Preprint
UCRL-JC-138267

Finite Element Output Bounds for Hyperbolic Problems

L. Machiels

This article was submitted to the16th International Association of Mathematics and Computers Simulation World Conference 2000, Lausanne, Switzerland, August 21-25, 2000

March 27, 2000

U.S. Department of Energy

Lawrence
Livermore
National
Laboratory

Finite Element Output Bounds for Hyperbolic Problems

L. Machiels*

Abstract

We propose a Neumann–subproblem *a posteriori* finite element error bound technique for linear stationary scalar advection problems. The method is similar in many respects to the previ..s output bound technique developed for elliptic problems. In the new approach, however, the primal residual is enhanced with a streamline diffusion term. We first formulate the bound algorithm, with particular emphasis on the proof of the bounding properties; then, we provide numerical results for an illustrative example.

Key words: output bounds, *a posteriori* error estimation, finite element methods, partial differential equations.

AMS subject classifications: 65N15, 65N30, 35L50.

1 Introduction

In a pioneering contribution [11], Paraschivoiu and Patera propose a new finite element error control strategy for linear coercive elliptic problems; a quadratic error equality ("energy equality") [14] and a duality argument are invoked to compute inexpensive lower and upper bounds for engineering quantities (outputs) of interest. The method is an implicit Aubin–Nitsche construction: first, inexpensive coarse mesh solutions of the original — primal — problem and an associated — dual — problem are computed; then, the error estimators are obtained by solving local Neumann subproblems on a conservative fine mesh. These subproblems are symmetric, positive (semi–) definite, and completely decoupled. Also, the new procedure does not involve unknown constant or function, and the bounds directly measure the error for the output of interest. Therefore, computational efficiency is preserved, and numerical uncertainty is considerably reduced.

The numerical analysis of the method for linear coercive elliptic problems is presented in [9]; output bounds can also be obtained for noncoercive semi–linear elliptic equations [13, 10] — including the Navier-Stokes equations [7] — and time dependent parabolic problems [6]. Finally, the bound approach is related to previous contributions in finite element *a posteriori* error estimation [5, 2, 1, 3].

This paper describes an output bound algorithm for linear advection problems; an earlier attempt [8] failed to yield optimal convergence of the bounds gap. In the new method, the primal residual is enhanced with a streamline diffusion term in the local subproblems, while the dual residual remains unchanged. Optimality is achieved thanks to this un–symmetric treatment of the primal and the dual residuals. The extension of the method to more general Friedrichs systems and non–linear equations is the subject of an ongoing research.

2 Problem Statement

Let Ω be an open polygonal domain in $\boldsymbol{R}^d$; we consider the stationary scalar linear advection equation

$$\begin{aligned} &(1) \qquad & \boldsymbol{\beta}\cdot\nabla u + \sigma u &= f, \quad \text{in} \quad \Omega, \\ &(2) \qquad & u_{|\Gamma_-} &= 0, \end{aligned}$$

*L-561, Lawrence Livermore National Laboratory, 7000 East Avenue, Livermore CA 94550, machiels1@llnl.gov

where $\beta \in (C^1(\overline{\Omega}))^d$, $\sigma > 0$, $f \in L^2(\Omega)$, and

$$\Gamma_- = \{x \in \partial\Omega \,|\, \beta(x) \cdot n(x) < 0\},$$

is the inflow boundary (n denotes the outer normal vector). We also assume that $\beta(x) \cdot n(x) \neq 0$ almost everywhere on $\partial\Omega$, and that $\exists \bar{\sigma} > 0$ such that $\sigma - \frac{1}{2}\nabla \cdot \beta \geq \bar{\sigma}$. We define the Hilbert space

$$V = \{v \in L^2(\Omega) \,|\, \beta \cdot \nabla u \in L^2(\Omega)\},$$

equipped with the norm

$$\|v\|_V = \left(\int_\Omega v^2 + \int_\Omega (\beta \cdot \nabla v)^2 + \int_{\partial\Omega} |\beta \cdot n| v^2\right)^{1/2}.$$

A strong solution of the above advection problem is a function $u \in V$ satisfying (1) and (2). Note that a function in V can be discontinuous across a characteristic surface.

