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Acoustic Wave Equations for a Linear Viscous Fluid and an Ideal Fluid

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ACOUSTIC WAVE EQUATIONS FOR A LINEAR VISCOUS FLUID AND AN IDEAL FLUID

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ABSTRACT

The mathematical description of acoustic wave propagation within a time- and space-varying, and moving, linear viscous fluid is formulated as a system of coupled linear equations. This system is rigorously developed from fundamental principles of continuum mechanics (conservation of mass, balance of linear and angular momentum, balance of entropy) and various constitutive relations (for stress, entropy production, and entropy conduction) by linearizing all expressions with respect to the small-amplitude acoustic wavefield variables. A significant simplification arises if the fluid medium is neither viscous nor heat conducting (i.e., an ideal fluid). In this case the mathematical system can be reduced to a set of five, coupled, first-order partial differential equations. Coefficients in the systems depend on various mechanical and thermodynamic properties of the ambient medium that supports acoustic wave propagation. These material properties cannot all be arbitrarily specified, but must satisfy another system of *nonlinear* expressions characterizing the dynamic behavior of the background medium. Dramatic simplifications in both systems occur if the ambient medium is simultaneously adiabatic and stationary.

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CONTENTS

1.0 INTRODUCTION

Accurate simulation of sound wave propagation within a three-dimensional environment that is spatially heterogeneous, time-varying, and/or moving has important scientific, military, and commercial applications. In this study, the fundamental mathematical equations governing acoustic wave propagation within such a complex and dynamic medium are developed. The expressions are rigorously derived from basic principles of continuum mechanics, relevant constitutive relations, and equations of state using a straightforward linearization process. Standard textbooks in acoustics (e.g., Morse and Ingard, 1968; Kinsler et al., 2000) or wave propagation (e.g., Tolstoy, 1973; Chew, 1990) do not treat this topic with the requisite degree of generality. However, a recent text by Ostashev (1997) does provide greater in-depth analysis.

Some previous efforts have been devoted to deriving a single, higher-order, partial differential equation containing a single acoustic wavefield variable. For example, Pierce (1990) obtains a second-order partial differential equation for acoustic particle velocity potential, and Ostashev (1997) develops a third-order equation for acoustic pressure. In contrast, the goal of the present mathematical development is to obtain a *system* of coupled, linear, first-order, partial differential equations amenable to numerical solution via explicit, time-domain, finite-difference techniques. Coupled, first-order systems possess favorable characteristics for finite-difference numerical algorithms, when compared with higher-order equations (or higher-order systems of equations).

In developing the system of first-order partial differential equations, an attempt is made to:

1) Minimize the number of dependent variables (i.e., the acoustic wavefield variables).
2) Minimize the number of ambient medium parameters.
3) Eliminate ambient medium parameters that are difficult to determine in practice.
4) Incorporate all acoustic wavefield source types.

If the number of dependent variables in the system is minimized, then computational storage is reduced and execution speed is enhanced. Similarly, minimizing the number of ambient medium parameters (appearing in the coefficients of the system) reduces the overall computational storage demand. Finally, the utility of the equations is enhanced by including a variety of acoustic energy source terms.

A particular objective of this study entails developing a system of equations appropriate for sound wave propagation in dynamic atmospheric environments. However, for pedagogical purposes, the analysis is initiated with a more general point of view than typically found in mathematical treatments of atmospheric acoustics. That is, the ambient medium supporting acoustic wave propagation is considered to be both viscous and heat conducting. The derived wave propagation equations are correspondingly complicated. Subsequent specialization to (i) an ideal and non-heat-conducting fluid, (ii) an adiabatic ambient medium, and (iii) divergence-free ambient fluid flow, yields a system of equations applicable to many atmospheric sound propagation situations. In this context, all three of the above assumptions are considered quite reasonable. However, the more general equations may find application to acoustic wave problems in viscous and/or heat-conducting fluids.

Indicial notation is used in the following mathematical development. All quantities associated with the ambient medium supporting acoustic wave propagation are superscripted with the symbol "0". All acoustic wavefield variables and sources are prefixed with the symbol "δ".

2.0 DEFINITIONS

Acoustic waves are small propagating fluctuations in pressure, mass density, entropy, temperature, particle velocity, and stress superimposed on larger and more uniform background values of these quantities. Let $p^0(\mathbf{r},t)$, $\rho^0(\mathbf{r},t)$, $\eta^0(\mathbf{r},t)$, and $\theta^0(\mathbf{r},t)$ refer to the (scalar-valued) pressure, mass density, specific entropy density (i.e., entropy per unit mass), and absolute temperature of the background (or ambient, or reference) medium, respectively. Then, the total pressure $p(\mathbf{r},t)$, mass density $\rho(\mathbf{r},t)$, specific entropy density $\eta(\mathbf{r},t)$, and absolute temperature $\theta(\mathbf{r},t)$ associated with a propagating acoustic disturbance are assumed to be

$$p(\mathbf{r},t) = p^0(\mathbf{r},t) + \delta p(\mathbf{r},t), \tag{2.1a}$$

$$\rho(\mathbf{r},t) = \rho^0(\mathbf{r},t) + \delta\rho(\mathbf{r},t), \tag{2.1b}$$

$$\eta(\mathbf{r},t) = \eta^0(\mathbf{r},t) + \delta\eta(\mathbf{r},t), \tag{2.1c}$$

$$\theta(\mathbf{r},t) = \theta^0(\mathbf{r},t) + \delta\theta(\mathbf{r},t), \tag{2.1d}$$

where the pressure, density, entropy, and temperature perturbations satisfy

$$\frac{|\delta p(\mathbf{r},t)|}{p^0(\mathbf{r},t)} << 1, \qquad \frac{|\delta\rho(\mathbf{r},t)|}{\rho^0(\mathbf{r},t)} << 1, \qquad \frac{|\delta\eta(\mathbf{r},t)|}{\eta^0(\mathbf{r},t)} << 1, \qquad \frac{|\delta\theta(\mathbf{r},t)|}{\theta^0(\mathbf{r},t)} << 1. \tag{2.2a,b,c,d}$$

Similarly, the total particle velocity of the fluid medium is given by

$$v_i(\mathbf{r},t) = v_i^0(\mathbf{r},t) + \delta v_i(\mathbf{r},t). \tag{2.3}$$

However, it is *not* permissible to assume that fluctuations in the particle velocity vector δv_i are small compared to the ambient medium velocity v_i^0. For example, the ambient medium may be at rest (i.e., $v_i^0(\mathbf{r},t) = 0$). In like manner, the stress tensor components in the fluid are assumed to be an additive superposition of ambient stress and perturbation stress components:

$$\sigma_{ij}(\mathbf{r},t) = \sigma_{ij}^0(\mathbf{r},t) + \delta\sigma_{ij}(\mathbf{r},t). \tag{2.4}$$

Since certain components of the ambient stress tensor σ_{ij}^0 may vanish, each stress perturbation $\delta\sigma_{ij}$ need not be small compared with the corresponding σ_{ij}^0.

The background medium is subject to body forces $\mathbf{f}^0(\mathbf{r},t)$, energy (alternately, heat) supply $e^0(\mathbf{r},t)$, and surface tractions $\mathbf{s}^0(\mathbf{r},t)$, which maintain the medium in its reference configuration. Propagating acoustic disturbances are initiated by small fluctuations in the body forces, heat supply, and/or surface tractions. In this study, surface tractions are neglected. The total body force density (i.e., body force per unit volume) applied to the medium is

$$f_i(\mathbf{r},t) = f_i^0(\mathbf{r},t) + \delta f_i(\mathbf{r},t), \tag{2.5}$$

where δf_i are perturbations that excite acoustic waves. Similarly, the total external energy density (i.e., energy per unit volume) imposed on the medium is assumed to be

$$e(\mathbf{r},t) = e^0(\mathbf{r},t) + \delta e(\mathbf{r},t)\,, \tag{2.6}$$

where e^0 is the external energy density supplied to the ambient medium., and δe is a small fluctuation.

3.0 CONTINUITY EQUATION

The local form of the principle of conservation of mass is expressed by the continuity equation:

$$\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0\,, \tag{3.1}$$

where it is assumed that no mass sources or mass sinks exist. In this and subsequent expressions, repeated subscripts imply summation. Evaluating (3.1) for the ambient state of the medium yields

$$\frac{\partial \rho^0}{\partial t} + v_i^0 \frac{\partial \rho^0}{\partial x_i} + \rho^0 \frac{\partial v_i^0}{\partial x_i} = 0\,. \tag{3.2}$$

The first-order variation of (3.1) gives the linearized form

$$\frac{\partial (\delta\rho)}{\partial t} + v_i^0 \frac{\partial (\delta\rho)}{\partial x_i} + \frac{\partial \rho^0}{\partial x_i}\delta v_i + \rho^0 \frac{\partial (\delta v_i)}{\partial x_i} + \frac{\partial v_i^0}{\partial x_i}\delta\rho = 0\,. \tag{3.3}$$

4.0 EQUATIONS OF MOTION

Let σ_{ij} ($= \sigma_{ji}$) be the stress tensor components and f_i be the force density vector components. Then, Cauchy's equations of motion for a continuum, expressed in indicial notation, are

$$\rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right] - \frac{\partial \sigma_{ij}}{\partial x_j} = f_i\,. \tag{4.1}$$

Equations (4.1) express conservation of linear and angular momentum for all material parts of a continuum. Evaluating these equations of motion for the reference state of the medium gives

$$\rho^0\left[\frac{\partial v_i^0}{\partial t} + v_j^0 \frac{\partial v_i^0}{\partial x_j}\right] - \frac{\partial \sigma_{ij}^0}{\partial x_j} = f_i^0\,. \tag{4.2}$$

Also, equations (4.1) are readily linearized by calculating the first-order variation:

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t}+v_j^0\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial v_i^0}{\partial x_j}\delta v_j\right]+\left[\frac{\partial v_i^0}{\partial t}+v_j^0\frac{\partial v_i^0}{\partial x_j}\right]\delta\rho-\frac{\partial(\delta\sigma_{ij})}{\partial x_j}=\delta f_i. \tag{4.3}$$

5.0 CONSTITUTIVE EQUATIONS AND EQUATIONS OF STATE

The constitutive relations and equations of state characterizing a linear viscous fluid are summarized in this section, without mathematical proof. The *constitutive equations* explicitly relate the stress tensor components σ_{ij}, the specific internal entropy production rate ξ, and the entropy flux vector components p_i, to the velocity vector components v_i and the absolute temperature θ. The material parameters that arise in the expressions are all functions of mass density ρ and absolute temperature θ. The specific functional forms are referred to as *equations of state*.

