Final Technical Report

Sensor Guided Control and Navigation with Intelligent Machines

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Abstract

This report discusses dynamical systems approach to problems in robust control of possibly time varying linear systems, problems in vision and visually guided control and finally applications of these control techniques to intelligent navigation with a mobile platform. Robust design of a controller for a time varying system essentially deals with the problem of synthesizing a controller that can adapt to sudden changes in the parameters of the plant and can maintain stability. The approach presented in this report is to design a compensator that simultaneously stabilizes each and every possible modes of the plant as the parameters undergo sudden and unexpected changes. Such changes can in fact be detected by a visual sensor and hence visually guided control problems are studied as a natural consequence. The problem here is to detect parameters of the plant and maintain stability in the closed loop using a c.c.d camera as a sensor. The main result discussed in the report is the role of perspective systems theory that was developed in order to analyze such a detection and control problem. The robust control algorithms and the visually guided control algorithms are applied in the context of a PUMA 560 robot arm control where the goal is to visually locate a moving part on a mobile turntable. Such problems are of paramount importance in manufacturing with a certain lack of structure. Sensor guided control problems are extended to problems in robot navigation using a NOMADIC mobile platform with a c.c.d and a laser range finder as sensors. The localization and map building problems are studied with the objective of navigation in an unstructured terrain.

Keywords: stabilization, machine vision, mobile robots, perspective control

1 Introduction

In this report we summarize research results that has been supported in part by US Department of Energy under grant number DE-FG02-90ER14140 between the years 1990 and 1999. This research has been essentially conducted by the PI in collaboration with another faculty member in the Department of Systems Science and Mathematics at the Washington University in Saint Louis, Missouri, USA with the help of 8 graduate students supported partially by this grant and one postdoctoral fellow, not supported by this grant. The research started as a theoretically endeavor and resulted in a long standing research collaboration with the Washington University Robotics and Automation Laboratory and partially supported in the creation of a new center for BioCybernetics and Intelligent Systems at the Washington University. It also resulted in a interuniversity collaboration with the Mathematics Department of the Texas Tech University, Lubbock, Texas and with the Department of Information Sciences at the Tokyo Denki University, Japan.

In recent years there has been a growing interest in the need for sensor fusion to solve problems in control and planning for robotic systems. The application of such systems would range from assembly tasks in industrial automation to material handling in hazardous environments and servicing tasks in space. Within the framework of an event-driven approach, robotics has found new application in automation, such as robot-assisted surgery and microfabrication, that pose new challenges to control, automation and manufacturing communities. To meet such challenges, it is important to develop planning and control systems that can integrate various types of sensory information and human knowledge in order to carry out tasks efficiently with or without the need for human intervention. The structure of a sensing, planning and control system and the computer architecture should be designed for a large class of tasks rather than for a specific task. User-friendliness of the interface is essential for human operators who

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pass their knowledge and expertise to the control system before and during task execution. Finally, robustness and adaptability of the system are also essential.

The control system we propose is able to perform in its environment on the basis of prior knowledge and real-time sensory information. We introduce a new task-oriented approach to sensing, planning and control. As a specific example of this approach, we discuss an event based method for system design. In order to design a specific control objective, we introduce the problem of combining task planning and three dimensional modelling in the execution of remote operations. Typical remote systems are teleoperated and work efficiencies that are on the order of 10 times slower than what is directly achievable by humans. Consequently, the effective integration of automation into teleoperated remote systems offers the potential to improve their work efficiency.

The research results can be broadly classified into three distinct categories. The first one has to do with robust feedback control of linear dynamical systems wherein the parameters of the plant may or may not vary but the parameters do suffer sudden and unexpected changes. The problem is to design a controller that would stabilize the entire class of plants. Our second problem has to do with the application of visual sensor in order to detect unknown parameters of the dynamic plant modelling changes in the visual world. A camera based sensor gives rise to a rational output map and the associated dynamical system is called a perspective system. State and parameter estimation problem for a perspective dynamical system has been studied. The third and the last problem studied is to apply sensor guided control to problems in robotic manipulation and to mobile robot navigation. The sensors used include vision, the force/torque and the laser range finder in various different modes.

1.1 Detection and estimation with visual sensing

In this part of our research we focus on motion and shape estimation of a moving body with the aid of a monocular camera. We show that the estimation problem reduces to a specific parameter estimation of a perspective dynamical system. Surprisingly, the above reduction is independent of whether or not the data measured is the brightness pattern which the object produces on the image plane or whether the data observed are points, lines or curves on the image plane produced as a result of discontinuities in the brightness pattern. Many cases of the perspective parameter estimation problem has been studied in our research. These include a fairly complete analysis of a planar textured surface undergoing a rigid flow and an affine flow. The two cases have been analyzed for orthographic, pseudo-orthographic and image centered projections. The basic procedure introduced for parameter estimation is to subdivide the problem into two modules, one for spatial averaging and he other for time averaging. The estimation procedure is carried out with the aid of a new realization theory for perspective systems, introduced for systems described both in discrete and in continuous time. Our research also emphasizes observability and identifiability problems that arise in linear dynamical systems with perspective observation function.

1.2 Motion and shape identification with vision and range

In this research we consider the problem of motion and shape estimation using a camera and a laser range finder. The object considered is a plane which is undergoing a riccati motion. The camera observes features on the moving plane perspectively or orthographically. The range finder camera is capable of obtaining the range of the plane along a given Laser-Plane which can either be kept fixed or can be altered in time. Finally we also assume that the identification is carried out as soon as the visual and range data isavailable or after a suitable temporal integration. In each of these various cases, we derive, to what extent motion and shape parameters are identifiable and characterize the results as orbit of a suitable group.

1.3 Sensor fusion in robotic manipulation

For a robot manipulator to operate properly in an unstructured environment, it is essential to employ a variety of sensors. A ccd camera based machine vision sensor is a typical noncontact sensor that provides feedback information to the manipulator controller. By including cameras inside the control servo loop of a manipulator, visual servoing can easily be achieved. There are three basic strategies of visual servoing. The first strategy uses a camera mounted on the end arm of a manipulator, which is commonly referred to as an eye in hand configuration. Because of the close proximity of the camera and the end effector to the workpiece, this technique is desirable for close inspection, gauging and automated part recognition. The second strategy employs an overhead camera. This technique is usually implemented in a carefully designed and controlled environment in which the depth of a scene is known or fixed. For example, it can be use for servo control of a manipulator used to grasp an non oriented workpiece on a workbench or a conveyor. The third strategy is a natural extension of the second strategy; it uses multiple cameras whose pose and zoom may be controlled to improve the viewing conditions. Because cameras are not mounted on the arms, more manipulators can be added to the system when needed for multi robot tasks without altering the overall system configuration. Visual servo control may assume different forms. Depending on the choice of feedback representation, a position based or image based scheme can be proposed. In a position based scheme, the three dimensional position and orientation information of the environment is first inferred from a set of derived image features and then used in the manipulator controller. On the other hand, image based visual servo control defines task reference configurations directly in the image space by using image features that are uniquely related to spatial position and orientation information.

Our research focuses on a new approach to the problem of tracking and grasping a moving object by a manipulator with multiple cameras. We propose a new object centered model for the motion of the object by defining a 3D reference point in the object. Tracking control can be designed with respect to the translational motion of this reference point. However, no attempt is made to recover the 3D feature points of the object on the basis of stereo matching. Instead, only the image of the reference point in each camera has been used and is determined from the 2D features of the object within the same camera. By properly defining an error function between the image of the robot gripper and that of the reference point, the image based tracking control law is obtained using nonlinear regulator theory.

We have also analyzed problems in robotic manipulation in an uncalibrated environment. The environment consists of a PUMA 560 robotic manipulator, a turntable rotating around a vertical axis equipped with an encoder that records the instantaneous angular displacement with respect to an axis chosen apriori, and a c.c.d camera-based vision sensor that is fixed permanently on the ceiling. It is assumed that a part with a known shape but unknown orientation is placed on the turn-table, which is rotating with unknown motion dynamics. Furthermore, the relative positions of the manipulator, turntable, and camera (the calibration parameters) are assumed to be unknown. Inspite of our lack of knowledge of the orientation of the part and the calibration parameters, the objective is to track the rotating part (with an a priori specified relative orientation) and grasp the part with an end effector of the manipulator.

In addition, we consider planning and control of a robotic manipulator for a class of constrained motions. Here the task under consideration is to control a robot so that a tool grasped by its end-effector follows a path on an unknown surface with the aid of a single-camera vision system. To accomplish the task, we propose a new planning and control strategy based on multi sensor fusion. Three different sensors – joint encoders, a wrist force-torque sensor, and a vision system with a single camera fixed above a work space – are employed. First, based on sensory information, we decouple control variables into two subspaces: one for force control and the other for control of constrained motion. This decoupling allows one to design control schemes for regulation of force and for constrained motion separately. Second, we develop a new scheme by means of sensor fusion to handle the uncertainties in an uncalibrated work space. The contact surface is assumed to be unknown but the trajectory to be followed is visible to the vision system, and the precise position and orientation of the camera with respect to the robot are also assumed to be unknown. Overall our research contributions include (1) multisensor fusion used for both force-torque and visual sensors with complimentary observed data, as opposed to many sensor fusion schemes in the literature with redundant data; (2) intelligent manipulation of a robot that can work in an uncalibrated work space with a camera that is not calibrated with respect to the robot.

Three of the important results of our research are now detailed in the next three sections.

2 Dynamic Models of Planar Algebraic Curves

There has been a growing interest in the use of implicit representations for curves and surfaces. Applications arise in numerous areas of physical sciences, such as engineering, robotics, vision, graphics, geometric optics, and in many areas of pure mathematics such as differential geometry, complex analysis, number theory and differential equations. *Implicit algebraic models* have proved very useful representations for 2D curves and 3D surfaces in several model-based applications including vision, graphics, computational geometry and CAD [1]- [11].

Motivated from problems in mobile robotics and computer vision, in this section we are interested in the study of how to model and describe the dynamics of planar algebraic curves. Such a curve might originate as features that are observed by the aid of a charged coupled device camera (CCD), or it might originate as the outline of a surface being observed with the aid of a laser range finder (LRF), see Fig. 1. In either of the two cases, one is typically

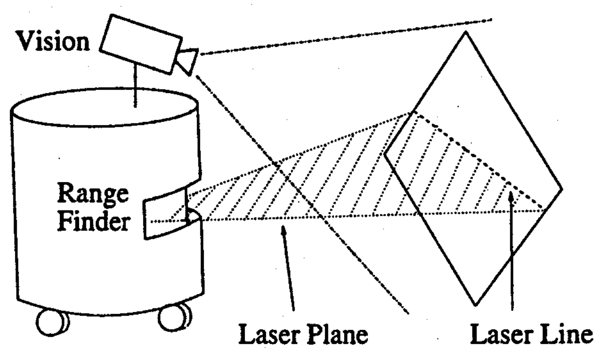

Figure 1: The Laser Range Finder and the CCD vision

faced with the problem of modelling two dimensional planar curves in motion. There has been a steadily growing literature in robotics on the problem of line correspondence for line features moving in $\mathbb{R}^3$, (see [12]- [16]). This section, we hope, would perhaps initiate a strategy for curve correspondence as well. For some other older references in the literature on the dynamics of curves, see [17]- [19].

2.1 Standard Facts About Algebraic Curves

Algebraic curves are defined implicitly by equations of the form $f(x,y)=0$, where $f(x,y)$ is a polynomial in the variables x,y, i.e. $f(x,y)=\sum_{ij} a_{ij}x^iy^j$ where $0 \le i+j \le n$ (n is finite) and the coefficients a_{ij} are real numbers [1]. Alternatively, the intersection of an explicit surface $z=f(x,y)$ with the $z=0$ plane yields an algebraic curve if $f(x,y)$ is a polynomial. Algebraic curves of degree 1, 2, 3, 4, ... are called *lines, conics, cubics, quartics,* ... etc.

In general, an *algebraic curve* of *degree* n can be defined by the *implicit polynomial (IP) equation*:

$$f_n(x,y) = \underbrace{a_{00}}_{H_0} + \underbrace{a_{10}x + a_{01}y}_{H_1(x,y)} + \underbrace{a_{20}x^2 + a_{11}xy + a_{02}y^2}_{H_2(x,y)} + \ldots$$

$$+ \underbrace{a_{n0}x^n + a_{n-1,1}x^{n-1}y + \ldots + a_{0n}y^n}_{H_n(x,y)} = \sum_{i=0}^{n} H_i(x,y) = 0, \tag{2.1}$$

where each binary *form* $H_r(x,y)$ is a *homogeneous polynomial* of degree r in the variables x and y. The number of terms in each $H_r(x,y)$ is $r+1$, so that the IP equation defined by (2.1) has one constant term, two terms of the first degree, three terms of the second degree, etc., up to and including $n+1$ terms of the (highest) n-th degree, for a total of $(n+1)(n+2)/2$ coefficients. Since the above equation can be multiplied by a non-zero constant without changing the zero set, an algebraic curve defined by $f_n(x,y)=0$ has $(n+1)(n+2)/2-1 = n(n+3)/2$ independent coefficients or *degrees of freedom* (DOF). A *monic* polynomial $f_n(x,y)=0$ will be defined by the condition that $a_{n0}=1$ in (2.1).

As shown by Unel and Wolovich [4, 6] algebraic curves can be decomposed as a *unique* sum of *line products.* The following theorem will enable us to study the dynamics of the planar curves through the dynamics of the lines.

Theorem [4, 6] A non-degenerate (monic) $f_n(x,y)$ can be uniquely decomposed as a finite sum of real and complex line products, namely

$$f_n(x,y) = \Pi_n(x,y) + \gamma_{n-2}[\Pi_{n-2}(x,y) + \gamma_{n-4}[\Pi_{n-4}(x,y) + \ldots]] \tag{2.2}$$

where each $\Pi_r(x,y) = \prod_{i=1}^{r} L_{ri}(x,y)$ and each line factor $L_{ri}(x,y)$ can be written as the (vector) dot product

$$L_{ri}(x,y) = x + l_{ri}y + k_{ri} = \underbrace{\begin{bmatrix}1 & l_{ri} & k_{ri}\end{bmatrix}}_{\stackrel{def}{=} L_{ri}^T} X = \underbrace{\begin{bmatrix}x & y & 1\end{bmatrix}}_{\stackrel{def}{=} X^T} L_{ri}$$

and $\gamma_{n-2}, \gamma_{n-4}, \ldots$ are real scalars.

For example, a (monic) conic, cubic and quartic curve can be (line) decomposed as

$$f_2(x,y) = L_1(x,y)L_2(x,y) + \alpha = 0,$$

$$f_3(x,y) = L_1(x,y)L_2(x,y)L_3(x,y) + \alpha L_4(x,y) = 0,$$

and

$$f_4(x,y) = L_1(x,y)L_2(x,y)L_3(x,y)L_4(x,y) + \alpha L_5(x,y)L_6(x,y) + \beta = 0, \tag{2.3}$$

respectively, where α and β are real scalars. Note that each line factor in these (and higher degree decompositions) has 2 DOF. Therefore, by including the multiplicative scalars, we verify that *a monic IP curve has* $n(n+3)/2$ *independent coefficients* or DOF.

In the sequel, we will restrict our attention to quartic curves and note that the results can easily be generalized to higher degree curves.

2.2 Rigid and Affine Motion of Planar Algebraic Curves in a Plane

We consider the problem of motion estimation using a laser range finder (LRF). We assume that the LRF is located on a mobile platform which moves rigidly on the horizontal floor. It is capable of obtaining the range of an object along a given *Laser Plane*, which is assumed to be parallel to the ground. The data points are provided on a planar curve obtained by the intersection of the Laser Plane and the object. Since the platform moves, so does the planar curve with respect to the coordinates on the LRF.

We propose to fit algebraic curves to data and study curve dynamics through the dynamics of the lines in the decomposition of the curve. It turns out that the *line dynamics* can be described by a Riccati Equation with parameters that depend on the motion of the platform. Each of the lines satisfy the same Riccati Equation initialized at different points on the state space.

The essential idea that *if points on a line moves following a rigid dynamics on a plane, then the slope and the intercept satisfy a Riccati Equation* has already been studied earlier (see [20] and many other references therein). What is new in this section is that the same line dynamics can essentially be used to study the motion of planar curves in a factored form.

To begin with, consider a line decomposed planar quartic curve

$$f_4(x,y) = \prod_{i=1}^{4} \begin{pmatrix} 1 & m_i & c_i \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + \alpha \prod_{i=1}^{2} \begin{pmatrix} 1 & n_i & e_i \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + \beta = 0 \tag{2.1}$$

We can use the following substitutions to homogenize the lines in the decomposition

$$x = \frac{\bar{x}}{\bar{w}}, \; y = \frac{\bar{y}}{\bar{w}} \tag{2.2}$$

$$m_i = \frac{\bar{m}_i}{\bar{l}_i}, \; c_i = \frac{\bar{c}_i}{\bar{l}_i}, \; n_i = \frac{\bar{n}_i}{\bar{k}_i}, \; e_i = \frac{\bar{e}_i}{\bar{k}_i} \tag{2.3}$$

and get

$$f_4(\bar{x},\bar{y},\bar{w}) = \prod_{i=1}^{4} \begin{pmatrix} \bar{l}_i & \bar{m}_i & \bar{c}_i \end{pmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{w} \end{bmatrix} + \alpha \left(\frac{\prod_{i=1}^{4} \bar{l}_i}{\prod_{i=1}^{2} \bar{k}_i}\right) \bar{w}^2 \prod_{i=1}^{2} \begin{pmatrix} \bar{k}_i & \bar{n}_i & \bar{e}_i \end{pmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{w} \end{bmatrix} + \beta \left(\prod_{i=1}^{4} \bar{l}_i\right) \bar{w}^4 = 0. \tag{2.4}$$

Now consider (2.4) when $t = 0$. We have

$$f_4(\bar{x}(0), \bar{y}(0), \bar{w}(0)) =$$

$$\prod_{i=1}^{4}(\bar{l}_i(0) \quad \bar{m}_i(0) \quad \bar{c}_i(0))\begin{bmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{w}(0)\end{bmatrix} + \alpha(0)(\frac{\prod_{i=1}^{4}\bar{l}_i(0)}{\prod_{i=1}^{2}\bar{k}_i(0)})\bar{w}^2(0)\prod_{i=1}^{2}(\bar{k}_i(0) \quad \bar{n}_i(0) \quad \bar{e}_i(0))\begin{bmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{w}(0)\end{bmatrix} + \beta(0)(\prod_{i=1}^{4}\bar{l}_i(0))\bar{w}^4(0) = 0 \tag{2.5}$$

Suppose the curve undergoes an affine motion described as

$$\frac{d}{dt}\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{w}\end{bmatrix} = \underbrace{\begin{bmatrix}a_1 & a_2 & b_1\\ a_3 & a_4 & b_2\\ 0 & 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{w}\end{bmatrix}. \tag{2.6}$$

In case the 2×2 matrix

$$M = \begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}$$

happens to be a skew-symmetric matrix, i.e. $M + M^T = 0$, the motion will be termed as a rigid motion. Note that in this case $e^{Mt} \in SO(2)$, the rotation group of 2×2 matrices.