We consider the following output

$$s = \int_\Omega \ell u + \int_{\Gamma_+} g u(\beta \cdot n),$$

where $\Gamma_+ = \partial\Omega \setminus \Gamma_-$, $\ell \in L^2(\Omega)$, and $g \in L^2(\Gamma_+)$. We also define a dual problem: find $\psi \in V$ such that

$$-\beta \cdot \nabla\psi + (\sigma - \nabla \cdot \beta)\psi = -\ell, \tag{3}$$
$$\psi_{|\Gamma_+} = -g; \tag{4}$$

the importance of this problem will appear shortly.

Let $\mathcal{T}_H$ be a quasi-uniform finite element triangulation of Ω; here H denotes the diameter of the mesh. We define the finite element space,

$$V_H = \{v \in C(\overline{\Omega}) \,|\, v_{|T_H} \in \boldsymbol{P}_1(T_H),\ \forall T_H \in \mathcal{T}_H\},$$

where $\boldsymbol{P}_1(T_H)$ denotes the space of linear polynomials defined on T_H. We use the streamline diffusion method [4] to write the following approximate problem: find $(u_H, s_H) \in V_H \times \boldsymbol{R}$ such that

$$\begin{aligned}(\beta \cdot \nabla u_H + \sigma u_H, v + H\beta \cdot \nabla v) - \langle u_H, v\rangle_{\Gamma_-} &= (f, v + H\beta \cdot \nabla v), \quad v \in V_H,\\ s_H &= (\ell, u_H) + \langle g, u_H\rangle_{\Gamma_+},\end{aligned} \tag{5}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product and

$$\langle v, w\rangle_{\Gamma_-} = \int_{\Gamma_-} vw(\beta \cdot n).$$

The approximate dual problem reads: find the adjoint $\psi_H \in V_H$ such that

$$(-\beta \cdot \nabla\psi_H + (\sigma - \nabla \cdot \beta)\psi_H, v - H\beta \cdot \nabla v) + \langle \psi_H, v\rangle_{\Gamma_+} = -(\ell, v - H\beta \cdot \nabla v) - \langle g, v\rangle_{\Gamma_+}.$$

If we define the "improved" approximate output

$$\tilde{s}_H = s_H + H(f - \beta \cdot \nabla u_H - \sigma u_H, \beta \cdot \nabla\psi_H),$$

we can show the following Aubin–Nitsche estimate for the error in the output

$$\begin{aligned}s - \tilde{s}_H &= \int_\Omega \beta \cdot \nabla\psi(u - u_H) - \int_\Omega (\sigma - \nabla \cdot \beta)\psi(u - u_H) - \langle \psi, (u - u_H)\rangle_{\Gamma_+} - H(f - \beta \cdot \nabla u_H - \sigma u_H, \beta \cdot \nabla\psi_H)\\ &= -\int_\Omega \psi\beta \cdot \nabla(u - u_H) - \int_\Omega \sigma\psi(u - u_H) + \langle \psi, u - u_H\rangle_{\Gamma_-} - H(f - \beta \cdot \nabla u_H - \sigma u_H, \beta \cdot \nabla\psi_H)\\ &= -\int_\Omega (\psi - \psi_H)\beta \cdot \nabla e - \int_\Omega (\psi - \psi_H)e + \langle \psi - \psi_H, e\rangle_{\Gamma_-}\\ &\leq C(\|\psi - \psi_H\|_0 + |\psi - \psi_H|)\|e\|_V,\end{aligned}$$

where we have used the definitions of s, s_H, and $\tilde{s}_H$, and equations (3) and (4) in the first line; the second line is obtained by application of the Green formula (which holds in V); the third line follows from (5) and the definition $e = u - u_H$; finally, in the last line $\|\cdot\|_0$ denote the $L^2(\Omega)$ norm and

$$|\psi - \psi_H| = \left(\int_{\partial\Omega} (\psi - \psi_H)^2 |\boldsymbol{\beta}\cdot\boldsymbol{n}|\right)^{1/2}.$$

For a smooth solution, and for H sufficiently small, one can prove [4] that

$$\|\psi - \psi_H\|_0 + |\psi - \psi_H| \leq CH^{3/2} \quad \text{and} \quad \|e\|_V \leq CH;$$

therefore, under these assumptions, we expect $|s - \tilde{s}_H| \leq CH^{5/2}$; by contrast, if s_H is used instead of $\tilde{s}_H$, we can only ensure $|s - s_H| \leq CH^2$.

