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Using Vector Spherical Harmonics to Compute Antenna Mutual Impedance from Measured or Computed Fields

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Using Vector Spherical Harmonics to Compute Antenna Mutual Impedance from Measured or Computed Fields

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Abstract

The mutual coupling that exists between the antenna elements in an antenna array can be described with a mutual impedance. The knowledge of this mutual impedance is critical to the successful design of the array. Computing the mutual impedance involves integrating vector products of fields over a surface, but the integrands can oscillate wildly over the integration surface, and are often difficult to integrate accurately. The method described here relies on the expansion of the fields in terms of vector spherical harmonics. The integrations over the closed surface are performed in closed form, leaving the mutual impedance expressed as a sum of products of expansion coefficients.

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Contents

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Introduction

The mutual impedance is a circuit-theory quantity associated with a network. It relates the current, i_k, flowing into one port of the network to the open-circuit voltage, v_j, at another port. The definition is

$$z_{jk} = \left. \frac{v_j}{i_k} \right|_{i_m=0;\, m=1,2,\cdots,\, m\neq k},$$

where j, k, and m are indices which designate the various ports of the network. Alternatively, a mutual admittance can be defined,

$$y_{jk} = \left. \frac{i_j}{v_k} \right|_{v_m=0;\, m=1,2,\cdots,\, m\neq k},$$

which relates the voltage, v_k, applied across one port to the short-circuit current, i_j, at another port. Mutual impedance (or admittance) exists between the antenna elements in an antenna array, and knowledge of this mutual impedance (or admittance) is critical to the successful design of the array. Because of the mutual coupling, the reflection coefficient looking into an element is different when it is embedded in an array with all the elements excited, compared to that for the isolated element. Thus, in order to tune the elements properly for minimum reflection in the active array, the mutual impedance (or admittance) is needed.

Obviously, one could build an array of antenna elements, and then measure the coupling between the elements. From this measurement, usually in the form of a scattering matrix, the mutual impedance (or admittance) is easily determined (see Appendix V). However, this is not very practical in many situations, when the number of elements is large.

As described below, the mutual impedance can be computed if the electric and magnetic fields for the elements are known. Often, especially when the array is composed of small, identical elements, it is practical to measure or compute the fields for the element. Ideally, these fields should be measured or computed in the array environment, with all other elements terminated in matched impedances, but not excited. However, this is not always practical, and, sometimes, useful information can be obtained with the element fields measured or computed in an environment where it is the only element present.

For many applications, the mutual impedance is needed for many different relative positions of the two antenna elements. When numerical methods, such as method of moments, finite-difference time-domain methods, and finite-element methods are used, it can be very time consuming to compute the fields at each new relative position of the antennas. Thus, a field representation that allows easy translation to new positions would be valuable for improving the efficiency of the computation as the relative position is iterated.

Computing the mutual impedance involves integrating vector products of fields over a surface. The integrands can oscillate wildly over the integration surface, and are often difficult to integrate accurately. The method described here relies on the expansion of the fields in terms of vector spherical harmonics. The integrations over the closed surface are performed in closed form, leaving the mutual impedance expressed as a sum of products of expansion coefficients.

The mutual impedance is described in terms of a physical observable called the reaction, introduced by Rumsey. The reaction theorem is related to the reciprocity theorem derived by Lorentz, and an understanding of the reciprocity theorem is helpful for understanding the reaction concept. In the following, the Lorentz reciprocity theorem is derived and examined in some detail. In order to understand better the generality and applicability of the theorem, it is derived in a very general form. The medium in which the elements are embedded is assumed linear and time-invariant, but not homogeneous or even isotropic. The validity of the theorem does place constraints on the medium, and these constraints will be stated.

After the discussion of the reciprocity theorem, an expression for the mutual impedance in terms of the reaction quantity is written. The reaction is a term contained in the mathematical statement of the reciprocity theorem, and thus the discussion of the reciprocity theorem is very relevant to understanding the mutual impedance.

Once the expression for mutual impedance is written, the fields can be expanded in vector spherical harmonics, and the mutual impedance is ultimately written in terms of the expansion coefficients. This process is somewhat tedious and is described in detail. Although the final result may appear unwieldy, it is straightforward to program a computer to perform the computation. The first advantage, of course, is the avoidance of the need to integrate a wildly oscillating integrand that is slow to converge. The second advantage is that mutual impedance can be computed for many sets of element positions, using a *single* measurement or computation of the fields around an element.

Lorentz reciprocity theorem

The reciprocity theorem derived by Lorentz [1, 2, 3] leads to a reaction concept [3, 4,5] that is useful for understanding and computing mutual coupling between two antennas, or more generally, between two sets of source currents. The reciprocity theorem is discussed below, but a particularly entertaining discussion of the reciprocity theorem is contained in Weeks [6]. The reaction quantity, which corresponds to terms contained in the statement of the reciprocity theorem, was introduced by Rumsey [5]. The reaction quantity is a physical observable associated with the reaction between the fields of two sources. In an electrostatic system, Rumsey's reaction corresponds to the force exerted by one source of charge on another. He shows that, for monochromatic electromagnetic fields, the reaction is the difference between the instantaneous and average rates (over one period) at which one source performs work against the other.

Following [1], the reciprocity theorem will be developed in a general form. It is important to realize that the theorem is obtained by simply applying certain mathematical operations to fields associated with two independent sets of sources. The fields are required to satisfy Maxwell's equations, but they are not required to be related to each other, or even to exist at the same time. However, they are required to be associated with the same region of space. In addition, one would expect the validity of the theorem to require that the media associated with each set of fields be the same. While this is true in the isotropic case, it will be shown below that a more general relationship between the two media must hold, and that for certain anisotropic media, the media will not be the same.

The fields will be assumed time-harmonic (monochromatic) with dependence $e^{j\omega t}$. An electric current, $\vec{\mathbf{J}}$, it the usual true physical source for the fields. A fictitious magnetic current, $\vec{\mathbf{M}}$, will also be included, because of its convenience in handling the equivalent sources often associated with the tangential electric-field of apertures in conducting surfaces. The medium of interest will be assumed linear and time-invariant. However, it will not be assumed homogeneous or even isotropic. Thus, the medium will be characterized by dyadic electric permittivity and magnetic permeability, which are not necessarily symmetric,

$$\bar{\bar{\varepsilon}} \neq \bar{\bar{\varepsilon}}^{\dagger}, \tag{1.1}$$

$$\bar{\bar{\mu}} \neq \bar{\bar{\mu}}^{\dagger}, \tag{1.2}$$

where † indicates the transposed dyadic. The electric displacement field and magnetic flux density are