Let us now consider (2.4) at time t and substitute the solution of (2.6). This will imply that

$$\prod_{i=1}^{4}(\bar{l}_i(t) \quad \bar{m}_i(t) \quad \bar{c}_i(t))\, e^{At}\begin{bmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{w}(0)\end{bmatrix} + \alpha(t)(\frac{\prod_{i=1}^{4}\bar{l}_i(t)}{\prod_{i=1}^{4}\bar{l}_i(0)}\frac{\prod_{i=1}^{2}\bar{k}_i(0)}{\prod_{i=1}^{2}\bar{k}_i(t)}\frac{\bar{w}^2(t)}{\bar{w}^2(0)})(\frac{\prod_{i=1}^{4}\bar{l}_i(0)}{\prod_{i=1}^{2}\bar{k}_i(0)}\bar{w}^2(0))$$

$$\prod_{i=1}^{2}(\bar{k}_i(t) \quad \bar{n}_i(t) \quad \bar{e}_i(t))\, e^{At}\begin{bmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{w}(0)\end{bmatrix} + \beta(t)(\frac{\prod_{i=1}^{4}\bar{l}_i(t)}{\prod_{i=1}^{4}\bar{l}_i(0)}\frac{\bar{w}^4(t)}{\bar{w}^4(0)})(\prod_{i=1}^{4}\bar{l}_i(0))\bar{w}^4(0) = 0 \tag{2.7}$$

is another description of the curve (2.5). Since (2.4) is a unique decomposition of $f_4(\bar{x}, \bar{y}, \bar{w})$, (2.5) and (2.7) imply that

$$\begin{bmatrix}\bar{l}_i(t)\\ \bar{m}_i(t)\\ \bar{c}_i(t)\end{bmatrix} = e^{-A^T t}\begin{bmatrix}\bar{l}_i(0)\\ \bar{m}_i(0)\\ \bar{c}_i(0)\end{bmatrix}, \tag{2.8}$$

$$\begin{bmatrix}\bar{k}_i(t)\\ \bar{n}_i(t)\\ \bar{e}_i(t)\end{bmatrix} = e^{-A^T t}\begin{bmatrix}\bar{k}_i(0)\\ \bar{n}_i(0)\\ \bar{e}_i(0)\end{bmatrix} \tag{2.9}$$

and

$$\alpha(t) = (\prod_{i=1}^{4}\frac{\bar{l}_i(0)}{\bar{l}_i(t)})(\prod_{i=1}^{2}\frac{\bar{k}_i(t)}{\bar{k}_i(0)})(\frac{\bar{w}(0)}{\bar{w}(t)})^2\alpha(0), \quad \beta(t) = (\prod_{i=1}^{4}\frac{\bar{l}_i(0)}{\bar{l}_i(t)})(\frac{\bar{w}(0)}{\bar{w}(t)})^4\beta(0) \tag{2.10}$$

Recall that

$$A = \begin{pmatrix}a_1 & a_2 & b_1\\ a_3 & a_4 & b_2\\ 0 & 0 & 0\end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix}\phi_1(t) & \phi_2(t) & \phi_3(t)\\ \phi_4(t) & \phi_5(t) & \phi_6(t)\\ 0 & 0 & 1\end{pmatrix} \Rightarrow e^{-A^T t} = \begin{pmatrix}\psi_1(t) & \psi_2(t) & 0\\ \psi_3(t) & \psi_4(t) & 0\\ \psi_5(t) & \psi_6(t) & 1\end{pmatrix} \tag{2.11}$$

It follows that $\bar{w}(t) = \bar{w}(0)$ for all t. As a result, (2.10) can be simplified to

$$\alpha(t) = (\prod_{i=1}^{4}\frac{\bar{l}_i(0)}{\bar{l}_i(t)})(\prod_{i=1}^{2}\frac{\bar{k}_i(t)}{\bar{k}_i(0)})\alpha(0), \quad \beta(t) = (\prod_{i=1}^{4}\frac{\bar{l}_i(0)}{\bar{l}_i(t)})\beta(0). \tag{2.12}$$

One can go even further and express $\alpha(t)$ and $\beta(t)$ using only motion and line parameters. In order to do that one need to use (2.8) and (2.11):

$$\bar{l}_i(t) = \psi_1(t)\bar{l}_i(0) + \psi_2(t)\bar{m}_i(0) \Rightarrow 1 = \psi_1(t)\frac{\bar{l}_i(0)}{\bar{l}_i(t)} + \psi_2(t)\underbrace{\frac{\bar{m}_i(0)}{\bar{l}_i(0)}}_{m_i(0)}\frac{\bar{l}_i(0)}{\bar{l}_i(t)} \Rightarrow \frac{\bar{l}_i(0)}{\bar{l}_i(t)} = \frac{1}{\psi_1(t) + m_i(0)\psi_2(t)}$$

Similarly,

$$\frac{\bar{k}_i(0)}{\bar{k}_i(t)} = \frac{1}{\psi_1(t) + n_i(0)\psi_2(t)}$$

Thus,

$$\alpha(t) = \left(\frac{\prod_{i=1}^{2}\psi_1(t) + n_i(0)\psi_2(t)}{\prod_{i=1}^{4}\psi_1(t) + m_i(0)\psi_2(t)}\right)\alpha(0), \quad \beta(t) = \left(\prod_{i=1}^{4}\frac{1}{\psi_1(t) + m_i(0)\psi_2(t)}\right)\beta(0) \tag{2.13}$$

Note that (2.8) and (2.9) imply the following line dynamics:

$$\frac{d}{dt}\begin{bmatrix}\bar{l}_i(t)\\ \bar{m}_i(t)\\ \bar{c}_i(t)\end{bmatrix} = -A^T\begin{bmatrix}\bar{l}_i(t)\\ \bar{m}_i(t)\\ \bar{c}_i(t)\end{bmatrix} = \begin{pmatrix}-a_1 & -a_3 & 0\\ -a_2 & -a_4 & 0\\ -b_1 & -b_2 & 0\end{pmatrix}\begin{bmatrix}\bar{l}_i(t)\\ \bar{m}_i(t)\\ \bar{c}_i(t)\end{bmatrix} \quad i = 1, 2, \ldots, 4 \tag{2.14}$$

and

$$\frac{d}{dt}\begin{bmatrix}\bar{k}_i(t)\\ \bar{n}_i(t)\\ \bar{e}_i(t)\end{bmatrix} = -A^T\begin{bmatrix}\bar{k}_i(t)\\ \bar{n}_i(t)\\ \bar{e}_i(t)\end{bmatrix} = \begin{pmatrix}-a_1 & -a_3 & 0\\ -a_2 & -a_4 & 0\\ -b_1 & -b_2 & 0\end{pmatrix}\begin{bmatrix}\bar{k}_i(t)\\ \bar{n}_i(t)\\ \bar{e}_i(t)\end{bmatrix} \quad i = 1, 2. \tag{2.15}$$

We next show that line parameters m_i, c_i, n_i, e_i in the decomposition of the original curve satisfy coupled Riccati equations. To this end we differentiate (2.3) with respect to time and use (2.14) and (2.15) to obtain the following Riccati equations:

$$\dot{m}_i = -a_2 + (a_1 - a_4)m_i + a_3 m_i^2 \tag{2.16}$$

$$\dot{c}_i = -b_1 - b_2 m_i + a_1 c_i + a_3 m_i c_i \tag{2.17}$$

$$\dot{n}_i = -a_2 + (a_1 - a_4)n_i + a_3 n_i^2 \tag{2.18}$$

$$\dot{e}_i = -b_1 - b_2 n_i + a_1 e_i + a_3 n_i e_i \tag{2.19}$$

The conclusion is that the dynamics of the planar quartic curve can be studied through the dynamics of 6 lines in the decomposition of the curve and the line parameters, i.e. slope and intercept, satisfy coupled Riccati equations.

2.3 Dynamics of Planar Algebraic Curves in 3-D Space

In this section we are interested in the dynamics of *planar space curves* as opposed to planar curves that are moving on a fixed plane. Assume that we have a feature curve on a plane in $\mathbb{R}^3$ and that the plane is moving in 3D together with the curve. The immediate question of interest is how can we decompose such a planar curve in space and how we can study the dynamics of these curves in terms of the *line dynamics* introduced in the previous section? (see Fig. 1)

Note that (2.4) at $t = 0$ can be rewritten as

$$\prod_{i=1}^{4}\begin{pmatrix}\bar{l}_i(0) & \bar{m}_i(0) & \bar{s}_i(0) & \bar{c}_i(0)\end{pmatrix}\begin{bmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{z}(0)\\ \bar{w}(0)\end{bmatrix}$$

$$+\alpha(0)\left(\frac{\prod_{i=1}^{4}\bar{l}_i(0)}{\prod_{i=1}^{2}\bar{k}_i(0)}\right)\bar{w}^2(0)\prod_{i=1}^{2}\begin{pmatrix}\bar{k}_i(0) & \bar{n}_i(0) & \bar{h}_i(0) & \bar{e}_i(0)\end{pmatrix}\begin{bmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{z}(0)\\ \bar{w}(0)\end{bmatrix}+\beta(0)\left(\prod_{i=1}^{4}\bar{l}_i(0)\right)\bar{w}^4(0)=0 \tag{2.1}$$

where $\bar{s}_i(0)=0$ and $\bar{h}_i(0)=0$. Consider a rigid motion of the xy plane in $\mathbb{R}^3$ as follows

$$\frac{d}{dt}\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\\ \bar{w}\end{pmatrix}=\underbrace{\begin{pmatrix}\Omega & b\\ 0 & 0\end{pmatrix}}_{A}\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\\ \bar{w}\end{pmatrix}\Rightarrow\begin{pmatrix}\bar{x}(t)\\ \bar{y}(t)\\ \bar{z}(t)\\ \bar{w}(t)\end{pmatrix}=e^{At}\begin{pmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{z}(0)\\ \bar{w}(0)\end{pmatrix} \tag{2.2}$$

where

$$\Omega=\begin{pmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{pmatrix} \tag{2.3}$$

is a skew-symmetric matrix, i.e. $\Omega+\Omega^T=0$, and

$$b=\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}$$

is a translation vector. As initial conditions we have $\bar{x}(0)=\bar{x}(0),\bar{y}(0)=\bar{y}(0),\bar{z}(0)=0$ and $\bar{w}(0)=1$. From (2.4) and (2.2), it follows that

$$\prod_{i=1}^{4}\begin{pmatrix}\bar{l}_i(t) & \bar{m}_i(t) & \bar{s}_i(t) & \bar{c}_i(t)\end{pmatrix}e^{At}\begin{bmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{z}(0)\\ \bar{w}(0)\end{bmatrix}$$

$$+\alpha(t)\left(\frac{\prod_{i=1}^{4}\bar{l}_i(t)}{\prod_{i=1}^{2}\bar{k}_i(t)}\right)\bar{w}^2(t)\prod_{i=1}^{2}\begin{pmatrix}\bar{k}_i(t) & \bar{n}_i(t) & \bar{h}_i(t) & \bar{e}_i(t)\end{pmatrix}e^{At}\begin{bmatrix}\bar{x}(0)\\ \bar{y}(0)\\ \bar{z}(0)\\ \bar{w}(0)\end{bmatrix}+\beta(t)\left(\prod_{i=1}^{4}\bar{l}_i(t)\right)\bar{w}^4(t)=0 \tag{2.4}$$

As before using uniqueness we get complimentary equations as:

$$\begin{bmatrix}\bar{l}_i(t)\\ \bar{m}_i(t)\\ \bar{s}_i(t)\\ \bar{c}_i(t)\end{bmatrix}=e^{-A^Tt}\begin{bmatrix}\bar{l}_i(0)\\ \bar{m}_i(0)\\ \bar{s}_i(0)\\ \bar{c}_i(0)\end{bmatrix}, \tag{2.5}$$

$$\begin{bmatrix}\bar{k}_i(t)\\ \bar{n}_i(t)\\ \bar{h}_i(t)\\ \bar{e}_i(t)\end{bmatrix}=e^{-A^Tt}\begin{bmatrix}\bar{k}_i(0)\\ \bar{n}_i(0)\\ \bar{h}_i(0)\\ \bar{e}_i(0)\end{bmatrix} \tag{2.6}$$

or,

$$\frac{d}{dt}\begin{bmatrix}\bar{l}_i(t)\\ \bar{m}_i(t)\\ \bar{s}_i(t)\\ \bar{c}_i(t)\end{bmatrix}=-A^T\begin{bmatrix}\bar{l}_i(t)\\ \bar{m}_i(t)\\ \bar{s}_i(t)\\ \bar{c}_i(t)\end{bmatrix}=\begin{pmatrix}0 & -\omega_3 & \omega_2 & 0\\ \omega_3 & 0 & -\omega_1 & 0\\ -\omega_2 & \omega_1 & 0 & 0\\ -b_1 & -b_2 & -b_3 & 0\end{pmatrix}\begin{bmatrix}\bar{l}_i(t)\\ \bar{m}_i(t)\\ \bar{s}_i(t)\\ \bar{c}_i(t)\end{bmatrix}\quad i=1,2,\ldots,4 \tag{2.7}$$

and

$$\frac{d}{dt}\begin{bmatrix}\bar{k}_i(t)\\ \bar{n}_i(t)\\ \bar{h}_i(t)\\ \bar{e}_i(t)\end{bmatrix} = -A^T\begin{bmatrix}\bar{k}_i(t)\\ \bar{n}_i(t)\\ \bar{h}_i(t)\\ \bar{e}_i(t)\end{bmatrix} = \begin{pmatrix}0 & -\omega_3 & \omega_2 & 0\\ \omega_3 & 0 & -\omega_1 & 0\\ -\omega_2 & \omega_1 & 0 & 0\\ -b_1 & -b_2 & -b_3 & 0\end{pmatrix}\begin{bmatrix}\bar{k}_i(t)\\ \bar{n}_i(t)\\ \bar{h}_i(t)\\ \bar{e}_i(t)\end{bmatrix} \quad i=1,2. \tag{2.8}$$

where $\bar{l}_i(0)=1$, $\bar{k}_i(0)=1$, $\bar{m}_i(0)=m_i(0)$, $\bar{s}_i(0)=0$, $\bar{c}_i(0)=c_i(0)$, $\bar{n}_i(0)=n_i(0)$, $\bar{h}_i(0)=0$ and $\bar{e}_i(0)=e_i(0)$.

Using (2.7) and (2.8), we obtain the following Riccati equations for the parameters of the planes in the decomposition:

$$\dot{m}_i = \omega_3 - \omega_1 s_i - \omega_2 s_i m_i + \omega_3 m_i^2 \tag{2.9}$$

$$\dot{s}_i = -\omega_2 + \omega_1 m_i + \omega_3 m_i s_i - \omega_2 s_i^2 \tag{2.10}$$

$$\dot{c}_i = -b_1 - b_2 m_i - b_3 s_i + \omega_3 m_i c_i - \omega_2 s_i c_i \tag{2.11}$$

$$\dot{n}_i = \omega_3 - \omega_1 h_i - \omega_2 h_i n_i + \omega_3 n_i^2 \tag{2.12}$$

$$\dot{h}_i = -\omega_2 + \omega_1 n_i + \omega_3 n_i h_i - \omega_2 h_i^2 \tag{2.13}$$

$$\dot{e}_i = -b_1 - b_2 n_i - b_3 h_i + \omega_3 n_i e_i - \omega_2 h_i e_i \tag{2.14}$$

Note that we have planes instead of lines since the curve in the xy plane undergoes the same rigid transformation as the xy plane does and therefore we have a surface at time t which is decomposed in terms of planes, and the intersection of this surface and the plane gives the planar space curve on that plane. In a sense we have created a surface from the motion of a planar curve and by construction this surface is decomposable in terms of planes although there is no decomposition for a general algebraic surface in $3D$ in terms of planes. It will be interesting to look at the geometry and/or topology of this surface but we will not do that in this work.

Note that the form of equations which describe the dynamics of m_i, s_i and c_i is the same as the form of equations which describe the dynamics of n_i, h_i and e_i. We conclude that the dynamics of the planar space curve can be studied by the dynamics of the planes in the decomposition of that curve, and moreover the same coupled Riccati equations determine the motion of all these planes. Finally we note that the recursion for α and β in (2.4) will still be provided by (2.10).

2.4 Dynamics of Planar Algebraic Curves Resulting out of Perspective Projection

If we view a moving planar curve in $\mathbb{R}^3$, using a CCD camera, as opposed to a LRF camera as was discussed in section 3, we are able to observe the perspective projection of the planar curve on the image plane as a function of time. In coordinates already introduced earlier, if we define

$$X = \frac{\bar{x}}{\bar{z}}, \quad Y = \frac{\bar{y}}{\bar{z}} \tag{2.1}$$

where X and Y are coordinates on the image plane, we have a planar curve defined on the image plane in these coordinates. In the spirit of the earlier sections, we would like to study the dynamics of the curve on the image plane, as the *planar space curve* moves in 3D.

The above described projection problem has already been discussed in [21]. If we have a planar curve of degree 2 described in the homogeneous coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ as

$$\begin{pmatrix}a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9\\ 0 & 0 & 0 & 0 & 0 & 0 & p & q & -1 & r\end{pmatrix}\theta^T = \begin{pmatrix}0\\0\end{pmatrix}. \tag{2.2}$$

where

$$\theta = (\bar{x}^2, \bar{y}^2, \bar{z}^2, \bar{x}\bar{y}, \bar{x}\bar{z}, \bar{y}\bar{z}, \bar{x}\bar{w}, \bar{y}\bar{w}, \bar{z}\bar{w}, \bar{w}^2)$$

under perspective projection, the projected curve has the equation given by

$$\eta_1 X^2 + \eta_2 Y^2 + \eta_3 XY + \eta_4 X + \eta_5 Y + \eta_6 = 0$$

where the parameters η_j can appropriately defined for $j = 1, ..., 6$. If the planar curve undergoes a rotational motion given by

$$\frac{d}{dt}\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\\ \bar{w}\end{pmatrix} = \begin{pmatrix}\Omega & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\\ \bar{w}\end{pmatrix} \tag{2.3}$$

it has been shown [21] that the coefficients of the projected curve satisfy the following

$$\frac{d}{dt}\begin{pmatrix}\eta_5\\ \eta_4\\ \eta_3\\ \eta_1\\ \eta_2\\ \eta_6\end{pmatrix} = \begin{pmatrix}\Omega & -2R^T\\ R & 0\end{pmatrix}\begin{pmatrix}\eta_5\\ \eta_4\\ \eta_3\\ \eta_1\\ \eta_2\\ \eta_6\end{pmatrix} \tag{2.4}$$

where Ω has already been defined in (2.3) and where R is given by

$$\begin{pmatrix}0 & -\omega_2 & \omega_3\\ \omega_1 & 0 & -\omega_3\\ -\omega_1 & \omega_2 & 0\end{pmatrix} \tag{2.5}$$

If on the other hand, the planar curve undergoes a rigid motion (as would typically be the case) given by

$$\frac{d}{dt}\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\\ \bar{w}\end{pmatrix} = \begin{pmatrix}\Omega & b\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\\ \bar{w}\end{pmatrix} \tag{2.6}$$

where the vector b is any 3×1 translational vector, a decription similar to (2.4) can be written [22], but involves 21 state dimensions. To the best of our knowledge, when the degree of the curve is greater than 2, dynamics of the projected curve under perspective projection, similar to (2.4) has not been written and studied.

Generalizing the computations presented in the earlier sections, we show that the dynamics of a planar curve obtained via perspective projection of a planar curve in $\mathbb{R}^3$ undergoing a rigid motion as described by (2.6) can be studied through a set of line dynamics. The line dynamics can be described by a Riccati Equation with time varying parameters. Indeed the parameters of the Riccati equation are functions of a set of shape parameters in addition to the motion parameters Ω and b, where the shape parameters satisfy a Riccati equation of its own.