3 Output Bounds

3.1 Preliminaries

We first introduce the broken spaces

$$\begin{aligned} \hat{V} &= \{v \in L^2(\Omega) \,|\, \boldsymbol{\beta}\cdot\nabla(v_{|T_H}) \in L^2(T_H), \forall T_H \in \mathcal{T}_H\}, \\ \hat{V}_H &= \{v \in \hat{V} \,|\, v_{|T_H} \in \boldsymbol{P}_1(T_H), \forall T_H \in \mathcal{T}_H\}. \end{aligned}$$

We then define the hybrid flux space

$$Z_H = \{v \in L^2(\Gamma(\mathcal{T}_H)) \,|\, v_{|\gamma} \in \boldsymbol{P}_1(\gamma), \forall \gamma \in \Gamma(\mathcal{T}_H))\},$$

where $\Gamma(\mathcal{T}_H)$ denotes the set of interior edges $\gamma \subset \Omega$ of the triangulation $\mathcal{T}_H$. We also introduce the form $b : \hat{V} \times Z_H \to \boldsymbol{R}$

$$b(v, z) = \sum_{\gamma \in \Gamma(\mathcal{T}_H)} \int_\gamma [v]_\gamma z(\boldsymbol{\beta}\cdot\boldsymbol{n}),$$

and $[v]_\gamma = v_{|\gamma^+} - v_{|\gamma^-}$ denotes the jump in v over the (arbitrarily oriented) edge γ. Finally, we define the local generalized residuals for the primal and the dual problems

$$\begin{aligned} \mathcal{R}^{pr}_{T_H}(v;\epsilon) &= \int_{T_H} (f - \boldsymbol{\beta}\cdot\nabla u_H - \sigma u_H)(v + \epsilon\boldsymbol{\beta}\cdot\nabla v) + \langle u_H, v\rangle_{\Gamma_-}, \\ \mathcal{R}^{du}_{T_H}(v;\epsilon) &= \int_{T_H} (-\ell + \boldsymbol{\beta}\cdot\nabla\psi_H - (\sigma - \nabla\cdot\boldsymbol{\beta})\psi_H)(v - \epsilon\boldsymbol{\beta}\cdot\nabla v) + \langle -g - \psi_H, v\rangle_{\Gamma_+}, \end{aligned}$$

and we write $\mathcal{R}^{pr} : \hat{V} \times \boldsymbol{R} \to \boldsymbol{R}$, $\mathcal{R}^{pr}(v;\epsilon) = \sum_{T_H \in \mathcal{T}_H} \mathcal{R}^{pr}_{T_H}(v;\epsilon)$ and $\mathcal{R}^{du} : \hat{V} \times \boldsymbol{R} \to \boldsymbol{R}$, $\mathcal{R}^{du}(v;\epsilon) = \sum_{T_H \in \mathcal{T}_H} \mathcal{R}^{du}_{T_H}(v;\epsilon)$

3.2 Bound Algorithm

The bound algorithm proceeds in five steps.

1. Compute $(u_H, s_H) \in V_H \times \boldsymbol{R}$ solution of the primal problem:

$$\begin{aligned} (\boldsymbol{\beta}\cdot\nabla u_H + \sigma u_H, v + H\boldsymbol{\beta}\cdot\nabla v) - \langle u_H, v\rangle_{\Gamma_-} &= (f, v + H\boldsymbol{\beta}\cdot\nabla v), \quad v \in V_H, \\ s_H &= (\ell, u_H) + \langle g, u_H\rangle_{\Gamma_+}. \end{aligned}$$

