$\alpha = \beta + 1 \tag{2.5}$$

is satisfied, the Richards equation (2.1) can be readily linearized as follows:

$$D\left[\frac{\partial^2 S_w}{\partial x^2} + \frac{\partial^2 S_w}{\partial y^2} + \frac{\partial^2 S_w}{\partial z^2}\right] = \frac{\partial S_w}{\partial t} \tag{2.6}$$

where D, called soil-water or moisture diffusivity [*Philip*, 1969], is defined by

$$D = \frac{k\,k_{rw}}{\phi\,\mu_w}\frac{\partial P_w}{\partial S_w} = \frac{k C_k C_p \beta}{\phi\,\mu_w (1 - S_{wr})} \tag{2.7}$$

with dimension of m^2/s.

Note that the linearization expressed in Equations (2.2) through (2.7) is different from simply assuming a constant moisture diffusivity, D [e.g., *Philip*, 1969], because relative permeability and capillary pressure are still nonlinear functions of saturation in Equations (2.2) and (2.3). In fact, they are quite generally used relations in modeling two-phase porous media flow [*Honarpour et. al.*, 1986], which are particularly useful in assessing the accuracy of numerical approaches that solve the nonlinear Richards equation (2.1).

3. Analytical Solutions

In this work, we are primarily interested in a cubic shape of matrix blocks: a rock matrix cube, surrounded by a 3-D orthogonal fracture network, as shown in Figure 1. Because fluid flow or pressure propagation in the highly permeable fractures are usually much more rapid than in the low-permeability matrix, we can assume that the water pressure at the surface of the matrix cube is constant everywhere at any given time. To facilitate analytical solutions, we will define the physical problem with the linearized Richards' Equation (2.6) associated with the initial and boundary conditions as follows:

Initial condition within matrix:

$$S_w = S_i \qquad \text{at } t = 0 \text{ within the matrix block} \tag{3.1}$$

The boundary conditions at the matrix surface (e.g., half the dimension of a 3-D matrix block) are:

$$S_w = S_b \qquad \text{on a matrix surface for } t > 0 \tag{3.2}$$

where S_i and S_b are constant initial and boundary saturations, respectively.

3.1 Exact 3-D Solution

Let us first introduce the following dimensionless variables. The dimensionless distances are defined as

$$X = \frac{x}{2a}, \quad Y = \frac{y}{2a}, \quad Z = \frac{z}{2a} \tag{3.3}$$

and the dimensionless time is

$$\tau = \frac{Dt}{a^2} \tag{3.4}$$

where a is half dimension of the 3-D cube. The normalized (or scaled) water saturation is

$$S_D = \frac{S_w - S_i}{S_b - S_i} \tag{3.5}$$

Under these transformations, the unsaturated water flow problem is mathematically equivalent to the heat transfer problem solved by *Carslaw* and *Jaeger* [1959]. Therefore, the solution, in terms of the normalized saturation, can be expressed as:

$$\begin{aligned} S_D(X,Y,Z,\tau) = 1 - \frac{64}{\pi^3} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{l+m+n}}{(2l+1)(2m+1)(2n+1)} \cos[(2l+1)\pi X] \\ \cos[(2m+1)\pi)\ ]\cos[(2n+1)\pi)\ ]e^{-\pi^2\left((2l+1)^2+(2m+1)^2+(2n+1)^2\right)\tau/4} \end{aligned} \tag{3.6}$$

The rate of mass flow into or from the cube through the matrix surface is given by

$$q(t) = 2aD\rho(S_b - S_i)q_D(\tau) \tag{3.7}$$

where ρ is the water density and $q_D(\tau)$ is a dimensionless mass flow rate, defined as

$$q_D(\tau) = \frac{1536}{\pi^4} \left\{ \sum_{n=0}^{\infty} e^{-\pi^2(2n+1)^2\tau/4} \right\} \left\{ \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} e^{-\pi^2(2m+1)^2\tau/4} \right\}^2 \tag{3.8}$$

The cumulative mass flow into or from the cube is given by

$$Q(t) = 8a^3\phi\rho(S_b - S_i)Q_D(\tau) \tag{3.9}$$

where $Q_D(\tau)$ is a dimensionless cumulative mass exchange, defined as

$$Q_D(\tau) = 1 - \frac{512}{\pi^6}\left\{\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}e^{-\pi^2(2m+1)^2\tau/4}\right\}^3 \tag{3.10}$$

3.2 Solution for 1-D Spherical Flow

In most modeling studies of fracture-matrix flow, 3-D inter-flow within matrix blocks is approximated as 1-D spherical flow when using a double-porosity concept [*Warren* and *Root,* 1963] or a multiple interacting continua (MINC) concept [*Pruess* and *Narasimhan*, 1985]. Using the term, 1-D "spherical" flow, is because the governing equation of such flow can be shown to be identical to that of radially symmetric 1-D spherical flow under the MINC approximation, i.e., thermodynamic variables (pressure, temperature, concentration, and etc) are the same spatially at an equal distance from the matrix surface [*Pruess* and *Narasimhan,* 1985]. The advantage of the 1-D flow approximation is that it significantly reduces the total numbers of meshes for discretizing matrix blocks. In practice, the 1-D flow approximation is perhaps the most commonly used approach in discretization for modeling fracture-matrix interactions. Therefore, an analytical solution for such a 1-D spherical flow problem is very useful.