5.1 Stress Tensor

For a linear viscous fluid, the stress tensor components are given by

$$\sigma_{ij}=\left[-p+\lambda\frac{\partial v_k}{\partial x_k}\right]\delta_{ij}+\mu\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right], \tag{5.1}$$

where p is the thermodynamic pressure and λ, μ are viscosity coefficients. Each is considered to be a function solely of the mass density ρ and the absolute temperature θ:

$$p=\bar{p}(\rho,\theta),\qquad \lambda=\bar{\lambda}(\rho,\theta),\qquad \mu=\bar{\mu}(\rho,\theta). \tag{5.2a,b,c}$$

The specific functional forms $\bar{p}$, $\bar{\lambda}$, and $\bar{\mu}$ are referred to as equations of state. The viscosity coefficients have physical dimension pressure - time (SI units P-s). Thermodynamic constraints require

$$\mu\geq 0,\qquad \lambda+\frac{2}{3}\mu\geq 0. \tag{5.3a,b}$$

Evaluating (5.1) for the ambient medium gives

$$\sigma_{ij}^0=\left[-p^0+\lambda^0\frac{\partial v_k^0}{\partial x_k}\right]\delta_{ij}+\mu^0\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right], \tag{5.4}$$

where

$$p^0=\bar{p}(\rho^0,\theta^0),\qquad \lambda^0=\bar{\lambda}(\rho^0,\theta^0),\qquad \mu^0=\bar{\mu}(\rho^0,\theta^0). \tag{5.5a,b,c}$$

Linearized versions of the stress constitutive equations, calculated from the first-order variation of (5.1), are

$$\delta\sigma_{ij} = \left[-\delta p + \lambda^0 \frac{\partial(\delta v_k)}{\partial x_k} + \frac{\partial v_k^0}{\partial x_k}\delta\lambda \right]\delta_{ij} + \mu^0 \left[\frac{\partial(\delta v_i)}{\partial x_j} + \frac{\partial(\delta v_j)}{\partial x_i} \right] + \left[\frac{\partial v_i^0}{\partial x_j} + \frac{\partial v_j^0}{\partial x_i} \right]\delta\mu \,. \tag{5.6}$$

Small fluctuations in the thermodynamic pressure and viscosity coefficients are related to the mass density perturbation $\delta\rho$ and absolute temperature perturbation $\delta\theta$ via first-order Taylor series expansions:

$$\delta p = a_\rho^0 \delta\rho + a_\theta^0 \delta\theta \,, \qquad \delta\lambda = b_\rho^0 \delta\rho + b_\theta^0 \delta\theta \,, \qquad \delta\mu = c_\rho^0 \delta\rho + c_\theta^0 \delta\theta \,, \tag{5.7a,b,c}$$

where the expansion coefficients are the partial derivatives:

$$a_\rho^0 \equiv \left. \frac{\partial\, \bar{p}(\rho,\theta)}{\partial\rho} \right|_{\rho=\rho^0,\,\theta=\theta^0} , \qquad a_\theta^0 \equiv \left. \frac{\partial\, \bar{p}(\rho,\theta)}{\partial\theta} \right|_{\rho=\rho^0,\,\theta=\theta^0} , \tag{5.8a,b}$$

$$b_\rho^0 \equiv \left. \frac{\partial\, \bar{\lambda}(\rho,\theta)}{\partial\rho} \right|_{\rho=\rho^0,\,\theta=\theta^0} , \qquad b_\theta^0 \equiv \left. \frac{\partial\, \bar{\lambda}(\rho,\theta)}{\partial\theta} \right|_{\rho=\rho^0,\,\theta=\theta^0} , \tag{5.9a,b}$$

$$c_\rho^0 \equiv \left. \frac{\partial\, \bar{\mu}(\rho,\theta)}{\partial\rho} \right|_{\rho=\rho^0,\,\theta=\theta^0} , \qquad c_\theta^0 \equiv \left. \frac{\partial\, \bar{\mu}(\rho,\theta)}{\partial\theta} \right|_{\rho=\rho^0,\,\theta=\theta^0} . \tag{5.10a,b}$$

Note that these expansion coefficients depend only on the ambient state of the fluid medium, and are thus superscripted with the symbol "0".

5.2 Entropy Production Rate

The constitutive relation for the specific internal entropy production rate ξ (physical dimension: entropy/mass/time; SI unit: J/ °K/kg/s) is expressed as

$$\rho\theta\xi = \lambda \frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j} + \frac{1}{2}\mu \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right]\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] + \kappa \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_i} , \tag{5.11}$$

where the coefficient κ has the equation of state

$$\kappa = \bar{\kappa}(\rho,\theta) . \tag{5.12}$$

That is, κ depends solely on mass density and absolute temperature.

Evaluating (5.11) and (5.12) for the reference state of the medium gives

$$\rho^0\theta^0\xi^0 = \lambda^0 \frac{\partial v_i^0}{\partial x_i}\frac{\partial v_j^0}{\partial x_j} + \frac{1}{2}\mu^0 \left[\frac{\partial v_i^0}{\partial x_j} + \frac{\partial v_j^0}{\partial x_i} \right]\left[\frac{\partial v_i^0}{\partial x_j} + \frac{\partial v_j^0}{\partial x_i} \right] + \kappa^0 \frac{\partial\theta^0}{\partial x_i}\frac{\partial\theta^0}{\partial x_i} , \tag{5.13}$$

and

$$\kappa^0 = \bar{\kappa}(\rho^0, \theta^0). \tag{5.14}$$

Linearization of equation (5.11) is postponed until after it is substituted into the entropy balance expression (Section 6.0 below). However, the linearized form of equation (5.12) is

$$\delta\kappa = d_\rho^0 \delta\rho + d_\theta^0 \delta\theta, \tag{5.15}$$

where the coefficients are the first-order partial derivatives

$$d_\rho^0 \equiv \left.\frac{\partial\, \bar{\kappa}(\rho,\theta)}{\partial \rho}\right|_{\rho=\rho^0,\, \theta=\theta^0}, \qquad d_\theta^0 \equiv \left.\frac{\partial\, \bar{\kappa}(\rho,\theta)}{\partial \theta}\right|_{\rho=\rho^0,\, \theta=\theta^0}. \tag{5.16a,b}$$

5.3 Entropy Flux Vector

The constitutive equations for the components of the entropy flux vector p_i (physical dimension: entropy/area/time; SI units: J /°K/m^2/s) are

$$p_i = -\kappa \frac{\partial \theta}{\partial x_i}, \tag{5.17}$$

where κ is given by equation (5.12). This expression indicates that κ is interpreted as a thermal conduction coefficient for entropy; thermodynamic constraints require $\kappa \geq 0$. The ambient state version and the linearized version of (5.17) are

$$p_i^0 = -\kappa^0 \frac{\partial \theta^0}{\partial x_i}, \tag{5.18}$$

and

$$\delta p_i = -\kappa^0 \frac{\partial(\delta\theta)}{\partial x_i} - \frac{\partial \theta^0}{\partial x_i} \delta\kappa, \tag{5.19}$$

respectively.

The entropy flux vector components p_i and the heat flux vector components q_i are related via $q_i = \theta\, p_i$. Thus, the product $\kappa\theta$ is interpreted as a thermal conduction coefficient for energy.

5.4 Entropy Density

In a linear viscous fluid, both the specific entropy density η and the thermodynamic pressure p may be derived from a (specific) Helmholtz free energy density function ψ:

$$\eta = -\frac{\partial \psi}{\partial \theta}, \qquad p = \rho^2 \frac{\partial \psi}{\partial \rho}, \tag{5.20a,b}$$

where ψ is strictly a function of ρ and θ: $\psi = \bar{\psi}(\rho,\theta)$. Then, equation (5.20a) implies that an equation of state for the specific entropy density has the general form

$$\eta = \bar{\eta}(\rho,\theta). \tag{5.21}$$

That is, the specific entropy density is solely a function of mass density and absolute temperature. Ambient and linearized versions of this equation of state are

$$\eta^0 = \bar{\eta}(\rho^0,\theta^0). \tag{5.22}$$

and

$$\delta\eta = g_\rho^0 \delta\rho + g_\theta^0 \delta\theta , \tag{5.23}$$

where the expansion coefficients depend on the ambient state of the viscous fluid via

$$g_\rho^0 \equiv \left.\frac{\partial \bar{\eta}(\rho,\theta)}{\partial \rho}\right|_{\rho=\rho^0,\theta=\theta^0}, \qquad g_\theta^0 \equiv \left.\frac{\partial \bar{\eta}(\rho,\theta)}{\partial \theta}\right|_{\rho=\rho^0,\theta=\theta^0}. \tag{5.24a,b}$$