2. Compute the adjoint $\psi_H \in V_H$ solution of the dual problem:

$$(-\boldsymbol{\beta}\cdot\nabla\psi_H + (\sigma - \nabla\cdot\boldsymbol{\beta})\psi_H, v - H\boldsymbol{\beta}\cdot\nabla v) + \langle\psi_H, v\rangle_{\Gamma_+} = -(\ell, v - H\boldsymbol{\beta}\cdot\nabla v) - \langle g, v\rangle_{\Gamma_+}.$$

3. Compute the hybrid fluxes, $z^{pr} \in Z_H$ and $z^{du} \in Z_H$, such that

$$\begin{aligned} b(v, z^{pr}) &= \mathcal{R}^{pr}(v; H), \quad \forall v \in \hat{V}_H, \\ b(v, z^{du}) &= \mathcal{R}^{du}(v; H), \quad \forall v \in \hat{V}_H. \end{aligned}$$

4. Compute the reconstructed errors $\hat{e}^{pr} \in \hat{V}$ and $\hat{e}^{du} \in \hat{V}$ such that

$$2\alpha(\beta \cdot \nabla \hat{e}^{pr}, \beta \cdot \nabla v) + 2\eta(\hat{e}^{pr}, v) = \mathcal{R}^{pr}(v;\alpha) - b(v, z^{pr}), \quad \forall v \in \hat{V}, \tag{6}$$
$$2\alpha(\beta \cdot \nabla \hat{e}^{du}, \beta \cdot \nabla v) + 2\eta(\hat{e}^{du}, v) = \mathcal{R}^{du}(v;0) - b(v, z^{du}), \quad \forall v \in \hat{V}, \tag{7}$$

where $\alpha > 0$ and $\eta > 0$ are chosen such that

$$\alpha < \min\left(\frac{2\bar{\sigma}}{M\sigma}, \frac{1}{\sigma}\right), \quad \eta \leq \bar{\sigma} - \frac{M\sigma\alpha}{2}, \quad \text{with} \quad M = \max_{x\in\overline{\Omega}}(\nabla \cdot \beta(x)), \tag{8}$$

if $\exists x \in \overline{\Omega}$ such that $\nabla \cdot \beta(x) > 0$; if $\nabla \cdot \beta(x) \leq 0$ for all $x \in \overline{\Omega}$ then $0 < \alpha < 1/\sigma$ and $\eta \leq \sigma$ suffice.

5. Compute the bound

$$s_H^{\pm} = \tilde{s}_H - 2\alpha(\beta \cdot \nabla \hat{e}^{pr}, \beta \cdot \nabla \hat{e}^{du}) - 2\eta(\hat{e}^{pr}, \hat{e}^{du}) \pm \Delta_H,$$

where the (half) bounds gap is given by

$$\Delta_H = 2\{\alpha(\beta \cdot \nabla \hat{e}^{pr}, \beta \cdot \nabla \hat{e}^{pr}) + \eta(\hat{e}^{pr}, \hat{e}^{pr})\}^{1/2}\{\alpha(\beta \cdot \nabla \hat{e}^{du}, \beta \cdot \nabla \hat{e}^{du}) + \eta(\hat{e}^{du}, \hat{e}^{du})\}^{1/2}.$$

We make the four following remarks. First, thanks to the equilibrium $\mathcal{R}^{pr}(v; H) = 0$, $\forall v \in V_H$, and $\mathcal{R}^{du}(v; H) = 0$, $\forall v \in V_H$, the hybrid fluxes in Step 3 can be (efficiently) computed using an equilibration procedure [5, 12]. Second, the solvability of the subproblems (6) and (7) is ensured by the $L^2(\Omega)$ stabilization term in the left–hand–side. The equilibration of the hybrid fluxes guarantees that the means of $\hat{e}^{pr}$ and $\hat{e}^{du}$ are zero since $1_{|T_H} \in \hat{V}_H$, and $\mathcal{R}^{pr}(1_{|T_H}; \epsilon) - b(1_{|T_H}, z^{pr}) = 0$, $\forall \epsilon \in \boldsymbol{R}$ (a similar argument applies for the dual subproblems) — note that a preliminary analysis suggests that this property is in fact not required and the hybrid fluxes can be set equal to zero. Third, in actual practice, the local subproblems are solved on a conservative decoupled fine mesh. Fourth, the cost–effectiveness follows from the decoupled nature of (6) and (7), their symmetry, and positive definiteness.