It is well known [23], that for a point $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ on the planar curve (2.2) undergoing rigid motion described by (2.6), the equation of the projected point on the image plane, under perspective projection (2.1) can be written in homogeneous coordinates

$$X = \frac{\bar{X}}{\bar{W}}, \quad Y = \frac{\bar{Y}}{\bar{W}}$$

as follows

$$\frac{d}{dt}\begin{pmatrix}\bar{X}\\ \bar{Y}\\ \bar{W}\end{pmatrix} = \begin{pmatrix}d_3 & d_4 & d_1\\ d_5 & d_6 & d_2\\ d_7 & d_8 & 1\end{pmatrix}\begin{pmatrix}\bar{X}\\ \bar{Y}\\ \bar{W}\end{pmatrix}. \tag{2.7}$$

In the cartesian coordinates (X, Y) of the image plane, the equation (2.7) can be written as a Riccati equation given by

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} d_3 & d_4 \\ d_5 & d_6 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} d_7X^2 + d_8XY \\ d_8Y^2 + d_7XY \end{pmatrix} \tag{2.8}$$

where

$$\begin{aligned} d_1 &= \omega_2 + \frac{b_1}{r} \\ d_2 &= -\omega_1 + \frac{b_2}{r} \\ d_3 &= -(\frac{b_3}{r}) + p\frac{b_1}{r} \\ d_4 &= -\omega_3 - q\frac{b_1}{r} \\ d_5 &= \omega_3 - p\frac{b_2}{r} \\ d_6 &= -(\frac{b_3}{r} + q\frac{b_2}{r}) \\ d_7 &= -p\frac{b_3}{r} - \omega_2 \\ d_8 &= -q\frac{b_3}{r} + \omega_1. \end{aligned}$$

Notice that the parameters p, q and r are parameters of the plane that contains the planar curve defined in (2.2). These parameters are called *shape parameters* and the associated dynamics is called *shape dynamics*. It is shown in [23] that the *shape dynamics* is a Riccati equation described as follows

$$\begin{aligned} \dot{p} &= -\omega_3 q - \omega_2 - \omega_2 p^2 + \omega_1 pq \\ \dot{q} &= \omega_3 p + \omega_1 - \omega_2 pq + \omega_1 q^2 \\ \dot{r} &= (\omega_1 q - \omega_2 p)r + (b_3 - b_2 q - b_1 p) \end{aligned}$$

The important question of interest in this section is *How do we describe the dynamics of a planar curve undergoing a Riccati Motion?*

Restricting attention only to quartic curves, without any loss of generality, we write the equation of a planar quartic curve on the image plane as

$$f_4(\bar{X}, \bar{Y}, \bar{W}) = \prod_{i=1}^{4}\begin{pmatrix} \bar{l}_i & \bar{m}_i & \bar{c}_i \end{pmatrix}\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{W} \end{bmatrix} + \alpha(\frac{\prod_{i=1}^{4}\bar{l}_i}{\prod_{i=1}^{2}\bar{k}_i})\bar{W}^2\prod_{i=1}^{2}\begin{pmatrix} \bar{k}_i & \bar{n}_i & \bar{e}_i \end{pmatrix}\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{W} \end{bmatrix} + \beta(\prod_{i=1}^{4}\bar{l}_i)\bar{W}^4 = 0. \tag{2.9}$$

Proceeding as before, it can be seen that the line dynamics can be written by the following pair of homogeneous equations

$$\frac{d}{dt}\begin{bmatrix} \bar{l}_i(t) \\ \bar{m}_i(t) \\ \bar{c}_i(t) \end{bmatrix} = \begin{pmatrix} -d_3 & -d_5 & -d_7 \\ -d_4 & -d_6 & -d_8 \\ -d_1 & -d_2 & -1 \end{pmatrix}\begin{bmatrix} \bar{l}_i(t) \\ \bar{m}_i(t) \\ \bar{c}_i(t) \end{bmatrix} \quad i = 1, 2, \ldots, 4 \tag{2.10}$$

and

$$\frac{d}{dt}\begin{bmatrix} \bar{k}_i(t) \\ \bar{n}_i(t) \\ \bar{e}_i(t) \end{bmatrix} = \begin{pmatrix} -d_3 & -d_5 & -d_7 \\ -d_4 & -d_6 & -d_8 \\ -d_1 & -d_2 & -1 \end{pmatrix}\begin{bmatrix} \bar{k}_i(t) \\ \bar{n}_i(t) \\ \bar{e}_i(t) \end{bmatrix} \quad i = 1, 2. \tag{2.11}$$

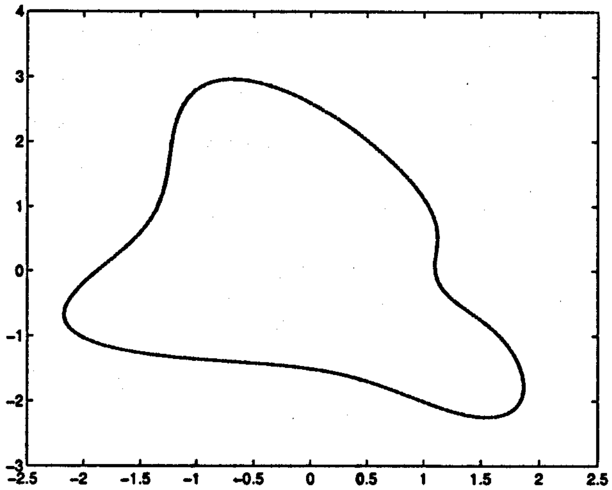

Figure 2: A Quartic IP Curve $f_4(x, y) = 0$

which reduces to the following pair of Riccati equations

$$\frac{d}{dt}\begin{pmatrix} m_i \\ c_i \end{pmatrix} = -\begin{pmatrix} d_4 \\ d_1 \end{pmatrix} + \begin{pmatrix} d_3 - d_6 & -d_8 \\ -d_2 & d_3 - 1 \end{pmatrix}\begin{pmatrix} m_i \\ c_i \end{pmatrix} + \begin{pmatrix} d_5 m_i^2 + d_7 m_i c_i \\ d_5 m_i c_i + d_7 c_i^2 \end{pmatrix} \tag{2.12}$$

and

$$\frac{d}{dt}\begin{pmatrix} n_i \\ e_i \end{pmatrix} = -\begin{pmatrix} d_4 \\ d_1 \end{pmatrix} + \begin{pmatrix} d_3 - d_6 & -d_8 \\ -d_2 & d_3 - 1 \end{pmatrix}\begin{pmatrix} n_i \\ e_i \end{pmatrix} + \begin{pmatrix} d_5 n_i^2 + d_7 n_i e_i \\ d_5 n_i e_i + d_7 e_i^2 \end{pmatrix} \tag{2.13}$$

2.5 Some Examples

Example 2.1. Consider the quartic curve $f_4(x, y) = 0$ depicted in Figure 2 at $t = 0$, as defined by the row vector

[1, −1.727, 3.546, 3.727, 1.727, −2.636, 5.0, −3.174, −2.136, 0, −1.327, −3.936, 10.782, −1.836, −9.827]

Using (2.2) the line decomposition of $f_4(x, y)$ implies four imaginary lines for $\Pi_4(x, y)$, namely

$$L_{41}^T X = x + \underbrace{(-1.285 + 1.934i)}_{m_1(0)}\, y \underbrace{-0.661 + 0.422i}_{c_1(0)},$$

$$L_{42}^T X = x + \underbrace{(-1.285 - 1.934i)}_{m_2(0)}\, y \underbrace{-0.661 - 0.422i}_{c_2(0)},$$

$$L_{43}^T X = x + \underbrace{(0.422 + 0.377i)}_{m_3(0)}\, y \underbrace{-0.657 - 0.053i}_{c_3(0)},$$

$$L_{44}^T X = x + \underbrace{(0.422 - 0.377i)}_{m_4(0)}\, y - \underbrace{0.657 + 0.053i}_{c_4(0)},$$

and two imaginary lines for $\Pi_2(x, y)$, namely

$$L_{21}^T X = x + \underbrace{(-0.5135 + 1.162i)}_{n_1(0)}\, y + \underbrace{(-2.1825 - 0.5139i)}_{e_1(0)},$$

$$L_{22}^T X = x + \underbrace{(-0.5135 - 1.162i)}_{n_2(0)} y + \underbrace{(-2.1825 + 0.5139i)}_{e_2(0)},$$

with scalars

$$\gamma_2 = -2.787, \quad \gamma_0 = -1.405.$$

Thus, $\alpha(0) = \gamma_2 = -2.787$ and $\beta(0) = \gamma_2\gamma_0 = (-2.787)(-1.405) = 3.916$.

Consider a rigid motion defined by

$$A = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One can compute e^{At} and $e^{-A^T t}$ as

$$e^{At} = \begin{pmatrix} \cos 2t & \sin 2t & (1/2)\sin 2t \\ -\sin 2t & \cos 2t & (-1/2)(1-\cos 2t) \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow e^{-A^T t} = \begin{pmatrix} \cos 2t & \sin 2t & 0 \\ -\sin 2t & \cos 2t & 0 \\ (-1/2)\sin 2t & (-1/2)(1-\cos 2t) & 1 \end{pmatrix}$$

Using (2.16)-(2.19) we get the following Riccati equations for the parameters of the lines:

$$\begin{aligned} \dot{m}_i &= -2(1+m_i^2) \quad i = 1,2,3,4 \\ \dot{c}_i &= -1 - 2m_i c_i \quad i = 1,2,3,4 \\ \dot{n}_i &= -2(1+n_i^2) \quad i = 1,2 \\ \dot{e}_i &= -1 - 2n_i e_i \quad i = 1,2 \end{aligned}$$

with initial conditions

$$m_1(0) = -1.285 + 1.934i, m_2(0) = -1.285 - 1.934i, m_3(0) = 0.422 + 0.377i, m_4(0) = 0.422 - 0.377i,$$

$$c_1(0) = -0.661 + 0.422i, c_2(0) = -0.661 - 0.422i, c_3(0) = -0.657 - 0.053i, c_4(0) = 0.657 + 0.053i,$$

$$n_1(0) = -0.5135 + 1.162i, n_2(0) = -0.5135 - 1.162i, e_1(0) = -2.1825 - 0.5139i, e_2(0) = -2.1825 + 0.5139i$$

Note that $\psi_1(t) = \cos 2t$ and $\psi_2(t) = \sin 2t$. Using (2.13) we obtain the following equations which describe $\alpha(t)$ and $\beta(t)$:

$$\alpha(t) = \left(\frac{\prod_{i=1}^{2} \cos 2t + n_i(0)\sin 2t}{\prod_{i=1}^{4} \cos 2t + m_i(0)\sin 2t}\right)(-2.787), \quad \beta(t) = \left(\prod_{i=1}^{4} \frac{1}{\cos 2t + m_i(0)\sin 2t}\right)(3.916)$$

The main message of this subsection is that - rigid or affine dynamics of curves defined by implicit polynomial equations can be represented in terms of Riccati dynamics of possibly complex lines. We illustrate the application of this principal for planar curves moving in $\mathbb{R}^3$ and for the perspective projections of these curves on the image plane of a ccd camera. We believe that these dynamic equations can be used in motion and structure estimation problems, particularly for moving bodies described by polynomial surfaces.

3 Simultaneous Localization and Mapping

This section focuses on two important problems in mobile robotics: the problem of map building and localization. The map building problem studied is to develop the map of an indoor environment with the aid of geometric features viz. line segments, circular arcs and point clusters. These features are typically obtained from a 2D laser range sensor that is capable of taking range measurements in a horizontal plane. Two kinds of localization problems have been studied: the local and the global localization. In the local localization problem, the location of the mobile robot is assumed to be roughly known via *dead reckoning*. In the global localization, the approximate position of the robot is assumed to be unknown and has to be estimated from the surrounding features. As a third important problem discussed in this section, we consider building a 3 dimensional map. Such a map is obtained in the neighborhood of the already obtained 2D map using monocular vision and 2D range information. The map estimation accuracy has also been improved by fusing monocular vision and 2D range in an Extended Kalman Filter scheme. The techniques proposed have been implemented on a Nomad XR4000 mobile robot

3.1 Introduction and Problem Formulation

Simultaneous map building and localization, of the position and orientation of a mobile robotic platform is perhaps a basic problem in mobile robotics. The basic difficulty arises from the fact that the two problems are interdependent and one cannot be solved without the other. For a platform to be able to navigate in a possibly unknown terrain, the importance of a map cannot be overemphasized. A map is a description of the environment based on which the robotic platform is able to autonomously perform *path planning* and *navigation* while avoiding obstacles. Localization, on the other hand is important in order that a team or a platoon of robots perform the above task of navigation with some amount of coordination and cooperation. Thus it is important to ascertain each robot coordinate with respect to a global task space coordinate.

The 2D map building process that we describe in this section is based on a Laser Range Finder that gathers range information along a fixed horizontal plane, called the laser plane, from a moving platform that moves parallel to this plane. The 2D map building problem is to describe the intersection between the laser plane and the objects in the environment, producing a set of intersection lines on a plane (see Fig. 2). A 3D map on the other hand is a description of the structure of the objects in the environment in the vicinity of the 2D map, utilizing added information from the monocular camera on board the mobile platform. An example of a 3D map is shown in Fig. 4, where the dotted line shows the intersection of the laser plane with the objects in the environment.

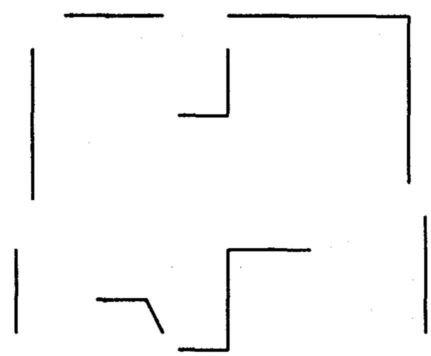

Figure 3: An example of a 2D map

There has been basically two types of 2D maps that are commonly used in the literature. The *occupancy grid map*, introduced by Elfes and Moravec [46], [34] is the most widely used [41], [47], [48], [58], [59]. A horizontal 2D space in the environment is represented as a two dimensional array of cells, each of which has a probability of being occupied. Fig. 5 shows an example of such an occupancy grid map. The white cells indicate free space (probability 0) and the black cells indicate occupied (probability 1). The gray cells indicate different levels of occupancy described by a nonzero probability.

The concept of an Occupancy Grid Map has been originally developed to utilize scarce and imprecise range measurements from SONARS. The advantage is that the cells can represent a wide distribution of measured data which can be easily updated, using Baysian rules, incorporating new measurements. The other advantage is that, without going into the specific details of the structure of the environment, it can describe the free space quite well. The disadvantage of using Occupancy Grid is that it is a poor descriptor of a structure and is therefore not particularly useful in 3D map building. For a somewhat more precise and dense range sensor, like Laser Range Sensor, a large number of small cells would be required to be compatible with the accuracy of the sensor, increasing the computational load.

As an alternative to Occupancy Grid based map building, the *Geometric Feature based map* building procedure has been explored by many researchers [27], [29], [30], [31], [38], [55]. In this procedure one extracts from the raw range data or from the occupancy map, the required geometric features to model the environment. This procedure is particularly suitable while utilizing a Laser Range Finder, since geometric features such as a line segment can be reliably extracted with sufficient accuracy from their dense and sufficiently accurate data. Geometric feature based algorithms are usually much faster since they utilize a lesser number of features.

A typical 2D feature map as extracted from a set of range data is not connected. The gaps between two features arise possibly because – there are some portions of the environment that cannot be reliably modelled as a geometric feature, of the class being looked for such as a line or an arc of a circle. Alternatively there exists gaps because it

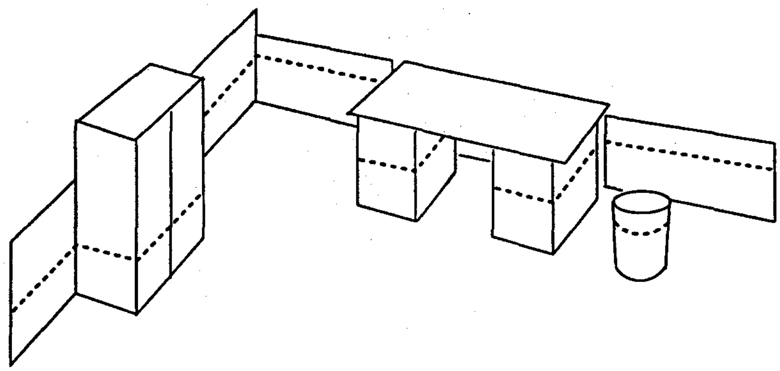

Figure 4: An example of a 3D map model

is an unoccupied free space. For the purpose of navigation, it is important to distinguish between free space and portions of the environment which is otherwise occupied but the structure of which is not reliably clear. The 2D feature map that we propose to build in this section consists of solid segments of lines and circular arcs, virtual line segments indicated by dashed lines and point clusters see Fig. 6. The solid segments indicate portions of the map that has already been built with confidence. The dashed line segments, also called Virtual Line Segments indicate portions of the map that are yet unexplored and represents a jump in the range data (possibly as a result of occlusion or otherwise) as observed by the Laser Range Finder. Virtual Line Segments indicate the boundary between free space and unexplored space. Point clusters, on the other hand, indicates collection of the range data to which no line segment or circular arc can be extracted, and is possibly a boundary between free space and occupied space.

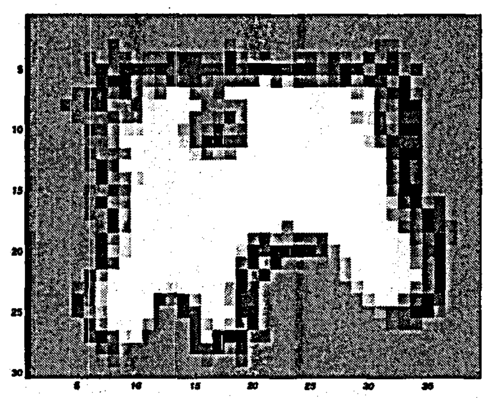

Figure 5: A sample occupancy grid map

The 3D map building problem has not been studied in any details in the literature. All the existing approaches are geometric feature based, because use of occupancy grids is prohibitive in terms of memory requirement, computational cost and usability. Many researchers have used stereo vision to construct 3D maps [26], [57], [64]. A typical procedure is to extract 3D line segments and use them to model the scene. A 3D range finder has also been used by some researchers [28], [53]. A typical approach here is to extract small plane patches from 3D range measurements and use them to form a mesh description for the scene. A 3D Laser Range Finder is expensive and scanning rates are very low (hundred times slower than cameras and 2D Laser Range Finders). So they are not at all common on mobile robots.

The problem of structure estimation, typically referred to as *Structure from X* problem, has been studied for decades in machine vision. These include structure from stereo vision [32], [49], [64]; shading [39], [63]; motion [50], [52], [54]; active vision [24], [45], [56]; and image aspect [33], [40], [44]. In this section we propose to build a 3D map based on a 2D range finder and a single monocular camera. The range finder is used for the purpose of building a 2D map reliably together with localizing the position of the mobile robot. The 2D geometric features on the map are used to hypothesize the 3D structures in the neighborhood of the laser plane and the monocular camera is used to estimate the structure.