6.0 ENTROPY BALANCE PRINCIPLE

The local form of the entropy balance principle may be expressed as

$$\rho\theta\left[\frac{\partial\eta}{\partial t} + v_i \frac{\partial\eta}{\partial x_i}\right] - \rho\theta\xi + \theta\frac{\partial p_i}{\partial x_i} = \frac{\partial e}{\partial t}, \tag{6.1}$$

where e is the external energy supply density (physical dimension: energy per volume; SI unit: J/m^3). The entropy balance principle expresses conservation of entropy for all material volumes of a continuum. Substituting the constitutive relations for entropy production ξ and entropy flux p_i into the above expression gives

$$\rho\theta\left[\frac{\partial\eta}{\partial t} + v_i \frac{\partial\eta}{\partial x_i}\right] - \lambda\frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j} - \frac{1}{2}\mu\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] - \frac{\partial}{\partial x_i}\left[\kappa\theta\frac{\partial\theta}{\partial x_i}\right] = \frac{\partial e}{\partial t}. \tag{6.2}$$

Recall that the product $\kappa\theta$ is the heat conduction coefficient for the linear viscous fluid. Evaluating (6.2) for the reference state of the medium yields:

$$\rho^0\theta^0\left[\frac{\partial\eta^0}{\partial t}+v_i^0\frac{\partial\eta^0}{\partial x_i}\right]-\lambda^0\frac{\partial v_i^0}{\partial x_i}\frac{\partial v_j^0}{\partial x_j}-\frac{1}{2}\mu^0\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]$$

$$-\frac{\partial}{\partial x_i}\left[\kappa^0\theta^0\frac{\partial\theta^0}{\partial x_i}\right]=\frac{\partial e^0}{\partial t}. \tag{6.3}$$

The first-order variation of (6.2) gives the linearized version:

$$\rho^0\theta^0\left[\frac{\partial(\delta\eta)}{\partial t}+v_i^0\frac{\partial(\delta\eta)}{\partial x_i}+\frac{\partial\eta^0}{\partial x_i}\delta v_i\right]+\left[\frac{\partial\eta^0}{\partial t}+v_i^0\frac{\partial\eta^0}{\partial x_i}\right]\left(\rho^0\delta\theta+\theta^0\delta\rho\right)$$

$$-2\lambda^0\left[\frac{\partial v_i^0}{\partial x_i}\frac{\partial(\delta v_j)}{\partial x_j}\right]-\left[\frac{\partial v_i^0}{\partial x_i}\frac{\partial v_j^0}{\partial x_j}\right]\delta\lambda$$

$$-\mu^0\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]\left[\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial(\delta v_j)}{\partial x_i}\right]-\frac{1}{2}\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]\delta\mu$$

$$-\frac{\partial}{\partial x_i}\left[\kappa^0\theta^0\frac{\partial(\delta\theta)}{\partial x_i}+\frac{\partial\theta^0}{\partial x_i}\left(\kappa^0\delta\theta+\theta^0\delta\kappa\right)\right]=\frac{\partial(\delta e)}{\partial t}. \tag{6.4}$$

7.0 SYSTEMS OF EQUATIONS

The set of mathematical expressions collectively representing conservation of mass, balance of momentum, balance of entropy, constitutive relations, and equations of state may be assembled together to provide a coupled system of equations that govern the dynamic behavior of a linear viscous fluid. Systems describing both the ambient and perturbed (i.e., wave propagation) state of the medium are given below.

7.1 Ambient Medium Equations

For the ambient state of the medium, the assembled system of equations is

$$\frac{\partial \rho^0}{\partial t} + v_i^0 \frac{\partial \rho^0}{\partial x_i} + \rho^0 \frac{\partial v_i^0}{\partial x_i} = 0, \tag{7.1a}$$

$$\rho^0 \left[\frac{\partial v_i^0}{\partial t} + v_j^0 \frac{\partial v_i^0}{\partial x_j} \right] - \frac{\partial \sigma_{ij}^0}{\partial x_j} = f_i^0, \tag{7.1b}$$

$$\sigma_{ij}^0 = \left[-p^0 + \lambda^0 \frac{\partial v_k^0}{\partial x_k} \right] \delta_{ij} + \mu^0 \left[\frac{\partial v_i^0}{\partial x_j} + \frac{\partial v_j^0}{\partial x_i} \right], \tag{7.1c}$$

$$\rho^0 \theta^0 \left[\frac{\partial \eta^0}{\partial t} + v_i^0 \frac{\partial \eta^0}{\partial x_i} \right] - \lambda^0 \frac{\partial v_i^0}{\partial x_i} \frac{\partial v_j^0}{\partial x_j} - \frac{1}{2} \mu^0 \left[\frac{\partial v_i^0}{\partial x_j} + \frac{\partial v_j^0}{\partial x_i} \right] \left[\frac{\partial v_i^0}{\partial x_j} + \frac{\partial v_j^0}{\partial x_i} \right]$$

$$- \frac{\partial}{\partial x_i} \left[\kappa^0 \theta^0 \frac{\partial \theta^0}{\partial x_i} \right] = \frac{\partial e^0}{\partial t}, \tag{7.1d}$$

$$p^0 = \bar{p}(\rho^0, \theta^0), \qquad \lambda^0 = \bar{\lambda}(\rho^0, \theta^0), \qquad \mu^0 = \bar{\mu}(\rho^0, \theta^0), \tag{7.1e,f,g}$$

$$\kappa^0 = \bar{\kappa}(\rho^0, \theta^0), \qquad \eta^0 = \bar{\eta}(\rho^0, \theta^0). \tag{7.1h,i}$$

Clearly, the ambient medium parameters cannot all be independently specified. Rather, system (7.1) constitutes a set of nonlinear constraints that these material parameters must satisfy.

7.2 Wave Propagation Equations

Collecting the linearized versions of the aforementioned equations together yields the system:

$$\frac{\partial(\delta\rho)}{\partial t}+v_i^0\frac{\partial(\delta\rho)}{\partial x_i}+\frac{\partial v_i^0}{\partial x_i}\delta\rho+\rho^0\frac{\partial(\delta v_i)}{\partial x_i}+\frac{\partial\rho^0}{\partial x_i}\delta v_i=0, \tag{7.2a}$$

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t}+v_j^0\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial v_i^0}{\partial x_j}\delta v_j\right]+\left[\frac{\partial v_i^0}{\partial t}+v_j^0\frac{\partial v_i^0}{\partial x_j}\right]\delta\rho-\frac{\partial(\delta\sigma_{ij})}{\partial x_j}=\delta f_i, \tag{7.2b}$$

$$\delta\sigma_{ij}=\left[-\delta p+\lambda^0\frac{\partial(\delta v_k)}{\partial x_k}+\frac{\partial v_k^0}{\partial x_k}\delta\lambda\right]\delta_{ij}+\mu^0\left[\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial(\delta v_j)}{\partial x_i}\right]+\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]\delta\mu, \tag{7.2c}$$

$$\rho^0\theta^0\left[\frac{\partial(\delta\eta)}{\partial t}+v_i^0\frac{\partial(\delta\eta)}{\partial x_i}+\frac{\partial\eta^0}{\partial x_i}\delta v_i\right]+\left[\frac{\partial\eta^0}{\partial t}+v_i^0\frac{\partial\eta^0}{\partial x_i}\right]\left(\theta^0\delta\rho+\rho^0\delta\theta\right)$$

$$-2\lambda^0\left[\frac{\partial v_i^0}{\partial x_i}\frac{\partial(\delta v_j)}{\partial x_j}\right]-\left[\frac{\partial v_i^0}{\partial x_i}\frac{\partial v_j^0}{\partial x_j}\right]\delta\lambda$$

$$-\mu^0\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]\left[\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial(\delta v_j)}{\partial x_i}\right]-\frac{1}{2}\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]\left[\frac{\partial v_i^0}{\partial x_j}+\frac{\partial v_j^0}{\partial x_i}\right]\delta\mu$$

$$-\frac{\partial}{\partial x_i}\left[\kappa^0\theta^0\frac{\partial(\delta\theta)}{\partial x_i}+\frac{\partial\theta^0}{\partial x_i}\left(\kappa^0\delta\theta+\theta^0\delta\kappa\right)\right]=\frac{\partial(\delta e)}{\partial t}, \tag{7.2d}$$

$$\delta p=a_\rho^0\delta\rho+a_\theta^0\delta\theta,\qquad \delta\lambda=b_\rho^0\delta\rho+b_\theta^0\delta\theta,\qquad \delta\mu=c_\rho^0\delta\rho+c_\theta^0\delta\theta, \tag{7.2e,f,g}$$

$$\delta\kappa=d_\rho^0\delta\rho+d_\theta^0\delta\theta,\qquad \delta\eta=g_\rho^0\delta\rho+g_\theta^0\delta\theta. \tag{7.2h,i}$$

Expressions (7.2a through i) constitute a set of 16 coupled linear equations containing the 16 dependent variables δv_i (i = 1,2,3), $\delta\sigma_{ij}$ (i,j = 1,2,3 with $\delta\sigma_{ij}=\delta\sigma_{ji}$), $\delta\rho$, δp, $\delta\lambda$, $\delta\mu$, $\delta\kappa$, $\delta\eta$, and $\delta\theta$. In principle, after boundary and initial conditions are specified, the system may be solved for these unknown acoustic wavefield variables, as functions of position **r** and time t.