3.3 Bounding Properties

We prove here that the estimators $s_H^{\pm}$ of the preceding section are in fact bounds for s. We first define

$$\kappa = \left\{\frac{\alpha(\beta \cdot \nabla \hat{e}^{du}, \beta \cdot \nabla \hat{e}^{du}) + \eta(\hat{e}^{du}, \hat{e}^{du})}{\alpha(\beta \cdot \nabla \hat{e}^{pr}, \beta \cdot \nabla \hat{e}^{pr}) + \eta(\hat{e}^{pr}, \hat{e}^{pr})}\right\}^{1/2},$$

and $\hat{e}^{\pm} = \hat{e}^{pr} \mp \hat{e}^{du}/\kappa$. We then write

$$\begin{aligned} 0 &\leq \kappa\{\alpha(\beta \cdot \nabla(e - \hat{e}^{\pm}), \beta \cdot \nabla(e - \hat{e}^{\pm})) + \eta(e - \hat{e}^{\pm}, e - \hat{e}^{\pm})\} \\ &= \kappa\alpha(\beta \cdot \nabla e, \beta \cdot \nabla e) + \kappa\eta(e, e) - \kappa\mathcal{R}^{pr}(e;\alpha) \pm \mathcal{R}^{du}(e;0) + \kappa\alpha(\beta \cdot \nabla \hat{e}^{\pm}, \beta \cdot \nabla \hat{e}^{\pm}) + \kappa\eta(\hat{e}^{\pm}; \hat{e}^{\pm}), \end{aligned} \tag{9}$$

where we have used the definition of $\hat{e}^{\pm}$, and (6) and (7). We now expand

$$\begin{aligned} \mathcal{R}^{pr}(e;\alpha) &= \int_\Omega (f - \beta \cdot \nabla u_H - \sigma u_H)(e + \alpha\beta \cdot \nabla e) + \langle u_H, e\rangle_{\Gamma_-} \\ &= \int_\Omega (\beta \cdot \nabla e + \sigma e)(e + \alpha\beta \cdot \nabla e) - \langle e, e\rangle_{\Gamma_-} \\ &= \alpha\int_\Omega (\beta \cdot \nabla e)^2 + \int_\Omega \sigma e^2 + \int_\Omega e(\beta \cdot \nabla e)(1 + \sigma\alpha) - \langle e, e\rangle_{\Gamma_-} \\ &= \alpha\int_\Omega (\beta \cdot \nabla e)^2 + \int_\Omega e^2\left\{\sigma - (\nabla \cdot \beta)\frac{1+\sigma\alpha}{2}\right\} + \int_{\Gamma_+} |\beta \cdot \boldsymbol{n}|e^2\left(\frac{1+\sigma\alpha}{2}\right) + \int_{\Gamma_-} |\beta \cdot \boldsymbol{n}|e^2\left(\frac{1-\sigma\alpha}{2}\right) \\ &\geq \alpha\int_\Omega (\beta \cdot \nabla e)^2 + \eta\int_\Omega e^2. \end{aligned} \tag{10}$$

In the above expression, the first line is the definition of $\mathcal{R}^{pr}(e;\alpha)$; in the second line, we use the definition of e and equations (1) and (2); the third line is simply a rearrangement of the terms; the Green formula yields the fourth line; the last lines follow from (8) which implies that $\sigma - (\nabla \cdot \beta)\frac{1+\sigma\alpha}{2} \geq \eta > 0$ and $\sigma\alpha < 1$.

