Representation of Random Shock
Via the Karhunen Loeve Expansion

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Shock excitations are normally random process realizations, and most of our efforts to represent them either directly or indirectly reflect this fact. The most common indirect representation of shock sources is the shock response spectrum. It seeks to establish the damage-causing potential of random shocks in terms of responses excited in linear, single-degree-of-freedom systems. This paper shows that shock sources can be represented directly by developing the probabilistic and statistical structure that underlies the random shock source. Confidence bounds on process statistics and probabilities of specific excitation levels can be established from the model. Some numerical examples are presented.

INTRODUCTION

The shock response spectrum (SRS) is the de facto standard for representation of the severity of shock sources. It was proposed by Biot in the 1930's (See Biot, 1943) to account for the fact that test excitations with arbitrary time histories could not be generated on shock test hardware available at that time. Today, though arbitrary time histories of shock excitation still cannot be generated for all test items, it is desirable to establish models for shock excitation sources that characterize the actual environment. These can be used in analysis and laboratory applications where an arbitrary time history can be generated for a particular test item. A general, stochastic model for shock might be used to rationally and systematically specify laboratory and analysis test shock environments that have computable and controllable levels of conservatism. In this study we develop a stochastic shock model whose parameters can be chosen to have any desired level of confidence (of not being surpassed) and which can be used to generate shocks with any desired probability of not being surpassed.

The Karhunen-Loeve (KL) expansion (Ghamen and Spanos, 1991) is a tool for the compact representation of arbitrary random processes. Masri, Smyth and Traina (1998) have shown that the KL expansion can be used to model the responses of deterministic structures to earthquake excitations. This investigation shows that the KL expansion can also be used to express arbitrary nonstationary random process sources, and, in addition, that the parameters of the KL expansion can be adjusted to model a source whose parameters reflect a preestablished confidence level, and whose realizations reflect a particular probability of being surpassed. Statistical analysis of the parameters in the KL expansion is performed using the bootstrap (Efron and Tibshirani, 1993, Paez and Hunter, 1998, Hunter and Paez, 1998).

A numerical example is presented to demonstrate how shock signals that are nonstationary random process realizations can be generated.

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THE KARHUNEN-LOEVE EXPANSION

The Karhunen-Loeve (KL) expansion is a tool for the compact representation of multi-variate, continuous-valued, continuous-parametered (i.e., continuous time) random processes. We consider in this investigation a special case: the representation of a zero-mean, univariate, continuous-valued, discrete time, nonstationary random process. Let X, an $n \times 1$ column vector of random variables, be such a random process. Its autocorrelation matrix, an $n \times n$ nonnegative definite matrix, is denoted R_{XX}. To establish the KL expansion, compute the eigenvalue decomposition of the autocorrelation matrix:

$$R_{XX} = VWV^T \tag{1}$$

V is the $n \times n$ matrix whose columns are the eigenvectors of the autocorrelation matrix. V^T is orthogonal with respect to V, and we can scale the columns of V to achieve orthonormality. W is the diagonal matrix of eigenvalues of the autocorrelation matrix. Its elements are nonnegative. (We sort the elements on the diagonal of W so that the largest ones appear at top left.) The smallest elements in W are eliminated along with the corresponding columns in V, leaving only positive values in W. (This may substantially reduce the dimensions of V and W.) When this is done Eq. (1) is approximated by:

$$R_{XX} \cong vwv^T \tag{2}$$

where w is an $m \times m$ ($m \leq n$) diagonal matrix of the largest values in W, and v is the $n \times m$ matrix containing the corresponding vectors from V.