$$\vec{\mathbf{D}} = \bar{\bar{\varepsilon}} \cdot \vec{\mathbf{E}}, \tag{1.3}$$

and

$$\vec{\mathbf{B}} = \bar{\bar{\mu}} \cdot \vec{\mathbf{H}}. \tag{1.4}$$

Suppose there exist two sets of independent sources, $(\vec{\mathbf{J}}_1, \vec{\mathbf{M}}_1)$, and $(\vec{\mathbf{J}}_2, \vec{\mathbf{M}}_2)$. The first set of sources is associated with the medium $(\bar{\bar{\varepsilon}}, \bar{\bar{\mu}})$, and the second source is associated with the "transposed" medium $(\bar{\bar{\varepsilon}}^{\dagger}, \bar{\bar{\mu}}^{\dagger})$. At least when the medium is symmetric ($\bar{\bar{\varepsilon}}^{\dagger} = \bar{\bar{\varepsilon}}$, $\bar{\bar{\mu}}^{\dagger} = \bar{\bar{\mu}}$), it is natural to ask whether the two sets of sources are present at the same time. The theorem to be developed will be valid regardless of whether the sources are present at the same time. The fields associated with each source satisfy

$$\nabla \times \vec{\mathbf{E}}_1 = -j\omega\bar{\bar{\mu}} \cdot \vec{\mathbf{H}}_1 - \vec{\mathbf{M}}_1 \tag{1.5}$$

$$\nabla \times \vec{\mathbf{H}}_1 = j\omega\bar{\bar{\varepsilon}} \cdot \vec{\mathbf{E}}_1 + \vec{\mathbf{J}}_1 \tag{1.6}$$

and

$$\nabla \times \vec{\mathbf{E}}_2 = -j\omega\bar{\bar{\mu}}^{\dagger} \cdot \vec{\mathbf{H}}_2 - \vec{\mathbf{M}}_2 \tag{1.7}$$

$$\nabla \times \vec{\mathbf{H}}_2 = j\omega\bar{\bar{\varepsilon}}^{\dagger} \cdot \vec{\mathbf{E}}_2 + \vec{\mathbf{J}}_2. \tag{1.8}$$

The reciprocity theorem is obtained by combining vector products of the fields and applying vector identities, with the fields subject to (1.5) through (1.8). Thus, the theorem begins as simply a mathematical relationship that is imposed because the fields are solutions of Maxwell's equations. We begin by forming the difference between the cross product between the electric field of the first source with the magnetic field of the second source and the cross product of the remaining electric and magnetic fields. The divergence of this difference is

$$\nabla\cdot\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)=\bar{\mathbf{H}}_2\cdot\left(\nabla\times\bar{\mathbf{E}}_1\right)-\bar{\mathbf{E}}_1\cdot\left(\nabla\times\bar{\mathbf{H}}_2\right)-\bar{\mathbf{H}}_1\cdot\left(\nabla\times\bar{\mathbf{E}}_2\right)+\bar{\mathbf{E}}_2\cdot\left(\nabla\times\bar{\mathbf{H}}_1\right). \quad (1.9)$$

Now, substitute (1.5) through (1.8) for the curl of the fields

$$\begin{aligned}\nabla\cdot\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)=&-j\omega\left(\bar{\mathbf{H}}_2\cdot\bar{\bar{\mu}}\cdot\bar{\mathbf{H}}_1-\bar{\mathbf{H}}_1\cdot\bar{\bar{\mu}}^{\dagger}\cdot\bar{\mathbf{H}}_2\right)\\&+j\omega\left(\bar{\mathbf{E}}_2\cdot\bar{\bar{\varepsilon}}\cdot\bar{\mathbf{E}}_1-\bar{\mathbf{E}}_1\cdot\bar{\bar{\varepsilon}}^{\dagger}\cdot\bar{\mathbf{E}}_2\right)\\&+\bar{\mathbf{E}}_2\cdot\bar{\mathbf{J}}_1-\bar{\mathbf{E}}_1\cdot\bar{\mathbf{J}}_2+\bar{\mathbf{H}}_1\cdot\bar{\mathbf{M}}_2-\bar{\mathbf{H}}_2\cdot\bar{\mathbf{M}}_1\end{aligned}\quad . \quad (1.10)$$

Since the transpose of a scalar is that same scalar,

$$\bar{\mathbf{a}}\cdot\bar{\bar{\mathbf{X}}}\cdot\bar{\mathbf{b}}=\left(\bar{\mathbf{a}}\cdot\bar{\bar{\mathbf{X}}}\cdot\bar{\mathbf{b}}\right)^{\dagger}=\bar{\mathbf{b}}\cdot\bar{\bar{\mathbf{X}}}^{\dagger}\cdot\bar{\mathbf{a}}$$

for all vectors $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ and all dyadics $\bar{\bar{\mathbf{X}}}$. Thus, (1.10) becomes

$$\nabla\cdot\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)=\bar{\mathbf{E}}_2\cdot\bar{\mathbf{J}}_1-\bar{\mathbf{E}}_1\cdot\bar{\mathbf{J}}_2+\bar{\mathbf{H}}_1\cdot\bar{\mathbf{M}}_2-\bar{\mathbf{H}}_2\cdot\bar{\mathbf{M}}_1, \quad (1.11)$$

which is the differential form of the Lorentz reciprocity theorem. Integrating (1.11) over the volume containing the sources,

$$\begin{aligned}\iiint_V\nabla\cdot\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)dV&=\oiint_\Sigma\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)\cdot d\bar{\mathbf{s}}\\&=\iiint_V\left(\bar{\mathbf{E}}_2\cdot\bar{\mathbf{J}}_1-\bar{\mathbf{E}}_1\cdot\bar{\mathbf{J}}_2+\bar{\mathbf{H}}_1\cdot\bar{\mathbf{M}}_2-\bar{\mathbf{H}}_2\cdot\bar{\mathbf{M}}_1\right)dV\end{aligned}, \quad (1.12)$$

where the closed surface Σ encloses the volume V, and the surface normal points out of the volume. The integral form of the Lorentz reciprocity theorem is given by (1.12).

For most situations of interest, the electric permittivity and the magnetic permeability are scalars or symmetric dyadics, and the two sets of sources are radiating in the same medium. However, even when the constitutive parameters are non-symmetric dyadics, (1.11) and (1.12) still hold, provided $\bar{\mathbf{E}}_2, \bar{\mathbf{H}}_2$ meet a very special condition: $\bar{\mathbf{E}}_2, \bar{\mathbf{H}}_2$ must correspond to the fields when the second set of sources are embedded in a medium whose constitutive properties are the transpose of the constitutive properties of the medium in which the first set of sources are embedded, that is

$$\bar{\bar{\varepsilon}}_2=\bar{\bar{\varepsilon}}_1^{\dagger}$$

and

$$\bar{\bar{\mu}}_2=\bar{\bar{\mu}}_1^{\dagger}.$$

When the source currents exist in a finite volume and radiate into unbounded space, the fields are subject to the radiation condition [1]. The radiation condition says that the electric and magnetic fields become transverse to each other and propagate outward, so that

$$\lim_{r\to\infty}r\left(\nabla\times\bar{\mathbf{E}}+jk_0\hat{\mathbf{r}}\times\bar{\mathbf{E}}\right)=0. \quad (1.13)$$