We turn now to the dual residual,

$$\begin{aligned}
\mathcal{R}^{du}(e;0) &= -\int_\Omega \ell e - \int_{\Gamma_+} ge(\beta\cdot n) + \int_\Omega (\beta\cdot\nabla\psi_H - (\sigma - \nabla\cdot\beta)\psi_H)e - \langle\psi_H, e\rangle_{\Gamma_+} \\
&= -\int_\Omega \ell e - \int_{\Gamma_+} ge(\beta\cdot n) - \int_\Omega (\beta\cdot\nabla e + \sigma e)\psi_H + \int_{\partial\Omega} \psi_H e(\beta\cdot n) - \langle\psi_H, e\rangle_{\Gamma_+} \\
&= -\int_\Omega \ell e - \int_{\Gamma_+} ge(\beta\cdot n) - \int_\Omega (f - \beta\cdot\nabla u_H - \sigma u_H)\psi_H - \int_{\Gamma_-} \psi_H u_H(\beta\cdot n) \\
&= -\int_\Omega \ell e - \int_{\Gamma_+} ge(\beta\cdot n) - \int_\Omega (f - \beta\cdot\nabla u_H - \sigma u_H)(\psi_H + H\beta\cdot\nabla\psi_H) - \int_{\Gamma_-} \psi_H u_H(\beta\cdot n) \\
&\quad + H\int_\Omega (f - \beta\cdot\nabla u_H - \sigma u_H)\beta\cdot\nabla\psi_H \\
&= -\int_\Omega \ell e - \int_{\Gamma_+} ge(\beta\cdot n) + H\int_\Omega (f - \beta\cdot\nabla u_H - \sigma u_H)\beta\cdot\nabla\psi_H, \qquad (11)
\end{aligned}$$

where we have invoked the definition of the dual residual in the first equality and the Green formula in the second equality; the third equality follows directly from the definition of e and (1) and (2); the fourth equality is simply obtained by adding and subtracting the streamline diffusion term; the last equality is an immediate consequence of equation (5).

We now collect (9), (10), and (11) to write

$$0 \le \mp\int_\Omega \ell e \mp \int_{\Gamma_+} ge \pm H\int_\Omega (f - \beta\cdot\nabla u_H - \sigma u_H)\beta\cdot\nabla\psi_H + \kappa\alpha\int_\Omega (\beta\cdot\nabla\hat{e}^{\pm})^2 + \kappa\eta\int_\Omega (\hat{e}^{\pm})^2,$$

from which we immediately deduce $s_H^- \le s \le s_H^+$ thanks to the definition of $s_H^{\pm}$, $\tilde{s}_H$, and the remark that

$$\begin{aligned}
\kappa\alpha\int_\Omega (\beta\cdot\nabla\hat{e}^{\pm})^2 + \kappa\eta\int_\Omega (\hat{e}^{\pm})^2 &= \mp 2\alpha\int_\Omega \beta\cdot\nabla\hat{e}^{pr}\beta\cdot\nabla\hat{e}^{du} + \kappa\alpha\int_\Omega (\beta\cdot\nabla\hat{e}^{pr})^2 + \frac{\alpha}{\kappa}\int_\Omega (\beta\cdot\nabla\hat{e}^{du})^2 \\
&\quad \mp 2\eta\int_\Omega \hat{e}^{pr}\hat{e}^{du} + \kappa\eta\int_\Omega (\hat{e}^{pr})^2 + \frac{\eta}{\kappa}\int_\Omega (\hat{e}^{du})^2 \\
&= \mp 2\alpha\int_\Omega \beta\cdot\nabla\hat{e}^{pr}\beta\cdot\nabla\hat{e}^{du} \mp 2\eta\int_\Omega \hat{e}^{pr}\hat{e}^{du} + \Delta_H,
\end{aligned}$$

where we have used the definitions of $\hat{e}^{\pm}$, κ, and Δ_H.

4 Numerical Results

We consider a simple one-dimensional example [8]

$$\begin{aligned}
\frac{du}{dx} + u &= f, \qquad 0 < x < 1, \\
u(0) &= 0,
\end{aligned}$$

with f given by

$$f(x) = \begin{cases} 1 - 3x & \text{for } 0 \le x \le 0.5 \\ 2.5(x - 0.7) & \text{for } 0.5 < x \le 0.7 \\ 0 & \text{for } 0.7 < x \le 1. \end{cases}$$