Clearly, the number of equations and dependent variables may be reduced by combining various equations within the system. Perhaps the simplest reduction entails eliminating the four perturbations δp, $\delta\lambda$, $\delta\mu$, and $\delta\kappa$ in favor of $\delta\rho$ and $\delta\theta$, by substituting expressions (7.2e through h) into (7.2c and d). None of these four perturbations occurs in a differentiated form within system (7.2), so the resulting system of 12 equations with 12 unknowns is not overly complicated. In contrast, eliminating the entropy fluctuation $\delta\eta$ by substituting (7.2i) into (7.2d) entails differentiating the expansion coefficients g_ρ^0 and g_θ^0 with respect to time t and spatial coordinates x_i. Similarly, combining the stress constitutive relations (7.2c) with the equations of motion (7.2b) involves differentiating the viscosity coefficients λ^0 and μ^0 with respect to coordinates x_i.

8.0 NON-HEAT-CONDUCTING IDEAL FLUID

An important special case of the linear viscous fluid occurs when the viscosity coefficients *and* the entropy conduction coefficient vanish. For $\lambda = \mu = 0$, the medium is referred to as an ideal fluid, or a perfect fluid, or an inviscid fluid. For $\kappa = 0$, the medium is non-heat-conducting. Setting $\lambda = \mu = \kappa = 0$ in the constitutive relations (5.1), (5.11), and (5.17) yield

$$\sigma_{ij} = -p\delta_{ij}, \tag{8.1}$$

$$\rho\theta\xi = 0, \tag{8.2}$$

$$p_i = 0. \tag{8.3}$$

Thus, in an ideal fluid, the stress tensor is isotropic. Equation (8.2) implies $\xi = 0$, since both the mass density ρ and the absolute temperature θ are intrinsically positive. Thus, there is no internal entropy production (and any subsequent entropy conduction) within an ideal fluid.

The coupled systems of equations characterizing an ideal and non-heat-conducting fluid are assembled in the following subsections. These systems are simplified by eliminating the stress tensor components σ_{ij} in favor of the pressure p, merely by combining the equations of motion with the stress constitutive relations.

8.1 Ambient Medium Equations

Setting $\lambda^0 = \mu^0 = \delta\lambda = \delta\mu = 0$ as well as $\kappa^0 = \delta\kappa = 0$ in the previous system (7.1) gives

$$\frac{\partial \rho^0}{\partial t} + v_i^0 \frac{\partial \rho^0}{\partial x_i} + \rho^0 \frac{\partial v_i^0}{\partial x_i} = 0, \tag{8.4a}$$

$$\rho^0 \left[\frac{\partial v_i^0}{\partial t} + v_j^0 \frac{\partial v_i^0}{\partial x_j} \right] + \frac{\partial p^0}{\partial x_i} = f_i^0, \tag{8.4b}$$

$$\rho^0\theta^0 \left[\frac{\partial \eta^0}{\partial t} + v_i^0 \frac{\partial \eta^0}{\partial x_i} \right] = \frac{\partial e^0}{\partial t}, \tag{8.4c}$$

$$p^0 = \overline{p}(\rho^0, \theta^0), \qquad \eta^0 = \overline{\eta}(\rho^0, \theta^0). \tag{8.4d,e}$$

This is a set of seven coupled equations linking the seven quantities v_i^0 ($i = 1,2,3$), ρ^0, p^0, η^0, and θ^0 (i.e, particle velocity vector components, mass density, pressure, specific entropy density, and absolute temperature of the ambient medium, respectively). Clearly, system (8.4) is nonlinear with respect to these variables. The nonhomogeneous terms f_i^0 (external force density vector components) and $\partial e^0/\partial t$ (external energy density supply rate) are considered prescribed functions of position **r** and time t.

8.2 Wave Propagation Equations

Similarly, setting $\lambda^0 = \mu^0 = \delta\lambda = \delta\mu = 0$ as well as $\kappa^0 = \delta\kappa = 0$ in the previous system (7.2) gives

$$\frac{\partial(\delta\rho)}{\partial t} + v_i^0 \frac{\partial(\delta\rho)}{\partial x_i} + \frac{\partial v_i^0}{\partial x_i}\delta\rho + \rho^0 \frac{\partial(\delta v_i)}{\partial x_i} + \frac{\partial \rho^0}{\partial x_i}\delta v_i = 0, \tag{8.5a}$$

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t} + v_j^0 \frac{\partial(\delta v_i)}{\partial x_j} + \frac{\partial v_i^0}{\partial x_j}\delta v_j\right] + \left[\frac{\partial v_i^0}{\partial t} + v_j^0 \frac{\partial v_i^0}{\partial x_j}\right]\delta\rho + \frac{\partial(\delta p)}{\partial x_i} = \delta f_i, \tag{8.5b}$$

$$\rho^0\theta^0\left[\frac{\partial(\delta\eta)}{\partial t} + v_i^0 \frac{\partial(\delta\eta)}{\partial x_i} + \frac{\partial \eta^0}{\partial x_i}\delta v_i\right] + \left[\frac{\partial \eta^0}{\partial t} + v_i^0 \frac{\partial \eta^0}{\partial x_i}\right]\left(\theta^0\delta\rho + \rho^0\delta\theta\right) = \frac{\partial(\delta e)}{\partial t}, \tag{8.5c}$$

$$\delta p = a_\rho^0 \delta\rho + a_\theta^0 \delta\theta, \tag{8.5d}$$

$$\delta\eta = g_\rho^0 \delta\rho + g_\theta^0 \delta\theta. \tag{8.5e}$$

This is a set of seven, coupled, linear equations containing the seven dependent variables δv_i (i = 1,2,3), $\delta\rho$, δp, $\delta\eta$, and $\delta\theta$. (i.e., fluctuations in particle velocity vector components, mass density, pressure, specific entropy density, and absolute temperature, respectively). The background medium is characterized by fluid velocity vector components v_i^0 (i=1,2,3), mass density ρ^0, specific entropy density η^0, and absolute temperature θ^0, as well as four coefficients a_ρ^0, a_θ^0, g_ρ^0, g_θ^0 that arise from linearizing various equations of state. Acoustic waves are initiated via the non-homogeneous terms in the system. These are fluctuations in the force density vector components δf_i (i=1,2,3) and fluctuations in the rate of external energy supply $\partial(\delta e)/\partial t$. All quantities may be functions of both position **r** and time t.

The number of equations and dependent variables in system (8.5) may be reduced by combining various expressions. The most straightforward reduction entails eliminating the absolute temperature fluctuation $\delta\theta$, since it does not appear in (8.5) in differentiated form (either with respect to time t or spatial coordinates x_i). Eliminating the absolute temperature perturbation $\delta\theta$ yields a system of six coupled equations with six dependent variables:

$$\frac{\partial(\delta\rho)}{\partial t} + v_i^0 \frac{\partial(\delta\rho)}{\partial x_i} + \frac{\partial v_i^0}{\partial x_i}\delta\rho + \rho^0 \frac{\partial(\delta v_i)}{\partial x_i} + \frac{\partial \rho^0}{\partial x_i}\delta v_i = 0, \tag{8.6a}$$

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t} + v_j^0 \frac{\partial(\delta v_i)}{\partial x_j} + \frac{\partial v_i^0}{\partial x_j}\delta v_j\right] + \left[\frac{\partial v_i^0}{\partial t} + v_j^0 \frac{\partial v_i^0}{\partial x_j}\right]\delta\rho + \frac{\partial(\delta p)}{\partial x_i} = \delta f_i, \tag{8.6b}$$

$$\rho^0\theta^0\left[\frac{\partial(\delta\eta)}{\partial t} + v_i^0 \frac{\partial(\delta\eta)}{\partial x_i} + \frac{\partial \eta^0}{\partial x_i}\delta v_i\right] + \left[\frac{\partial \eta^0}{\partial t} + v_i^0 \frac{\partial \eta^0}{\partial x_i}\right]\left(r^0\delta\rho + s^0\delta\eta\right) = \frac{\partial(\delta e)}{\partial t}, \tag{8.6c}$$

$$\delta p = \left(c^0\right)^2 \delta\rho + h^0 \delta\eta \,, \tag{8.6d}$$

where new coefficients are defined as

$$\left(c^0\right)^2 \equiv a_\rho^0 - a_\theta^0 \frac{g_\rho^0}{g_\theta^0}, \qquad h^0 \equiv \frac{a_\theta^0}{g_\theta^0}, \tag{8.7a,b}$$

$$r^0 \equiv \theta^0 - \rho^0 \frac{g_\rho^0}{g_\theta^0}, \qquad s^0 \equiv \frac{\rho^0}{g_\theta^0}. \tag{8.7c,d}$$

Equation (8.6d) indicates that the pressure perturbation is a linear combination of the mass density and specific entropy density perturbations. The coefficients in this superposition are readily interpreted. From equations (5.20a,b), both the pressure and the specific entropy density are considered to be functions of mass density and absolute temperature, that is $p = \overline{p}(\rho,\theta)$ and $\eta = \overline{\eta}(\rho,\theta)$. Inverting the second of these expressions for absolute temperature gives $\theta = \tilde{\theta}(\rho,\eta)$. Substituting this into the first expression indicates that the pressure may be expressed as a *different* function of mass density and specific entropy density:

$$p = \overline{p}(\rho,\theta) = \overline{p}\left(\rho,\tilde{\theta}(\rho,\eta)\right) = \tilde{p}(\rho,\eta). \tag{8.8}$$

Equation (8.8) may be considered an alternative equation of state for pressure. A first-order Taylor series expansion of (8.8) gives

$$p^0 + \delta p = \tilde{p}\left(\rho^0 + \delta\rho, \eta^0 + \delta\eta\right) = p^0 + \left(c^0\right)^2 \delta\rho + h^0 \delta\eta \,, \tag{8.9}$$

where

$$p^0 \equiv \tilde{p}\left(\rho^0,\eta^0\right), \qquad \left(c^0\right)^2 \equiv \left.\frac{\partial\,\tilde{p}(\rho,\eta)}{\partial\rho}\right|_{\rho=\rho^0,\,\eta=\eta^0}, \qquad h^0 \equiv \left.\frac{\partial\,\tilde{p}(\rho,\eta)}{\partial\eta}\right|_{\rho=\rho^0,\,\eta=\eta^0}. \tag{8.10a,b,c}$$

Thus, $(c^0)^2$ is the squared adiabatic sound speed (change in pressure produced by a change in mass density, for fixed specific entropy density) evaluated for the background medium. Coefficient h^0 quantifies the change in pressure produced by a change in specific entropy density (for fixed mass density), also evaluated for the background medium. Coefficients r^0 and s^0 [in equation (8.6c)] do not appear to have such straightforward interpretations.