Thus, if the surface Σ is taken to be the surface of the sphere at $r\to\infty$, then, assuming the medium does not change as $r\to\infty$,

$$\lim_{r\to\infty}\oint\!\!\!\oint_{\Sigma}\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)\cdot d\bar{s}=\lim_{r\to\infty}\oint\!\!\!\oint_{\Sigma}\begin{pmatrix}\bar{\mathbf{E}}_1\times\left(\dfrac{k_0}{\omega}\bar{\bar{\mu}}^{\dagger-1}\cdot\left(\hat{\mathbf{r}}\times\bar{\mathbf{E}}_2\right)\right)\\-\bar{\mathbf{E}}_2\times\left(\dfrac{k_0}{\omega}\bar{\bar{\mu}}^{-1}\cdot\left(\hat{\mathbf{r}}\times\bar{\mathbf{E}}_1\right)\right)\end{pmatrix}\cdot d\bar{s} \tag{1.14}$$

Since the inverse of the transpose of a dyadic is the same as the transpose of the inverse, we can write

$$\bar{\bar{\mu}}^{-1}=\bar{\eta}_L\bar{\eta}_R, \tag{1.15}$$

so that

$$\left(\bar{\bar{\mu}}^{\dagger}\right)^{-1}=\left(\bar{\bar{\mu}}^{-1}\right)^{\dagger}=\bar{\eta}_R\bar{\eta}_L. \tag{1.16}$$

Substituting (1.15) and (1.16) into (1.14)

$$\lim_{r\to\infty}\oint\!\!\!\oint_{\Sigma}\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)\cdot d\bar{s}=-\frac{k_0}{\omega}\lim_{r\to\infty}\oint\!\!\!\oint_{\Sigma}\begin{pmatrix}\left(\bar{\mathbf{E}}_1\times\bar{\eta}_R\left(\bar{\eta}_L\times\bar{\mathbf{E}}_2\right)\cdot\hat{\mathbf{r}}\right)\\-\left(\bar{\mathbf{E}}_2\times\bar{\eta}_L\left(\bar{\eta}_R\times\bar{\mathbf{E}}_1\right)\cdot\hat{\mathbf{r}}\right)\end{pmatrix}\cdot d\bar{s}. \tag{1.17}$$

Thus, when the magnetic permeability is either a scalar or a symmetric dyadic,

$$\lim_{r\to\infty}\oint\!\!\!\oint_{\Sigma}\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)\cdot d\bar{s}=0. \tag{1.18}$$

Inserting (1.18) into (1.12) we see that

$$\iiint_V\left(\bar{\mathbf{E}}_2\cdot\bar{\mathbf{J}}_1-\bar{\mathbf{E}}_1\cdot\bar{\mathbf{J}}_2+\bar{\mathbf{H}}_1\cdot\bar{\mathbf{M}}_2-\bar{\mathbf{H}}_2\cdot\bar{\mathbf{M}}_1\right)dV=0, \tag{1.19}$$

when currents contained in a finite volume radiate into unbounded space, and the magnetic permeability is either a scalar or a symmetric dyadic.

Suppose each set of sources is localized and the sets are contained in non-overlapping, finite, closed volumes V_1 and V_2. The volume integral can be broken into two pieces

$$\begin{aligned}\iiint_V\left(\bar{\mathbf{E}}_2\cdot\bar{\mathbf{J}}_1-\bar{\mathbf{E}}_1\cdot\bar{\mathbf{J}}_2+\bar{\mathbf{H}}_1\cdot\bar{\mathbf{M}}_2-\bar{\mathbf{H}}_2\cdot\bar{\mathbf{M}}_1\right)dV&=\iiint_{V_1}\left(\bar{\mathbf{E}}_2\cdot\bar{\mathbf{J}}_1-\bar{\mathbf{H}}_2\cdot\bar{\mathbf{M}}_1\right)dV\\&\quad-\iiint_{V_2}\left(\bar{\mathbf{E}}_1\cdot\bar{\mathbf{J}}_2-\bar{\mathbf{H}}_1\cdot\bar{\mathbf{M}}_2\right)dV\end{aligned}, \tag{1.20}$$

where V is a closed volume containing all of the sources, V_1 is the closed volume containing only sources $\bar{\mathbf{J}}_1,\bar{\mathbf{M}}_1$, and V_2 is the closed volume containing only sources $\bar{\mathbf{J}}_2,\bar{\mathbf{M}}_2$. Under this condition, (1.41) becomes

$$\iiint_{V_1}\left(\bar{\mathbf{E}}_2\cdot\bar{\mathbf{J}}_1-\bar{\mathbf{H}}_2\cdot\bar{\mathbf{M}}_1\right)dV=\iiint_{V_2}\left(\bar{\mathbf{E}}_1\cdot\bar{\mathbf{J}}_2-\bar{\mathbf{H}}_1\cdot\bar{\mathbf{M}}_2\right)dV. \tag{1.21}$$

Suppose we choose to integrate (1.12) over the source-free volume, the volume V less the two closed volumes containing the sources, V_1 and V_2. The surface integral in (1.12) will now contain three separate parts,

$$\oint\!\!\!\oint_{\Sigma}\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)\cdot d\bar{s}+\oint\!\!\!\oint_{\Sigma_1}\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)\cdot d\bar{s}+\oint\!\!\!\oint_{\Sigma_2}\left(\bar{\mathbf{E}}_1\times\bar{\mathbf{H}}_2-\bar{\mathbf{E}}_2\times\bar{\mathbf{H}}_1\right)\cdot d\bar{s} \tag{1.22}$$

where Σ is the surface of volume V, and Σ_1, Σ_2 are the surfaces of volumes V_1 and V_2, respectively. In each of the integrals, the direction of $d\bar{\mathbf{s}}$ is outward from the enclosed source-free volume. This means that in the integrals over Σ_1, Σ_2, $d\bar{\mathbf{s}}$ is pointing into the volume containing the sources. Now, the volume integral on the right side of (1.12) will be zero since no sources are contained within the volume. Also, as previously shown, the integral over the outer surface Σ will be zero as we allow $r \to \infty$. In this case, the Lorentz reciprocity theorem reduces to

$$\oiint_{\Sigma_1} \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) \cdot d\bar{\mathbf{s}} + \oiint_{\Sigma_2} \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) \cdot d\bar{\mathbf{s}} = 0 . \tag{1.23}$$

Reaction and mutual impedance

Now suppose the volume of integration is the closed volume containing only sources $\bar{\mathbf{J}}_1, \bar{\mathbf{M}}_1$, or sources $\bar{\mathbf{J}}_2, \bar{\mathbf{M}}_2$. In the first case, we have, from (1.12)

$$\oiint_{\Sigma_1} \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) \cdot d\bar{\mathbf{s}} = \iiint_{V_1} \left(\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1\right) dV , \tag{1.24}$$

while in the second case,

$$\oiint_{\Sigma_2} \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) \cdot d\bar{\mathbf{s}} = -\iiint_{V_2} \left(\bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 - \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2\right) dV . \tag{1.25}$$