Now factor the right side of Eq. (2):

$$R_{XX} \cong vw^{1/2}Iw^{1/2}v^T \tag{3}$$

where I is the $m \times m$ identity matrix. Recognize that the autocorrelation on the left is $E\left[XX^T\right]$ (where the symbol $E[.]$ denotes the operation of mathematical expectation) and that the identity matrix on the right can be interpreted as the autocorrelation matrix, $R_{UU} = E\left[UU^T\right]$, of an $m \times 1$ vector of zero-mean, uncorrelated random variables, U. Then Eq. (3) can be rewritten:

$$E\left[XX^T\right] = vw^{1/2}E\left[UU^T\right]w^{1/2}v^T = E\left[vw^{1/2}UU^Tw^{1/2}v^T\right] \tag{3a}$$

Association of the terms in brackets with one another leads to an approximation for X:

$$X \cong vw^{1/2}U \tag{4}$$

This expression is the KL expansion of the random process X. Computation of the autocorrelation matrix $E\left[XX^T\right]$ confirms this representation by yielding the approximation in Eq. (2). In this representation of the random process X, the columns of v assume the role of shape functions characterizing underlying temporal components that appear in realizations of X. The elements in the diagonal matrix w assume the role of amplitudes characterizing the relative contributions of the individual shapes in v. The elements in the mean zero, unit variance, vector of uncorrelated random variables U are the randomizing factors that combine the shapes times their corresponding amplitudes. Clearly, components associated with small amplitudes tend to have only slight influence on the overall representation.

Because w is diagonal with positive elements, and the columns of v are mutually orthonormal, Eq. (4) can be inverted to obtain:

$$U = w^{-1/2} v^T X \tag{5}$$

If the joint probability distribution of the elements in the vector X is known, then it can (in principle) be used to establish the joint probability distribution of the elements in the vector U. The fact that the elements in the vector U are uncorrelated does not necessarily imply that the elements are indepeendent, except in the case where X and U are normally distributed. When the random vector X is jointly normal, theen the elements in U are jointly normal and uncorrelated (i.e., independent). Experience indicates that even when thae joint probability distribution of the random process X is non-Gaussian, the marginal probability distributions oɔf the elements in U will often be close to Gaussian.

In practice we will often assume that the elements of U are independdent, and seek to satisfy ourselves that this assumption is justified. Further, we will also often assume that the elemaents of U have a standard normal distribution (i.e., Gaussian with zero mean and unit variance). Some comments on ɽparameter estimation and practical statistical issues follow.

Assume that a collection of realizations of the random process X hhas been measured and the elements of the measured ensemble are denoted x_j, $j = 1,...,M$. (Each measured reealization of X is stored in a column vector.) Combine the measured realizations in a matrix as follows:

$$x = [x_1, x_2,...,x_M] \tag{6}$$

The estimator for the autocorrelation matrix is:

$$\hat{R}_{XX} = \frac{1}{M} xx^T \tag{7}$$

Use the estimated autocorrelation matrix as in Eqs. (1) and (2) to ɔobtain the parameters of the estimated KL expansion, $\hat{w}$ and $\hat{v}$. Use these parameters and the form of Eq. (5) ɯto transform the measured realizations of the random process X into realizations of the random vector U:

$$u = \hat{w}^{-1/2} \hat{v}^T x \tag{8}$$

There are M columns in the matrix u, u_j, $j = 1,...,M$, each one ɔ an m-vector corresponding to a realization x_j, $j = 1,...,M$. The collection of i(th) elements in all the vectors u_j, $j = 1,...,M$, are realizations of the i(th) random variable in the vector U, i.e., U_i. When M is sufficiently largee, these quantities can be used to estimate the probability distribution of the random variable U_i. When M is smmall (say, less than ten) the only practical alternative is to is to assume a distribution for the elements of U. Typiczally, we would assume that the elements in U are jointly Gaussian. Alternatives will be developed in later papeɽrs, however, for the present we adopt this simplifying assumption.

GENERATION OF RANDOM PROCESS REALIZATIONS

The model developed here can be used to generate accurate realizationsɜ of the random process X. To accomplish this we refer to Eqs. (4) and (8). Equation (8) can be inverted to establish a rɽealization-based version of Eq. (4). It is:

$$x_r = \hat{v}\hat{w}^{1/2}u_r \qquad (9)$$

This expression indicates that if we generate a realization u_r of the random vector U and use it as shown, we obtain a realization x_r of the random process X. Given the assumptions made above, it is easy to generate a realization of U. We simply generate a sequence of m standard normal random variates (using a standard procedure or a software package), and use them to populate u_r.

