

MEASUREMENT OF RESIDUAL STRESSES USING  
FRACTURE MECHANICS WEIGHT FUNCTIONS

Y. Fan

USDOE Contract No. DE-AC11-98PN38206

**NOTICE**

This report was prepared as an account of work sponsored by the United States Government. Neither the United States, nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

BETTIS ATOMIC POWER LABORATORY

WEST MIFFLIN, PENNSYLVANIA 15122-0079

Operated for the U.S. Department of Energy  
by Bechtel Bettis, Inc.

## **DISCLAIMER**

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, make any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

## **DISCLAIMER**

**Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.**

# MEASUREMENT OF RESIDUAL STRESSES USING FRACTURE MECHANICS WEIGHT FUNCTIONS

Y. Fan

Bettis Atomic Power Laboratory, P.O. Box 79,  
West Mifflin, PA 15122-0079  
U. S. A.

KEYWORDS: Methods, Measures, Structure

A residual stress measurement method has been developed to quantify through-the-thickness residual stresses. Accurate measurement of residual stresses is crucial for many engineering structures. Fabrication processes such as welding and machining generate residual stresses that are difficult to predict. Residual stresses affect the integrity of structures through promoting failures due to brittle fracture, fatigue, stress corrosion cracking, and wear.

In this work, the weight function theory of fracture mechanics is used to measure residual stresses. The weight function theory is an important development in computational fracture mechanics. Stress intensity factors for arbitrary stress distribution on the crack faces can be accurately and efficiently computed for predicting crack growth. This paper demonstrates that the weight functions are equally useful in measuring residual stresses. In this method, an artificial crack is created by a thin cut in a structure containing residual stresses. The cut relieves the residual stresses normal to the crack-face and allows the relieved residual stresses to deform the structure. Strain gages placed adjacent to the cut measure the relieved strains corresponding to incrementally increasing depths of the cut. The weight functions of the cracked body relate the measured strains to the residual stresses normal to the cut within the structure. The procedure details, such as numerical integration of the singular functions in applying the weight function method, will be discussed.

## Introduction

For fracture mechanics, Mode I (stress normal to the crack face) stress intensity factor at crack depth  $a$  for arbitrary stress distributions can be calculated by integrating the product of the weight function,  $m(x, a)$ , and the stress distribution normal to the crack,  $\sigma(x)$ .

$$K_I(a) = \int_0^a m(x, a) \sigma(x) dx \quad (1)$$

For residual stress determination, an artificial crack is created by a thin cut in a structure containing residual stresses. The thin cut can be made by the electric discharge machining process (EDM). In terms of Equation (1),  $m(x, a)$  is the weight function of the EDM cut;  $\sigma(x)$  is the residual stress distribution normal to the cut;  $K_I$  is the mode I stress intensity factor caused by the residual stress.

From Equation (1), the residual stress distribution  $\sigma(x)$  is the unknown. It must be determined from  $m(x, a)$  and  $K_I$ . Weight function  $m(x, a)$  is intrinsic to the specific crack geometry. It is independent of the stress distribution. It is available from the literature or can be computed for the cracked geometry.  $K_I$  is dependent on the stress distribution. It is measured for each residual stress distribution.  $K_I$  however is an inferred quantity. It can not be directly measured.  $K_I$  is computed from the measured displacement or strain. The displacement and strain at a point when a crack is introduced into a body subjected to residual stress can be directly measured. The relationship between  $K_I$  and measured displacement or strain can be derived from the Castigliano's theorem (Reference (1)) which states that the partial derivative of the strain energy with respect to a selected virtual force is equal to the displacement at the point of application of that force in the direction of that force. The derivation for calculating displacements from strain energy and stress intensity factors is given in detail in Reference (1).