Rumsey [5] defines the right hand sides of (1.24) and (1.25) as the reaction, $\langle 1,2 \rangle$, between source 1 and 2, and $\langle 2,1 \rangle$ between sources 2 and 1, respectively. In Rumsey's notation, the first designator in $\langle :,: \rangle$ indicates the source located inside the volume of integration. Specifically

$$\langle 1,2 \rangle = \iiint_{V_1} \left(\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1\right) dV , \tag{1.26}$$

and

$$\langle 2,1 \rangle = \iiint_{V_2} \left(\bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 - \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2\right) dV . \tag{1.27}$$

From (1.24) and (1.25), we also have

$$\langle 1,2 \rangle = \oiint_{\Sigma_1} \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) \cdot d\bar{\mathbf{s}} , \tag{1.28}$$

and,

$$\langle 2,1 \rangle = \oiint_{\Sigma_2} \left(\bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1 - \bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2\right) \cdot d\bar{\mathbf{s}} . \tag{1.29}$$

Richmond [4] has shown that the reaction can also be written

$$v_{jk} i_j = -\langle j,k \rangle , \tag{1.30}$$

where v_{jk} is the voltage induced across the open-circuited terminals of source j in the presence of the fields, $\left(\bar{\mathbf{E}}_k, \bar{\mathbf{H}}_k\right)$, due to current i_k at the terminals of source k. The fields $\left(\bar{\mathbf{E}}_j, \bar{\mathbf{H}}_j\right)$ are the result of applying terminal current i_j at source j.

In a multiport network, the currents at each port are related to the port voltages by an impedance matrix, as follows,

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z_{n1} & \cdots & \cdots & z_{nn} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix}. \tag{1.31}$$

Thus, from (1.30) and (1.31), the mutual impedance between port j and port k is

$$\begin{aligned} z_{jk} &= \left.\frac{v_j}{i_k}\right|_{i_m=0;\, m\neq k} \\ &= \frac{v_{jk}}{i_k} = \frac{-\langle j,k\rangle}{i_j i_k} = -\frac{1}{i_j i_k} \oint\!\!\!\oint_{\Sigma_j} \left(\bar{\mathbf{E}}_j \times \bar{\mathbf{H}}_k - \bar{\mathbf{E}}_k \times \bar{\mathbf{H}}_j\right) \cdot d\bar{\mathbf{s}} \end{aligned} . \tag{1.32}$$

When the fields associated with two antennas are known, then the mutual impedance between them can be found from (1.32). In this case, the currents on the antennas need not be known. Only the fields radiated when each antenna is excited with a known terminal current are necessary. If necessary, these fields can be obtained through measurement.

Fields associated with different media

Suppose the two sets of sources are contained within different media, $\bar{\bar{\mu}}_1, \bar{\bar{\varepsilon}}_1$ for source 1, and $\bar{\bar{\mu}}_2, \bar{\bar{\varepsilon}}_2$ for source 2. With this situation, (1.10) becomes

$$\begin{aligned} \nabla \cdot \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) = &-j\omega\left(\bar{\mathbf{H}}_2 \cdot \bar{\bar{\mu}}_1 \cdot \bar{\mathbf{H}}_1 - \bar{\mathbf{H}}_1 \cdot \bar{\bar{\mu}}_2 \cdot \bar{\mathbf{H}}_2\right) \\ &+ j\omega\left(\bar{\mathbf{E}}_2 \cdot \bar{\bar{\varepsilon}}_1 \cdot \bar{\mathbf{E}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\bar{\varepsilon}}_2 \cdot \bar{\mathbf{E}}_2\right) \\ &+ \bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1 \end{aligned} . \tag{1.33}$$

As shown previously, in order for (1.33) to reduce to the usual forms of Lorentz's reciprocity theorem, (1.11) and (1.12), the media must have the transpose relationship

$$\bar{\bar{\varepsilon}}_2 = \bar{\bar{\varepsilon}}_1^{\dagger}, \tag{1.34}$$

$$\bar{\bar{\mu}}_2 = \bar{\bar{\mu}}_1^{\dagger}. \tag{1.35}$$

However, when the fields are associated with media that do not satisfy the transpose relationship, the reciprocity theorem is not as simple, but still it can be stated that

$$\begin{aligned} \oint\!\!\!\oint_{\Sigma} \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) \cdot d\bar{\mathbf{s}} = &\iiint_V \left(\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1\right) dV \\ &+ j\omega \iiint_V \left(\bar{\mathbf{E}}_2 \cdot \bar{\bar{\varepsilon}}_1 \cdot \bar{\mathbf{E}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\bar{\mu}}_1 \cdot \bar{\mathbf{H}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\bar{\varepsilon}}_2 \cdot \bar{\mathbf{E}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\bar{\mu}}_2 \cdot \bar{\mathbf{H}}_2\right) dV \end{aligned} . \tag{1.36}$$

The additional volume integral compensates for the different materials, but now the integration must be extended to the entire volume where the media properties differ.

When the media are isotropic (scalar permittivity and permeability), then

$$\oint\!\!\!\oint_{\Sigma} \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) \cdot d\bar{s} = \iiint_V \left(\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1\right) dV + j\omega \iiint_V \left(\varepsilon' \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{E}}_2 - \mu' \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{H}}_2\right) dV, \tag{1.37}$$

where

$$\varepsilon' = \varepsilon_1 - \varepsilon_2, \tag{1.38}$$

and

$$\mu' = \mu_1 - \mu_2. \tag{1.39}$$

Consider the situation as $r \to \infty$. Using (1.13) in the left-hand expression of (1.37), we see that

$$\lim_{r\to\infty} \oint\!\!\!\oint_{\Sigma} \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1\right) \cdot d\bar{s} = \lim_{r\to\infty} \oint\!\!\!\oint_{\Sigma} \begin{pmatrix} \bar{\mathbf{E}}_1 \times \left(\frac{k_0}{\omega} \bar{\bar{\mu}}_2^{-1} \cdot \left(\hat{\mathbf{r}} \times \bar{\mathbf{E}}_2\right)\right) \\ -\bar{\mathbf{E}}_2 \times \left(\frac{k_0}{\omega} \bar{\bar{\mu}}_1^{-1} \cdot \left(\hat{\mathbf{r}} \times \bar{\mathbf{E}}_1\right)\right) \end{pmatrix} \cdot d\bar{s} \tag{1.40}$$

In general, the surface integral does not go to zero as $r \to \infty$ when the different media extend to $r \to \infty$. However, it is reasonable to assume that the region of differing media is finite, so as $r \to \infty$, $\bar{\bar{\mu}}_1 \to \mu_0$ and $\bar{\bar{\mu}}_2 \to \mu_0$. With this assumption, the surface integral does go to zero at $r \to \infty$. Thus, in the general case where the currents are contained in a finite volume of differing media and radiate into unbounded space

$$\iiint_V \left(\bar{\mathbf{E}}_2 \cdot \bar{\mathbf{J}}_1 - \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2 + \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{M}}_2 - \bar{\mathbf{H}}_2 \cdot \bar{\mathbf{M}}_1\right) dV = j\omega \iiint_V \left(\bar{\mathbf{E}}_1 \cdot \bar{\bar{\varepsilon}}_2 \cdot \bar{\mathbf{E}}_2 - \bar{\mathbf{H}}_1 \cdot \bar{\bar{\mu}}_2 \cdot \bar{\mathbf{H}}_2\right) dV - j\omega \iiint_V \left(\bar{\mathbf{E}}_2 \cdot \bar{\bar{\varepsilon}}_1 \cdot \bar{\mathbf{E}}_1 - \bar{\mathbf{H}}_2 \cdot \bar{\bar{\mu}}_1 \cdot \bar{\mathbf{H}}_1\right) dV. \tag{1.41}$$