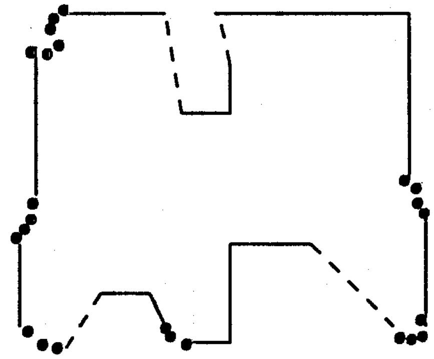

Figure 6: The 2D feature map showing solid segments of lines, virtual line segments and point clusters

3.2 Two Dimensional Feature Map Building

In this section we discuss an incremental process to build a comprehensive 2D geometric feature map. Note that the platform has to move around to observe an entire environment and at any particular location, geometric features are extracted based on range measurements. Subsequently, local maps are built using these geometric features and global maps are updated using the local maps. We begin this section with a discussion of the feature extraction process.

The mobile robot we use is Nomad XR4000 from Nomadic Technologies. It is equipped with a Sick LMS 200 2D laser range finder in the front and obtains a dense and accurate range measurement in a plane. The range finder is equipped with a rotating mirror to scan a large field of view. The range finder is installed on the robot in such a way that the scanning plane is horizontal. A sample laser scan is shown in Fig. 7. Since the scan is dense and relatively accurate, line segments [60], [61] and circular arcs are robustly extractable from the range data. We introduce point clusters as a special geometric feature. A point cluster consists of a number of range points from which no line segments and circular arcs can be extracted and which has a certain maximal size in terms of supremal distance between points.

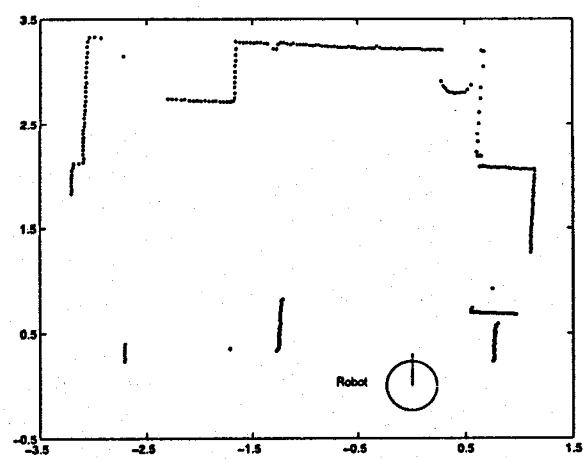

Figure 7: A sample laser scan

We now describe the local map building procedure. A local map is built based on the current range measurements in a coordinate frame attached to the robot. As has been narrated before, one extracts line segments, circular arcs and point clusters from the laser data. A local map also contains *Virtual Line Segments* which we now explain. Due to occlusion, the range data from the laser range finder may have jumps. These jumps do not belong to any features that we have already included in the map so far. Notice that these jumps are located roughly at the boundary between free space and unexplored space and that they can be modelled by straight line segments. We introduce a new geometric feature, called the *Virtual Line Segments* to model these jumps by connecting the two end points of a jump.

The SICK LMS 200 laser range finder has 180 degrees of scanning angle. It can only see the environment in front of the mobile robot. To make the map close, a virtual line segment is created to connect the first and the last range points passing through the center of the laser range finder. This virtual line segment has exactly the same meaning as other virtual line segments, viz. the boundary between free and unexplored space. Fig. 8 shows an example local map. The virtual line segments are drawn as dashed lines in the figure. Since virtual line segments are located at the boundary between free space and unexplored space, they also indicate where the unexplored spaces are, on the current map (see Fig. 8). In this way, these features provide an efficient exploring strategy – always search for the virtual line segments on the partially built global map, to be described later, and explore them until no virtual line segments exist.

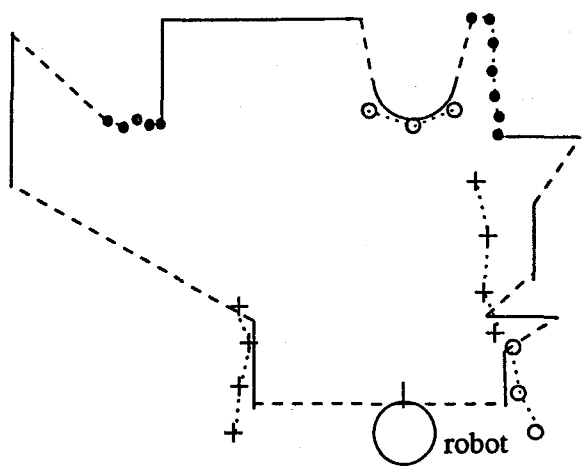

Figure 8: Example of a local map showing the virtual line segments

We now describe the global map building procedure which aggregates the local maps obtained at each new position of the mobile robot. Each of the new positions of the robot are computed using a localization technique described in [61] which provides an accurate estimate of the position together with an estimate of the uncertainty. Once the localization data are available for each positions of the robot, the local map are transformed to a global coordinate.

A global map is updated by taking the union between the existing global map with the computed local map suitably transformed to the global coordinates. Note that both the global and the local maps are closed regions and the union operation results in an enlarged closed region. This is illustrated in Fig. 9. The features of the global map are described by a start and end points using $a, b, c, \cdots$, while the local map uses $a', b', c', \cdots$. To see both map clearly, the local map is shifted up and right. All the matching line segments, circular arcs, and point clusters between the local and the global maps are fused into new ones. In Fig. 9, the local features $a'b'$, $b'c'$, $e'f'$ and $f'g'$ are fused into the global features ab, bc, de and ef respectively. The result of such a feature fusion usually expands the previous set of global features. New features in the local map, like $d'e'$, are added directly into the updated map.

The expansion of the closed region is mainly achieved by modifying the virtual line segments, because they are the only portion of the global map that is expandable. The virtual line segments of one map (either local or global) have a set of three alternative situations with respect to the other map: completely lie inside of the other map, completely lie outside of the other map, or intersect with virtual line segments of the other map (partially inside and partially outside). Only the virtual line segments or portions of them (due to intersection) that are outside of the other map are kept as a feature in the updated global map. In Figure 9, $c'd'$ is kept in the updated map since it is outside of the global map, while cd is discarded because it is inside of the local map. fa and $g'a'$ intersect at point i. Based on the above rule, $g'i$ and ia are kept, while fi and ia' are discarded.

Finally we describe an experimental implementation of the map building procedure. The setup is shown in Fig. 10. The room to be mapped is about $5m \times 6m$ in size. There are several tables and boxes, a chair and a cylindrical trash bin. The mobile robot incrementally builds a complete geometric feature map by moving around in a stop-and-go

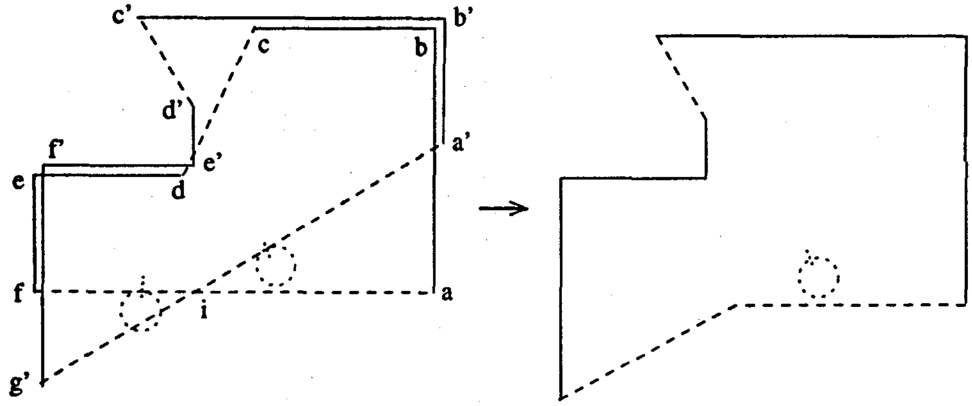

Figure 9: Union operation between local and global maps

fashion.

Figs. 11 and 12 show the process of using local map to update the global map. In Fig. 11, on the left side is the first local map, and thus is treated as the global map. On the right side is the new local map after one movement. The current pose of the mobile robot is computed by localization based on these two maps. In Fig. 12, on the left side, the local map is transformed into the global coordinate and is drawn (shifted for clearer view) together with the previous global map. The updated global map is shown on the right side of Fig. 12.

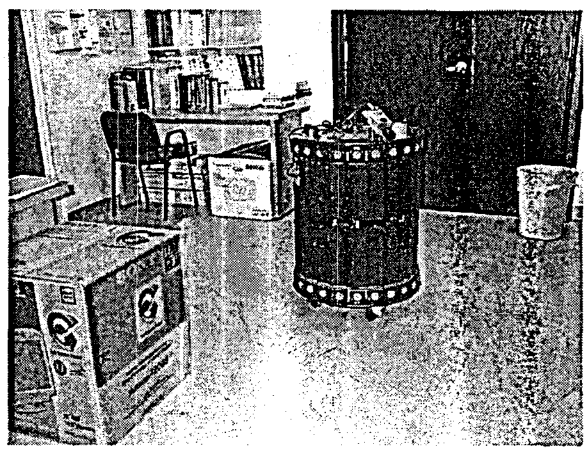

Figure 10: Experimental setup for mapping and localization

In these maps, the dashed lines are the virtual line segments and small circles are the point clusters. The final map is shown in Figure 13. It takes 29 movements to finish and contains 36 line segments, 1 circular arc, 10 point clusters, and 30 virtual line segments that are not accessible.

4 Detection and estimation with visual sensing

In this section, we study homogeneous dynamical systems and essentially introduce the section in two parts. In the first part we introduce a controlled dynamical system and study *controllability* and *observability in the presence of a control.* In the second part, we consider an uncontrolled dynamical system and study problems in *observability* and *realizability*. Connections with the Riccati Flow, the PBH rank condition and the Exponential Interpolation Problem will be established at various parts of the section.

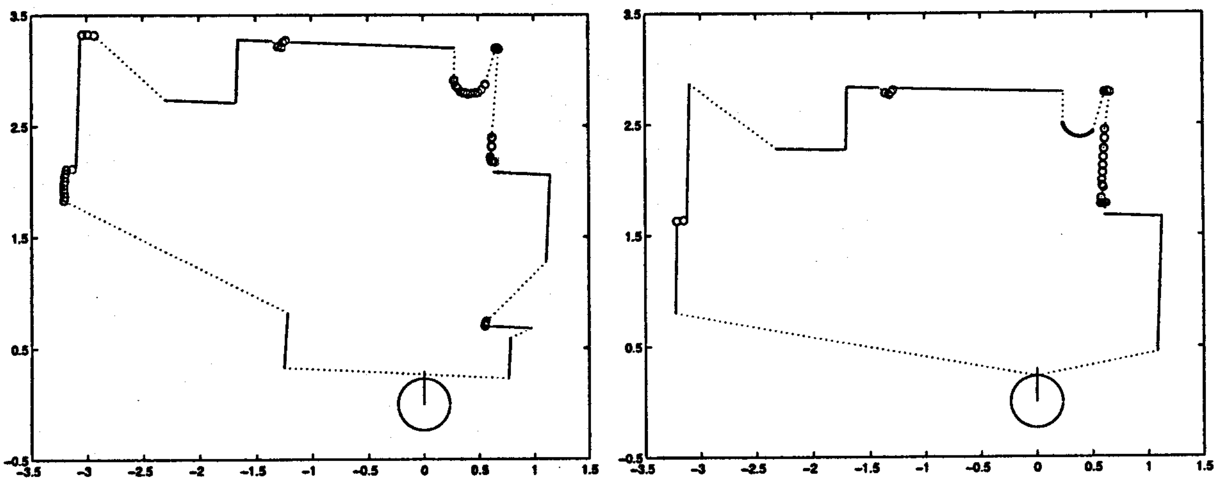

Figure 11: Left: previous global map. Right: current local map

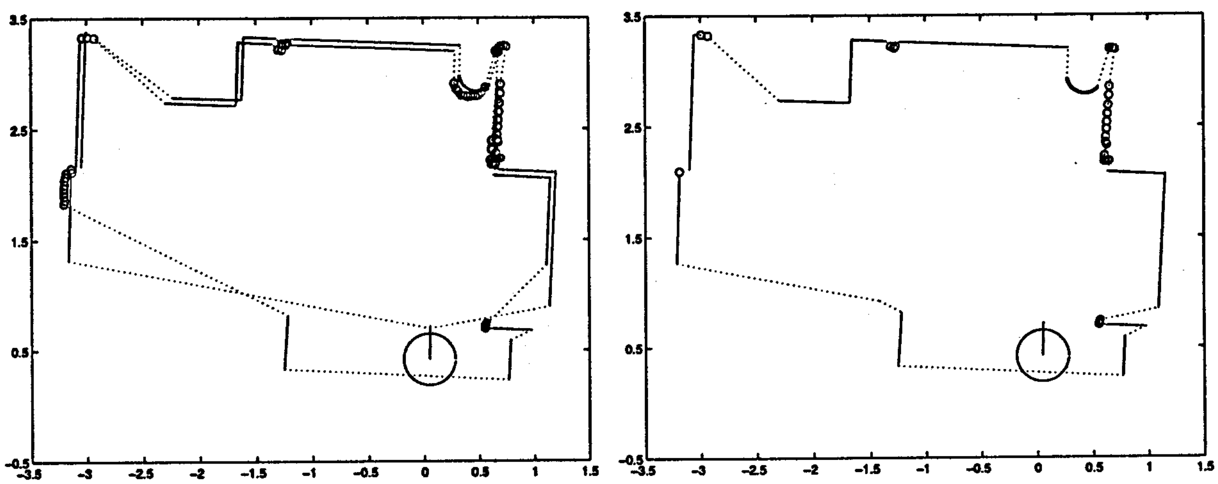

Figure 12: Left: local map (shifted) and global map. Right: updated global map

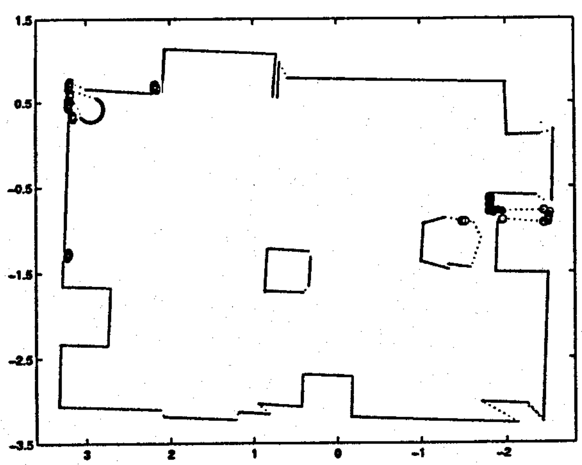

Figure 13: The final global map

4.1 Introduction to homogeneous dynamical systems

The class of problem we study in this section has to do with the problem of controlling and observing the orientation of a state vector in $\mathbb{R}^n$. Of course, the problem of "orientation-control" has been a subject of study in the non-linear control literature for at least the last two decades [65]. A typical example of such a control problem is "satellite orientation control" by active means such as gas jets, magnetic torquing etc. see [66]. In recent years, an important example of the orientation control problem arises in Biology and is known as the 'gaze-control' problem. The 'gaze-control' problem has close connection with the problem of "eye-movement" (see [67, 68] and [69]), where the problem is to orient the eye so that a target is in the field of view. An example of the dynamical system we study in this section is described as follows

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} u \tag{4.1}$$

$$y = \frac{c_{11}x_1 + c_{12}x_2 + c_{13}x_3}{c_{21}x_1 + c_{22}x_2 + c_{23}x_3} \tag{4.2}$$

where the scalar output $y(t)$ may be considered to be the slope of the line spanned by the vector $(y_1, y_2)^T$ and where

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.3}$$

For a dynamical system of the form (4.1), (4.2) one is interested in controlling only the direction of the state vector $(x_1, x_2, x_3)^T$ and such problems are therefore of interest in gaze control. To generalize the control problem, we consider a dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= [Cx] \end{aligned} \tag{4.4}$$

in continuous time where we assume that $x \in \mathbb{R}^n$, $n > 1$, $u \in \mathbb{R}^m$ and $Y \in \mathbb{RP}^{p-1}$, $p > 1$. The observation function $y(t)$ is projective valued and is defined as

$$\begin{aligned} y: \ & R^n - S \rightarrow RP^{p-1} \\ x \ & \longmapsto [Cx] \end{aligned} \tag{4.5}$$

where $[Cx]$ is the homogeneous line spanned by the non-zero vector $Cx \in \mathbb{R}^p$. The set S is defined as

$$S = \{x : Cx = 0\}.$$

Note that $\mathbb{RP}^{p-1}$ denotes the $p-1$ dimensional projective space, the space of all homogeneous lines in $\mathbb{R}^p$. The pair (4.4), (4.5) is a linear dynamical system with a homogeneous observation function and has been introduced in [70]-[73] as an example of a perspective dynamical system. The following two problems would initiate two of the important questions discussed in this section.

Problem 4.1. *(Perspective Control Problem) Let $[\xi_1]$ and $[\xi_2]$ be two distinct elements of RP^{n-1}, does there exist a $T > 0$ and $u(t)$, $t \in [0, T]$ such that*

$$[\xi_2] = [e^{AT}\xi_1^* + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau] \tag{4.6}$$

where ξ_1^* is a non-zero vector in R^n such that $[\xi_1^*] = [\xi_1]$?

Problem 4.2. *(Perspective Observation Problem) Let* $[\xi_1]$ *and* $[\xi_2]$ *be two distinct elements of* RP^{n-1}, *does there exist* $T > 0$ *and* $u(t)$, $t \in [0, T]$ *such that*

$$[Ce^{At}\xi_1^* + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau] \neq [Ce^{At}\xi_2^* + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau] \tag{4.7}$$

for all $t \in [T_1, T_2]$ *where* $0 \leq T_1 < T_2 \leq T$, *and where* ξ_1^* *and* ξ_2^* *are two non-zero vectors in* R^n *such that* $[\xi_1^*] = [\xi_1]$, $[\xi_2^*] = [\xi_2]$*?*

The two problems 4.1 and 4.2 refer to the extent the state vector $x(t)$ can be controlled upto its direction using a control input $u(t)$ and can be observed upto its direction using a projective valued observation $y(t)$. The proposed class of problems can be motivated from machine vision and robotics where the goal is to guide a robot, possibly a mobile robot, with C.C.D. cameras as sensors. These tasks are typically known as *Visual Servoing* and has been of interest for at least the last two decades (see [74], [75] for an early and more recent reference). The problem 4.2 arises in computer vision as *Observability Problem* [76], [77] from visual motion, where the goal is to estimate the location of a moving object from the image data and may be more, viz. structure and motion parameters as well. Many often the image data is produced by a line [78].

The problem of *visual servoing* typically deals with the problem of controlling the movement of a robot arm or a mobile platform guided only by a visual sensor, such as a C.C.D. camera. In the last decade, this problem has been studied in sufficient details and the associated literature is large. A major difference in the servoing scheme has been *passive servoing* wherein the camera is assumed to be held permanently fixed to the ceiling and *active servoing* [79], [80], where the manipulator moves with the camera. The problem of active camera manipulation is of particular importance in mobile and walking robots, see [81], [82].

The *observability problem* or perhaps the *identifiability problem*, if the parameters are changing in time, deals with the problem of ascertaining the location of a target at the very least and subsequently estimating the structure and motion parameters, if possible (see [73], [83], [84] for some new references on this problem). What makes the *perspective observation problem* interesting is that typically a single camera is unable to observe the precise position of a target exactly. Thus it becomes imperative to observe the targets upto their directions with the hope that eventually multiple cameras can precisely locate the position. This point of view is in sharp contrast with stereo based algorithms, wherein multiple cameras are also used, but one requires feature correspondence between various cameras.