With the 1-D spherical flow MINC approximation, the unsaturated flow toward the center of a cubic matrix block can be generally described by [e.g., *Lai et al.,* 1983],

$$D\left[\frac{\partial^2 S_w}{\partial x^2} + \frac{2}{x}\frac{\partial S_w}{\partial x}\right] = \frac{\partial S_w}{\partial t} \tag{3.11}$$

where x is the distance from a nested cross-sectional surface within the matrix block (having an equal distance to the matrix surface) to the center of the cube (see Figure 1).

Using the same dimensionless variables, defined by Equations (3.3) - (3.5), the analytical solution of Equation (3.11), subject to (3.1) and (3.2), is given [*Carslaw* and *Jaeger*, 1959] as

$$S_D(X,\tau) = 1 + \frac{1}{\pi X}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(2n\pi X)\ e^{-n^2\pi^2\tau} \qquad (3.12)$$

The rate of mass flow into or from the cube through the matrix surface and the cumulative mass of flow into or from the cube are given by Equations (3.7) and (3.9), respectively, while the dimensionless mass flow rate, is given as

$$q_D(\tau) = 24\sum_{n=1}^{\infty} e^{-n^2\pi^2\tau} \qquad (3.13)$$

The dimensionless cumulative mass exchange is given as

$$Q_D(\tau) = 1 - \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n^2\pi^2\tau} \qquad (3.14)$$

4. Discussion and Application

With the proposed linearization to Richards' equation in Section 3, many more analytical solutions can be easily derived for 1-D, 2-D and 3-D problems in a finite, semi-finite, or infinite flow domain (e.g., in analogy with the corresponding heat conduction problems [*Carslaw* and *Jaeger*, 1959]). Note that the assumptions used in deriving such analytical solutions will impose limitations on their applicability. Specifically, the requirements of specially correlated relations between relative permeability and capillary functions and no gravity effects are critical to deriving analytical solutions. These assumptions are the necessary conditions to linearizing the flow governing equation or for the existence of analytical solutions. Despite these restrictions, relative permeability and capillary functions of Equations (2.2) and (2.3) are among the most widely used relations in unsaturated flow. On the other hand, in most field studies capillary forces in the matrix system are generally dominant in controlling fracture-matrix flow relative to the effect of gravity, as long as matrix block sizes are relatively small. For these reasons, gravity effects have been ignored in almost all dual-continuum models for studies of multiphase or unsaturated flow in fractured reservoirs.

4.1 Type Curves of Transient Fracture-Matrix Flow

Here, we use the analytical solutions presented above to generate several type curves for transient flow into rock matrix. These type curves are calculated and presented in terms of the dimensionless variables for saturation, flow rate, and cumulative mass exchanges to make the plots independent of the size of matrix blocks and specific sets of rock or fluid properties and more convenient to use. In addition, the type curves give us some insight into transient flow processes through unsaturated fractured rock and can be used directly for verifying simulation results of numerical models to particular applications.

Spatial distributions of normalized water saturation within the matrix, as a function of dimensionless time and distance, are shown in Figure 2, calculated using Equation (3.12), i.e., the 1-D flow approximation solution. The 1-D spherical approximation for flow inside the matrix has been the commonly used conceptual model for handling fracture-matrix interactions with the dual-continuum approach. The type curves in Figure 2 cover the entire transient flow period of imbibition or drainage processes between fracture and matrix systems.

Figure 3 presents type curves of dimensionless flow rate and cumulative mass exchange, respectively, using both 3-D solutions [Equations (3.8) and (3.10)] and 1-D approximations [(3.13) and (3.14)]. Note that there is a very small difference between 3-D and the 1-D solutions in describing flow into matrix blocks. A previous modeling study [*Wu* and *Pruess*, 1988] also concludes that the 1-D spherical inside-matrix flow assumption using a 1-D nested mesh provides a good approximation to two-phase oil-water, counter-flow imbibition problems. The comparison of 1-D and 3-D inside-matrix flow results in Figure 3 indicates that the simplified 1-D, MINC-type approximation may be accurate enough for estimating mass exchanges between fracture and matrix systems in practical applications.

4.2 Evaluation of Numerical-Modeling Results

Numerical modeling approaches are widely used for simulating interactions between fractures and rock matrix in variably saturated fractured porous media. The key issue in numerical-modeling efforts is how to handle fracture-matrix interactions under different flow conditions. Among the commonly used methods for dealing with such interactions is the dual-continuum method, including double- and multiporosity models [e.g., *Barenblatt et al.*, 1960; *Warren* and *Root*, 1963; *Kazemi*, 1969; *Pruess* and *Narasimhan*, 1985; *Wu* and *Pruess*, 1988]. In the double-porosity concept [*Barenblatt et al*., 1960; *Warren* and *Root*, 1963], a flow domain is composed of matrix blocks with low permeability embedded in a network of interconnected fractures. Global flow in the formation occurs only through the fracture system, conceptualized as an effective continuum, and matrix blocks are treated as spatially distributed source/sink terms, based on a quasi-steady-state assumption of inter-porosity flow.

As a generalization to the *Warren-Root* model, a more rigorous dual-continuum method, the MINC concept [*Pruess* and *Narasimhan*, 1985], takes into account gradients of pressures, temperatures and concentrations between fractures and matrix by appropriate subgridding of the matrix blocks. This approach provides a better approximation to transient fracture-matrix interactions than the one-block representation of fractures or the matrix in a double-porosity model. In comparison, however, the double-porosity model may produce inaccurate modeling results when gradients of pressures or moisture condition are large or changing rapidly at or near fracture-matrix interfaces. This section presents efforts to qualify such numerical errors introduced by the dual-continuum conceptual model in handling fracture-matrix flow.