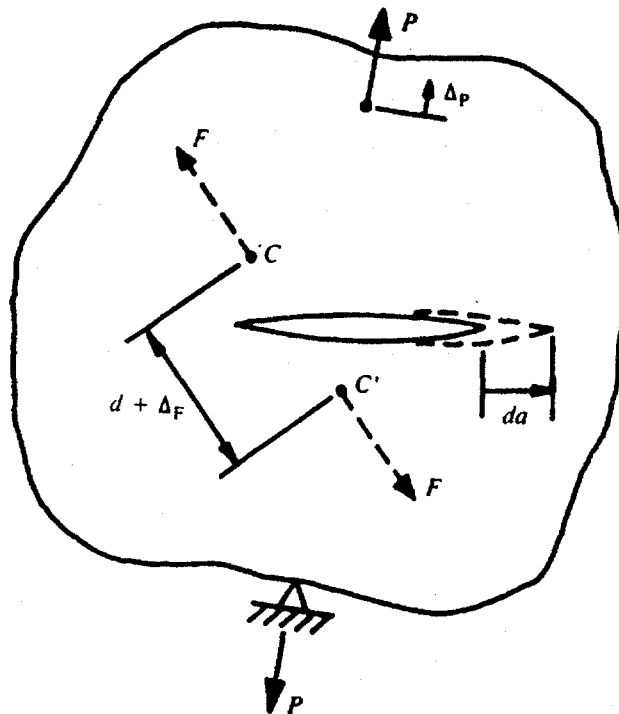


Figure 1. A Body Loaded by Forces,  $P$ , and Virtual Forces,  $F$ .

In Reference (1), assuming no out-of-plane deformation, the displacement at any point of the cracked body (Figure 1) can be found by applying a virtual force  $F$  at the location and direction of the displacement:

$$\Delta_F = \Delta_{F0} + \frac{2}{E'} \int_0^a (K_{IP} \frac{\partial K_{IF}}{\partial F} + K_{IIP} \frac{\partial K_{IIF}}{\partial F}) da \quad (2)$$

Where,

$\Delta_F$  is the displacement at the location and in the direction of the virtual force;

$\Delta_{F0}$  is the displacement if no crack were present;

$E' = E$  for plane stress,  $E' = E/(1-\nu^2)$  for plane strain,  $E$  is the modulus of elasticity;  $\nu$  is Poisson's ratio;

$K_{IF}$  is the mode I stress intensity factor caused by the virtual force  $F$ ;

$K_{IIP}$  is the mode II stress intensity factor caused by the actual force  $P$ ;

$K_{IP}$  is the mode I stress intensity factor caused by the actual force  $P$ ;

$K_{IIF}$  is the mode II stress intensity factor caused by the virtual force  $F$ ;

$a$  is the length of the crack.

Differentiating Equation (2) with respect to the distance,  $d$ , between  $C$  and  $C'$ , the strain at  $C$  and  $C'$  is:

$$\varepsilon_F = \frac{\partial \Delta_F}{\partial d} = \frac{\partial \Delta_{F0}}{\partial d} + \frac{2}{E'} \int_0^a (K_{IP} \frac{\partial^2 K_{IF}}{\partial d \partial F} + K_{IIP} \frac{\partial^2 K_{IIF}}{\partial d \partial F}) da \quad (3)$$

If the residual stress field is normal to the crack (i.e., no shear), only the mode I stress intensity factor need be considered. Furthermore, for the geometry considered herein,  $\Delta_{F0}$  is zero. Equation (2) becomes:

$$\Delta_F = \frac{2}{E'} \int_0^a K_{IP} \frac{\partial K_{IF}}{\partial F} da \quad (4)$$

By the same token, taking partial derivative of the displacement with respect to  $d$ , Equation (3) becomes:

$$\varepsilon_F = \frac{\partial \Delta_F}{\partial d} = \frac{2}{E'} \int_0^a K_{IP} \frac{\partial^2 K_{IF}}{\partial d \partial F} da \quad (5)$$

In Equation (4), the displacement  $\Delta_F$  can be measured and the stress intensity factor due to  $F$ ,  $K_{IF}$ , can be computed directly by knowing the weight function and the location

of the virtual force  $F$ .  $K_{IP}$ , the stress intensity factor due to residual stress (actual force  $P$ ), depends on the residual stress distribution normal to the crack. A solution scheme for the residual stress distribution according to Equation (4) needs to be constructed based on the measured displacement values at various depths and the weight function. The solution scheme can also be based on measured strains as indicated by Equation (5).

#### **Residual stress distribution in a single edge notched specimen:**

The weight function for the single edge notched specimen is available in Reference (2). The geometry of the single edge notched specimen (Figure 2) can be used for many residual stress measurement conditions. The weight function of the single edge notched specimen from Reference (2) is as follows:

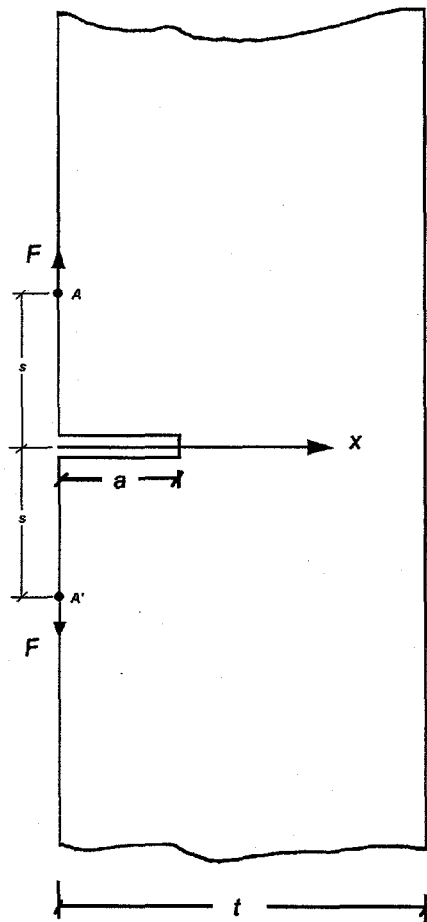


Figure 2. A Single Edge Notched Specimen.

$$m(x, a) = \frac{2}{\sqrt{\pi a}} \cdot \frac{G(x/a, a/t)}{(1-a/t)^{3/2} \sqrt{1-(x/a)^2}} \quad (6)$$

where

$t$  is the thickness of the single edge notched specimen;

$a$  is the length of the crack;

$x$  is the coordinate from the edge of the specimen in the plane of the crack;

$G$  is a function that is the combination of individual  $g_i$  functions:

$$G(x/a, a/t) = g_1(a/t) + g_2(a/t) \cdot \frac{x}{a} + g_3(a/t) \cdot \left(\frac{x}{a}\right)^2 + g_4(a/t) \cdot \left(\frac{x}{a}\right)^3 \quad (7)$$

where,

$$g_1(a/t) = 0.46 + 3.06 \frac{a}{t} + 0.84 \left(1 - \frac{a}{t}\right)^5 + 0.66 \left(\frac{a}{t}\right)^2 \left(1 - \frac{a}{t}\right)^2$$

$$g_2(a/t) = -3.52 \left(\frac{a}{t}\right)^2$$

$$g_3(a/t) = 6.17 - 28.22 \frac{a}{t} + 34.54 \left(\frac{a}{t}\right)^2 - 14.39 \left(\frac{a}{t}\right)^3 - \left(1 - \frac{a}{t}\right)^{3/2} - 5.88 \left(1 - \frac{a}{t}\right)^5 - 2.64 \left(\frac{a}{t}\right)^2 \left(1 - \frac{a}{t}\right)^2$$

$$g_4(a/t) = -6.63 + 25.16 \frac{a}{t} - 31.04 \left(\frac{a}{t}\right)^2 + 14.41 \left(\frac{a}{t}\right)^3 + 2 \left(1 - \frac{a}{t}\right)^{3/2} + 5.04 \left(1 - \frac{a}{t}\right)^5 + 1.98 \left(\frac{a}{t}\right)^2 \left(1 - \frac{a}{t}\right)^2$$

The stress intensity factor due to a pair of virtual forces,  $F$ , applied at point A and A' where the strains can be measured (Figure 2) is:

$$K_{IF}(a, s) = \int_0^a m(x, a) \sigma_F(x) dx \quad (8)$$



where  $s$  is the distance from point A or A' to the edge of the crack opening,  $a$  is the crack length,  $x$  is the coordinate axis from the crack opening through the crack tip and  $\sigma_F$  is the stress distribution due to  $F$  on the crack face with no crack present:

$$\sigma_F(x) = \frac{4F}{\pi s} \cdot \frac{1}{\{1 + (x/s)^2\}^2} + \frac{F}{t} \left(3 - \frac{6x}{t}\right) \cdot \left\{-0.0256875 + 1.4278859 \frac{s}{t} - 0.4780998 \left(\frac{s}{t}\right)^2\right\} \quad (9)$$

The first term in the equation represents the solution for a plate with semi-infinite width and the second term is the correction for the bending effect due to the width of the plate (Reference (3)). The coefficients in the equation are obtained by the finite element method in Reference (3). Discussion on this expression can be found in Reference (4). The bending term by closed form solution given in Reference (4) is:

$$\sigma_b(x) = \frac{3F}{\pi t} (1 - 2x) \left[ \tan^{-1} \left( \frac{s}{2} - \frac{1}{s} \right) + \frac{\pi}{2} + \frac{2s}{s^2 + 1} \right] \quad (10)$$

Reference (4) states that the error in the second term of Equation (9) becomes significant as the  $x$  and  $s$  increase. Replacing the second term of Equation (9) with Equation (10) would result in a more accurate solution.