In the last ten years, there has been many excellent books, tutorials and surveys written on the topic of *vision based control and observation* (see [85]-[91]). To summarize the main content of this section, we analyze problem 4.1 and show that the dynamical system (4.1) is *perspectively controllable* in between two states iff the two states are in the same orbit of a Riccati flow. We also analyze 4.2 and show that the *perspective observability* of the dynamical system (4.1), (4.2) can be ascertained via a suitable generalization of the *Popov-Belevitch-Hautus* rank test, (see [92]). Such rank tests have already been derived in [93] and [94] assuming that the control input $u(t)$ is not present. We introduce *perspective realizability* problems and establish connection between these realization problems and the well known *exponential interpolation* problems [96].

4.2 Perspective control problem and the Riccati flow

We start this section with the following definition of *perspective controllability*.

Definition 4.3. *We shall say that the dynamical system (4.4) is perspective controllable if problem 4.1 has an affirmative answer for almost every pair of points* $[\xi_1^*]$, $[\xi_2^*] \in RP^{n-1} \times RP^{n-1}$ *in the usual product topology.*

Let us define

$$H_i = column\ span\{\xi_i^*, B, AB, \ldots, A^{n-1}B\} \tag{4.1}$$

for $i = 1, 2$. The following theorem characterizes the main result in perspective control.

bf Theorem The dynamical system (4.4) is perspective controllable iff for almost every pair of points $[\xi_1^*], [\xi_2^*]$ in RP^{n-1}, the subspaces H_1 and H_2 have the same dimension and there exists a real number $T > 0$ such that

$$e^{AT} H_1 = H_2. \tag{4.2}$$

Proof of Theorem 4.2:
Initializing the dynamical system (4.4) at $x(0) = x_0$ we have

$$e^{-At}x(t) - x_0 = \int_0^t e^{-A\tau}Bu(\tau)d\tau. \tag{4.3}$$

It is well known (see Brockett [95]) that the set of all vectors in the right hand side of (4.3) for various choice of $u(\tau)$, is given by all vectors in the subspace

$$H = span\{B, AB, \ldots, A^{n-1}B\}. \tag{4.4}$$

Assume that (4.4) is perspective controllable, it follows that for almost every pair $[\xi_1^*], [\xi_2^*]$

$$\alpha e^{-AT}\xi_2^* - \beta\xi_1^* = \int_0^T e^{-A\tau}Bu(\tau)d\tau \tag{4.5}$$

for some scalars $\alpha \neq 0$, $\beta \neq 0$ and $T > 0$ and for some $u(\tau)$, $\tau \in [0, T]$. Thus we have

$$e^{-AT}\xi_2^* \subset span\{\xi_1^*, B, AB, \ldots, A^{n-1}B\} \tag{4.6}$$

from which we infer that $e^{-AT}H_2 \subset H_1$. By interchanging the role of ξ_1^* and ξ_2^* we can deduce likewise that $H_1 \subset e^{-AT}H_2$. Thus the condition (4.2) is satisfied.

Conversely, because H_1 and H_2 have the same dimension it follows that either ξ_1 and ξ_2 both belong to H or they both do not. In the former case it is trivial to find α, β and $T > 0$ such that (4.5) is satisfied. In the latter case it follows from (4.2) that there exists a $T > 0$ such that $e^{-AT}\xi_2^* \subset H_1$, i.e.

$$e^{-AT}\xi_2^* = \gamma\xi_1^* + v \tag{4.7}$$

where $v \in H$. Note also that $\gamma \neq 0$ for otherwise it would follow that $\xi_2^* \in H$ violating the assumption that it is not. It now follows easily from (4.7) that one can satisfy (4.5) by choosing $\alpha = 1$, $\beta = \gamma$ and an appropriate choice of $u(\tau)$. (Q.E.D.)

Note that theorem 4.2 may be viewed as a criterion for checking perspective controllability of a homogeneous system (4.4), in between the two directions $[\xi_1]$ and $[\xi_2]$. The following is an important corollary for perspective control.

Corollary 4.4. *If $n \geq 3$, the dynamical system (4.4) is perspective controllable iff*

$$\dim H \geq n - 1 \tag{4.8}$$

Proof of Corollary 4.4:
Assume (4.8), it follows that for all ξ_1^*, ξ_2^* that are not in H, we have $H_1 = H_2$ and hence (4.2) is trivially satisfied. Thus from theorem 4.2 it follows that the dynamical system is perspective controllable for every pair of vectors ξ_1^*, ξ_2^* not in H. These vectors would give rise to a generic pair of points in RP^{n-1}, hence the dynamical system (4.4) is perspective-controllable.

Conversely, assume that $\dim H = n - 2$. For $n \geq 3$ there exists two vectors ξ_1^*, ξ_2^* not in H and are such that

$$e^{At}\xi_1^* \notin H_2 \tag{4.9}$$

for all $t \in [0, \infty)$. Hence (4.4) is not perspective controllable in between ξ_1^*, and ξ_2^*. Moreover there exists an open neighborhood of $[\xi_1^*]$ and $[\xi_2^*]$, such that (4.9) is satisfied. Hence a generic pair is not perspective controllable. (Q.E.D.)

Remark 4.5. *For $n = 2$, the above corollary 4.4 is not true. If $\dim H = 0$ it would follow that for two non-zero vectors ξ_1^* and ξ_2^* in R^2, we would have $\dim H_i = 1$ for $i = 1,2$. Perspective controllability would depend upon, whether or not (4.2) is satisfiable for some $T \geq 0$. For certain choice of A and B (4.3) is satisfiable for every pair of nonzero vectors. One such choice is when $B = 0$ and when*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Remark 4.6. *In this remark we point out that the perspective controllability problem has connection with a flow on an associated Grassmannian which is described by a Riccati equation. Let us assume that*

$$\dim H_1 = \dim H_2 = n_1 \tag{4.10}$$

and let us represent by Grass (n_1, n), the space of all n_1 dimensional homogeneous planes in R^n. Clearly the equation

$$\dot{x} = Ax \tag{4.11}$$

defines a flow in Grass (n_1, n) described by

$$\begin{aligned} \chi : \quad & Grass(n_1, n) \times R \rightarrow Grass(n_1, n) \\ & (H, T) \longmapsto e^{AT} H. \end{aligned} \tag{4.12}$$

The flow (4.11) can also be described via a Riccati Equation as follows. Let us denote

$$H = span \ \ of \ \begin{bmatrix} \Theta_0 \\ \Theta_1 \end{bmatrix}. \tag{4.13}$$

Where Θ_0 is a $n_1 \times n_1$ nonsingular matrix and where Θ_1 is a $(n - n_1) \times n_1$ matrix. Writing

$$X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{4.14}$$

Where X is a $n \times n_1$ matrix, and defining $W = X_1 X_0^{-1}$, we have the equation

$$\dot{W} = A_{21} + A_{22}W - WA_{11} - WA_{12}W, \tag{4.15}$$

$$W(0) = \Theta_1 \Theta_0^{-1}.$$

It is trivial to verify that the Riccati Flow (4.15) is an equivalent representation of the homogeneous flow (4.12) upto time t when $X_0(t)$ is singular.

For many other properties of the phase portrait of a Riccati Equation we refer to Shayman [97]. We now have the following restatement of theorem 4.2 which we state without proof.

Theorem The dynamical system (4.4) is perspective controllable in between the two directions $[\xi_1]$ and $[\xi_2]$ in RP^{n-1} iff the subspaces H_1 and H_2 have the same dimension, assume $= n_1$ are in the same orbit of the homogeneous flow (4.12)

4.3 Perspective observability in the presence of a control

In this section and in the next, we would consider the *perspective observation problem* 4.2. The essential question is that if the state $x(t)$ of the dynamical system (4.4), (4.5) is not observable from the output function $y(t)$ in (4.5), with $u(t) = 0$, can it be made observable by a proper choice of a control signal $u(t)$. It is perhaps clear that if we have two nonzero vectors $\xi_1^*, \xi_2^* \in R^n$ such that $Ce^{A\sigma}\xi_1^* = Ce^{A\sigma}\xi_2^*$ for all $\sigma \geq 0$, (4.7) can never be satisfied for any control input $u(t)$. In order to state the main result we assume that these two nonzero vectors satisfy

$$Ce^{A\sigma}\xi_1^* \neq Ce^{A\sigma}\xi_2^* \tag{4.1}$$

for $\sigma \in [T_1, T_2]$, where $T_2 > T_1 \geq 0$ and prove the following lemma.

Lemma 4.7. *Consider the dynamical system (4.4), (4.5) and the pair of vectors ξ_1^* and ξ_2^* satisfying (4.1). The pair of points $[\xi_1^*]$ and $[\xi_2^*]$ in RP^{n-1} are perspectively unobservable iff*

$$\dim\ Ce^{A\sigma}(H_1 + H_2) \leq 1 \tag{4.2}$$

for every σ in R.

Proof of Lemma 4.7:

Assume that the pair of points $[\xi_1^*]$ and $[\xi_2^*]$ are perspectively unobservable. It follows that for every $\sigma \in R$, every $\bar{\xi}_1, \bar{\xi}_2 \in R^n : [\xi_1^*] = [\bar{\xi}_1], [\xi_2^*] = [\bar{\xi}_2]$, and for every control signal $u(t)$ we have

$$\begin{aligned} Ce^{A\sigma}\bar{\xi}_1 \quad &+ \int_0^\sigma Ce^{A(\sigma-\tau)}Bu(\tau)d\tau = \\ &\alpha\left(Ce^{A\sigma}\bar{\xi}_2 + \int_0^\sigma Ce^{A(\sigma-\tau)}Bu(\tau)d\tau\right) \end{aligned} \tag{4.3}$$

for some non-zero scalar α, which may be dependent on σ. It follows, by choosing $u(t) = 0$ in particular, that for every fixed σ, there must exist a homogeneous line ℓ, possibly dependent on σ, in R^n such that

$$Ce^{A\sigma}\bar{\xi}_1 \in \ell, \quad Ce^{A\sigma}\bar{\xi}_2 \in \ell. \tag{4.4}$$

Additionally it follows that

$$Ce^{A\sigma}\int_0^\sigma e^{-A\tau}Bu(\tau)d\tau \tag{4.5}$$

is a subset of ℓ for otherwise we can always choose $\bar{\xi}_1 = \xi_1^*$, $\bar{\xi}_2 = \xi_2^*$ and can conclude that under the restriction (4.1), there would always exist a control $u(t)$ such that (4.2) is not satisfied for σ in some interval $[T_1, T_2]$. Thus we infer that

$$Ce^{A\sigma}H \subset \ell. \tag{4.6}$$

Combining (4.4) and (4.6) we have

$$Ce^{A\sigma}(H_1 + H_2) \subset \ell \tag{4.7}$$

which implies (4.2).

Conversely, assume that (4.2) is satisfied for every σ in R. It follows that there exists a homogeneous line ℓ, possibly dependent on σ such that (4.7) is satisfied and that for every $\bar{\xi}_1$, $\bar{\xi}_2$ such that $[\bar{\xi}_1] = [\xi_1^*]$ and $[\bar{\xi}_2] = [\xi_2^*]$ we have (4.4) and (4.6) satisfied for every σ. We conclude that the vectors

$$Ce^{A\sigma}\bar{\xi}_1 + \int_0^\sigma Ce^{A(\sigma-\tau)}Bu(\tau)d\tau$$

and

$$Ce^{A\sigma}\bar{\xi}_2 + \int_0^\sigma Ce^{A(\sigma-\tau)}Bu(\tau)d\tau$$

in R^p are linearly dependent for any choice of control input $u(t)$ and for every σ. Thus $[\xi_1^*]$ and $[\xi_2^*]$ are perspectively unobservable.

(Q.E.D.)

Before we state the main result on perspectively observability, we consider the following definition.

Definition 4.8. *We shall say that the dynamical system (4.4) is perspectively observable if problem 4.2 has an affirmative answer for every pair of distinct points $[\xi_1], [\xi_2] \in RP^{n-1} \times RP^{n-1}$.*

We are now in a position to state and prove one of the main theorems of this section which generalizes an earlier result reported in [93] and [94] wherein the control input was not present. The next theorem is about a PBH rank condition to test perspective observability of a dynamical system in the presence of a control.

Theorem Assume that the matrix pair (C,A) is an observable pair i.e.

$$rank(C^T,(CA)^T,(CA^2)^T,...,(CA^{n-1})^T)^T = n \tag{4.8}$$

The dynamical system (4.4), (4.5) is perspectively unobservable over the base field C iff there exists a pair of eigenvalues λ_0, λ_1 of A, such that for all pairs of complex numbers μ_0, μ_1 (may be the same) in the set $\{\lambda_0, \lambda_1, \delta_1, \ldots, \delta_s\}$ one has

$$rank\begin{pmatrix} (A-\mu_0 I)(A-\mu_1 I) \\ C \end{pmatrix} < n \tag{4.9}$$

where $\delta_1, \ldots, \delta_s$ is the set of eigenvalues of A such that the subspace spanned by the corresponding eigenvectors or generalized eigenvectors of A is H.

Remark 4.9. *Note that the existence of $\delta_1, \ldots, \delta_s$ follows from the fact that H is A-invariant and the property in question is true for any such subspace. Note also that if the above observability rank condition (4.8) is not satisfied, then the system (4.4) is perspectively unobservable.*

Proof of Theorem 4.3:

Before we sketch the formal proof, we note the following. If we assume that (4.9) is satisfied it follows that there exist a non-zero vector $v \in C^n$ such that

$$\begin{aligned} (A-\mu_0 I)(A-\mu_1 I)v = 0 \\ Cv = 0. \end{aligned} \tag{4.10}$$

Let S be the eigenspace spanned by the eigenvectors or generalized eigenvectors u_0, u_1 of A corresponding to eigenvalues μ_0, μ_1 it follows that v is an element of S and can be written as

$$v = \alpha_0 u_0 + \alpha_1 u_1 \tag{4.11}$$

for some scalars α_0 and α_1. Finally we have $Cv = 0 \Rightarrow \alpha_0 Cu_0 + \alpha_1 Cu_1 = 0$. It follows that

$$Ce^{A\sigma}v \;=\; \alpha_0 e^{\mu_0\sigma}Cu_0 + \alpha_1 e^{\mu_1\sigma}Cu_1 = (e^{\mu_0\sigma} - e^{\mu_1\sigma})\alpha_0 Cu_0.$$

Hence we conclude that

$$\dim Ce^{A\sigma}(S) \leq 1. \tag{4.12}$$

Sufficiency: Assume that (4.9) is satisfied for a pair of eigenvalues λ_0, λ_1 of A. It follows from (4.12) that if S is the subspace spanned by the eigenvectors or generalized eigenvectors of A corresponding to eigenvalues $\lambda_0, \lambda_1, \delta_1, \ldots, \delta_s$, we have

$$\dim Ce^{A\sigma}(S) \leq 1. \tag{4.13}$$

Let ξ_1^* and ξ_2^* be two linearly independent vectors in S, it follows from (4.13) that (4.2) would be satisfied. Thus the pair $[\xi_1^*], [\xi_2^*]$ cannot be observed.

Necessity: Assume that (4.4), (4.5) is perspectively unobservable. It follows that there exist two independent vectors ξ_1^*, ξ_2^* such that $[\xi_1^*]$ and $[\xi_2^*]$ cannot be observed by (4.4), (4.5). Moreover (4.1) is automatically satisfied by ξ_1^*, ξ_2^* because of the rank assumption (4.8). Using Lemma 4.7, we conclude that (4.2) is satisfied, for every σ in R. It follows that there exists a homogeneous line ℓ, possibly depending on σ such that

$$Ce^{A\sigma}[\xi_i^*] \subset \ell, i = 1,2 \tag{4.14}$$

and

$$Ce^{A\sigma}H \subset \ell. \tag{4.15}$$

In what follows we show that from (4.14) and (4.15) we can infer the following – *There exists a pair of eigenvectors v_r and v_s of A with associated eigenvalues λ_r and λ_s such that for all pairs of complex numbers μ_0, μ_1 in the set*

$$\{\lambda_r, \lambda_s, \delta_1, \ldots, \delta_s\} \tag{4.16}$$

the rank condition (4.9) is satisfied, which would complete the proof.

Let $v_1, v_2, \ldots, v_s$ be a set of eigenvectors or generalized eigenvectors of A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ such that

$$\begin{aligned} \xi_1^* &= \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_s v_s \\ \xi_2^* &= \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_r v_r \end{aligned} \tag{4.17}$$

where $\alpha_s \neq 0$ and $\beta_r \neq 0$ and where we assume that $r < s$ without any loss of generality (if not, replace ξ_2^* by $\xi_2^* + a\xi_1^*$ for some choice of a. One can order the eigenvalues in such a way that

$$\lambda_i + \lambda_j = \lambda_r + \lambda_s, \iff \lambda_i = \lambda_r, \lambda_j = \lambda_s,$$

and when $\lambda_i + \lambda_j = \lambda_r + \lambda_s$, we have

$$p_i + p_j = p_r + p_s, \iff p_i = p_r, p_j = p_s,$$

where p_i is the multiplicity of the eigenvalue λ_i. For details on the existence of this ordering see [93]. It follows from (4.14) that $Ce^{A\sigma}v_s$ and $Ce^{A\sigma}v_r$ are linearly dependent for all values of σ. In particular since Cv_s and Cv_r are linearly dependent, there would exist a scalar θ such that $v_s + \theta v_r$ would be in the null space of the matrix

$$rank \begin{pmatrix} (A - \lambda_s I)(A - \lambda_r I) \\ C \end{pmatrix}. \tag{4.18}$$

The above argument can actually be repeated for each and every pair of eigenvectors in $H_1 + H_2$ indicating that for every pair of eigenvalues μ_0, μ_1 in the set (4.16), the rank condition (4.9) would be satisfied. (Q.E.D.)

To end this section, it may not be a bad idea to write down the condition for perspective observability explicitly.

Remark 4.10. *Assuming the observability rank condition (4.8) for the matrix pair (C, A), the dynamical system (4.4), (4.5) is perspectively observable, over the base field C iff for every pair of eigenvalues λ_1, λ_2 of A, there exist some pair μ_0, μ_1 in the set $\{\lambda_1, \lambda_2, \delta_1, \ldots, \delta_s\}$ such that*

$$rank \begin{pmatrix} (A - \mu_0 I)(A - \mu_1 I) \\ C \end{pmatrix} = n \tag{4.19}$$

Over the base field R, the rank condition (4.19) is only sufficient.

The following corollary of the Theorem 4.3 is perhaps surprising.

Corollary 4.11. *Assume that the dynamical system (4.4), (4.5) is such that*

$$rank(B, AB, A^2B, \ldots, A^{n-1}B) \geq 2, \tag{4.20}$$

then the same dynamical system is perspectively observable over the base field C iff the observability rank condition *(4.8) is satisfied.*

Proof of Corollary 4.11:

If the observability rank condition is not satisfied, the dynamical system is clearly perspectively unobservable. Conversely, it follows from (4.8), (4.20) that

$$dim\, Ce^{A\sigma}H \geq 2.$$

Thus from Lemma 4.7, it would follow that every pair of vectors would be perspectively observable. (Q.E.D.)