We anticipate that when the elements of u_r are large, then the realization x_r will tend to be severe, in the sense that its peak values will be large. This is not a simple issue because (1) the elements of $\hat{w}^{1/2}$ diminish down the diagonal, and (2) the values of u_r can be both positive and negative, therefore, the relative values of the elements of u_r affect the phasing of components in $\hat{v}$ and this affects the actual peaks realized in x_r. In view of these things, the leading values of the random variates in u_r are most important in establishing the severity of x_r, and when all the values in u_r tend to be small, x_r tends to be less severe.

Because we have assumed the elements of U to be jointly standard normal, the square of the cartesian length of U (i.e., the square root of the sum of the squares of the elements in U) is a chi squared distributed random variable with m degrees of freedom. (See Mood, Graybill and Boes, 1974.) Denote this length $|U|$. It is possible to specify the length of u_r such that it has a precise probability of not being surpassed. We express this mathematically as:

$$P\left(|U| \le |u_r|\right) = F_{\chi_m^2}\left(|u_r|^2\right) = 1 - \alpha \qquad (10)$$

where $F_{\chi_m^2}(.)$ is the cumulative distribution function of a chi squared distributed random variable with m degrees of freedom, and $1-\alpha$ is the probability that the length $|u_r|$ is not surpassed by $|U|$. The quantity $|u_r|^2$ is the $(1-\alpha)\times 100\%$ percentage point of a chi squared distribution with m degrees of freedom. For example, when $m=5$ and $\alpha=0.01$, the ninety-nine percent percentage point of the chi squared distribution with five degrees of freedom is 15.0863, and there is probability of 0.99 that when a vector u_r is formed by choosing its five components as uncorrelated standard normal variates, its length is less than $\sqrt{15.0863} = 3.8841$.

This has a special meaning with regard to the formation of realizations x_r of the random process X. When we choose realizations u_r of the random vector U to be random, yet to have a particular length $|u_r|$, we are in essence choosing the realizations at random from the surface of a hypersphere in a space of dimension m. When $m=2$ the hypersphere is a circle. When $m=3$ the hypersphere is a sphere in three dimensional space, etc. This hypersphere is the boundary of a region within which a fraction equal to $F_{\chi_m^2}\left(|u_r|^2\right)$ of the realizations of U lie. Realizations x_r formed with such vectors u_r are $F_{\chi_m^2}\left(|u_r|^2\right)\times 100\%$ conservative in a probabilistic sense. The realizations all have different severities that depend on the definition of severity. For example, we might take the absolute maximum acceleration of x_r as our definition of shock severity, or we might take the shock with the highest SRS in some preestablished frequency range as the most severe. The definition of severity is application dependent. When shock severity has been defined, we can generate many shocks and calculate their severities. Normally, we accept the one with the greatest severity as the representative of the shock source.

The steps to forming a vector u_r with cartesian length L are: (1) Generate a vector z of uncorrelated standard normal random variates. (2) Multiply each element in z by the ratio $L/|z|$, where $|z|$ is the cartesian length of z. (3) Set u_r equal to the resulting vector.

INTERVAL ANALYSIS OF PARAMETERS OF THE KL EXPANSION

It now remains to establish confidence intervals for the parameters of the KL expansion. Because the columns of the matrix $\hat{v}$ serve as shape functions of the random process X, and the diagonal elements of $\hat{w}^{1/2}$ serve as their corresponding amplitudes, the interval analysis that relates to amplitude conservatism of the random process must treat the elements of $\hat{w}^{1/2}$. Because the elements of $\hat{w}^{1/2}$ do not follow a standard probability distribution, we perform a confidence analysis of these elements using the bootstrap.

The bootstrap (Efron and Tibshirani, 1993, Paez and Hunter, 1998, Hunter and Paez, 1998) is a technique for statistical analysis of measured data. The data may be non-Gaussian (and not Gaussian-related). The data may be limited in quantity (as is often the case with shock data where data are expensive to obtain or naturally limited in quantity). The statistical measure of interest may be one of the classical measures of random data behavior or something more complicated (as in the following application).