Variable  $s$  in Equation (8) and (9) is equivalent to  $d/2$  in Equation (5). The stress intensity factors  $K_{IP}$  and  $K_{IF}$  can be computed.  $K_{IF}$  can be computed from Equation (8), (9) and (6). Residual stress distribution is the unknown to be solved. The residual stress distribution can be represented as orthogonal polynomials ( $P_i(x)$ ) as in Reference (4). Namely,

$$\sigma_y(x) = \sum_{i=0}^n A_i P_i(x)$$

$A_i$ 's are the polynomial coefficients,  $P_i(x)$ 's are the orthogonal polynomials. The forms of the polynomials can be power series, Legendre polynomials (Reference 5), trigonometry series (Reference 6), or piecewise overlapping polynomials (Reference 7). The effort in developing the computer program differs very little if the numerical integration is used.

Or, in matrix form,

$$\{\sigma\} = [P]\{A\} \quad (11)$$

Substitute the polynomial expressions, Equation (11), into Equation (1) and (5), the coefficients of the polynomials become the unknowns to be solved. The measured strains become:

$$\varepsilon_x(a_j) = \sum_{i=0}^n A_i C_i(a_j) \quad (12)$$

$C_i(a)$ 's are defined as the compliance functions. They can be evaluated through carrying out the integration in Equation (5).

Suppose that there are more strain measurements than the number of coefficients, Equation (12) becomes an over-determined system of linear equations. The least squares method for an over-determined system can be expressed as:

$$\frac{\partial}{\partial A_i} \sum_{j=1}^m [\varepsilon_x(a_j) - \sum_{k=0}^n A_k C_k(a_j)]^2 = 0 \quad i = 0, \dots, n$$

Or in matrix form the coefficients  $A$ 's can be solved by:

$$\{A\} = ([C]^T [C])^{-1} [C]^T \{\varepsilon\} \quad (13)$$

The compliance coefficients  $C_{ij}$  in Equation (13) can be obtained by integrating Equation (1) and Equation (5). The compliance for coefficient  $A_i$  at crack depth  $a_j$  is:

$$C_{ij} = \frac{2}{E'} \frac{\partial}{\partial A_i} \int_0^{a_j} K_{IP} \frac{\partial^2 K_{IF}}{\partial F \partial d} da \quad (14)$$

Substituting Equation (13) into Equation (11) yields the residual stress distribution.

### Implementation:

There are two interesting issues in implementing the computer program: one is that the double integrals in Equation (14) are complex; the other is that the stress intensity factors,  $K_{IP}$  and  $K_{IF}$  in Equation (14) involve singular terms at the crack tip as shown in Equation (6). The Gauss-LeGendre quadrature scheme can be used for integrating Equation (14). Accurate results can be obtained by dividing the integration interval into many subintervals. The rate of convergence is slow for such an integration scheme. It takes hours of execution time on a Silicon Graphics Octane R12000 workstation for the

integral to converge. A Gauss-Chebyshev quadrature scheme was also tried. The convergence rate is even slightly slower than the Gauss-LeGendre scheme. In this method, a few hundred terms of  $C_{ij}$  need to be integrated. The slow rate of convergence becomes a concern.

The convergence rate for the Gauss-LeGendre quadrature scheme, ignoring the singularity, can be demonstrated with an example. For the special case with  $t = \infty$ , symbolic integration for Equation (14) can be performed using Mathematica (Reference 8). The convergence of the Gauss-LeGendre quadrature is shown in Figure 3 by plotting the error calculated from the difference in computing strains between Gauss-LeGendre quadrature and symbolic integration divided by the strain from symbolic integration against the number of subdivisions within the crack length.