The only output considered is the mean of u over the domain

$$s = \int_0^1 u(x)\,dx.$$

We use uniform meshes. The Neumann subproblems (6) and (7) are solved on a (decoupled) mesh of diameter $h = 1/1000$. In Table 1, we first examine the convergence of $u_h - u_H$ and $\psi_h - \psi_H$ in the $H_1(\Omega)$ semi-norm (denoted $|\cdot|_1$) and $L^2(\Omega)$ norm, respectively; we note that $|u_h - u_H|_1 = O(H)$ and $\|\psi_h - \psi_H\|_0 = O(H^2)$. Turning now to the error estimators $s_H^{\pm}$, we have verified that they are bounds for any choice of H, and we observe, in Table 1, that the bound gap converges at the expected rate, $O(H^3)$. In [8], for the same problem, only $O(H^2)$ was achieved for the bound gap. We have also computed the bounds for several different choices of α in the range of admissible values, and the bound gap appears to be relatively independent of this choice.

H	$\|\psi_H - \psi_h\|_0$	$\|u_H - u_h\|_1$	Δ_H/s_h
0.100	0.0749	2.5512e-04	6.2698e-04
0.050	0.0373	6.2526e-05	7.5004e-05
0.025	0.0186	1.5475e-05	9.1719e-06
0.010	0.0074	2.4606e-06	5.6676e-07
0.005	0.0036	6.1339e-07	6.4093e-08

Table 1: Numerical results.

References

[1] M. Ainsworth and J. T. Oden. A posteriori error estimation in finite element analysis. *Comp. Meth. in Appl. Mech. and Engrg.*, 142:1–88, 1997.

[2] R. E. Bank and A. Weiser. Some a posteriori error estimators for elliptic partial differential equations. *Math. Comp.*, 44(170):283–301, 1985.

[3] R. Becker and R. Rannacher. A feedback approach to error control in finite element method: Basic analysis and examples. *East-West J. Numer. Math.*, 4:237–264, 1996.

[4] C. Johnson, U. Navert, and J. Pitkaranta. Finite element methods for linear hyperbolic problems. *Comp. Meth. Appl. Mech. Engrg.*, 45:285–312, 1984.

[5] P. Ladeveze and D. Leguillon. Error estimation procedures in the finite element method and applications. *SIAM J. Numer. Anal.*, 20:485–509, 1983.

[6] L. Machiels. A posteriori finite element bounds for output functionals of discontinuous Galerkin discretizations of parabolic problems. Comp. Meth. in Appl. Mech. and Engrg., to appear.

[7] L. Machiels, A. T. Patera, and J. Peraire. A posteriori finite element bounds for output functionals of the incompressible Navier-Stokes equations; application to a natural convection problem. Submitted to J. Comput. Phys., 1999.

[8] L. Machiels, A. T. Patera, J. Peraire, and Y. Maday. A general framework for finite element a posteriori error control: application to linear and nonlinear convection–dominated problems. Proceedings of ICFD Conference, Oxford, 1998.

[9] Y. Maday and A. T. Patera. Numerical analysis of a posteriori finite element bounds for linear-functional outputs. Mathematical Models and Methods in Applied Science, to appear.

[10] Y. Maday, A. T. Patera, and J. Peraire. A general formulation for a posteriori bounds for output functionals of partial differential equations; application to the eigenvalue problem. *C.R. Acad. Sci. Paris*, t. 328, Série I:823–828, 1999.

[11] M. Paraschivoiu and A. T. Patera. A hierarchical duality approach to bounds for the outputs of partial differential equation. *Comp. Meth. Appl. Mech. and Engrg.*, 158:389–407, 1998.

[12] M. Paraschivoiu, J. Peraire, and A. T. Patera. A posteriori finite element bounds for linear-functional outputs of elliptic partial differential equations. *Comp. Meth. Appl. Mech. and Engrg.*, 150:289–312, 1997.

[13] J. Peraire and A. T. Patera. Asymptotic a posteriori finite element bounds for the outputs of noncoercive problems: the Helmholtz and Burgers equations. Comp. Meth. in Appl. Mech. and Engrg., to appear.

[14] J. Peraire and A. T. Patera. Bounds for linear-functional outputs of coercive partial differential equations: local indictors and adaptive refinement. In P. Ladeveze and J. T. Oden, editors, *On New Advances in Adaptive Computational Methods in Mechanics*, 1997.

This work was performed under the auspices of the U. S. Department of Energy by the University of California, Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48.