The fundamental idea behind the bootstrap is that data measured from a random source are treated, for purposes of statistical estimation, as though they completely describe the random source. For example, when the data $x_j, j = 1,...,n$, are measured from a scalar random source we may be interested in estimating some parameter θ of the probability distribution from which the data emanate. We might use a formula $\hat{\theta} = g(x)$ (where $x = \{x_1, x_2,...,x_n\}$, and the formula is obtained, for example, through maximum likelihood estimation) to establish an estimator of the parameter θ. Normally, we are interested in assessing the quality of the estimator $\hat{\theta}$, by approximating its bias, standard error, or we may wish to use it to estimate confidence intervals on θ. When the data $x_j, j = 1,...,n$, come from a Gaussian source, and when the parameter is compatible with classical analysis, this assessment can be done using the classical theory of statistics. In other situations, the bootstrap is most useful.

In the current application we seek to extend the use of the bootstrap to the assessment of statistics of the KL expansion. We do so in the following way. As above, assume that a collection of realizations of the random process X has been measured and the elements of the measured ensemble are denoted $x_j, j = 1,...,M$. Create a bootstrap sample of the random process realizations by selecting M integer-valued random numbers from the interval [1,M]. Denote these numbers $j_i, i = 1,...,M$, and use them as indices. (The sequence of numbers $j_i, i = 1,...,M$, may contain duplicates of one or more digits in [1,M], and some of the values in [1,M] may not be present.) Create a bootstrap sample based on these indices.

$$x_b^* = \left\{ x_{j_1}, x_{j_2},..., x_{j_M} \right\} \tag{11}$$

Now use the bootstrap sample x_b^* as in Eqs. (7), (2) and (3) to obtain bootstrap replicates of the autocorrelation matrix estimate, $\left(\hat{R}_{XX} \right)_b^* = \frac{1}{M} x_b^* \left(x_b^* \right)^T$, and the KL parameters, $\hat{w}_b^*$ and $\hat{v}_b^*$. Repeat this procedure B times to generate an ensemble of bootstrap replicates of the KL parameters, $\hat{w}_b^*$ and $\hat{v}_b^*$, $b=1,...,B$. These parameters form the basis for the approximate statistical analysis of the shock random process.

In this investigation we choose to statistically gage the severity of the shock model using the magnitudes of the diagonal elements in $\left(w^{1/2}\right)_b^*, b = 1,...,B$. (Recall that these elements are all positive valued.) When these quantities are large, Eq. (9) indicates that shock magnitudes will be great. Further, because the diagonal elements in every $\hat{w}_b^*$ are organized in descending order, the first few values are most important in establishing the severity of the shock model. In view of this, we create a parameter

$$\hat{\theta}_b^* = \sum_{k=1}^{m} \left(w_k^{1/2}\right)_b^* \qquad b = 1,...,B \tag{12}$$

to use as a metric of the severity of the shock model. The expression $\left(w_k^{1/2}\right)_b^*$ is the (k)th diagonal element in the matrix $\left(w^{1/2}\right)_b^*$.

In order to use this information to obtain a $(1-\alpha)\times100$ percent conservative shock model, we plot the order statistic, defined,

$$F_b = \frac{b - 0.5}{B} \qquad b = 1,...,B \tag{13}$$

as a function of the values of $\hat{\theta}_b^*, b = 1,...,B$. This yields an empirical cumulative distribution function of θ, the parameter estimated by the $\hat{\theta}_b^*$. We identify the value of b for which $F_b = 1 - \alpha$, and denote this β. Then the $(1-\alpha)\times100$ percent conservative shock model is the one whose KL parameters are $\hat{w}_\beta^*$ and $\hat{v}_\beta^*$. (Some slightly and much more complicated, and accurate, approaches can be taken to the identification of parameters. See Efron and Tibshirani, 1993.) When we use these parameters in Eq. (9) to generate a shock realization, that realization can be characterized as one that is $(1-\alpha)\times100$ percent conservative, in the sense described here.

NUMERICAL EXAMPLE

Consider a shock source that generates non-Gaussian random process realizations like the one shown in Figure 1. Assume that $M=10$ realizations have been measured from that source. Use the approach described in the previous section to identify the parameters of a 90 percent conservative shock model with $B=1000$ bootstrap samples. Use this model to generate shocks with a 95 percent probability of not being exceeded. Take the worst of the shocks as a test shock. The result is the shock shown in Figure 2. Its greatest absolute peak value is 350g compared to a peak absolute value of the most severe underlying shock of 250g. This is a factor of 1.4. Its SRS compares to the SRS of the shock shown in Figure 1 as shown in Figure 3.