Many different conceptual models for fracture-matrix interaction have been evaluated for the Yucca Mountain site characterization studies [*Doughty*, 1999]. Currently, the most widely used model is based on the dual-continuum or dual-permeability concept, in which fractured rocks in different hydrogeological units are approximated as two globally connected and interacting fracture and matrix continua [e.g., *Wu et al.,* 2002]. We use the analytical solution for 1-D spherical flow into a cubic matrix to examine the numerical simulation results. The test problem concerns both imbibition (flow into matrix) and drainage (flow out of matrix). The numerical simulations were performed using a numerical reservoir simulator [*Pruess*, 1991; *Wu et al*., 1996]. Note that the governing equation solved in numerical modeling is still the original Richards' equation (2.1) instead of the linearized forms of Equations (2.6) or (3.11).

The example problem deals with transient flow processes into a 1 × 1 × 1 meter cube of matrix, which is discretized into 2, 5, 10, 30, and 500 nested cells, respectively, using volume fractions of the MINC concept for 1-D spherical flow toward or from the matrix center. The basic parameters used for the example are listed in Table 1. Saturation distribution within matrix at four different times, calculated using the analytical and numerical solutions of the water imbibition problem, are displayed in Figure 4 with three different discretizations of 10, 30, and 500 cells, respectively. As shown in Figure 4, the numerical results with refined grids (30 and 500) cells are in excellent agreement with the analytical solution during the entire transient imbibition period. In contrast, the simulation using the coarser 10-cell grid cannot in general match the analytical solution well, except near the matrix surface.

It should be mentioned that the scheme of MINC subgridding of matrix normally uses a set of volume fractional values, leading to nested cells with approximately equal volumes. In general, an equal-volume mesh results in smaller grid spacings near matrix surface or fractures and thus gives better numerical accuracy for estimating fracture-matrix interactions [*Pruess* and *Narasimhan*, 1985]. However, it creates larger grid spacing at or near the matrix center because of the requirement of equal mesh volume. This explains why the results using a 10-cell grid cannot well match the analytical solution inside the matrix block. On the other hand, a two-cell (or double-porosity) model using only one gridblock average to represent the matrix system cannot match saturation distribution at all. Only after a long time (100 days for this case), do all the numerical and analytical solutions converge to a steady-state solution of S_w=0.8.

Figure 5 shows the saturation distribution within the matrix for a case of water drainage from the matrix, simulated using the same matrix subgriddings, respectively, as in the imbibition case. Similarly, the model results using refined grids of 30 and 500 cells match well with the analytical solution. However, the 10-cell model results are in worse agreement with the analytical solution than the imbibition case (Figure 4). This is primarily because of the change in flow directions in the two cases, since the identical griddings were used for the same cell grids. In the previous imbibition case, the upstream of flow at the matrix surface (or fracture), specified with fixed pressure/saturation flow condition, is simulated using relatively refined subgrids. In comparison, for the drainage situation, only the downstream condition at the matrix surface is physically fixed. The upstream of the drainage flow is located at the center of the matrix block, which is very transient — with pressure head declining rapidly with time. The temporal discretization errors by the fully-implicit time-stepping scheme are worsened by coarser gridding near the matrix center, leading to large numerical errors.

Figure 6 shows the change of water-imbibing rate at the matrix surface and cumulative imbibition mass into the matrix over time. In this case, all the numerical results are in good agreement with the analytical results. In contrast, simulated drainage flow rates and cumulative mass exchanges in Figure 7 show very different results for different matrix subgriddings. In this case, due to the same reasons as for saturation distributions, two-cell or double porosity and five-cell discretizations give extremely large errors to estimated drainage rates, compared with the results from 30- or 500-cell models or analytical solution. Furthermore, the "humps" in drainage rate versus time curves, for five-cell, even ten-cell discretizations, reflect numerical errors of the coarse-grid models in estimating potential or saturation gradients near the matrix surface, which is time-dependent. This indicates that the accuracy in modeling drainage processes is highly dependent on matrix grid resolutions and may require more refined grids than modeling imbibition processes, because of the more transient nature at the upstream condition of flow. This example demonstrates the usefulness of the linearization approach and the resulting analytical solutions in assessing numerical methods that solve the nonlinear Richards' equation.

5. Concluding Remarks

This paper presents a new linearization approach to Richards' equation, based on the assumptions of (1) negligible gravitational effect and (2) a special correlation of capillary pressure and relative permeability functions. Under such simplifications, a set of analytical solutions for transient flow into unsaturated matrix is derived. The analytical-solution approach of this work can be easily extended to different flow geometries, such as cylindrical, radial, and other multidimensional unsaturated flow.

The new analytical solutions, though limited by the assumptions for their applications, can be used to obtain some insight into the physics of transient imbibition and drainage processes of fracture-matrix interactions. In particular, several dimensionless type curves, specifically spatial saturation distributions and flow rates for mass exchange crossing fracture-matrix interfaces, are provided in this work. These type curves are independent of matrix-block size and specific parameters of fluid and rock, and can be useful in verifying numerical models and their results in modeling flow through unsaturated fractured rock using a dual-continuum approach. Note that under the linearization approach proposed here, the relative permeability and capillary pressure remain to be nonlinear functions of saturation, an important feature that proved to be useful in assessing numerical methods that solve the highly nonlinear Richards equation.