Implementation of the computation of the mutual impedance

In order to compute the mutual impedance between two antennas, we will assume that the near fields associated with each antenna have already been obtained in some manner. Perhaps, the fields have been obtained through spherical-near-field measurement, method-of-moments computation, finite-element computation, or some other means that results in the complex-frequency-domain phasor representation of the spatial dependence of the time-harmonic field associated with each antenna. Regardless of how the fields have been obtained, it will be convenient to write the fields as expansions in a set of orthogonal vector harmonics. The convenient set associated with spherical coordinates is the set of vector spherical harmonics [3, 7, 8, 9, 10]. The vector spherical harmonics and their use in expansions of electromagnetic fields are described in Appendix I.

Initially, it may seem that this approach unnecessarily complicates the formulation of the mutual impedance. The motivation lies in the fact that considerable effort is required to obtain the near electric (or magnetic) field for a particular antenna element. However, if this effort is expended once for the element of interest, then the procedure described here will allow the mutual impedance with another identical element to be obtained easily, for any number of different locations of the second element. The second element can be translated to any position relative to the first element, but we will not consider rotation. (The method can be extended to include rotation, however.) In addition, the integrand of (1.32) can oscillate wildly, causing difficulty in obtaining an accurate value for the mutual impedance by simply evaluating the integral. The use of vector spherical harmonics allows the integrations to be performed in closed form, and the expression for the mutual impedance is reduced to sums containing products of expansion coefficients.

In the expression for mutual impedance, (1.32), the fields associated with each element must be obtained in an environment that is consistent with the presence of the other antenna. For example, if the fields are obtained for an isolated element, the computed mutual impedance will be approximate, to the extent that the presence of the second element perturbs the fields away from the isolated-element fields.

Expansion of the antenna's field in vector spherical harmonics

We will assume that region around the antenna, in which we wish to expand the field, is characterized by scalar permittivity, ε, and permeability, μ. The electric field is written as an expansion in the normalized vector spherical harmonics

$$\bar{\mathbf{E}} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[b_{n,m}^{TE} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) + b_{n,m}^{TM} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \right], \tag{2.1}$$

where the normalized vector spherical harmonics are

$$\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = jC_{n,m} \frac{e^{jm\phi}}{2\sqrt{\pi}} \left[\frac{jm}{\sin\theta} z_n^{(i)}(kr) P_n^m(\cos\theta)\hat{\theta} + \sin\theta z_n^{(i)}(kr) \frac{d}{dx} P_n^m(x)\bigg|_{x=\cos\theta} \hat{\phi} \right], \tag{2.2}$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = jC_{n,m} \frac{e^{jm\phi}}{2\sqrt{\pi}} \left\{ \begin{aligned} &\frac{z_n^{(i)}(kr)}{kr} n(n+1) P_n^m(\cos\theta)\hat{\mathbf{r}} \\ &+ \frac{1}{kr}\frac{\partial}{\partial r}\left[r z_n^{(i)}(kr) \right] \left[-\sin\theta \frac{d}{dx} P_n^m(x)\bigg|_{x=\cos\theta} \hat{\theta} + \frac{jm}{\sin\theta} P_n^m(\cos\theta)\hat{\phi} \right] \end{aligned} \right\}, \tag{2.3}$$

where

$$C_{n,m} \equiv \sqrt{\frac{(2n+1)}{n(n+1)} \frac{(n-m)!}{(n+m)!}}. \tag{2.4}$$

The expansion in (2.1) is often referred to as a multipole expansion [7]. The $n=1$ terms are the dipole terms, while $n=2$ corresponds to the quadrupole terms, etc. In (2.2) and (2.3), $z_n^{(i)}(kr)$ is one of the spherical Bessel's functions

$$z_n^{(i)}(kr) = \begin{cases} h_n^{(1)}(kr); i=1 \\ h_n^{(2)}(kr); i=2 \\ j_n(kr); i=3 \\ y_n(kr); i=4 \end{cases}, \tag{2.5}$$

and $P_n^m(x)$ is the associated Legendre function of the first kind. The notation used here is consistent with the notation used by Jackson [7] and by Abramowitz and Stegun [8].

The vector spherical harmonics are described in detail in Appendix I. With the $e^{j\omega t}$ time dependence, the $\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}})$ and $\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}})$ represent outwardly propagating waves.

The magnetic field intensity is obtained by substituting (2.1) into curl equation for the electric field,

$$\bar{\mathbf{H}} = j\frac{k}{\omega\mu}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[b_{n,m}^{TE}\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) + b_{n,m}^{TM}\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}})\right]. \tag{2.6}$$

The coefficients $b_{n,m}^{TE}$ describe the strength of the transverse-electric (TE) components of the radiated field, while coefficients $b_{n,m}^{TM}$ describe the strength of the transverse-magnetic (TM) components. When the antenna can be enclosed in a sphere of radius a, the series usually can be truncated at degree $n \approx ka$ [11], but in critical cases, it may be advisable to use $n \approx ka + 10$ [12].

Using the asymptotic expansion of the spherical Hankel's function, the vector harmonics in the far-field region are approximated at large radius, r, by

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \cong \frac{j^{n+1}e^{-jkr}}{kr}\bar{\mathbf{X}}_{n,m}(\theta,\phi), \tag{2.7}$$

and

$$\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \cong \frac{j^{n}e^{-jkr}}{kr}\hat{\mathbf{r}}\times\bar{\mathbf{X}}_{n,m}(\theta,\phi), \tag{2.8}$$

where the radially independent vector spherical harmonic, $\bar{\mathbf{X}}_{n,m}$, is given by

$$\begin{aligned}\bar{\mathbf{X}}_{n,m}(\theta,\phi) &= \frac{1}{z_n^{(i)}(kr)}\bar{\mathbf{M}}_{n,m}^{(i)} \\ &= jC_{n,m}\frac{e^{jm\phi}}{2\sqrt{\pi}}\left[\frac{jm}{\sin\theta}P_n^m(\cos\theta)\hat{\theta} + \sin\theta\frac{d}{dx}P_n^m(x)\bigg|_{x=\cos\theta}\hat{\phi}\right]\end{aligned}. \tag{2.9}$$