CONCLUSIONS

A direct technique for the probabilistic and statistical specification of shock tests based on measured shock data has been developed. The technique is based on the Karhunen-Loeve expansion. It has two important features. First, it permits the selection of parameters in the KL expansion to satisfy a confidence criterion. That is, it permits us to specify a level of confidence that the shock model chosen is $(1-\alpha)\times100$ percent more conservative than other models that might have been chosen. Second, it permits the generation of test shocks that have a specified probability of being exceeded. This means that there is some controllable level of probability that a shock more severe than the

generated shock will occur. This combination of probabilistic plus statistical analyses yields what is commonly known as tolerance limit on shock test specification.
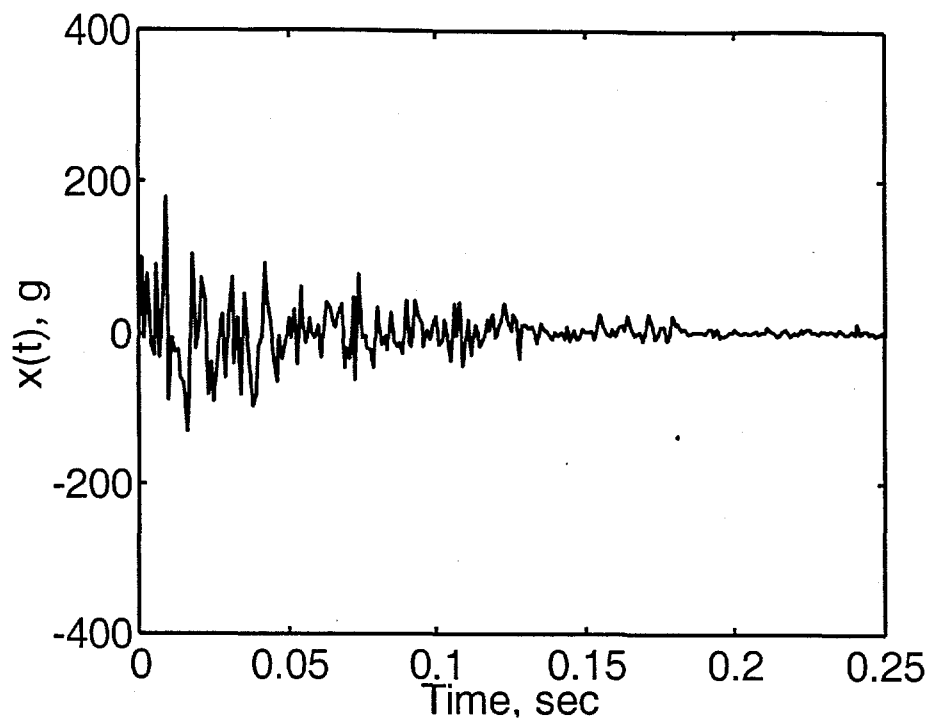