4.4 Perspective observability in the absence of a control

As a special case of the perspective observability problem 4.2 considered in the introduction, we now consider the dynamical system (4.4) under the assumption that $B = 0$, i.e. there is no influence of the control. We continue to assume that the observation function $y(t)$ is projective valued and is given by (4.5). The perspective observation problem is described as follows:

Problem 4.12. *Let ξ_1^* and ξ_2^* be two linearly independent vectors in R^n, does there exist $T_2 > T_1 \geq 0$ such that*

$$[Ce^{At}\xi_1^*] \neq [Ce^{At}\xi_2^*]$$

for all $t \in [T_1, T_2]$.

The above problem 4.12 already has a satisfactory answer over C^n and has been studied in [93] and [94]. It has been shown that a necessary and sufficient condition for perspective observability over C^n is that the rank of the matrix

$$rank \begin{pmatrix} (A-\lambda_1 I)(A-\lambda_2 I) \\ C \end{pmatrix} = n \tag{4.1}$$

for all pairs of eigenvalues λ_1, λ_2 (may be the same) of the matrix A. Over R^n the rank condition (4.1) is only sufficient. In order to state the main observability result of this section, we introduce the following *hat system* that has already been considered in [93]. Define the vector space $R^n \wedge R^n$, [99] and consider the linear map

$$\hat{A} : R^n \wedge R^n \rightarrow R^n \wedge R^n$$

given by

$$x \wedge y \longmapsto Ax \wedge y + x \wedge Ay.$$

We also consider the linear map

$$\hat{C} : R^n \wedge R^n \rightarrow R^m \wedge R^m$$

given by

$$x \wedge y \longmapsto Cx \wedge Cy.$$

We now define the promised hat system as follows.

$$\dot{\hat{x}} = \hat{A}\hat{x}, \hat{z} = \hat{C}\hat{x}, \tag{4.2}$$

where $\hat{x} \in R^n \wedge R^n$ and $\hat{z} \in R^m \wedge R^m$. The main result of this section is summarized in the following theorem.

Theorem 4.13. *The following three conditions are equivalent over R.*

1. *For $B = 0$, the dynamical system (4.4) is perspectively unobservable.*
2. *There exists a real number $\hat{\lambda}$ and a real decomposable vector $\theta_1 \wedge \theta_2$, $\theta_1, \theta_2 \in R^n$ such that*

$$\begin{pmatrix} \hat{A} - \hat{\lambda} I \\ \hat{C} \end{pmatrix} \theta_1 \wedge \theta_2 = 0 \tag{4.3}$$

3. *There exist two numbers λ_1, λ_2 which are either both real (may be the same) or complex conjugates of each other such that*

$$rank \begin{pmatrix} (A-\lambda_1 I)(A-\lambda_2 I) \\ C \end{pmatrix} < n \tag{4.4}$$

Remark 4.14. *The first two equivalent conditions of the theorem 4.13 basically says that the perspective unobservability of (4.4), assuming $B = 0$, is equivalent to the regular unobservability (in the sense of a linear system) of the hat system (4.3) with the additional requirement that the unobseravbility subspace of the hat system must contain a decomposable vector. Decomposability of a vector in $R^n \wedge R^n$ is hard to check. The third condition of the theorem 4.13 provides a computationally feasible solution, which involves checking the rank a matrix for every real or complex conjugate pairs of eigenvalues of the matrix A.*

Proof of Theorem 4.13 ($1 \iff 2$):

(Sufficiency) Assume that there exists a vector $x \wedge y$ such that (4.3) is satisfied. It follows that

$$\begin{aligned} \hat{A}(x \wedge y) &= \hat{\lambda}(x \wedge y) \\ \hat{C}(x \wedge y) &= 0. \end{aligned} \tag{4.5}$$

From (4.5) we conclude the following

$$\begin{aligned} Ce^{At}x \wedge Ce^{At}y &= \hat{C}(e^{At}x \wedge e^{At}y) \\ &= \hat{C}e^{\hat{A}t}(x \wedge y) \\ &= \hat{C}e^{\hat{\lambda}t}(x \wedge y) \\ &= e^{\hat{\lambda}t}\hat{C}(x \wedge y) \\ &= 0 \end{aligned} \tag{4.6}$$

Thus as a vector in R^m, $Ce^{At}x$ and $Ce^{At}y$ are linearly independent implying that (4.4) is perspectively unobservable for $B = 0$.

(Necessity) Assume that (4.4) is perspectively unobservable for $B = 0$. It follows that there exists two vectors $\theta_1, \theta_2 \in R^n$ such that $\theta_1 \wedge \theta_2 \neq 0$ and

$$\begin{aligned} & Ce^{At}\theta_1 \wedge Ce^{At}\theta_2 &= 0 \\ \iff\ & \hat{C}e^{\hat{A}t}(\theta_1 \wedge \theta_2) &= 0. \end{aligned} \tag{4.7}$$

In order to show (4.4) we need to show that there is a real decomposable eigenvector of $\hat{A}$ in the kernel of $\hat{C}$.

Define two complex conjugate vectors $\bar{\theta}_1$ and $\bar{\theta}_2$ as follows:

$$\begin{aligned} \bar{\theta}_1 &= \theta_1 + i\theta_2 \\ \bar{\theta}_2 &= \theta_1 - i\theta_2. \end{aligned} \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\hat{C}e^{\hat{A}t}(\bar{\theta}_1 \wedge \bar{\theta}_2) = 0. \tag{4.9}$$

The above relation (4.9) follows easily from the fact that

$$\bar{\theta}_1 \wedge \bar{\theta}_2 = 2i\, \theta_2 \wedge \theta_1.$$

We now proceed by expanding the vectors $\bar{\theta}_1$ and $\bar{\theta}_2$ in terms of the eigenvectors (generalized eigenvectors) v_i of A as has already been done (4.17) in the previous section. Because of (4.8), the integers r and s can be assumed to be equal and the set of eigenvectors (generalized eigenvectors)

$$\{v_1, v_2, \ldots, v_r\} \tag{4.10}$$

can be assumed to be a self conjugate set. Expanding $\bar{\theta}_1 \wedge \bar{\theta}_2$ in terms of the eigenvectors (generalized eigenvectors) $v_i \wedge v_j$ of $\hat{A}$ for $i, j = 1, \ldots, r, i \neq j$ we write

$$\bar{\theta}_1 \wedge \bar{\theta}_2 = \sum_{i,j=1, i\neq j}^{r} \alpha_{ij} v_i \wedge v_j,$$

where we can assume that $\alpha_{12} \neq 0$ and either v_1, v_2 are both real or are complex conjugates of each other. We can also assume that the vectors in the set (4.10) have been ordered so that we claim that

$$\lambda_i + \lambda_j = \lambda_1 + \lambda_2, \iff \lambda_i = \lambda_1, \lambda_j = \lambda_2,$$

and when $\lambda_i + \lambda_j = \lambda_1 + \lambda_2$, we have

$$p_i + p_j = p_1 + p_2, \iff p_i = p_1, p_j = p_2,$$

where p_i is the multiplicity of the eigenvalue λ_i corresponding to v_i. Such an ordering is always possible and has been detailed in [93]. It now follows from (4.9) that

$$\hat{C}e^{\hat{A}t}(v_1 \wedge v_2) = 0, \tag{4.11}$$

where $v_1 \wedge v_2$ is an eigenvector of $\hat{A}$ corresponding to the eigenvalue $\lambda_1 + \lambda_2$.

If v_1 and v_2 are both real, then the proof is over. If they are complex conjugates of each other, we can write

$$\begin{array}{rcl} v_1 & = & \eta_1 + i\eta_2 \\ v_2 & = & \eta_1 - i\eta_2 \end{array}, \eta_1, \eta_2 \in R^n. \tag{4.12}$$

It is easy to check that $v_1 \wedge v_2 = 2i\eta_2 \wedge \eta_1$. It follows from (4.11) that

$$\begin{pmatrix} \hat{A} - (\lambda_1 + \lambda_2)I \\ \hat{C} \end{pmatrix} v_1 \wedge v_2 = 0. \tag{4.13}$$

Thus we have

$$\begin{pmatrix} \hat{A} - (\lambda_1 + \lambda_2)I \\ \hat{C} \end{pmatrix} \eta_1 \wedge \eta_2 = 0. \tag{4.14}$$

Since $\lambda_1 + \lambda_2$ is always real, this completes the proof.

Proof of Theorem 4.13 (2 $\iff$ 3):

(Sufficiency) When λ_1 and λ_2 are both real, there is nothing else to prove. If v is a vector such that $(A - \lambda_1 I)(A - \lambda_2 I)v = 0$ and $Cv = 0$, we define $\theta_1 = v$ and $\theta_2 = (A - \lambda_2 I)v$. It is easy to see that $(\hat{A} - (\lambda_1 + \lambda_2)I)(\theta_1 \wedge \theta_2) = 0$ and $\hat{C}(\theta_1 \wedge \theta_2) = 0$ which implies (4.4).

When $\lambda_1 and \lambda_2$ are complex conjugates of each other we write

$$\begin{array}{l} \lambda_1 = \lambda_1^* + i\lambda_2^* \\ \lambda_2 = \lambda_1^* - i\lambda_2^*. \end{array} \tag{4.15}$$

Let v be as before, we define

$$\begin{array}{l} x = (A - \lambda_1 I)v \\ x = (A - \bar{\lambda}_1 I)v \end{array} \tag{4.16}$$

It is easy to verify that

$$\begin{array}{l} (\hat{A} - (\lambda_1 + \lambda_2)I)(x \wedge y) \\ = Ax \wedge y + x \wedge Ay - \lambda_1 x \wedge y - \lambda_2 x \wedge y \\ = ((A - \lambda_2 I)x) \wedge y + x \wedge ((A - \lambda_1 I)y \\ = 0 \wedge y + x \wedge 0 = 0. \end{array} \tag{4.17}$$

Moreover

$$\begin{array}{l} \hat{C}(x \wedge y) \\ = Cx \wedge Cy \\ = (CAv - \lambda_1 Cv) \wedge (CAv - \bar{\lambda}_1 Cv) \\ = CAv \wedge CAv \ since \ Cv = 0 \\ = 0. \end{array} \tag{4.18}$$

It is an easy calculation to show that

$$\frac{i}{2\lambda_2^*} x \wedge y = v \wedge Av.$$

It would therefore follow that

$$(\hat{A} - (\lambda_1 + \lambda_2)I)(v \wedge Av) = 0$$

and

$$\hat{C}(v \wedge Av) = 0.$$

We define $\theta_1 = v$ and $\theta_2 = Av$ and the proof is complete.

(Necessity) By assumption, we have $\theta_1, \theta_2 \in R^n$ such that (4.3) is satisfied. By decomposing the vector $\theta_1 \wedge \theta_2$ in terms of the eigenvectors (generalized eigenvectors) of the matrix $\hat{A}$, we can repeat the necessity proof of the first part of this theorem to show that there exist two eigenvectors (or generalized eigenvectors) v_1 and v_2 of A such that $\hat{C}e^{\hat{A}t}(v_1 \wedge v_2) = 0$ for all $t \geq 0$. In particular we have $Cv_1 \wedge Cv_2 = 0$. Define a scalar β such that

$$C(v_1 + \beta v_2) = 0.$$

For $v = v_1 + \beta v_2$ it would follow that $Cv = 0$ and $(A - \lambda_1 I)(A - \lambda_2 I)v = 0$ completing the proof. (Q.E.D.)

4.5 Realizability via the rational exponential interpolation

So far in the previous sections we have considered the problem of perspective control and observation assuming that the parameters of the homogeneous dynamical system (4.4) are known. If on the other hand, the parameters of the system are unknown, it is important to be able to identify the parameters from a record of the observation over a certain interval of time. In particular, it is important to realize homogeneous dynamical systems with minimum state dimension, if possible, and ascertain if the choice of the parameters are unique. If not, it is important to be able to classify the extent of the non-uniqueness. The interpolation problem considered in this section is a step in that direction.

Roughly speaking, the problem we consider is described as follows. Assume $B = 0$ and that the output of a dynamical system (4.4) has been observed over a finite interval of time. The problem is to identify the parameters of the system from the observed output. We shall see that the parameter identification problem is connected with a certain class of interpolation problem. In particular, for a linear dynamical system, one considers an *exponential interpolation problem.* On the other hand, for a homogeneous dynamical system, of the kind described in (4.4) one obtains a *rational exponential interpolation problem.*

Before we describe the "rational exponential interpolation" problem, let us consider the exponential interpolation problem already described in [96]. We consider a linear autonomous system

$$\dot{x} = Ax, \; y = Cx, \; x(0) = x_0 \tag{4.1}$$

solution of which can be expressed as follows, assuming distinct eigenvalues of A,

$$y(t) = Ce^{At}x_0 = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t}. \tag{4.2}$$

Let us assume that $y(t)$ is given at discrete data points $y(t)$, $t = t_0 + hk$, $k = 0, 1, 2, \ldots$; t_0, h are given fixed constants, we have

$$y(t_0 + hk) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t_0 + \lambda_i hk}. \tag{4.3}$$

Defining

$$\begin{aligned} \gamma_i &= \alpha_i e^{\lambda_i t_0}, \; \xi_i = e^{\lambda_i h}, y_k = y(t_0 + hk) \text{ we have} \\ y_k &= \sum_{i=1}^{n} \gamma_i \xi_i^k, \;\; k = 0, 1, 2, \ldots \end{aligned} \tag{4.4}$$

Writing this in matrix form we have

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ x_1^3 & x_2^3 & \ldots & x_n^3 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{pmatrix}. \tag{4.5}$$

We propose to break the above equation up into a series of square matrices. We begin by defining

$$V_k = \begin{pmatrix} x_1^k & x_1^{k+1} & \dots & x_1^{k+n-2} & x_1^{k+n-1} \\ x_2^k & x_2^{k+1} & \dots & x_2^{k+n-2} & x_2^{k+n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{n-1}^k & x_{n-1}^{k+1} & \dots & x_{n-1}^{k+n-2} & x_{n-1}^{k+n-1} \\ x_n^k & x_n^{k+1} & \dots & x_n^{k+n-2} & x_n^{k+n-1} \end{pmatrix} \tag{4.6}$$

$$\hat{y_k} = \begin{pmatrix} y_k^T & y_{k+1}^T & \cdots & y_{k+n-1}^T \end{pmatrix}^T \tag{4.7}$$

$$\hat{\alpha} = \begin{pmatrix} \alpha_1^T & \alpha_2^T & \alpha_3^T & \dots & \alpha_n^T \end{pmatrix}^T \tag{4.8}$$

and obtain an infinite set of equations

$$V_k\hat{\alpha} = \hat{y}_k. \tag{4.9}$$

We now make the following two observations. The first is that

$$V_{k+1}V_k^{-1} = V_0DV_0^{-1}$$

where D is a diagonal matrix with $x_1, \dots, x_n$ on the diagonal. The second observation is that

$$V_{k+1}V_k^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ \rho_1 & \rho_2 & \rho_3 & \dots & \rho_n \end{pmatrix} \tag{4.10}$$

We now eliminate $\hat{\alpha}$ from (4.9) by substitution and we have

$$V_{k+1}V_k^{-1}\hat{y_k} = \hat{y}_{k+1} \tag{4.11}$$

and from (4.11) we have

$$\rho_1 y_k + \rho_2 y_{k+1} + \dots + \rho_n y_{k+n-1} = y_{k+n}.$$

Thus we can find the ρ's using Hankel techniques. Finally we have the polynomial

$$x_n - \rho_n x^{n-1} - \dots - \rho_2 x - \rho_1 = \prod_{i=1}^{n}(x - x_i)$$

that determines the exponentials and we then use the exponentials to recover the coefficients. This is a very brief explanation of Prony's method. In the form described here it is obvious that the technique is numerically unstable. However it can be implemented in a more stable fashion [96]. A standard reference for Prony's method is the numerical analysis text by Hildebrand [98].

In order to consider the exponential rational interpolation problem we consider the homogeneous dynamical system (4.4), (4.5) and write

$$y(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} / \sum_{i=1}^{n} \beta_i e^{\lambda_i t}, \tag{4.12}$$

where we assume, as before, that the eigenvalues of A are all distinct. We assume that $y(t)$ is known and that the parameters α_i, β_i, λ_i, $i = 1, \dots, n$ are all unknown. We assume furthermore that we are given equally spaced data points of the form $(t_0 + hk, y(t_0 + hk))$ with t_0 and h known values. The objective is to identify the coefficients α_i, β_i

and the exponents λ_i. We make no restrictions on whether or not the coefficients and exponents are real or complex. Substituting the value of $t_0 + hk$ into $y(t)$ we have the following simplification

$$y(t_0 + hk) = \frac{\sum_{i=1}^{n} \alpha_i e^{\lambda_i (t_0 + hk)}}{\sum_{i=1}^{n} \beta_i e^{\lambda_i (t_0 + hk)}} \tag{4.13}$$

$$= \frac{\sum_{i=1}^{n} \left(\alpha_i e^{\lambda_i t_0}\right) \left(e^{\lambda_i h}\right)^k}{\sum_{i=1}^{n} \left(\beta_i e^{\lambda_i t_0}\right) \left(e^{\lambda_i h}\right)^k}. \tag{4.14}$$

Thus we see that the problem reduces to an interpolation problem.