Thus, the far-field expressions for the outward-propagating fields are

$$\bar{\mathbf{E}} = \frac{e^{-jkr}}{kr}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} j^n\left[jb_{n,m}^{TE}\bar{\mathbf{X}}_{n,m}(\theta,\phi) + b_{n,m}^{TM}\hat{\mathbf{r}}\times\bar{\mathbf{X}}_{n,m}(\theta,\phi)\right], \tag{2.10}$$

and

$$\bar{\mathbf{H}} = \frac{e^{-jkr}}{kr}\frac{k}{\omega\mu}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} j^n\left[jb_{n,m}^{TE}\hat{\mathbf{r}}\times\bar{\mathbf{X}}_{n,m}(\theta,\phi) - b_{n,m}^{TM}\bar{\mathbf{X}}_{n,m}(\theta,\phi)\right] = \frac{k}{\omega\mu}\hat{\mathbf{r}}\times\bar{\mathbf{E}}. \tag{2.11}$$

Suppose we have obtained $\bar{\mathbf{E}}(\bar{\mathbf{r}})$ at $\bar{\mathbf{r}} = \bar{\mathbf{r}}_0$. Then,

$$b_{n,m}^{TE} = \frac{1}{\mathscr{I}_{\bar{\mathbf{M}}\cdot\bar{\mathbf{M}}}(m,n;m,n)}\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta d\theta\bar{\mathbf{E}}(\bar{\mathbf{r}}_0)\cdot\left(\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}_0)\right)^*, \tag{2.12}$$

and

$$b_{n,m}^{TM} = \frac{1}{\mathscr{I}_{\bar{\mathbf{N}}\cdot\bar{\mathbf{N}}}(m,n;m,n)} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \bar{\mathbf{E}}(\bar{\mathbf{r}}_0) \cdot \left(\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}_0)\right)^*, \tag{2.13}$$

where (see Appendix I)

$$\mathscr{I}_{\bar{\mathbf{M}}\cdot\bar{\mathbf{M}}}(m,n;m',n';r_0) = \left|z_n^{(i)}(kr_0)\right|^2 \delta_{nn'}\delta_{mm'},$$

and

$$\mathscr{I}_{\bar{\mathbf{N}}\cdot\bar{\mathbf{N}}}(m,n;m',n';r_0) = \left[\left|z_n^{(i)}(kr_0)\right|^2 + \frac{1}{k^2 r_0^2}\frac{\partial}{\partial r}\left[r_0 z_n^{(i)}(kr_0)\frac{\partial}{\partial r}\left[r_0 z_n^{(i)^*}(kr_0)\right]\right]\right]\delta_{nn'}\delta_{mm'}, \tag{2.14}$$

and

$$\delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}. \tag{2.15}$$

If the magnetic field is obtained instead of the electric field,

$$b_{n,m}^{TE} = -j\frac{\omega\mu}{k\mathscr{I}_{\bar{\mathbf{N}}\cdot\bar{\mathbf{N}}}(m,n;m,n;r_0)} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \bar{\mathbf{H}}(\bar{\mathbf{r}}_0) \cdot \left(\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}_0)\right)^*, \tag{2.16}$$

and

$$b_{n,m}^{TM} = -j\frac{\omega\mu}{k\mathscr{I}_{\bar{\mathbf{M}}\cdot\bar{\mathbf{M}}}(m,n;m,n;r_0)} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \bar{\mathbf{H}}(\bar{\mathbf{r}}_0) \cdot \left(\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}_0)\right)^*. \tag{2.17}$$

Translation of the fields

Although the expression for mutual impedance (1.32) is valid for any two arbitrary antennas, the array problem is concerned with the mutual impedance between identical elements. Thus, we will only address the problem of computing the mutual impedance between identical elements. The translation of the fields is accomplished through the application of an appropriate addition theorem. The addition theorem for vector spherical harmonics is described in Appendix III.

Antenna 1 is located at the origin of coordinate system 1 (unprimed), and antenna 2 is located at the origin of coordinate system 2 (primed). The origin of coordinate system 2 is located from the origin of antenna 1 by position vector $\bar{\mathbf{r}}''$,

$$\bar{\mathbf{r}}'' = r''\left[\sin\theta''\cos\phi''\hat{\mathbf{x}}_1 + \sin\theta''\sin\phi''\hat{\mathbf{y}}_1 + \cos\theta''\hat{\mathbf{z}}_1\right]. \tag{2.18}$$

The geometry is illustrated in Figure 1. To obtain z_{21}, the integration in (1.32) will be performed over a sphere that surrounds antenna 2.

Note that, as illustrated in Figure 1, $\bar{\mathbf{r}}$ locates the field point on the sphere of integration from the origin of coordinate system 1 (unprimed), while $\bar{\mathbf{r}}'$ locates the same field point from the origin of coordinate system 2 (primed). In fact, during the integration, $|\bar{\mathbf{r}}'|$ will be constant at the value of the radius chosen for the integration, even though the length of $\bar{\mathbf{r}}$ changes. Also, note that $\bar{\mathbf{r}}''$, which locates origin 2 from coordinate system 1, is a constant vector; it does not move during the integration. These three vectors are related as

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}'' + \bar{\mathbf{r}}' \tag{2.19}$$

The fields for antenna 1 will be expanded in vector spherical harmonics as follows:

$$\bar{\mathbf{E}}_1 = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[b_{n,m}^{TE} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) + b_{n,m}^{TM} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \right], \tag{2.20}$$

and

$$\bar{\mathbf{H}}_1 = j \frac{k}{\omega\mu} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[b_{n,m}^{TE} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) + b_{n,m}^{TM} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) \right], \tag{2.21}$$

using vector spherical harmonics associated with the unprimed (antenna 1) coordinate system. However, in order to integrate over a sphere around antenna 2, it is convenient to express these fields in terms of vector spherical harmonics associated with the primed coordinate system.

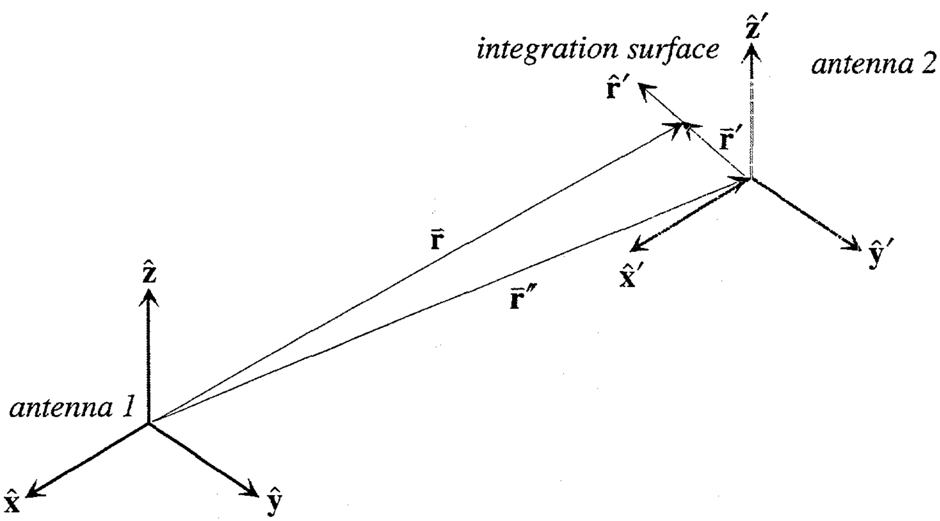

Figure 1 Geometry describing the relationship between the two antennas.