As an application example, the analytical solutions were used to examine numerical solutions for modeling both transient imbibition and drainage flow processes within rock matrix. The test indicates that numerical approaches, in particular, the errors associated with numerical temporal and spatial discretization, generally need to be checked before their application to field studies of unsaturated flow in fractured rock. The 1-D MINC-type flow approximation can provide a good approximation to unsaturated flow inside the matrix, while it may need very detailed grid refinements to model transient drainage flow behavior in fractured rocks.

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Captions of figures

Figure 1. Schematic of 3-D orthogonal fractures and concept of 3-D and 1-D flow approximation inside matrix blocks.

Figure 2. Type curves of normalized saturation within matrix versus dimensionless time and distance.

Figure 3. Type curves of dimensionless flow rate and cumulative mass exchanges on matrix surface versus dimensionless time.

Figure 4. Comparison of calculated saturation distributions from analytical and numerical solutions for imbibition into a cubic matrix block.

Figure 5. Comparison of calculated saturation distributions from analytical and numerical solutions for drainage from a cubic matrix block.

Figure 6. Comparison of calculated water imbibing rates and cumulative imbibition from analytical and numerical solutions into a cubic matrix block.

Figure 7. Comparison of calculated water drainage rates and cumulative drainage from analytical and numerical solutions from a cubic matrix block.

Table 1. Parameters for the comparison problem.

Parameter	Value	Unit
Half dimension of the matrix cube	$a = 0.5$	m
Effective porosity	$\phi = 0.30$	
Matrix permeability	$k = 1.0 \times 10^{-15}$	m^2
Water density	$\rho = 1{,}000$	kg/m^3
Water viscosity	$\mu_w = 1.0 \times 10^{-3}$	Pa•s
Residual saturation	$S_{wr} = 0.2$	

Initial saturation	S_i= 0.8 and 0.2	
Saturation on matrix surface	S_b= 0.2 and 0.8	
Coefficient of permeability function	$C_k = 1.0$	
Exponential of permeability function	$\alpha = 2.0$	
Coefficient of capillary function	$C_p = 1.0\times 10^4$	Pa
Exponential of capillary function	$\beta = 1.0$	

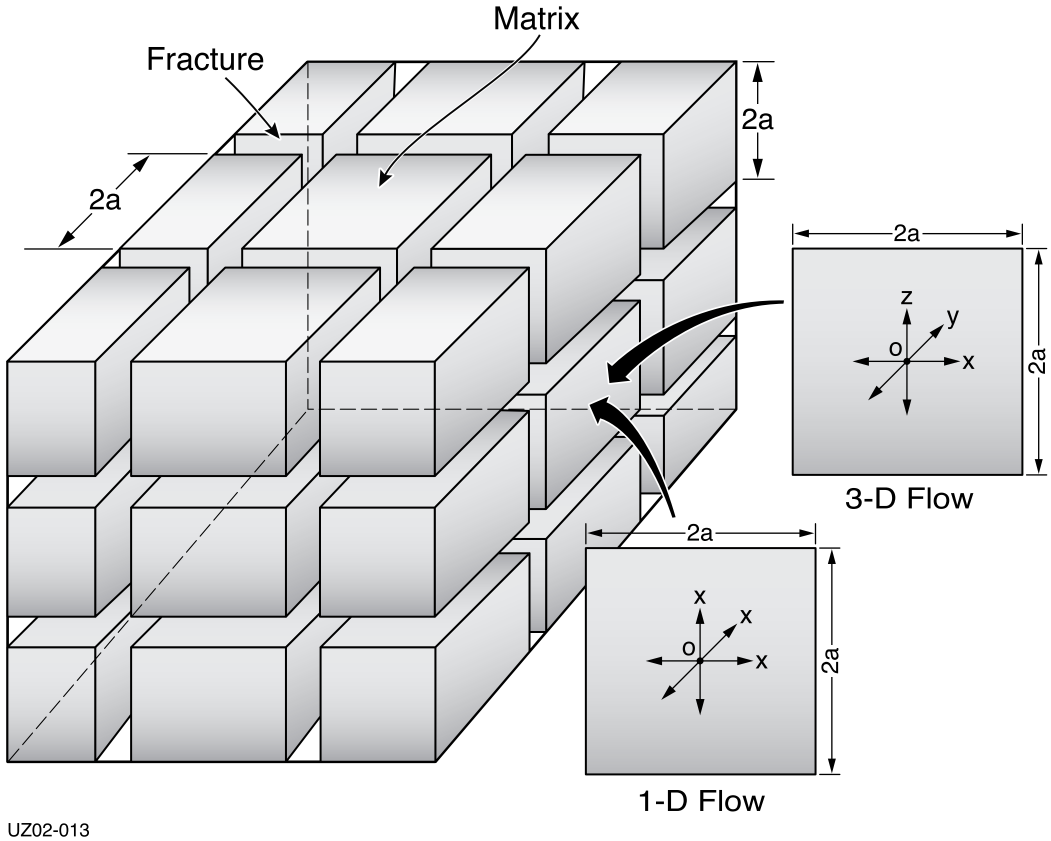

Figure 1. Schematic of 3-D orthogonal fractures and concept of 3-D and 1-D flow approximation inside matrix blocks.