9.0 ADIABATIC AMBIENT MEDIUM

In acoustic wave propagation problems, the ambient state is commonly assumed to be *adiabatic*. That is, the material derivative of the specific entropy density of the background medium vanishes:

$$\frac{\partial \eta^0}{\partial t} + v_i^0 \frac{\partial \eta^0}{\partial x_i} = 0. \tag{9.1}$$

If the ambient state is adiabatic, then equation (8.4c) implies that the rate of external energy supply to the ambient medium equals zero:

$$\frac{\partial e^0}{\partial t} = 0. \tag{9.2}$$

The adiabatic assumption simplifies the linearized acoustic wave propagation system (8.6) slightly:

$$\frac{\partial(\delta\rho)}{\partial t} + v_i^0 \frac{\partial(\delta\rho)}{\partial x_i} + \frac{\partial v_i^0}{\partial x_i}\delta\rho + \rho^0 \frac{\partial(\delta v_i)}{\partial x_i} + \frac{\partial \rho^0}{\partial x_i}\delta v_i = 0, \tag{9.3a}$$

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t} + v_j^0 \frac{\partial(\delta v_i)}{\partial x_j} + \frac{\partial v_i^0}{\partial x_j}\delta v_j\right] + \left[\frac{\partial v_i^0}{\partial t} + v_j^0 \frac{\partial v_i^0}{\partial x_j}\right]\delta\rho + \frac{\partial(\delta p)}{\partial x_i} = \delta f_i, \tag{9.3b}$$

$$\frac{\partial(\delta\eta)}{\partial t} + v_i^0 \frac{\partial(\delta\eta)}{\partial x_i} + \frac{\partial \eta^0}{\partial x_i}\delta v_i = \frac{1}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}, \tag{9.3c}$$

$$\delta p = \left(c^0\right)^2 \delta\rho + h^0 \delta\eta. \tag{9.3d}$$

[Equations (9.3a,b,d) are identical to (8.6a,b,d)]. These are still six coupled equations containing six unknown wavefield variables. However, the two ambient medium parameters r^0 and s^0 have disappeared, leaving the acoustic model characterized by the eight quantities v_i^0 (i=1,2,3), ρ^0, c^0, h^0, η^0, and θ^0. It should be noted that spatial derivatives of mass density and specific entropy density are related via the equation of state (8.10a):

$$\frac{\partial p^0}{\partial x_i} = \left(c^0\right)^2 \frac{\partial \rho^0}{\partial x_i} + h^0 \frac{\partial \eta^0}{\partial x_i}.$$

Thus, one may be eliminated in favor of the other in system (9.3), although the pressure derivative $\partial p^0/\partial x_i$ is then introduced.

Often, an additional adiabatic assumption is adopted: the material derivative of the *total* entropy density associated with the wave propagation process vanishes. Thus

$$\frac{\partial \eta}{\partial t} + v_i \frac{\partial \eta}{\partial x_i} = 0. \tag{9.4}$$

Substituting $\eta = \eta^0 + \delta\eta$ and $v_i = v_i^0 + \delta v_i$, subtracting the ambient medium adiabatic condition (9.1), and neglecting the second-order term gives the linearized form

$$\frac{\partial(\delta\eta)}{\partial t}+v_i^0\frac{\partial(\delta\eta)}{\partial x_i}+\frac{\partial\eta^0}{\partial x_i}\delta v_i=0. \tag{9.5}$$

Comparing this expression with equation (9.3c) above indicates that the total adiabatic assumption is equivalent to neglecting energy density sources of acoustic waves. Blokhintzev (1946) obtains a related version of the set of four coupled equations (9.3a,b,d) and (9.5), assuming additionally that $\delta f_i = 0$ (i.e., no acoustic force sources) as well as $f_i^0 = 0$, $\partial e^0/\partial t = 0$ (i.e., no force or energy sources active in the ambient medium).

9.1 Purely Mechanical System

Adiabatic assumption (9.1) facilitates the derivation of a coupled system of first-order partial differential equations containing only the five "mechanical" dependent variables δv_i (i=1,2,3), δp, and $\delta\rho$. The following, somewhat lengthy, derivation develops this particular system. Eliminating the fluctuation in specific entropy density $\delta\eta$ between equations (9.3c and d) yields

$$\left[\frac{\partial(\delta p)}{\partial t}+v_i^0\frac{\partial(\delta p)}{\partial x_i}\right]-\left(c^0\right)^2\left[\frac{\partial(\delta\rho)}{\partial t}+v_i^0\frac{\partial(\delta\rho)}{\partial x_i}\right]+h^0\frac{\partial\eta^0}{\partial x_i}\delta v_i$$

$$-\left[\frac{\partial(c^0)^2}{\partial t}+v_i^0\frac{\partial(c^0)^2}{\partial x_i}\right]\delta\rho-\left[\frac{\partial h^0}{\partial t}+v_i^0\frac{\partial h^0}{\partial x_i}\right]\left(\frac{\delta p-\left(c^0\right)^2\delta\rho}{h^0}\right)=\frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}. \tag{9.6}$$

Next, eliminating the spatial derivative of the ambient specific entropy density by using

$$\frac{\partial p^0}{\partial x_i}=\left(c^0\right)^2\frac{\partial\rho^0}{\partial x_i}+h^0\frac{\partial\eta^0}{\partial x_i}, \tag{9.7}$$

[from the chain rule of partial differentiation applied to equation of state (8.10a)] gives

$$\left[\frac{\partial(\delta p)}{\partial t}+v_i^0\frac{\partial(\delta p)}{\partial x_i}+\frac{\partial p^0}{\partial x_i}\delta v_i\right]-\left(c^0\right)^2\left[\frac{\partial(\delta\rho)}{\partial t}+v_i^0\frac{\partial(\delta\rho)}{\partial x_i}+\frac{\partial\rho^0}{\partial x_i}\delta v_i\right]$$

$$-\left[\frac{\partial(c^0)^2}{\partial t}+v_i^0\frac{\partial(c^0)^2}{\partial x_i}\right]\delta\rho-\left[\frac{\partial h^0}{\partial t}+v_i^0\frac{\partial h^0}{\partial x_i}\right]\left(\frac{\delta p-(c^0)^2\delta\rho}{h^0}\right)=\frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}. \tag{9.8}$$

The terms in (9.8) involving material derivatives of the ambient medium parameters $(c^0)^2$ and h^0 are now examined. Applying the chain rule to the definition $\left(c^0\right)^2=\partial\tilde{p}\left(\rho^0,\eta^0\right)/\partial\rho^0$ gives

$$\frac{\partial(c^0)^2}{\partial t}+v_i^0\frac{\partial(c^0)^2}{\partial x_i}=k^0\left[\frac{\partial\rho^0}{\partial t}+v_i^0\frac{\partial\rho^0}{\partial x_i}\right]+l^0\left[\frac{\partial\eta^0}{\partial t}+v_i^0\frac{\partial\eta^0}{\partial x_i}\right]. \tag{9.9a}$$

Similarly, applying the chain rule to the definition $h^0=\partial\tilde{p}\left(\rho^0,\eta^0\right)/\partial\eta^0$ gives

$$\frac{\partial h^0}{\partial t}+v_i^0\frac{\partial h^0}{\partial x_i}=l^0\left[\frac{\partial \rho^0}{\partial t}+v_i^0\frac{\partial \rho^0}{\partial x_i}\right]+m^0\left[\frac{\partial \eta^0}{\partial t}+v_i^0\frac{\partial \eta^0}{\partial x_i}\right]. \tag{9.9b}$$

Coefficients in the above two expressions are defined by second-order partial derivatives of the equation of state $p=\tilde{p}(\rho,\eta)$:

$$k^0 \equiv \left.\frac{\partial^2 \tilde{p}(\rho,\eta)}{\partial \rho^2}\right|_{\rho=\rho^0,\,\eta=\eta^0}, \qquad m^0 \equiv \left.\frac{\partial^2 \tilde{p}(\rho,\eta)}{\partial \eta^2}\right|_{\rho=\rho^0,\,\eta=\eta^0}, \tag{9.10a.,b}$$

$$l^0 \equiv \left.\frac{\partial^2 \tilde{p}(\rho,\eta)}{\partial \rho \partial \eta}\right|_{\rho=\rho^0,\,\eta=\eta^0} = \left.\frac{\partial^2 \tilde{p}(\rho,\eta)}{\partial \eta \partial \rho}\right|_{\rho=\rho^0,\,\eta=\eta^0}. \tag{9.10c}$$

Note that a Maxwell relation (i.e., equality of mixed second-order partial derivatives) is used in (9.10c). If the ambient state is adiabatic, then expressions (9.9a,b) reduce to

$$\frac{\partial (c^0)^2}{\partial t}+v_i^0\frac{\partial (c^0)^2}{\partial x_i}=k^0\left[\frac{\partial \rho^0}{\partial t}+v_i^0\frac{\partial \rho^0}{\partial x_i}\right], \tag{9.11a}$$

and

$$\frac{\partial h^0}{\partial t}+v_i^0\frac{\partial h^0}{\partial x_i}=l^0\left[\frac{\partial \rho^0}{\partial t}+v_i^0\frac{\partial \rho^0}{\partial x_i}\right]. \tag{9.11b}$$