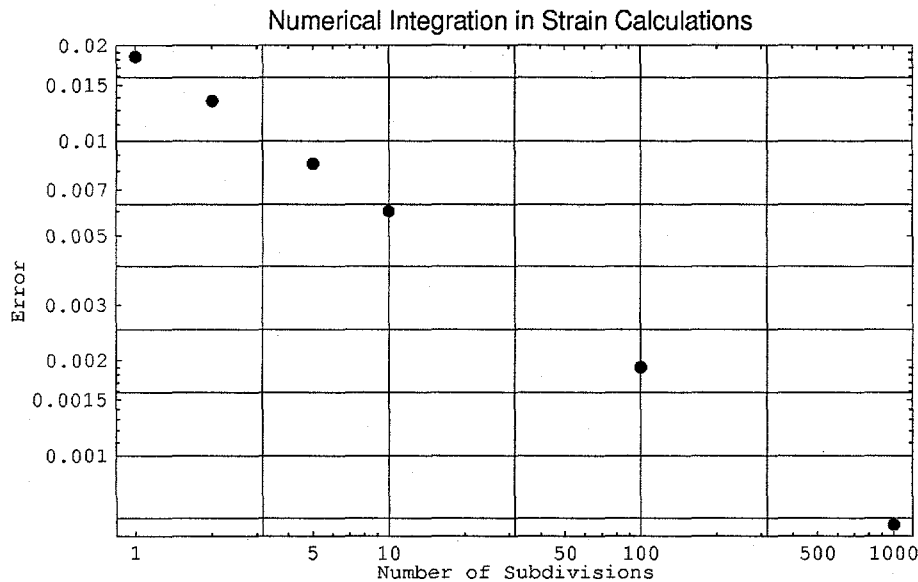


Figure 3. Monotonic Convergence of the Gauss-LeGendre quadrature.

Reference (9) offers a solution to this problem by dividing the integration interval into two subintervals. The crack length subinterval within 5% of the crack length employs a special integration formula. The remaining 95% of the crack length subinterval away from the crack tip uses regular numerical integration. The special integration formula is based on closed form integration. The weight functions near the crack tip can be simplified to a general form that can be integrated analytically.

The method in Reference (10) offers a better alternative. The singularity can be removed by a change of variable. For a function  $f(x)$  that has an inverse square root singularity at  $b$ , the identity:

$$\int_a^b f(x) dx = \int_0^{\sqrt{b-a}} 2t \cdot f(b-t^2) dt \quad (15)$$

can be used to remove the singularity. With the singularity removed, convergence is fast. Accurate results can be obtained using a single interval. Separation of the interval into a singular part and a non-singular part is unnecessary.

The effectiveness of this method can be demonstrated with a simple example. Applying Equation (15), the identity:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^1 \frac{2}{\sqrt{2-t^2}} dt$$

can be used to remove the singularity at  $x = 1$ . The closed form solution of the integral is  $\pi/2$  or 1.5707963267948966192.... Using a Gauss quadrature formula with a 5<sup>th</sup> degree LeGendre polynomial, integrating the left-hand-side directly yields 1.4761047... which represents about 6% error from the exact solution. Using the same quadrature scheme, integrating the right-hand-side yields 1.57079618... which is accurate to the 7<sup>th</sup> significant figure. For this example, a simple change of variable can reduce the numerical error from 6% to  $9 \times 10^{-8}$ .

The speed and accuracy improvements for the change of variables are more remarkable when integrating Equation (14). Accurate results can be obtained with a single integration interval. The computation can be achieved almost instantaneously.

The form of Equation (13) occurs in a typical least squares method. Gaussian elimination algorithm with pivots for solving the resulting set of linear equations yields satisfactory results.

The steps of the implementation are summarized as follows:

1. Compute  $K_{IF}$  from Equation (1) or (8).
2. Define form of the residual stress distribution according to Equation (11).
3. Calculate strains at strain gage location according to Equation (5).
4. Use Equation (14) to determine terms in C matrix in Equation (13).
5. Solve for coefficients  $\{A\}$  in Equation (13).

6. Calculate through-the-thickness residual stress distribution in Equation (11).

### Verification:

The correctness of the strain computations given in Equation (5) needs to be checked. With a given residual stress profile the computer program must be able to compute strains relieved by the cut at the given strain gage locations. A typical method is to compare the results of the weight function computations against the results using a different approach such as the finite element method. A special case is constructed for that purpose. The special case is to calculate surface strains located at 2.54 mm from the edge of the crack for a notch in half space subject to 68.95 MPa uniform pressure on the crack face. The strains at various crack depths computed by the weight function method are compared with the strains computed by finite element method. The comparison is shown in Figure 4.