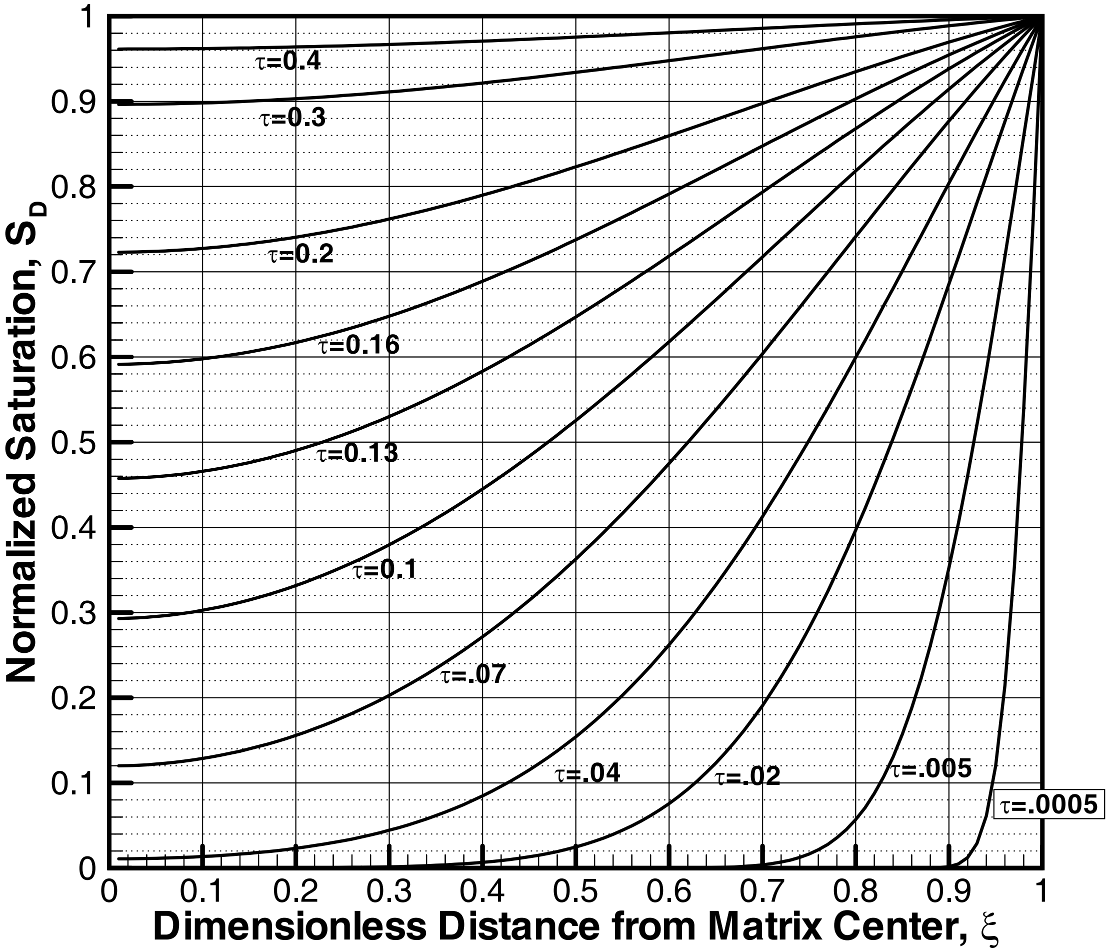

Figure 2. Type curves of normalized saturation within matrix versus dimensionless time and distance.