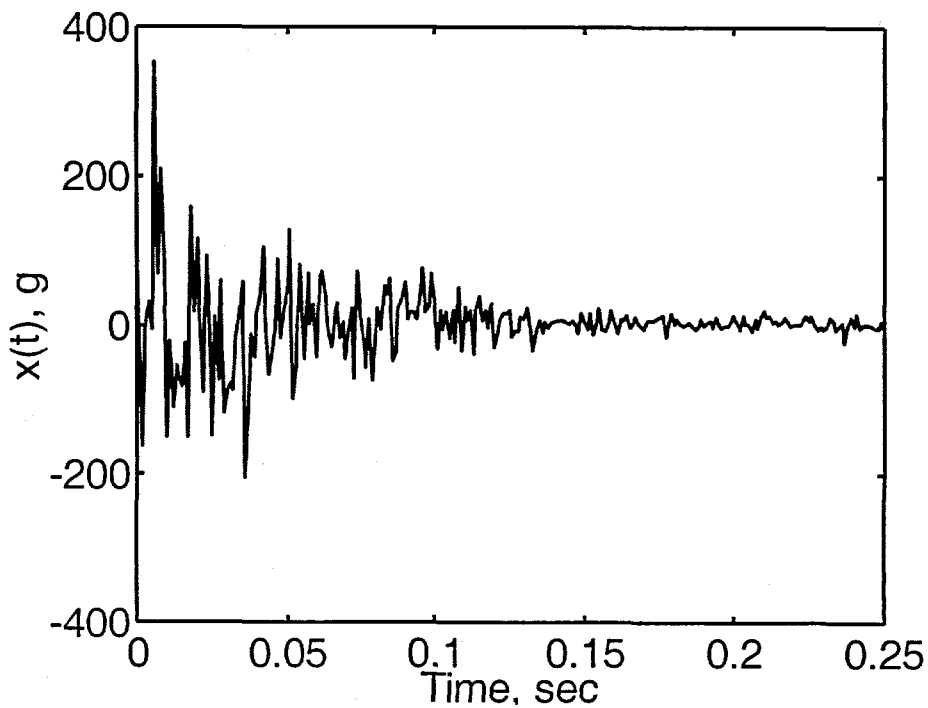Figure 1. Shock measured from a nonstationary random process source.

Figure 2. Test shock representing the ensemble of measured shocks at 90% confidence level and 95% probability of non-exceedance.

Though the technique established here introduces some new and useful capabilities, it involves some assumptions that require heed and further consideration. Most important, we have assumed that the random vector U, implicitly defined in Eq. (4) is composed of jointly standard normal, uncorrelated random variables. This will generally not be exactly true, in practice. Methods exist to transform arbitrarily distributed and dependent random vectors into the jointly standard normal, uncorrelated space, and they should be applied to this problem.
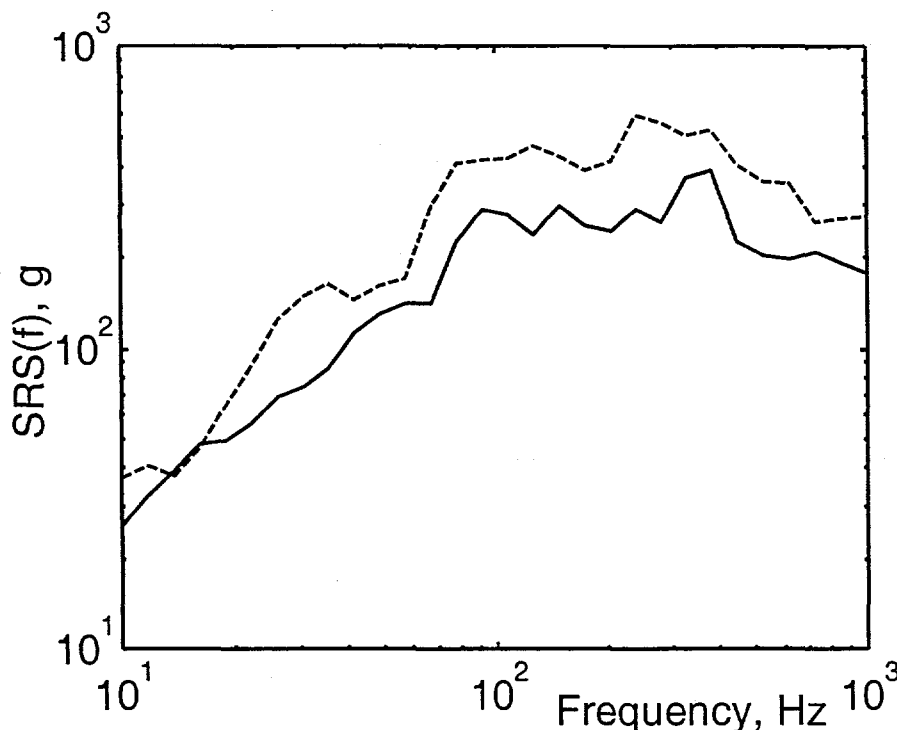

Figure 3. SRS of a typical shock in the measured ensemble (solid) and the test shock (dashed). SRS damping factor = 0.05.

ACKNOWLEDGMENT

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