The vector-spherical-harmonic addition theorem says (see Appendix III) that for the additive relationship $\bar{\mathbf{r}} = \bar{\mathbf{r}}' + \bar{\mathbf{r}}''$,

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \sum_{n',m'} A_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'), \tag{2.22}$$

and

$$\bar{\mathbf{N}}_{n',m'}^{(2)}(\bar{\mathbf{r}}) = \sum_{n',m'} A_{n',m';n,m} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'), \tag{2.23}$$

where $A_{n',m';n,m}$ and $B_{n',m';n,m}$ are given in Appendix III. Both the vector spherical harmonics $\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')$ and $\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')$ contain spherical Bessel's functions $z_{n'}^{(i)}(kr')$, the specific kind of which is determined by the relative size of r' and r'' as follows

$$\text{in } \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \text{ and } \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}'): \; z_{n'}^{(i)}(kr') = \begin{cases} j_{n'}(kr'), & r' < r'' \\ h_{n'}^{(2)}(kr'), & r' > r'' \end{cases}. \tag{2.24}$$

Similarly, both sets of coefficients $A_{n',m';n,m}$ and $B_{n',m';n,m}$ contain spherical Bessel's functions evaluated at r'', the specific kind of which is determined according to

$$\text{in } A_{n',m';n,m} \text{ and } B_{n',m';n,m}: \; z_{n'}^{(i')}(kr'') = \begin{cases} h_{n'}^{(2)}(kr''), \, r' < r'' \\ j_{n'}(kr''), \, r' > r'' \end{cases}. \tag{2.25}$$

It should be emphasized that, for the integration over the spherical surface around antenna 2 (Figure 1), these Bessel's functions are constant, since r' and r'' are constant.

Substituting (2.22) and (2.23) into (2.20) and (2.21) gives the appropriate expressions for the fields associated with antenna 1

$$\bar{\mathbf{E}}_1 = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\begin{bmatrix} b_{n,m}^{TE}\sum_{n',m'}\left(A_{n',m';n,m}\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m}\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\right) \\ +b_{n,m}^{TM}\sum_{n',m'}\left(A_{n',m';n,m}\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m}\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\right)\end{bmatrix}, \tag{2.26}$$

and

$$\bar{\mathbf{H}}_1 = j\frac{k}{\omega\mu}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\begin{bmatrix} b_{n,m}^{TE}\sum_{n',m'}\left(A_{n',m';n,m}\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m}\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\right) \\ +b_{n,m}^{TM}\sum_{n',m'}\left(A_{n',m';n,m}\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + B_{n',m';n,m}\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\right)\end{bmatrix}, \tag{2.27}$$

or

$$\bar{\mathbf{E}}_1 = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{n',m'}\begin{pmatrix}\left(b_{n,m}^{TE}A_{n',m';n,m} + b_{n,m}^{TM}B_{n',m';n,m}\right)\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \\ +\left(b_{n,m}^{TE}B_{n',m';n,m} + b_{n,m}^{TM}A_{n',m';n,m}\right)\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\end{pmatrix}, \tag{2.28}$$

and

$$\bar{\mathbf{H}}_1 = j\frac{k}{\omega\mu}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{n',m'}\begin{pmatrix}\left(b_{n,m}^{TE}A_{n',m';n,m} + b_{n,m}^{TM}B_{n',m';n,m}(\bar{\mathbf{r}}')\right)\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \\ +\left(b_{n,m}^{TE}B_{n',m';n,m} + b_{n,m}^{TM}A_{n',m';n,m}\right)\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\end{pmatrix}. \tag{2.29}$$

Exchanging the order of summation

$$\bar{\mathbf{E}}_1 = \sum_{n'=1}^{\infty}\sum_{m'=-n'}^{n'}\left({}_1b_{n',m'}^{TE}\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + {}_1b_{n',m'}^{TM}\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\right), \tag{2.30}$$

and

$$\bar{\mathbf{H}}_1 = j\frac{k}{\omega\mu}\sum_{n'=1}^{\infty}\sum_{m'=-n'}^{n'}\left({}_1b_{n',m'}^{TM}\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + {}_1b_{n',m'}^{TE}\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\right), \tag{2.31}$$

where

$${}_1b_{n',m'}^{TE} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(b_{n,m}^{TE}A_{n',m';n,m} + b_{n,m}^{TM}B_{n',m';n,m}\right), \tag{2.32}$$

and

$${}_1b_{n',m'}^{TM} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left(b_{n,m}^{TE}B_{n',m';n,m} + b_{n,m}^{TM}A_{n',m';n,m}\right). \tag{2.33}$$

The pre-subscript is used to designate the coefficients as belonging to the field expansion for antenna 1.

The fields from antenna 2 are simply

$$\bar{\mathbf{E}}_2 = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[{}_2b_{n',m'}^{TE} \bar{\mathbf{M}}_{n',m'}^{(2)}(\bar{\mathbf{r}}') + {}_2b_{n',m'}^{TM} \bar{\mathbf{N}}_{n',m'}^{(2)}(\bar{\mathbf{r}}') \right], \tag{2.34}$$

$$\bar{\mathbf{H}}_2 = j\frac{k}{\omega\mu} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[{}_2b_{n',m'}^{TE} \bar{\mathbf{N}}_{n',m'}^{(2)}(\bar{\mathbf{r}}') + {}_2b_{n',m'}^{TM} \bar{\mathbf{M}}_{n',m'}^{(2)}(\bar{\mathbf{r}}') \right], \tag{2.35}$$

where

$${}_2b_{n,m}^{TE} = b_{n,m}^{TE} \tag{2.36}$$

and

$${}_2b_{n,m}^{TM} = b_{n,m}^{TM}. \tag{2.37}$$

Mutual impedance

The fields for antenna 1 and antenna 2 have been expanded in vector spherical harmonics as follows:

$$\bar{\mathbf{E}}_1 = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[{}_1b_{n',m'}^{TE} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + {}_1b_{n',m'}^{TM} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right], \tag{2.38}$$

$$\bar{\mathbf{H}}_1 = j\frac{k}{\omega\mu} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[{}_1b_{n',m'}^{TE} \bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') + {}_1b_{n',m'}^{TM} \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right], \tag{2.39}$$

$$\bar{\mathbf{E}}_2 = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[{}_2b_{n,m}^{TE} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') + {}_2b_{n,m}^{TM} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right], \tag{2.40}$$

$$\bar{\mathbf{H}}_2 = j\frac{k}{\omega\mu} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[{}_2b_{n,m}^{TE} \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') + {}_2b_{n,m}^{TM} \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right]. \tag{2.41}$$

The pre-subscripts on the coefficients correspond to the antenna designation. The superscript showing the type of spherical Bessel's function used in the vector harmonic is designated with the superscript (i) in the expressions for $\bar{\mathbf{E}}_1$ and $\bar{\mathbf{H}}_1$. It is assumed that the harmonics have already been translated using the translation theorem. Also, the coefficients ${}_1b_{n',m'}^{TE}$ and ${}_1b_{n',m'}^{TM}$ have already been translated using the translation theorem. As noted above, the types of Bessel's functions contained in (2.38) and (2.39) depend on the relative size of the distance from the old coordinate origin to the new coordinate origin, r'', and the distance from the new origin to the field point, r'.