Dependence on the coefficient m^0 disappears. Combining these expressions with the equation of continuity for the ambient medium (equation (8.4a) above) yields the forms

$$\frac{\partial (c^0)^2}{\partial t}+v_i^0\frac{\partial (c^0)^2}{\partial x_i}=-k^0\rho^0\frac{\partial v_i^0}{\partial x_i}, \tag{9.12a}$$

$$\frac{\partial h^0}{\partial t}+v_i^0\frac{\partial h^0}{\partial x_i}=-l^0\rho^0\frac{\partial v_i^0}{\partial x_i}. \tag{9.12b}$$

Thus, the material derivatives of the ambient medium parameters $(c^0)^2$ and h^0 are proportional to the divergence of the fluid velocity field. Expressions (9.12a,b) are substituted into equation (9.8) to obtain

$$\left[\frac{\partial(\delta p)}{\partial t}+v_i^0\frac{\partial(\delta p)}{\partial x_i}+\frac{\partial p^0}{\partial x_i}\delta v_i\right]-\left(c^0\right)^2\left[\frac{\partial(\delta \rho)}{\partial t}+v_i^0\frac{\partial(\delta \rho)}{\partial x_i}+\frac{\partial \rho^0}{\partial x_i}\delta v_i\right]$$

$$+\rho^0\frac{\partial v_i^0}{\partial x_i}\left[\frac{l^0}{h^0}\delta p+\left(k^0-\frac{l^0}{h^0}\left(c^0\right)^2\right)\delta \rho\right]=\frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}. \tag{9.13}$$

Finally, the temporal derivative of the mass density fluctuation is eliminated by using equation (9.3a) above, yielding:

$$\frac{\partial(\delta p)}{\partial t}+v_i^0\frac{\partial(\delta p)}{\partial x_i}+\rho^0\left(c^0\right)^2\frac{\partial(\delta v_i)}{\partial x_i}+\frac{\partial p^0}{\partial x_i}\delta v_i$$

$$+\frac{\partial v_i^0}{\partial x_i}\left[\frac{\rho^0 l^0}{h^0}\delta p+\left(c^0\right)^2\left(\frac{\rho^0 k^0}{\left(c^0\right)^2}+1-\frac{\rho^0 l^0}{h^0}\right)\delta\rho\right]=\frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}. \tag{9.14a}$$

This expression, together with equations (9.3a and b), constitute a coupled linear system containing the five dependent variables δv_i (i=1,2,3), δp, and $\delta\rho$. Equations (9.3a and b) are repeated here:

$$\frac{\partial(\delta\rho)}{\partial t}+v_i^0\frac{\partial(\delta\rho)}{\partial x_i}+\frac{\partial v_i^0}{\partial x_i}\delta\rho+\rho^0\frac{\partial(\delta v_i)}{\partial x_i}+\frac{\partial\rho^0}{\partial x_i}\delta v_i=0, \tag{9.14b}$$

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t}+v_j^0\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial v_i^0}{\partial x_j}\delta v_j\right]+\left[\frac{\partial v_i^0}{\partial t}+v_j^0\frac{\partial v_i^0}{\partial x_j}\right]\delta\rho+\frac{\partial(\delta p)}{\partial x_i}=\delta f_i. \tag{9.14c}$$

Although the variable $\delta\eta$ (perturbation in specific entropy density) has been eliminated, system (9.14) still retains some dependence on thermodynamic quantities (i.e., h^0, l^0, and θ^0) in equation (9.14a). Further reductions occur for specific, simple motions of the background medium.

9.1.1 Stationary Background

If the ambient medium is not moving, then $v_i^0 = 0$ and system (9.14) reduces to

$$\rho^0\frac{\partial(\delta v_i)}{\partial t}+\frac{\partial(\delta p)}{\partial x_i}=\delta f_i, \tag{9.15a}$$

$$\frac{\partial(\delta p)}{\partial t}+\rho^0\left(c^0\right)^2\frac{\partial(\delta v_i)}{\partial x_i}+\frac{\partial p^0}{\partial x_i}\delta v_i=\frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}, \tag{9.15b}$$

$$\frac{\partial(\delta\rho)}{\partial t}=-\frac{\partial(\rho^0\delta v_i)}{\partial x_i}. \tag{9.15c}$$

Equations (9.15a,b) are a system of four, coupled, first-order partial differential equations containing the four dependent variables δv_i (i=1,2,3) and δp. After solution, the mass density perturbation $\delta\rho$ may be found by integrating (9.15c) in time. Thus, the solution for the particle velocity and pressure fluctuations decouples from the solution for the mass density fluctuation.

If the background pressure field is uniform (i.e., $\partial p^0/\partial x_i = 0$), then equations (9.15a,b) are identical to the velocity-pressure system commonly used for seismic wave propagation modeling. The nonhomogeneous term in (9.15b) is interpreted as a moment density source (physical dimension: moment per volume; SI units: N-m/m^3) differentiated with respect to time.

Interestingly, the twin assumptions of an *adiabatic* and *stationary* background medium imply that the ambient state material parameters are all time-invariant. Equations (9.12a,b) indicate that $\partial c^0/\partial t = \partial h^0/\partial t = 0$, whereas (8.4a) indicates $\partial\rho^0/\partial t = 0$. The adiabatic assumption (9.1) reduces to $\partial\eta^0/\partial t = 0$. Applying the chain rule of partial differentiation to the equations of state (8.4d,e) indicate that $\partial p^0/\partial t = \partial\theta^0/\partial t = 0$. However, all of these quantities may depend on the position vector **r**, i.e., the background medium may be heterogeneous.

A single, second-order (in space and time) partial differential equation for acoustic pressure is obtained by combining expressions (9.15a,b). Thus:

$$\rho^0 \frac{\partial}{\partial x_i}\left[\frac{1}{\rho^0}\frac{\partial(\delta p)}{\partial x_i}\right] + \frac{1}{\rho^0 \left(c^0\right)^2}\frac{\partial p^0}{\partial x_i}\frac{\partial(\delta p)}{\partial x_i} - \frac{1}{\left(c^0\right)^2}\frac{\partial^2(\delta p)}{\partial t^2}$$

$$= \rho^0 \frac{\partial}{\partial x_i}\left[\frac{1}{\rho^0}\delta f_i\right] + \frac{1}{\rho^0\left(c^0\right)^2}\frac{\partial p^0}{\partial x_i}\delta f_i - \frac{\partial}{\partial t}\left[\frac{h^0}{\rho^0\left(c^0\right)^2\theta^0}\frac{\partial(\delta e)}{\partial t}\right]. \tag{9.16}$$

An alternative way of writing the combined body source terms on the right-hand-side is

$$\frac{\partial(\delta f_i)}{\partial x_i} + \frac{h^0}{\rho^0\left(c^0\right)^2}\left[\frac{\partial\eta^0}{\partial x_i}\delta f_i - \frac{1}{\theta^0}\frac{\partial^2(\partial e)}{\partial t^2}\right].$$

Thus, body sources of acoustic waves involve (i) the divergence of the force density fluctuation, (ii) the inner product of the ambient entropy gradient with the force density fluctuation, and (iii) the second time derivative of the energy density fluctuation. This illustrates the enhanced complexity in source characterization that arises when first-order equations are combined [e.g., compare with system (9.15a,b) where body source terms are characterized by "lower-order" differentiations, and no pressure or entropy gradients are involved].

For uniform pressure *and* mass density in the ambient medium, equation (9.16) reduces to the well-known "variable velocity wave equation" traditionally used in seismic wave propagation modeling. Pierce (1990) states that the first published derivation of the homogeneous version of (9.16), for uniform pressure and *non-uniform* mass density, is given by Bergmann (1946).

9.1.2 Steady Uniform Flow

Another simple reduction occurs when the background velocity is independent of both temporal and spatial coordinates: $\mathbf{v}^0(\mathbf{r},t) = \mathbf{v}^0$. System (9.14) becomes

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t} + v_j^0 \frac{\partial(\delta v_i)}{\partial x_j}\right] + \frac{\partial(\delta p)}{\partial x_i} = \delta f_i, \tag{9.17a}$$

$$\frac{\partial(\delta p)}{\partial t} + v_i^0 \frac{\partial(\delta p)}{\partial x_i} + \rho^0\left(c^0\right)^2 \frac{\partial(\delta v_i)}{\partial x_i} + \frac{\partial p^0}{\partial x_i}\delta v_i = \frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}, \tag{9.17b}$$

$$\frac{\partial(\delta\rho)}{\partial t} + v_i^0 \frac{\partial(\delta\rho)}{\partial x_i} = -\frac{\partial(\rho^0\delta v_i)}{\partial x_i}. \tag{9.17c}$$

Once again, solution for the particle velocity and pressure perturbations is uncoupled from the mass density perturbation. The four equations (9.17a,b) are solved for δv_i and δp, and then $\delta\rho$ is obtained from (9.17c). Careful inspection reveals that system (9.17) may be obtained from system (9.15) by replacing all partial time derivatives of dependent variables with material time derivatives.

For steady (time-invariant) and uniform (space-invariant) flow, the *material derivatives* of all ambient medium parameters vanish. If the material derivative operator for the background medium is defined as $d/dt \equiv \partial/\partial t + v_i^0\, \partial/\partial x_i$, then previous equations imply

$$\frac{dc^0}{dt} = 0, \qquad \frac{dh^0}{dt} = 0, \qquad \frac{d\rho^0}{dt} = 0, \qquad \frac{d\eta^0}{dt} = 0, \qquad \frac{dp^0}{dt} = 0, \qquad \frac{d\theta^0}{dt} = 0.$$

Of course, dv_i^0/dt vanishes (trivially).