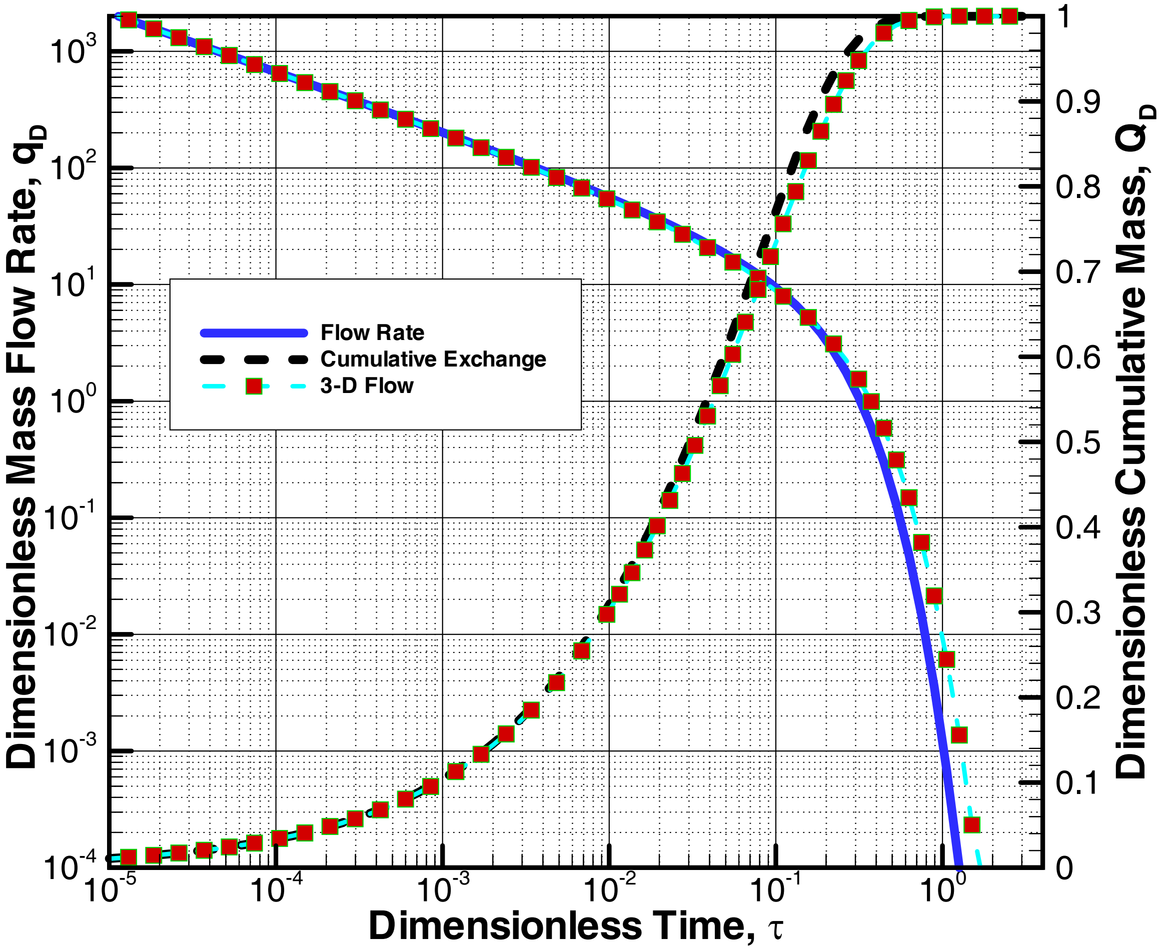

Figure 3. Type curves of dimensionless flow rate and cumulative mass exchanges on matrix surface versus dimensionless time.

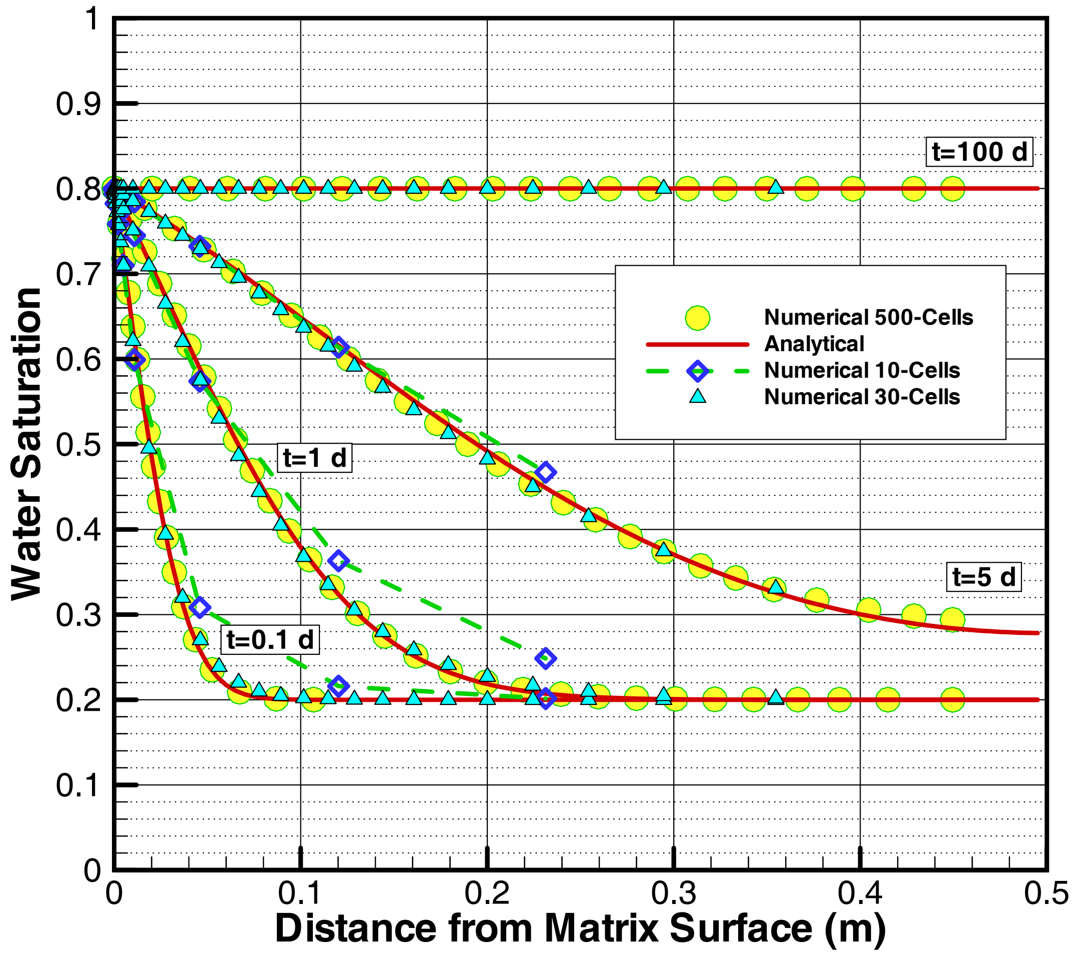

Figure 4. Comparison of calculated saturation distributions from analytical and numerical solutions for imbibition into a cubic matrix block.