Now we assume that we are given the data (k, τ_k) for $k = 0, 1, 2, \ldots$. We then have the following system of nonlinear equations to solve

$$\sum_{i=1}^{n} \alpha_i e^{\lambda_i hk} = \tau_k \left(\sum_{i=1}^{n} \beta_i e^{\lambda_i hk} \right) \tag{4.15}$$

for $k = 0, 1, 2, \ldots$. (Here we assume that α_i, β_i have been scaled by $e^{\lambda_i t_0}$). We now let $x_i = e^{\lambda_i h}$ and substituting we have

$$\sum_{i=1}^{n} \alpha_i x_i^k = \tau_k \left(\sum_{i=1}^{n} \beta_i x_i^k \right) \tag{4.16}$$

so the problem now becomes a polynomial problem of solving for the $3n$ unknowns α_i, β_i and x_i. Our goal is to develop a Prony like technique for solving this set of equations. Let $\hat{a}$ denote the vector $(\alpha_1, \ldots, \alpha_n)$ and let $\hat{b}$ denote the vector $(\beta_1, \ldots, \beta_n)$. We denote by $\hat{x}^k$ the vector $(x_1^k, \ldots, x_n^k)^T$. We can now rewrite the equations as

$$\hat{a}\hat{x}^k = \tau_k \hat{b}\hat{x}^k \tag{4.17}$$

by grouping n consecutive equations starting with k we can write

$$\hat{a}V_k = \hat{b}V_k D_k \tag{4.18}$$

where

$$D_k = \begin{pmatrix} \tau_k & 0 & \ldots & 0 & 0 \\ 0 & \tau_{k+1} & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & \tau k + n - 1 & 0 \\ 0 & 0 & & 0 & \tau_{k+n} \end{pmatrix} \tag{4.19}$$

We note that we can write

$$V_k = X_k V_0$$

where X_k is a diagonal matrix with $\hat{x}^k$ as its diagonal. We now eliminate the vector $\hat{a}$ by noting that $\hat{a} = \hat{b}V_k D_k V_k^{-1}$ and as in Prony's method we have then that

$$\hat{b}V_k D_k V_k^{-1} = \hat{b}V_{k+1} D_{k+1} V_{k+1}^{-1}$$

Now we factor to obtain

$$\hat{b}V_k (D_k V_k^{-1} V_{k+1} - V_k^{-1} V_{k+1} D_{k+1}) V_{k+1}^{-1} = 0$$

We now calculate the product $V_k^{-1}V_{k+1}$. We first note that $V_k^{-1}V_{k+1} = V_0^{-1}V_1$ and we calculate the product to obtain

$$V_0^{-1}V_1 = \begin{pmatrix} 0 & 0 & \ldots & 0 & f_1(\hat{x}) \\ 1 & 0 & & 0 & f_2(\hat{x}) \\ 0 & 1 & \ldots & 0 & f_3(\hat{x}) \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & f_n(\hat{x}) \end{pmatrix}$$

To calculate the functions $f_i(\hat{x})$ we first calculate the explicit inverse of the vandermonde matrix V_0 in terms of the Lagrange interpolation polynomials. Recall that

$$l_i(t) = \frac{(t-x_1)\dots(t-x_{i-1})(t-x_{i+1})\dots(t-x_n)}{(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$

and that

$$V_0^{-1} = \begin{pmatrix} l_1(0) & l_2(0) & \dots & {}_n-1(0) & l_n(0) \\ l_1^{(1)}(0) & l_2^{(1)}(0) & \dots & l_{n-1}^{(1)}(0) & l_n^{(1)}(0) \\ l_1^{(2)}(0) & l_2^{(2)}(0) & \dots & l_{n-1}^{(2)}(0) & l_n^{(2)}(0) \\ \vdots & & & & \vdots \\ l_1^{(n-1)}(0) & l_2^{(n-1)}(0) & \dots & l_{n-1}^{(n-1)}(0) & l_n^{(n-1)}(0) \end{pmatrix}$$

and hence we have

$$f_i(\hat{x}) = \sum_{k=1}^{n} l_k^{(i)}(0) x_k^n$$

We now use the equation

$$\hat{b} V_k (D_k V_k^{-1} V_{k+1} - V_k^{-1} V_{k+1} D_{k+1}) V_{k+1}^{-1} = 0$$

and after some tedious calculation we have that

$$\hat{b} \begin{pmatrix} \sum_{i=1}^{n} (\alpha_{k+i-1} - \alpha_{k+n}) f_i(\hat{x}) x_1^{i-1+k} \\ \sum_{i=1}^{n} (\alpha_{k+i-1} - \alpha_{k+n}) f_i(\hat{x}) x_2^{i-1+k} \\ \vdots \\ \sum_{i=1}^{n} (\alpha_{k+i-1} - \alpha_{k+n}) f_i(\hat{x}) x_n^{i-1+k} \end{pmatrix} = 0$$

for $k = 0, 1, 2, \dots$. This now allows us to form n determinants which must vanish if there is to be a nonzero $\hat{b}$. The algorithm described below is now the same as the classical Prony algorithm.

Exponential Interpolation Algorithm

Step 1 Collect data α_k, $k = 0, 1, 2, \dots, 3n-2$

Step 2 Form from the data the n determinants each of which is $n \times n$

Step 3 Solve the resulting n equations in n unknowns for non zero solutions for the x_i

Step 4 Solve the linear equation for the vector $\hat{b}$

Step 5 Solve the linear equation for the vector $\hat{a}$

Step 6 Solve the equation $x_i = e^{\lambda_i}$ for λ_i

Step 7 Construct the rational function

Using the algorithm descibed above, one is able to identify the parameters of a homogeneous dynamical system, given that the output of the system is known over a certain prescribed interval of time. Note in particular that the algorithm presupposes an apriori knowledge of the state dimension of the dynamical system.

4.6 Summary

To summarize, the important contributions of this section is three fold. First of all, it introduces a new Perspective Control Problem which is important in steering a vector in R^n upto its direction. It is not totally surprising that these problems have a connection with the Riccati Flow. What would be important in this context of gaze control is to introduce dynamical systems that are somewhat more general than (4.4) (see for example [100]) and possibly makes contact with the dynamics of mechanical systems with a hope to be able to visually actuate these systems upto direction. Second of all, we revisit the perspective observability problem [93], [94] and obtain a necessary and sufficient condition over the real base field when there is no control. We also obtain a rank condition (4.9) as an observability criterion in the presence of a control. The rank condition is necessary and sufficient over C and is claimed to be sufficient over R. The third contribution that we present in this section is that if the output vector function is known only upto its direction, how to synthesize a perspective system, that would match this data. We show that this problem is equivalent to a rational exponential interpolation problem. Without claiming any merit of the specific algorithm described in this section, we would like to emphasize the importance of this problem.

5 Concluding Remarks

As has been remarked in the introduction, the DOE support for nine years have resulted in theoretical understanding of the robust control problem and parameter estimation using a c.c.d. camera. This has lead to a theoretical development of perspective systems theory that has found application in dynamic estimation of algebraic curves. The control and estimation problems have been applied to robotic manipulation problems (not detailed in this report) and to problems in mobile robotic manipulation.

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6 Ph.D. students partly supported by the grant

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Zhenyu Yu Vision Guided Robot Motion Planning and Control, 1995.
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Mingqi Kong Motion Estimation using Spatio-Temporal Filtering, 1999.
Li Zhang Map Building, Localization and Structure Estimation Problems in Mobile Robotics, 2000.

7 Grant-supported workshops organized at various international conferences

1. Sensor-Referenced Control and Planning: Theory and Applications, IEEE International Conference on Decision and Control, New Orleans, USA, 1995 (with T. J. Tarn and Ning Xi)

2. Dynamics and Control Problems in Vision, Mathematical Theory of Networks and Systems, Saint Louis, USA, 1996 (with Clyde Martin)

3. Event Driven Sensing, Planning and Control of a Robotic System: An Integrated Approach, IROS'96 IEEE/RSJ International Conference on Intelligent Robots and Systems, Osaka, Japan, 1996 (with T. J. Tarn and Ning Xi)

8 Grant-supported invited sessions organized at various international conferences

1. Systems and Control Problems in Machine Vision, Thirty-Second Annual Allerton Conference on Communication, Control and Computing, 1994 (with R. Sharma, Univ. of Illinois at Urbana-Champaign.)

2. Identification and Control problems in Computer Vision, NOLCOS '95, Symposium on Nonlinear Control Systems Design, Tahoe City, California, USA, 1995. (with Clyde Martin, Texas Tech University, Lubbock, Texas)

3. Sensor-Referenced Planning and Control of Robotic Systems: Autonomous vs. Telerobotic, IROS'95 IEEE/RSJ International Conference on Intelligent Robots and Systems at Pittsburgh, Pennsylvania, USA, 1995. (with N. Papanikolopoulos, University of Minnesota and Ning Xi, Washington University)

4. Systems and Control Problems in Autonomous Vision and Robotic Systems, The 34^{th} IEEE Conference on Decision and Control, December 13-15, New Orleans, USA, 1995. (with G. Picci and N. Xi)

5. System Theoretic Methods in Machine Vision, Mathematical Theory of Networks and Systems-96, June 24-28, 1996, The Ritz-Carlton, Saint Louis, Missouri, USA. (with Georgio Picci)

6. Modeling and Control of Hybrid Systems, Mathematical Theory of Networks and Systems-96, June 24-28, 1996, The Ritz-Carlton, Saint Louis, Missouri, USA. (with Liyi Dai)

7. Modelling and Design Methods for Nonlinear Systems with Applications in Robotics and Visionics, The 13^{th} IFAC World Congress, June 30-July 5, 1996, San Francisco, California, USA. (with Clyde Martin and Ning Xi)
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9. Dynamic Vision and Control I and II, Mathematical Theory of Networks and Systems-98, Padova, Italy, July 6-10,1998. (with S. Soatto)
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9 Grant supported major publications

Books

1. Control in Robotics and Automation: Sensor Based Integration, Academic Press, 1999 (with T.J. Tarn, Ning Xi)

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1. Modelling issues in visual sensing, Special issue of the Journal of Mathematical and Computer Modelling, Volume 24, No. 5/6, September 1996, Elsevier Science Ltd, Oxford. (with N. Papanikolopoulos)

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1. "Simultaneous Coefficient Assignment of Discrete Time, Multi-Input Multi-Output, Linear Time Varying Systems - A New Approach to Compensator Design," SIAM J. Control and Optimization, vol. 31, No. 5, November 1993, pp. 1438-1461. (with P. Bouthellier)
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4. "Visually Guided Ranging from Observations of Points, Lines and Curves via an Identifier Based Nonlinear Observer," Systems and Control Letters, 25, (1995), pp. 63-73. (with M. Jankovic)
5. "A Perspective Theory for Motion and Shape Estimation in Machine Vision," SIAM J. Control and Optimization, vol. 33, No. 5, September 1995, pp. 1530-1559. (with E. P. Loucks)
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8. "Dynamical Systems Approach to Target Motion Perception and Ocular Motion Control" Systems and Control in the Twenty-First Century, C.I. Byrnes, B. N. Datta, D. S. Gilliam and C. F. Martin, Editors, Progress in Systems and Control Theory, 22, pp. 185-204, Birkhauser Boston, 1997. (with E. P. Loucks, C. F. Martin and L. Schovanec)
9. "A New Robust Control for a Class of Uncertain Discrete-Time Systems," IEEE Transactions on Automatic Control, vol. 42, No. 9, pp. 1252-1254, September 1997. (with Haiyan Wang).
10. "Robotic Motion Planning and Manipulation in an Uncalibrated Environment", IEEE Robotics and Automation Magazine, Special Issue on Applied Visual Servoing, Vol. 5, No. 4, pp. 50-57, December, 1998. (with T. J. Tarn, Ning Xi, Zhenyu Yu and Di Xiao)

11. "Multisensor Based Robotic Manipulation in an Uncalibrated Manufacturing Workcell" Special Issue of the Journal of Franklin Institute on Information/Decision Fusion with Engineering Applications, vol. 336, No. 2, pp. 237-255, March 1999. (with Di Xiao, Ning Xi and T. J. Tarn)

12. "Visually Guided Tracking and Manipulation", *Control in Robotics and Automation: Sensor Based Integration*, pp. 115-144, Academic Press, 1999. (with Ming Li)

13. "Complimentary Sensor Fusion in Robotic Manipulation, *Control in Robotics and Automation: Sensor Based Integration*, pp. 147-181, Academic Press, 1999. (with Zhenyu Yu, Di Xiao, Ning Xi and T. J. Tarn)

14. "Integration of Real-Time Planning and Control in an Unstructured Manufacturing Workcell", *Advanced Robotics*, vol. 13, no. 4, pp. 473-492, 1999. (with D. Xiao, N. Xi and T. J. Tarn)

15. "Sufficient Conditions for Generic Simultaneous Pole Assignment and Stabilization of Linear MIMO Dynamical Systems", IEEE Trans. on Automatic Control, vol. 45, no. 4, April 2000, pp. 734-738. (with A. Wang).

16. "Identification of Riccati Dynamics under Perspective and Orthographic Observations", IEEE Trans. on Automatic Control, vol. 45, no. 7, pp. 1267-1278, July 2000. (with H. Inaba and S. Takahashi).

Refereed Papers in International Conferences

1. "Observability of Perspective Systems: A New Approach to Computer Vision," Computation and Control II, Proceedings of the Second Bozeman Conference, Bozeman, Montana, August 1-7, 1990, K.L. Bowers and J. Lund Editors, Birkhauser, Boston, 1991, pp. 125-134 (with Y.T. Wu).

2. "Observability and Identifiability Problems in Perspective Systems: A New Approach to Computer Vision," SPIE Vol. 1607 Intelligent Robots and Computer Vision X: Algorithms and Techniques (1991), pp. 589-600. (with Y.T. Wu).

3. "Some New Results in Perspective System Theory and its Application to Computer Vision," Proceedings of the ninth symposium on energy engineering sciences - Fluid and Dynamical Systems, May 13-15, 1991, at the Argonne National Laboratory, Argonne, Illinois, Report No. CONF-9105116, pp. 85-92.

4. "New Geometric Methods in Computing the Motion Parameters of a Rigid Body Using Straight Line Correspondences," Proc. 1992 American Control Conference, Chicago, pp. 1500-1504, June 1992. (with M. Lei).

5. "An adaptive controller for systems with unmeasurable disturbance," Proc. 1992 American Control Conference, Chicago, June 1992. (with W. Lin).

6. "Some problems in perspective system theory and its application to machine vision," IROS'92, 1992 IEEE/RSJ International Conference on Intelligent Robots and Systems, Raleigh, North Carolina, USA, July 7-10, 1992 (with M. Jankovic and Y.T. Wu).

7. "Dynamical Systems approach to Computer Vision," SIAM Conference on Applications of Dynamical Systems, Salt Lake City, Utah, October 15-19, 1992.

8. "An optical flow based approach for motion and shape parameter estimation in computer vision," Proceedings of the 31st IEEE Conference on Decision and Control, 1992. (with E. P. Loucks and J. Lund).

9. "Visually guided robotic motion tracking," Proceedings of the Thirtieth Annual Allerton Conference on Communication, Control and Computing, September 30-Oct 2, 1992. (with M. Lei).

10. "On the problem of parameter identification in perspective systems and its application to motion estimation problems in Computer Vision," Computation and Control III, Proceedings of the Third Bozeman Conference, Bozeman, Montana. K.L. Bowers and J. Lund Editors, Birkhauser, Boston, 1993. (with E. P. Loucks).

11. "Estimation of Angular Velocity of a Moving Object Using Line Correspondences," Proceedings of the American Control Conference, San Francisco, June 2-4, 1993. (with M. Lei).

12. "Image Based Estimation Problems in System Theory: Motion and Shape Estimation of a Planar Textured Surface Undergoing a Rigid Flow," Proceedings of the American Control Conference, San Francisco, June 2-4, pp. 1322-1326, 1993. (Invited)

13. "On the Problem of Coefficient Assignment of Discrete Time Multi-Input Multi-Output Linear Time Varying Systems," Proceedings of the American Control Conference, San Francisco, June 2-4, 1993. (Invited)

14. "A Recursive Approach for Coefficient Assignment of Discrete Time MIMO Linear Time Varying Systems," Proceedings of the 12th IFAC World Congress, Sydney, Australia, July 19- 23, 1993. (Invited)

15. "Visionics: A New Vision Guided Estimation of a Dynamical System," Proceedings of the 12th IFAC World Congress, Sydney, Australia, July 19-23, 1993. (Invited)

16. "Some new results in discrete time motion and shape estimation in machine vision," Proceedings of the Tenth International Symposium on the Mathematical Theory of Networks and Systems, Regensburg, August 2-6, 1993, Mathematical Research, vol. 79, pp. 775-780. (Invited) (with E. P. Loucks).

17. "On the Realization of Perspective Systems and its Application to Motion and Shape Estimation Problems in Machine Vision," Proceedings of the 32nd IEEE Conference on Decision and Control, San Antonio, Texas, December 15-17, 1993, pp. 1233-1236. (with E. P. Loucks).

18. "Visually Guided Robotic Tracking and Grasping of a Moving Object," Proceedings of the 32nd IEEE Conference on Decision and Control, San Antonio, Texas, December 15-17, 1993, pp. 1604-1609. (with M. Lei).

19. "Visually Guided Control Systems: Present Technology and Future Prospects," Proceedings of the twelvth symposium on energy engineering sciences, Argonne National Laboratory, Argonne, Illinois, April 27-29, 1994, Conf-9404137, pp. 154-160.

20. "A Nonrecursive Coefficient Assignment and Stabilization Scheme for Linear Time Varying Systems," Proceedings of the Thirty-Second Annual Allerton Conference on Communication, Control and Computing, September 28-September 30, 1994, pp. 21-30. (with P. R. Bouthellier).

21. "On-Line Collision Avoidance for Robot in a Non-Stationary Environment," Proceedings of the Thirty-Second Annual Allerton Conference on Communication, Control and Computing, September 28-September 30, 1994, pp. 651-660. (with Z. Yu, T. J. Tarn and C. Guo).

22. "On the Problem of Observing Motion and Shape," Proceedings of the ICARCV'94, Third International Conference on Automation, Robotics and Computer Vision, Shangri-La, Singapore, November 9 - 11, 1994, pp. 653-657. (with M. Jankovic and E. P. Loucks).

23. "Temporal and Spatial Sensor Fusion in a Robotic Manufacturing Workcell," Proceedings of the 1994 Hongkong International Workshop on New Directions of Control and Manufacturing, Hotel Victoria, Hongkong Nov. 7-9, 1994, pp. 317-324. (with Zhenyu Yu, Ning Xi and T. J. Tarn).

24. "An Introduction to Perspective Observability and Recursive Identification Problems in Machine Vision," Proceedings of the 33rd IEEE Conference on Decision and Control, Lake Buena Vista, Florida, December 14-16, 1994, pp. 3229-3234. (with M. Jankovic and E. P. Loucks).

25. "Temporal and Spatial Sensor Fusion in a Robotic Manufacturing Workcell," IEEE International Conference on Robotics and Automation, Nagoya Congress Center, Nagoya, Japan, May 21-27, 1995, pp. 160-165. (with Zhenyu Yu, Ning Xi and T. J. Tarn).

26. "Multi-Sensor Based Planning and Control for Robotic Manufacturing Systems," Proceedings of the 1995 IEEE/RSJ International Conference on Intelligent Robots and Systems, August 5-9, 1995, Pittsburgh, Pennsylvania, USA, vol. 3, pp. 222-227.

27. "Calibration Free Visually Controlled Manipulation of Parts in a Robotic Manufacturing Workcell", Proceedings of the 1996 IEEE International Conference on Robotics and Automation, April 22-28, 1996, Minneapolis, Minnesota, USA., pp. 3197-3202. (with N. Xi, T. J. Tarn, Zhenyu Yu and Di Xiao).

28. "Multistage Nonlinear Estimation with applications to image based parameter estimation," Proceedings of the 13^{th} IFAC World Congress, June 30-July 5, 1996, San Francisco, California, USA, vol. F: Nonlinear Systems II, pp. 447-452. (with M. Jankovic and E.P. Loucks).

29. "A Calibration Free Multi Sensor Fusion Scheme for Motion Estimation, Tracking and Grasping in a Manufacturing Workcell " Proc. of the workshop on Foundations of Information/Decision Fusion: Applications to Engineering Problems, August 7-9, 1996, Washington D.C. (with D. Xiao, N. Xi and T. J. Tarn).

30. "Calibration Free Vision Based Control of a Robotic Manipulator", Proceedings of the 34^{th} Annual Allerton Conference on Communication, Control and Computing, 1996.

31. "Multisensor Based Intelligent Planning and Control for Robotic Manipulators on a Mobile Platform" Proceedings of Robot and Human Communication RO-MAN'96, November 11-14, 1996. (with D. Xiao, N. Xi and T. J. Tarn)

32. "Multi-Sensor Fusion Scheme for Calibration-Free Stereo Vision in a Manufacturing Workcell" Proceedings of the 1996 IEEE/SICE/RSJ International Conference on Multisensor Fusion and Integration for Intelligent Systems, December 8 - 11, 1996, Washington D.C., U.S.A., pp. 416-423 (with M. Song, N. Xi and T. J. Tarn)

33. "A Single Camera, Calibration Free Estimation and Tracking Scheme with Multi-Sensor Fusion in a Manufacturing Workcell" Proceedings of the 1996 IEEE/SICE/RSJ International Conference on Multisensor Fusion and Integration for Intelligent Systems, December 8 - 11, 1996, Washington D.C., U.S.A., pp. 679-686 (with D. Xiao, N. Xi and T. J. Tarn).