Combining (9.17a and b) gives a single, second-order partial differential equation for the acoustic pressure perturbation δp:

$$\rho^0 \frac{\partial}{\partial x_i}\left[\frac{1}{\rho^0}\frac{\partial(\delta p)}{\partial x_i}\right] + \frac{1}{\rho^0\left(c^0\right)^2}\frac{\partial p^0}{\partial x_i}\frac{\partial(\delta p)}{\partial x_i} - \frac{1}{\left(c^0\right)^2}\frac{d^2(\delta p)}{dt^2}$$

$$= \rho^0\frac{\partial}{\partial x_i}\left[\frac{1}{\rho^0}\delta f_i\right] + \frac{1}{\rho^0\left(c^0\right)^2}\frac{\partial p^0}{\partial x_i}\delta f_i - \frac{d}{dt}\left[\frac{h^0}{\rho^0\left(c^0\right)^2\theta^0}\frac{\partial(\delta e)}{\partial t}\right]. \tag{9.18}$$

This expression bears a remarkable similarity to equation (9.16) above. Contrary to speculation by Pierce (1990), it is valid for spatially variable mass density ρ^0 *and* sound speed c^0.

9.1.3 Steady, Laterally Invariant, Horizontal Flow

Suppose that the background fluid flow is strictly horizontal (i.e., $v_3^0 = 0$), as well as independent of time t and the two horizontal coordinates x_1 and x_2:

$$\mathbf{v}^0(\mathbf{r},t) = v_1^0(x_3)\mathbf{e}_1 + v_2^0(x_3)\mathbf{e}_2 . \tag{9.19}$$

Vertical variation in the two horizontal velocity components is allowed. Then, system (9.14) may be put into the form

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t} + v_1^0\frac{\partial(\delta v_i)}{\partial x_1} + v_2^0\frac{\partial(\delta v_i)}{\partial x_2} + \frac{\partial v_i^0}{\partial x_3}\delta v_3\right] + \frac{\partial(\delta p)}{\partial x_i} = \delta f_i , \tag{9.20a}$$

$$\frac{\partial(\delta p)}{\partial t} + v_1^0\frac{\partial(\delta p)}{\partial x_1} + v_2^0\frac{\partial(\delta p)}{\partial x_2} + \rho^0\left(c^0\right)^2\frac{\partial(\delta v_i)}{\partial x_i} + \frac{\partial p^0}{\partial x_i}\delta v_i = \frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t} , \tag{9.20b}$$

$$\frac{\partial(\delta\rho)}{\partial t} + v_1^0\frac{\partial(\delta\rho)}{\partial x_1} + v_2^0\frac{\partial(\delta\rho)}{\partial x_2} = -\frac{\partial(\rho^0\delta v_i)}{\partial x_i} . \tag{9.20c}$$

Once again, the particle velocity δv_i and pressure δp perturbations are obtained by solving a coupled system of four, first-order, partial differential equations (9.20a,b). The mass density perturbation $\delta\rho$ is obtained subsequently solving (9.20c).

9.1.4 Divergence-Free Flow

The previous three background fluid velocity situations are all *particular* cases of divergence-free fluid flow. That is, $\partial v_i^0/\partial x_i = 0$. For *general* divergence-free flow, with no additional restrictions, system (9.14) simplifies slightly:

$$\rho^0\left[\frac{\partial(\delta v_i)}{\partial t} + v_j^0\frac{\partial(\delta v_i)}{\partial x_j} + \frac{\partial v_i^0}{\partial x_j}\delta v_j\right] + \left[\frac{\partial v_i^0}{\partial t} + v_j^0\frac{\partial v_i^0}{\partial x_j}\right]\delta\rho + \frac{\partial(\delta p)}{\partial x_i} = \delta f_i , \tag{9.21a}$$

$$\frac{\partial(\delta p)}{\partial t} + v_i^0\frac{\partial(\delta p)}{\partial x_i} + \rho^0\left(c^0\right)^2\frac{\partial(\delta v_i)}{\partial x_i} + \frac{\partial p^0}{\partial x_i}\delta v_i = \frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t} , \tag{9.21b}$$

$$\frac{\partial(\delta\rho)}{\partial t} + v_i^0\frac{\partial(\delta\rho)}{\partial x_i} + \rho^0\frac{\partial(\delta v_i)}{\partial x_i} + \frac{\partial\rho^0}{\partial x_i}\delta v_i = 0 . \tag{9.21c}$$

Pressure and particle velocity fluctuations no longer decouple from the mass density fluctuation. System (9.21) must be solved as a set of five coupled equations with five dependent variables. However, the divergence-free flow assumption eliminates two ambient medium parameters (k^0 and l^0) from the system. If, additionally, there are no energy sources (i.e., $\partial(\delta e)/\partial t = 0$), then *all* thermodynamic parameters (i.e., including h^0 and θ^0) disappear. The ambient medium is characterized by particle velocity vector components v_i^0 (i=1,2,3), mass density ρ^0, and adiabatic sound speed c^0. [The pressure gradient $\partial p^0/\partial x_i$ in equation (9.21b) may be exchanged for force density f_i^0 and other ambient medium parameters by using expression (8.4b).]

The divergence-free flow assumption, together with adiabaticity, imply that material derivatives of all scalar-valued ambient medium parameters vanish:

$$\frac{dc^0}{dt}=0, \qquad \frac{dh^0}{dt}=0, \qquad \frac{d\rho^0}{dt}=0, \qquad \frac{d\eta^0}{dt}=0, \qquad \frac{dp^0}{dt}=0, \qquad \frac{d\theta^0}{dt}=0.$$

However, the three material derivatives of the background particle velocity vector components are *not* necessarily zero:

$$\frac{dv_i^0}{dt}\neq 0.$$

If this material derivative vanishes, *then* equations (9.21a,b) in the above system decouple from (9.21c), and the pressure and particle velocity perturbations are obtained by solving four coupled partial differential equations.

10.0 THREE DIVERGENCE–FREE SYSTEMS

The "purely mechanical system" of five, coupled, first-order, partial differential equations (9.14) is obtained by eliminating the perturbation in specific entropy density $\delta\eta$ from the more general system (9.3). Obviously, alternative systems may be obtained by eliminating the mass density fluctuation $\delta\rho$ or the pressure fluctuation δp. Each such system also consists of five, coupled, first-order, partial differential equations. In this section, all three systems are summarized and compared, with the goal of inferring an advantageous system for subsequent numerical solution.

In addition to an adiabatic ambient medium [as is assumed in the derivation of the parent system (9.3)], the present derivations assume that the particle velocity of the background medium is divergence-free: $\partial v_i^0/\partial x_i = 0$. This is, admittedly, a restricting assumption. However, it is considered a realistic condition characterizing fluid motion in the atmosphere or the ocean. Equation (8.4a) then implies that the material derivative of the ambient mass density vanishes; that is, the ambient fluid motion is *incompressible.*

The particular "purely mechanical system" (9.21), appropriate for an adiabatic and divergence-free background medium, is repeated here in slightly modified form:

$$\frac{\partial(\delta v_i)}{\partial t}+v_j^0\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial v_i^0}{\partial x_j}\delta v_j+\frac{1}{\rho^0}\frac{\partial(\delta p)}{\partial x_i}+\frac{1}{\rho^0}\left[\frac{\partial v_i^0}{\partial t}+v_j^0\frac{\partial v_i^0}{\partial x_j}\right]\delta\rho=\frac{1}{\rho^0}\delta f_i, \tag{10.1a}$$

$$\frac{\partial(\delta p)}{\partial t}+v_i^0\frac{\partial(\delta p)}{\partial x_i}+\rho^0\left(c^0\right)^2\frac{\partial(\delta v_i)}{\partial x_i}+\frac{\partial p^0}{\partial x_i}\delta v_i=\frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}, \tag{10.1b}$$

$$\frac{\partial(\delta\rho)}{\partial t}+v_i^0\frac{\partial(\delta\rho)}{\partial x_i}+\rho^0\frac{\partial(\delta v_i)}{\partial x_i}+\frac{\partial\rho^0}{\partial x_i}\delta v_i=0. \tag{10.1c}$$

There are five dependent variables (δv_i , δp, $\delta\rho$) and six medium parameters (v_i^0, ρ^0, c^0, p^0). The reason for *not* counting the two thermodynamic quantities h^0 and θ^0 in (10.1b) as "medium parameters" is explained below.

Alternately, the mass density fluctuation δp may be eliminated from system (9.3), giving

$$\frac{\partial(\delta v_i)}{\partial t}+v_j^0\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial v_i^0}{\partial x_j}\delta v_j+\frac{1}{\rho^0}\frac{\partial(\delta p)}{\partial x_i}+\frac{1}{\rho^0\left(c^0\right)^2}\left[\frac{\partial v_i^0}{\partial t}+v_j^0\frac{\partial v_i^0}{\partial x_j}\right]\left(\delta p-h^0\delta\eta\right)=\frac{1}{\rho^0}\delta f_i, \quad (10.2a)$$

$$\frac{\partial(\delta p)}{\partial t}+v_i^0\frac{\partial(\delta p)}{\partial x_i}+\rho^0\left(c^0\right)^2\frac{\partial(\delta v_i)}{\partial x_i}+\frac{\partial p^0}{\partial x_i}\delta v_i=\frac{h^0}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}, \quad (10.2b)$$

$$\frac{\partial(\delta\eta)}{\partial t}+v_i^0\frac{\partial(\delta\eta)}{\partial x_i}+\frac{\partial\eta^0}{\partial x_i}\delta v_i=\frac{1}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}. \quad (10.2c)$$

There are five dependent variables (δv_i, δp, $\delta\eta$) and eight medium parameters (v_i^0, ρ^0, c^0, h^0, p^0, η^0) [with θ^0 not counted]. However, the spatial derivative of the ambient specific entropy density $\partial\eta^0/\partial x_i$ may be exchanged for similar derivatives of background mass density and pressure by using the equation of state in the differentiated form (9.7). Hence, the number of ambient medium parameters is reduced to seven.