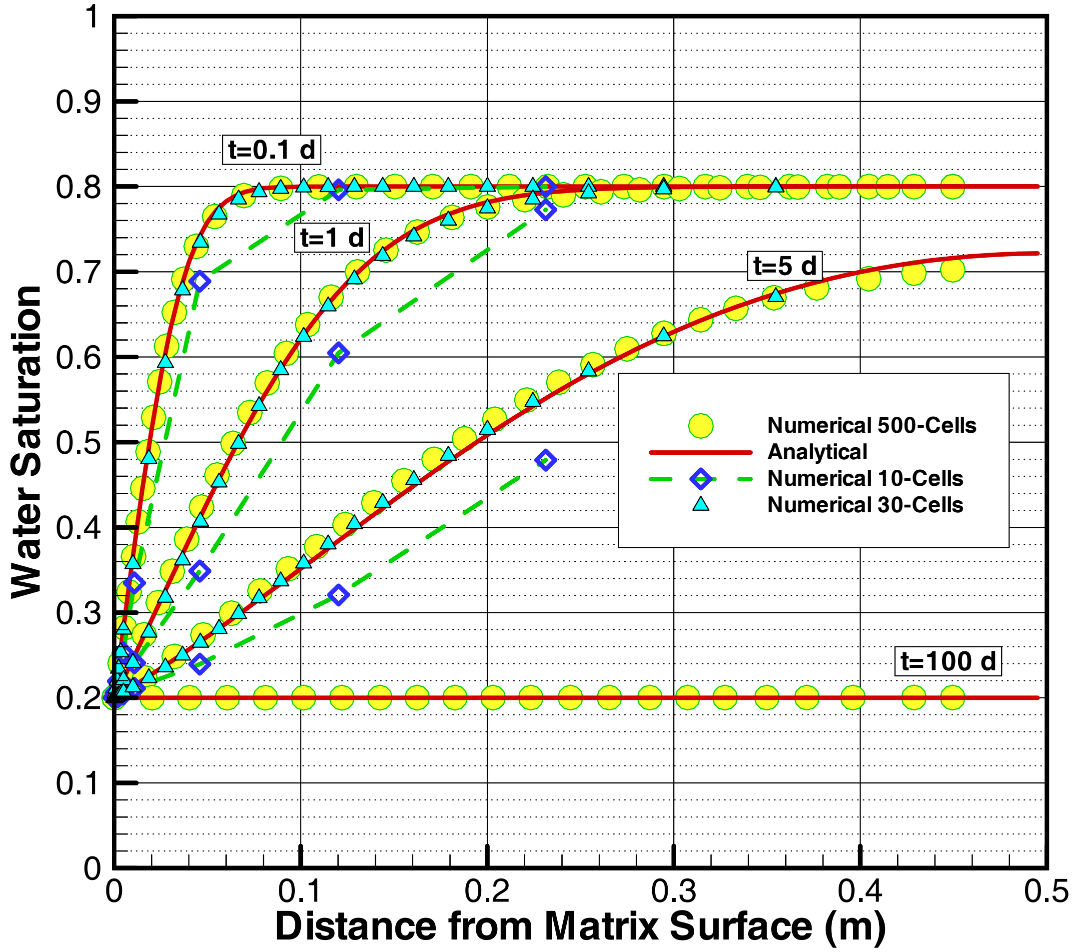

Figure 5. Comparison of calculated saturation distributions from analytical and numerical solutions for drainage from a cubic matrix block.