34. "Planning and Control of Self Calibrated Manipulation for a Robot on a Mobile Platform" Proceedings of the IEEE International Conference on Robotics and Automation, April 20-25, 1997, Albuquerque, New Mexico, USA. (with D. Xiao, N. Xi and T. J. Tarn).

35. "Visionics: The Design of Intelligent Machines " Fifteenth Symposium on Energy Engineering Sciences, Argonne National Laboratory, May 14-16, 1997. (with D. Xiao, N. Xi and T. J. Tarn).

36. "Planning and Control in Reconfigurable Manufacturing Workcell" Proceedings of the 1997 IEEE/ASME International Conference on Advanced Intelligent Mechatronics, June 16-20, 1997, Waseda University, Japan, (with D. Xiao, N. Xi and T. J. Tarn).

37. "Sensor Guided Manipulation in a Manufacturing Workcell" Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems, September 8-12, 1997, Grenoble, France, (with D. Xiao, N. Xi and T. J. Tarn).

38. "3D Part Manipulation Aided by Uncalibrated Mono-Camera" Proceedings of the Fifth Symposium on Robot Control, September 3-5, 1997, Nantes, France, (with D. Xiao, N. Xi and T. J. Tarn).

39. "Cross-Ratio Dynamics and its Application to Calibration Free Motion Estimation" Proceedings of the Conference on Decision and Control, December 1997.

40. "Simultaneous Pole Placement of Linear MIMO Dynamical Systems," Proceedings of the Conference on Decision and Control, December 1997, (with Alex Wang).

41. "Controllability, Observability and Realizability of Perspective Dynamic Systems," Proceedings of the Conference on Decision and Control, December 1997, (with E. P. Loucks, C. F. Martin).

42. "Parameter identifiability of Riccati dynamics under perspective and orthographic projections," Proceedings of the Mathematical Theory of Networks and Systems, July 1998, pp. 1011-1014. (with H. Inaba and S. Takahashi).

43. "Identification of motion and shape parameters using extended Kalman Filters," Proceedings of the Mathematical Theory of Networks and Systems, July 1998, pp. 1039-1042. (with H. Kano and H. Kanai).

44. "Integration of real-time planning and control in an unstructured workspace" Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems, Oct. 13-17, 1998, Victoria Conference Center, Victoria B.C., Canada, (with Di Xiao, Ning Xi and T. J. Tarn).

45. "A Note on Parameter Identifiability of Riccati Dynamics under Perspective Projection", Paper A8-2, Proceedings of the 30^{th} ISCIE International Symposium on Stochastic Systems Theory and its Applications, November 4-6, 1998, Kyodai Kaikan Hall, Kyoto University, Kyoto, Japan, (with S. Takahashi and H. Inaba)

46. "Estimation of Motion and Shape Parameters of a Moving Rigid Body by Extended Kalman Filter", Paper A8-3, Presented at the 30^{th} ISCIE International Symposium on Stochastic Systems Theory and its Applications, November 4-6, 1998, Kyodai Kaikan Hall, Kyoto University, Kyoto, Japan, (with H. Kano and H. Kanai).

47. "Homogeneous dynamical systems theory" Proceedings of the 14^{th} IFAC World Congress, July 5-9, 1999, Beijing, China, Paper No. 2a-12-2, pp. 399-404 (in the special invited session on *Sensor Guided Control and Learning with Applications*, Session Number 2a-12, 13:00 - 15:00, July 8) (with Clyde Martin and E. P. Loucks).

48. "Real-time planning and control for robot manipulator in unknown workspace" Proceedings of the 14^{th} IFAC World Congress, July 5-9, 1999, Beijing, China, Paper No. 2a-12-6, pp. 423-428 (in the special invited session on *Sensor Guided Control and Learning with Applications*, Session Number 2a-12, 13:00 - 15:00, July 8) (with Di Xiao, Ning Xi and T. J. Tarn).

49. "Hybrid position and force control of a robot manipulator", Proceedings of the 1999 IEEE Hongkong Symposium on Robotics and Control, July 2-3, pp. 367-372, 1999. (with D. Xiao, N. Xi and T. J. Tarn.)

50. "A multisensor fusion approach to shape estimation using a mobile platform with uncalibrated position", Proceedings of the 1999 IEEE/SICE/RSJ International Conference on Multisensor Fusion and Integration for Intelligent Systems MFI'99, August 15 - 18, 1999, The Grand Hotel, Taipei, Taiwan, ROC, pp. 205-210. (with L. Zhang)

51. "Line segment based map building and localization using 2D laser rangefinder", IEEE International Conference on Robotics and Automation, San Francisco Hilton and Towers, San Francisco, USA, April 22 - 28, 2000, pp. 2538-2543. (with L. Zhang)

52. "Orbits and canonical forms for perspective systems", 2000 American Control Conference, The Hyatt Regency Hotel, Chicago, Illinois, USA, June 28 - 30, 2000. (with S. Takahashi)

53. "Parameter identification using Kronecker canonical forms with applications to motion estimation ", 14^{th} International Symposium of Mathematical Theory of Networks and Systems, Perpignan, France, June 19-23, 2000. (with S. Takahashi)

54. "Representation and reconstruction of spatio-temporal signals using on/off cells", 2000 American Control Conference, The Hyatt Regency Hotel, Chicago, Illinois, USA, June 28 - 30, 2000. (with Zoran Nenadic and Peng Li)

55. "Geometric feature based $2\frac{1}{2}$D map building and planning with laser, sonar and tactile sensors", IEEE/RSJ International Conference on Intelligent Robots and Systems, October 30 - November 5, 2000, Kagawa University, Takamatsu, Japan.

56. "Canonical forms and orbit identification problems in machine vision", The 39^{th} IEEE Conference on Decision and Control, Sydney Convention and Exhibition Center, Australia, Dec. 12-15, 2000, pp. 5175-5181. (with S. Takahashi)

57. "Three Dimensional Structure Estimation and Planning with Vision and Range", The 39^{th} IEEE Conference on Decision and Control, Sydney Convention and Exhibition Center, Australia, Dec. 12-15, 2000, Invited Session at the Recent Advances in Vision Based Control, pp. 2515-2520. (with L. Zhang)

58. "Observability of perspective dynamical systems", The 39^{th} IEEE Conference on Decision and Control, Sydney Convention and Exhibition Center, Australia, Dec. 12-15, 2000, pp. 5157-5162. (with Hiroshi Inaba, Akifumi Yoshida and Rixat Abdursul)

59. "Identification of relative position and orientation of two cameras from motion and shape parameters of moving rigid body", The 39^{th} IEEE Conference on Decision and Control, Sydney Convention and Exhibition Center, Australia, Dec. 12-15, 2000, pp. 5169-5174. (with Hiroyuki Kano)

60. "A note on observability of perspective dynamical system", SICE Symposium on Dynamical System Theory, Nov. 6-8,2000, Nagaoka City, Japan. (with Hiroshi Inaba and Rixat Abdursul)

Invited Presentations at Conferences and Universities

1. "Some New Results in Computer Vision," Invited Presentation at the second conference on Computation and Control, Montana State University, Bozeman, Montana, August 1-7, 1990.

2. "Convergence Analysis of Linear Dynamical Systems by High Gain and High Dynamic Compensator," 2nd SIAM Conference on Linear Algebra in Signals, Systems and Control, San Francisco, California, Nov. 5-8, 1990.

3. "Estimation of Motion and Shape Parameters of a Rigid Body from its Orthogonal and Perspective Projections," 2nd SIAM Conference on Linear Algebra in Signals, Systems and Control, San Francisco, California, Nov. 5-8, 1990.

4. "Global Stabilization of Discrete Time Nonlinear Systems," 2nd SIAM Conference on Linear Algebra in Signals, Systems and Control, San Francisco, California, Nov. 5-8, 1990.

5. "Simultaneous design problems in linear system theory," Invited Presentation at the Center for applied mathematics, University of Notre Dame, Notre Dame, Indiana 46556, April 16th, 1991.

6. "Perspective System Theory: A new perspective in machine vision," Invited Presentation at the Center for applied mathematics, University of Notre Dame, Notre Dame, Indiana 46556, April 18th, 1991.

7. "Observability problems in perspective system theory and its application to computer vision," Presented at the 2nd NIU Conference on Linear Algebra, Numerical Linear Algebra, and Applications,Northern Illinois University, DeKalb, Illinois, May 5, 1991.

8. "Some New Results in Observer Design and its Application to Perspective Systems," Presented at the 9th Symposium on Energy Engineering Sciences, Argonne National Laboratory, Argonne, IL, May 13, 1991.

9. "A survey of simultaneous stabilization problems for linear time invariant systems and linear time varying systems," Presented at the Department of Control Engineering, Tokyo Institute of Technology, Japan, June 12, 1991, and at the Department of Information Sciences, Tokyo Denki University, Hatoyama-Machi, Hikigun, Saitama, Japan 350-03, 24th June, 1991.

10. "Some new perspective on machine vision," Invited presentation at the the Department of Mathematical Engineering and Information Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan, June 10, 1991, the Department of Mechanical Engineering, Nagoya University, Japan on 14th June, 1991 and at the Department of Information Sciences, Tokyo Denki University, Hatoyama-Machi, Hikigun, Saitama, Japan 350-03, 25th June, 1991.

11. "Problems in perspective system theory and its application to correspondence problems in machine vision," Invited presentation at the Beckmann Institute, University of Illinois at Urbana, Champaign. Dec. 18, 1991. Invited presentation at the General Robotics and Active Sensory Perception Laboratory (GRASP), University of Pennsylvania, 3401 Walnut Street, Philadelphia, PA 19104 on Feb. 19, 1992

12. "Perspective Problems and its Application to Computer Vision and System Theory," Invited presentation at the Department of Mechanical Engineering for Computer-Controlled Machinery, Faculty of Engineering, Osaka University, 2-1 Yamadaoka, Suita, Osaka 565, Japan, 25th July, 1992.

13. "Algebraic Geometric Methods in the Study of Line Based Correspondence Problems in Computer Vision" Invited presentation at the Department of Mechanical Engineering for Computer-Controlled Machinery, Faculty of Engineering, Osaka University, 2-1 Yamadaoka, Suita, Osaka 565, Japan, 27th July, 1992.

14. "A new nonlinear feedback controller for visually guided robotic motion tracking." Invited presentation at the Department of Mechanical Engineering for Computer-Controlled Machinery, Faculty of Engineering, Osaka University, 2-1 Yamadaoka, Suita, Osaka 565, Japan, 1st August, 1992.

15. "Some New Results in Computer Vision," Invited Presentation at the Third Conference on Computation and Control, Montana State University, Bozeman, Montana, August, 1992.

16. "Some New Problems in Computer Vision and its Connection to Perspective System Theory," Invited Presentation organized by the Department of Electrical Engineering, Indian Institute of Technology, Delhi, India on 5th August, 1993 and by the IEEE Kharagpur Chapter, Indian Institute of Technology, Kharagpur, India on 26th August, 1993 and by the Centre for Artificial Intelligence and Robotics, Bangalore, India on 1st September, 1993.

17. "On the Problem of Simultaneous Stabilization and Simultaneous Pole Assignment," "On Output Feedback Regulation and Disturbance Decoupling" and "On the Problem of Visually Guided Control of a Robot Arm," Invited Presentation in the Department of Electrical Engineering at Indian Institute of Technology, Kharagpur, India on 12th August, 1993, 17th August, 1993 and 24th August, 1993 respectively.

18. "Visually Guided Ranging from Observations of Points, Lines and Curves via an Identifier Based Nonlinear Observer," Invited Presentation in the Department of Mechanical Engineering at Indian Institute of Technology, Kharagpur, India on 25th August, 1993.

19. "Current Trends in the Field of Systems and Control" and "Visually Guided Control Problems in Robotics," Invited Presentation at the Institute of Armanent Technoligy, Girinagar, Pune, India on 2nd September, 1993 and on 3rd September, 1993 respectively.

20. "Perspective Problems in Systems Thoery and their Applications to Machine Vision," Invited Presentation in the Coordinated Science Lab., University of Illinois at Urbana- Champaign, Urbana, Illinois 61801 on 16th February, 1994.

21. "Nonlinear Estimation Schemes for Visual Servoing," Presentation at the Workshop on Visual Servoing: Achievements, Applications and Open Problems organized at the 1994 IEEE International Conference on Robotics and Automation, May 8-13, 1994, San Diego, California, USA.

22. "Visually Guided Control Problems: Present Technology and Future Prospects," Computation and Control IV, Montana State University, Bozeman, Montana, August 3-9, 1994.

23. "Parameter Identification Problems in Computer Vision," Thirty-Second Annual Allerton Conference on Communication, Control and Computing, September 28-September 30, 1994.

24. "On the Problem of Active Vision and Spatial Reasoning," Third Siam Conference on Control and its Applications, April 27-29, 1995, Saint Louis, Missouri.

25. "A Method of Parameter Identification for Perspective Systems and its Application to Machine Vision," Third Siam Conference on Control and its Applications, April 27-29, 1995, Saint Louis, Missouri. (with E. P. Loucks)

26. "Perspective Systems Theory and its Application to Machine Vision," Invited Presentation at the Department of Control Systems Engineering, Tokyo Institute of Technology, Oh-Okayama, Meguro-Ku, Tokyo 152, Japan, May 12, 1995.

27. "Observer Design and Identification of Systems with Internal Structure," Invited Presentation at the Department of Control Systems Engineering, Tokyo Institute of Technology, Oh-Okayama, Meguro-Ku, Tokyo 152, Japan, May 15, 1995.

28. "Temporal and Spatial Sensor Fusion in a Robotic Manufacturing Workcell," Invited Presentation at the Institute of Industrial Science, University of Tokyo, Roppongi, Minato-Ku, Tokyo 106, Japan, May 18, 1995.

29. "Some New Results in Nonlinear Systems Identification and Observer Design," Invited Presentation at the Graduate School of Science and Engineering, Tokyo Denki University, Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan, May 19, 1995.

30. "Visually Guided Control and Tracking with some New Approaches to Sensor-Fusion," Invited Presentation at the Department of Mechanical Engineering, Okayama University, Tsushima-Naka, Okayama 700, Japan, May 22, 1995.

31. "A Realization Theory for Perspective Systems," Invited Presentation at NOLCOS '95, Tahoe City, USA, 26-28 June, 1995.

32. "Role of Dynamics in Machine Vision with applications to Parameter Estimation and Image Segmentation," Invited Presentation at Society of Engineering Science 32^{nd} Annual Technical Meeting, New Orleans, USA, October 29 - November 2, 1995.

33. "Visually Controlled Manipulation," Invited presentation at the workshop on 'Sensor-Referenced Control and Planning: Theory and Applications', IEEE International Conference on Decision and Control, New Orleans, USA, 1995.

34. "Perspective Systems Theory and Machine Vision," Invited presentation at the workshop on 'Sensor-Referenced Control and Planning: Theory and Applications', IEEE International Conference on Decision and Control, New Orleans, USA, 1995.

35. "Visually controlled manipulation of parts in a manufacturing workcell using a robotic manipulator," Invited Presentation at the 34^{th} IEEE Conference on Decision and Control, New Orleans, December 15, 1995.

36. Some Problems in Simultaneous System Design with a view towards Hybrid Control, Mathematical Theory of Networks and Systems-96, June 24-28, 1996, The Ritz-Carlton, Saint Louis, Missouri, USA.

37. Observation and Control in a Perspective Framework, Symposium on Current and Future Directions in Applied Mathematics, The University of Nortre Dame, Indiana, USA, April 18 - 21, 1996.

38. A Theory of Perspective Systems with applications Machine Vision, Invited Presentation at the Department of Mathematical Engineering and Information Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113 Japan, Oct. 29, 1996.

39. A Theory of Perspective Systems with applications Machine Vision, Invited Presentation at the Department of Control Systems Engineering, Tokyo Institute of Technology, Oh-Okayama, Meguro-Ku, Tokyo 152, Japan, Oct. 30, 1996.

40. A Theory of Perspective Systems with applications Machine Vision, Invited Presentation at the Division of Applied Systems Science, Kyoto University, Kyoto, 606 Japan, Nov. 1, 1996.

41. Some Recent Results in Perspective Control and its Connection to the Riccati Flow, Invited Presentation at the Graduate School of Science and Engineering, Tokyo Denki University, Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan, Nov. 5, 1996.

42. Problems in Perspective Observability and its Connection to Popov-Belevitch-Hautus Test of Observability, Invited Presentation at the Graduate School of Science and Engineering, Tokyo Denki University, Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan, Nov. 6, 1996.

43. Planning and Control of Self Calibrated Manipulation for a Robot on a Mobile Platform, Invited Presentation at the Graduate School of Science and Engineering, Tokyo Denki University, Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan, Nov. 7, 1996.

44. A Theory of Perspective Systems with applications Machine Vision, Invited Presentation at the Graduate School of Science and Engineering, Tokyo Denki University, Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan, Nov. 8, 1996.

45. "Cross-Ratio Dynamics and its Application to Problems in Visually Guided Control," Invited presentation at the Department of Mathematics, Texas Tech University, Lubbock, Texas, February 3, 1997.

46. "Some New Results in Perspective Control", Invited Presentation at the Royal Institute of Technology, Stockholm, Sweden, May 30, 1997.

47. "On Controllability and Observability of Perspective Systems", Invited Presentation in the Department of Electrical Engineering at Indian Institute of Technology, Kharagpur, India on July 3, 1997.

48. "Robotic Manipulation in an Uncalibrated Environment", Invited Presentation in the Department of Electrical Engineering at Indian Institute of Technology, Kharagpur, India on July 4, 1997.

49. Cross Ratio Dynamics and Controllability Problems in Perspective Systems, Invited Presentation at the Graduate School of Science and Engineering, Tokyo Denki University, Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan, Oct. 28, 1997.

50. Perspective Systems Theory with Applications to Machine Vision and Control, Laboratory for Information Representation, Frontier Research Program, The Institute of Physical and Chemical Research, RIKEN, Hirosawa 2-1, Wako-shi, Saitama 351-01, Japan, Oct. 30, 1997.

51. Intelligent Robotic Manipulation with Hybrid Position/Force Control in an Uncalibrated Workspace, Invited Presentation at the IEEE Tokyo Chapter RAS meeting, Nov. 6, 1997. Invited Presentation at the Department of Machine Intelligence and Systems Engineering, Tohoku University, Aza-aoba, Aramaki, Sendai 980, Japan, Nov. 20, 1997. Invited Presentation at the Department of Robotics, Ritsumeikan University, Noji-higashi 1-1-1, Kusatsu, Shiga 525-77, Japan, Nov. 27, 1997.

52. Robotic Manipulation with Visually Guided Position and Force Feedback, Invited Presentation at the Department of Control Systems Engineering, Tokyo Institute of Technology, Oh-Okayama, Meguro-Ku, Tokyo 152, Japan, Nov. 13, 1997.

53. Vision Guided Estimation, Control and Tracking, Invited Presentation at the Department of Organismal Biology and Anatomy, 1027 East 57th Street, The University of Chicago, Chicago, Illinois 60637, Jan. 8, 1998.

54. A Cyclopian View on Machine Vision, Plenary Presentation at the 'Mathematics of the Life Sciences' conference on 1/30/1998, Texas Tech University, Jan 30, 1998.

55. "A Cyclopian View towards Observing a Homogeneous Dynamical System," Invited Presentation at the sixth conference Computation and Control, Montana State University, Bozeman, Montana, August 4-7, 1998.

56. Kronecker indices and canonical forms with application to motion estimation, Invited Presentation at the Graduate School of Science and Engineering, Tokyo Denki University, Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan, Nov. 13, 1998.