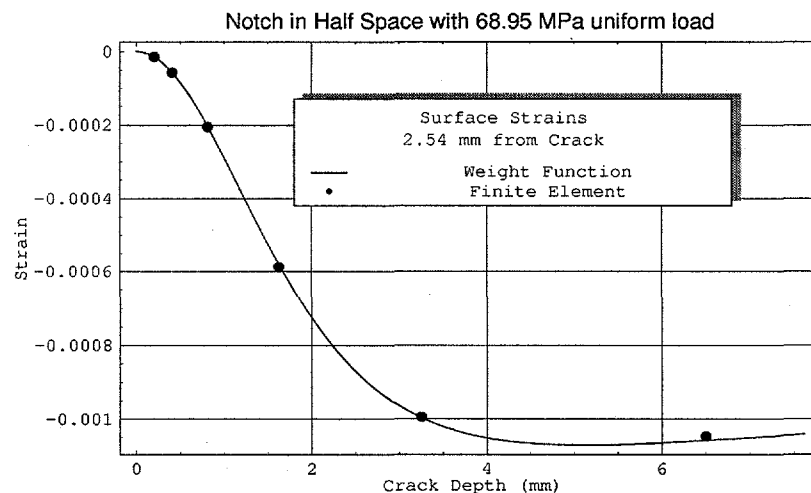


Figure 4. Notch in Half Space Subjected to a 68.95 MPa Uniform Crack Face Load

The weight function represents a sharp crack. The typical mechanical cutting method creates a slot with a width. The finite element model was modified to change the zero width crack to a flat bottom slot with a width of 0.1 mm which is typical of an EDM cut. The differences of the two finite element results are calculated by taking the absolute value of the difference divided by the absolute value of the larger one. The differences at crack depths of 0.2032 mm, 0.4064 mm, 0.8128 mm, 1.6256 mm, 3.2512 mm and 6.5024 mm are 17.5%, 9.4%, 4.7%, 0.9%, 0.1%, and 0.4%, respectively. The difference is small for crack depth beyond 1 mm which is adequate for most applications. If desire more accuracy at surface, a weight function including the effect of the width of the cut or finite element method needs to be used.

## Conclusions:

A weight function method in measuring residual stresses in a body is described. In this method, a cut is created in the body containing residual stresses. A strain gage is mounted near the cut to record the strains relieved by the cut. The method relates the strain response to the residual stresses normal to the cut within the body by fracture mechanics principles.

The weight function for a single edged notch specimen is used for the implementation. This weight function can be applied to many practical situations in residual stress measurements. The computations herein are verified by comparing results with finite element method. The effect of the width of the electric discharge machining cut should be insignificant for measuring residual stresses beyond 1 mm in depth.

The singularity at the crack tip for the weight function presents a difficulty for numerical integration. The singularity can be removed by change of variables. The effectiveness of the method is demonstrated by an example.

## REFERENCES

1. Tada, H., Paris, P., and Irwin, G., "The Stress Analysis of Cracks Handbook", (2<sup>nd</sup> Edition), 1973.
2. Glinka, G., and Shen, G., "Universal Features of Weight Functions for Cracks in Mode I, Engineering Fracture Mechanics", Vol., 40, No. 6, pp. 1135-1146, 1991.
3. Kang, K.J., Song, J.H., and Earmme, Y.Y., "A Method for the Measurement of Residual Stresses Using a Fracture Mechanics Approach", Journal of Strain Analysis, Vol. 24, No. 1, pp. 23-30, 1989.
4. Cheng, W., Finnie, I., and Vardar, Ö, "Deformation of an Edge-cracked Strip Subjected to Normal Surface Traction on the Crack Faces", Engineering Fracture Mechanics, Vol., 42, No. 1, pp. 97-107, 1992.
5. Cheng, W. and Finnie, I., "A Method for Measurement of Axisymmetric Axial Residual Stresses in Circumferentially Welded Thin-Walled Cylinders", Journal of Engineering Materials and Technology, 107, 181-185, 1985.
6. Fett, T., "Determination of Residual Stresses in Components Using the Fracture Mechanics Weight Function", Engineering Fracture Mechanics, Vol. 55, No. 4, pp. 571-576, 1996.
7. Gremaud, M., Cheng, W., Finnie, I., and Prime, M.B., "The Compliance Method for Measurement of Near Surface Residual Stresses – Analytical Background", Journal of Engineering Materials and Technology, 116, 550-555, 1994.
8. Wolfram, S., "Mathematica, A System for Doing Mathematics by Computer", Second Edition, Addison-Wesley Publishing Company, Inc., 1991.

9. Moftakhar, A.A. and Glinka, G., "Calculation of Stress Intensity Factors by Efficient Integration of Weight Functions", Engineering Fracture Mechanics, Vol.43, No. 5, pp. 749-756, 1992.
10. Press, W.H., Teukolsky, S.A., Vetterling, W.T., and Flannery, B.P., "Numerical Recipes in C – The Art of Scientific Computing", Second Edition, Cambridge University Press, 1992.