Finally, eliminating the pressure fluctuation δp from system (9.3) gives

$$\frac{\partial(\delta v_i)}{\partial t}+v_j^0\frac{\partial(\delta v_i)}{\partial x_j}+\frac{\partial v_i^0}{\partial x_j}\delta v_j+\frac{\left(c^0\right)^2}{\rho^0}\frac{\partial(\delta\rho)}{\partial x_i}+\frac{1}{\rho^0}\left[\frac{\partial v_i^0}{\partial t}+v_j^0\frac{\partial v_i^0}{\partial x_j}+\frac{\partial(c^0)^2}{\partial x_i}\right]\delta\rho$$
$$+\frac{h^0}{\rho^0}\frac{\partial(\delta\eta)}{\partial x_i}+\frac{1}{\rho^0}\frac{\partial h^0}{\partial x_i}\delta\eta=\frac{1}{\rho^0}\delta f_i, \quad (10.3a)$$

$$\frac{\partial(\delta\rho)}{\partial t}+v_i^0\frac{\partial(\delta\rho)}{\partial x_i}+\rho^0\frac{\partial(\delta v_i)}{\partial x_i}+\frac{\partial\rho^0}{\partial x_i}\delta v_i=0, \quad (10.3b)$$

$$\frac{\partial(\delta\eta)}{\partial t}+v_i^0\frac{\partial(\delta\eta)}{\partial x_i}+\frac{\partial\eta^0}{\partial x_i}\delta v_i=\frac{1}{\rho^0\theta^0}\frac{\partial(\delta e)}{\partial t}. \quad (10.3c)$$

There are five dependent variables (δv_i, $\delta\rho$, $\delta\eta$) and seven medium parameters (v_i^0, ρ^0, c^0, h^0, η^0) [with θ^0 not counted]. Once again, the gradient of the ambient specific entropy density may be exchanged for a linear combination of gradients of ambient mass density and pressure.

A straightforward visual inspection of the above three sets of equations suggests that system (10.1) is most favorable for numerical solution purposes. The four principal reasons for this conclusion are:

1) The coefficients in (10.1) are somewhat simpler than in the alternative systems. For example, equation (10.3a) possesses gradients of the material parameters $(c^0)^2$ and h^0, in addition to gradients of ρ^0 and v_i^0. The gradient of ambient specific entropy density appears in (10.2c) and (10.3c). Although $\partial\eta^0/\partial x_i$ may be expressed in terms of $\partial p^0/\partial x_i$ and $\partial\rho^0/\partial x_i$, the resulting coefficients are more complicated, and the "difficult to determine" parameter h^0 is introduced [see point 3) below].

2) System (10.2) contains three non-homogeneous terms representing body sources of acoustic waves. The other systems have only two such terms.

3) The thermodynamic parameter h^0 (quantifying the change in acoustic pressure induced by change in specific entropy density) is probably the most difficult ambient medium parameter to determine in practice. System (10.1) contains h^0 *only* in direct association with a non-homogeneous term, and not as a coefficient of a major term on the left-hand-side of the equations. If the acoustic energy source $\partial(\delta e)/\partial t$ is restricted to be a *point* source (i.e., isolated at a single location in space), then parameters h^0 and θ^0 in (10.1b) may be incorporated into the source magnitude factor. They need *not* be known throughout the three-dimensional region of space where the system of partial differential equations is numerically solved. This is the reason that h^0 and θ^0 are not considered medium parameters in system (10.1). In contrast, h^0 appears on the left-hand-side of systems (10.2) and (10.3). Hence, it must be known (and stored in a numerical algorithm) throughout the three-dimensional domain where the equations are solved. The same argument applies to the absolute temperature θ^0 in all three systems: for a point source (or set of point sources), it may be incorporated directly into the energy source magnitude factor(s).

4) System (10.1) contains fewer ambient medium parameters than the two alternatives. Thus, a numerical algorithm for solving the system requires less computational storage space.

Finally, note that numerical solution of systems (10.1) and (10.2) directly yields the acoustic pressure fluctuation, whereas δp must be calculated after solution of (10.3) for $\delta \rho$ and $\delta \eta$, using the linearized equation of state (9.3d). In atmospheric and oceanic sound propagation problems, the pressure fluctuation is probably the most useful acoustic wavefield variable.

11.0 CONCLUSION

The mathematical equations developed herein constitute the foundation of a numerical algorithm for simulating acoustic wave propagation through a variety of realistic atmospheric, oceanic, and/or laboratory environments. The expressions are rigorously derived from fundamental principles of continuum mechanics, pertinent constitutive relations, and equations of state. Wave propagation equations are obtained by linearizing the relevant expressions with respect to all acoustic wavefield variables. No mathematical approximations beyond first-order linearization are utilized; hence, the expressions are considered "exact" within this context. The utility of the equations is enhanced by their generality. Thus, the equations govern three-dimensional acoustic wave propagation within media that may be spatially heterogeneous, time-varying, and/or moving (in any or all of three spatial dimensions). In contrast to numerous alternative developments, no limitations are imposed on the temporal or spatial scales of ambient medium variability. Finally, acoustic waves are activated by two distinct body source types: fluctuations in applied force density and applied energy density. As previously indicated in section 2.0, time-varying boundary conditions are neglected in this study.

A reduced set of linearized wave propagation equations is obtained if the ambient medium is assumed to be (i) an ideal (i.e., non-viscous and non-heat-conducting) fluid, (ii) adiabatic (i.e., contains no energy sources) and (iii) executing divergence-free (i.e., incompressible) motion. These are reasonable assumptions for many atmospheric acoustic wave propagation scenarios. This reduced system (called the "purely mechanical system" herein) consists of five, coupled, first-order, partial differential equations containing the dependent variables δv_i, δp, and $\delta \rho$ (fluctuations in particle velocity vector components, pressure, and mass density). Coefficients in the system depend on six ambient medium parameters v_i^0, ρ^0, c^0, and p^0 (i.e., three fluid velocity vector components, mass density, adiabatic sound speed, and pressure, respectively). It is emphasized that these parameters cannot be chosen arbitrarily, but must satisfy a different *nonlinear* system of equations governing the dynamic behavior of the background medium.

The preferred system of equations appears to be amenable to numerical solution using an explicit, time-domain, finite-difference methodology. Provided spatial and temporal gridding intervals are chosen appropriately, the finite-difference approach will simulate all acoustic arrival types (direct waves, reflections, refractions, diffractions, etc.) with fidelity, because no additional mathematical or physical approximations (like paraxial propagation, high frequencies, weak scattering, stratified ambient motion, etc.) are adopted.

Finally, two particular aspects of this work, involving acoustic energy sources, require additional clarifying research:

1) **Mass source**: Conventional treatments of classical (i.e., non-relativistic) continuum mechanics (e.g., Malvern, 1969; Narasimhan, 1993) do not admit the existence of mass sources or mass sinks. However, certain texts on acoustic wave propagation (e.g., Morse and Ingard, 1968, p. 322; Ostashev, 1997, p. 27; Kinsler, et al., 2000, p. 141) introduce a mass source as a non-homogeneous term in the continuity equation [expression (3.1) above]. These two points of view require reconciliation. Presently, mass sources are inserted in an *ad hoc* sense, with no evaluation of their impact in Cauchy's equations of motion and/or the entropy balance principle. Nevertheless, they may be useful mathematical representations of certain types of acoustic energy sources.

2) **Body force representation**: In the current work, body force density (f_i^0 in the ambient medium equations and δf_i in the wave propagation equations) is specified as force per unit volume (SI units: N/m^3). This choice is motivated by prior experience in seismic wave propagation studies. Alternatively, body force density may be specified as force per unit mass (SI units: N/kg). The mathematical relationship between the two representations is

$$f_i = \rho g_i,$$

where ρ is mass density and g_i are the specific body force density vector components. Hence, background medium and linearized versions are

$$f_i^0 = \rho^0 g_i^0, \qquad \text{and} \qquad \delta f_i = \rho^0 \delta g_i + g_i^0 \delta\rho,$$

respectively. Curiously, use of the specific body force components g_i^0 and δg_i complicates many of the previous mathematical expressions governing acoustic wave propagation. For example, equation (9.15a) above (appropriate for an adiabatic and stationary ambient medium) becomes

$$\rho^0 \frac{\partial(\delta v_i)}{\partial t} + \frac{\partial(\delta p)}{\partial x_i} - g_i^0 \delta\rho = \rho^0 \delta g_i,$$

[as in Bergmann (1946)]. The presence of the dependent variable $\delta\rho$ (mass density fluctuation) in this expression implies that system (9.15a,b,c) does not decouple into a set of four partial differential equations for the pressure and particle velocity variables! Rather, system (9.15) must be solved as a set of *five* coupled equations. Clearly, a numerical solution algorithm is more complicated. A related example involves the single partial differential equation (9.16) for acoustic pressure. This expression is *significantly* simpler than the analogous equation given by Bergmann (1946), where the derivation employs the specific body force density representation.

It is emphasized that all of the expressions developed in the current study, which utilize the volumetric body force density formalism, are rigorously correct in a mathematical sense. However, it is presently unclear whether the appropriate *physical* representation of an acoustic wave source entails specification of a volumetric or a specific body force density.

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