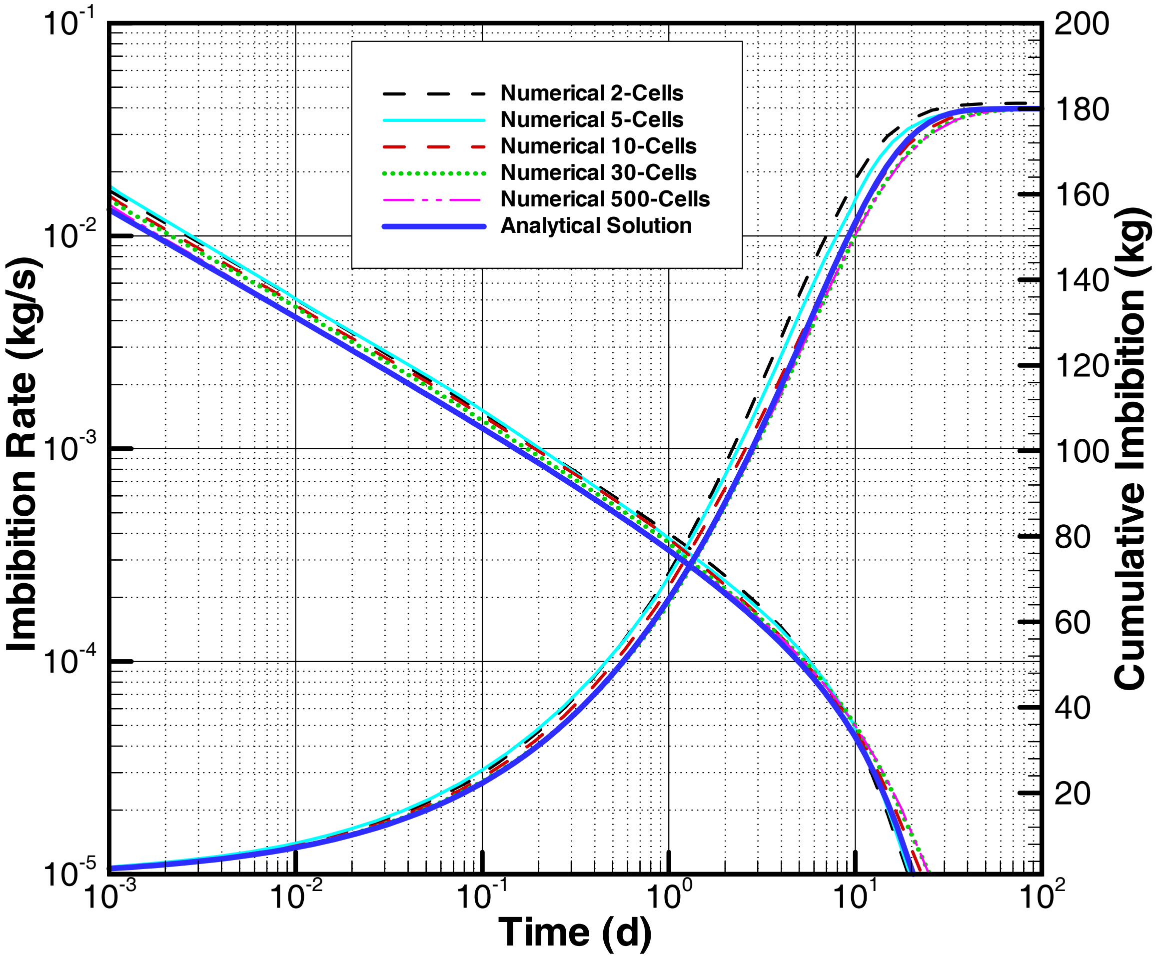

Figure 6. Comparison of calculated water imbibing rates and cumulative imbibition from analytical and numerical solutions into a cubic matrix block.

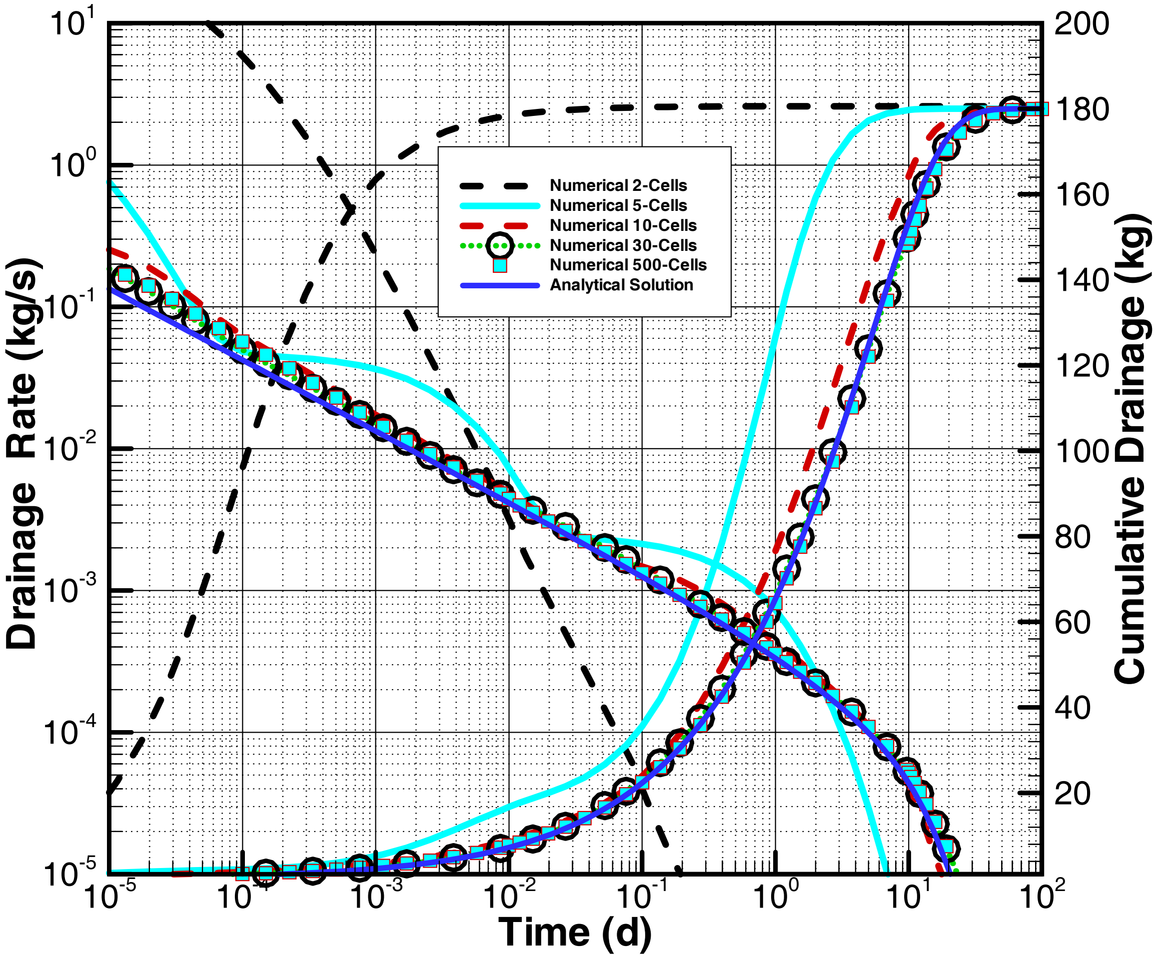

Figure 7. Comparison of calculated water drainage rates and cumulative drainage from analytical and numerical solutions from a cubic matrix block.