The mutual impedance between the antennas is, from (1.32)

$$z_{12} = z_{21} = -\frac{1}{i_1 i_2} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{E}}_1 \times \bar{\mathbf{H}}_2 - \bar{\mathbf{E}}_2 \times \bar{\mathbf{H}}_1 \right) \cdot \hat{\mathbf{r}}'. \tag{2.42}$$

The integrals in (2.42) are of the forms

$$\begin{aligned} &\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = \\ &jC_{n',m'} C_{n,m} \frac{h_n^{(2)}(kr') z_{n'}^{(i)}(kr')}{4\pi} \int_0^{2\pi} e^{j(m+m')\phi'} d\phi' \int_{-1}^{1} \left[m P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) - m' P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right] dx \end{aligned} \tag{2.43}$$

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' =$$
$$-C_{n',m'} C_{n,m} \frac{h_n^{(2)}(kr')}{4\pi kr'} \frac{\partial}{\partial r'} \left[r' z_{n'}^{(i)}(kr') \right] \int_0^{2\pi} e^{j(m+m')\phi'} d\phi' \int_{-1}^{1} \left(\frac{mm'}{1-x^2} P_n^m(x) P_{n'}^{m'}(x) + (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) \right) dx \tag{2.44}$$

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' =$$
$$-C_{n',m'} C_{n,m} \frac{z_{n'}^{(i)}(kr')}{4\pi kr'} \frac{\partial}{\partial r'} \left[r' h_n^{(2)}(kr') \right] \int_0^{2\pi} e^{j(m+m')\phi'} d\phi' \int_{-1}^{1} \left(\frac{mm'}{1-x^2} P_n^m(x) P_{n'}^{m'}(x) + (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) \right) dx \tag{2.45}$$

and

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' =$$
$$jC_{n',m'} C_{n,m} \frac{\frac{\partial}{\partial r'} \left[r' h_n^{(2)}(kr') \right] \frac{\partial}{\partial r'} \left[r' z_{n'}^{(i)}(kr') \right]}{4\pi k^2 r'^2} \int_0^{2\pi} e^{j(m+m')\phi'} d\phi' \int_{-1}^{1} \left[m P_n^m(x) \frac{d}{dx} P_{n'}^{m'}(x) - m' P_{n'}^{m'}(x) \frac{d}{dx} P_n^m(x) \right] dx \tag{2.46}$$

Each of these integrals contains the factor

$$\int_0^{2\pi} e^{j(m+m')\phi'} d\phi' = 2\pi \delta_{m,-m'}, \tag{2.47}$$

so each integral is zero unless $m' = -m$. Examination of (2.43) through (2.46) indicates that we only need to evaluate three integrals:

$$I_1(n,n',m) = \int_{-1}^{1} (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) dx \tag{2.48}$$

$$I_2(n,n',m) = m \int_{-1}^{1} \left[P_n^m(x) \frac{d}{dx} P_{n'}^{-m}(x) \right] dx \tag{2.49}$$

$$I_3(n,n',m) = -m^2 \int_{-1}^{1} \frac{1}{1-x^2} P_n^m(x) P_{n'}^{-m}(x) dx \tag{2.50}$$

The integrals (2.48) and (2.50) possess obvious symmetry properties:

$$\begin{aligned} I_1(n,n',m) &= I_1(n',n,-m) \\ I_1(n,n',-m) &= I_1(n',n,m) \end{aligned}, \tag{2.51}$$

and

$$\begin{aligned} I_3(n,n',m) &= I_3(n',n,-m) \\ I_3(n,n',-m) &= I_3(n',n,m) \end{aligned}. \tag{2.52}$$

It is convenient to express the integrals (2.48)–(2.50) in terms of integrals containing associated Legendre functions of only positive order, $m \geq 0$. Since [7]

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x) \tag{2.53}$$

we have

$$\begin{aligned} I_1(n,n',m) &= (-1)^m \frac{(n'-m)!}{(n'+m)!} \int_{-1}^{1} (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^m(x)\,dx \\ &= (-1)^m \frac{(n'-m)!}{(n'+m)!} \mathscr{I}_1(n,n',m) \end{aligned} \tag{2.54}$$

$$\begin{aligned} I_2(n,n',m) &= (-1)^m \frac{(n'-m)!}{(n'+m)!} m \int_{-1}^{1} \left[P_n^m(x) \frac{d}{dx} P_{n'}^m(x) \right] dx \\ &= (-1)^m \frac{(n'-m)!}{(n'+m)!} m \mathscr{I}_2(n,n',m) \end{aligned} \tag{2.55}$$

$$\begin{aligned} I_3(n,n',m) &= (-1)^{m+1} \frac{(n'-m)!}{(n'+m)!} m^2 \int_{-1}^{1} \frac{1}{1-x^2} P_n^m(x) P_{n'}^m(x)\,dx \\ &= (-1)^{m+1} \frac{(n'-m)!}{(n'+m)!} m^2 \mathscr{I}_3(n,n',m) \end{aligned} \tag{2.56}$$

It can be shown that the integrals in (2.54)–(2.56) can be expressed in closed form as follows (see Appendix I),

$$\begin{aligned} \mathscr{I}_1(n,n',m) &= \int_{-1}^{1} (1-x^2) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^m(x)\,dx \\ &= \begin{cases} 0; \; n'+n \text{ odd} \\ \dfrac{2n(n+1)}{(2n+1)} \dfrac{(n+m)!}{(n-m)!} \delta_{n,n'} - m \dfrac{(\min(n,n')+m)!}{(\min(n,n')-m)!}; \; n'+n \text{ even}, m \geq 0 \\ \dfrac{2n(n+1)}{(2n+1)} \dfrac{(n+m)!}{(n-m)!} \delta_{n,n'} + m \dfrac{(\max(n,n')+m)!}{(\max(n,n')-m)!}; \; n'+n \text{ even}, m < 0 \end{cases}, \end{aligned} \tag{2.57}$$

$$\mathcal{I}_2(n,n',m) = \int_{-1}^{1}\left[P_n^m(x)\frac{d}{dx}P_{n'}^m(x)\right]dx$$

$$= \begin{cases} 0; & \begin{cases} n'+n \text{ even, for all } m \\ n+n' \text{ odd, } n>n', \text{ and } m=0 \end{cases} \\ 2; & n+n' \text{ odd, } n'>n, \text{ and } m=0 \\ \dfrac{1}{m(2n'+1)}\left(\begin{array}{l} (n'+1)(n'+m)\dfrac{(\min(n,n'-1)+m)!}{(\min(n,n'-1)-m)!} \\ -n'(n'+1-m)\dfrac{(\min(n,n'+1)+m)!}{(\min(n,n'+1)-m)!} \end{array}\right); & \begin{cases} n'+n \text{ odd} \\ \text{and } m>0 \end{cases} \\ -\dfrac{1}{m(2n'+1)}\left(\begin{array}{l} (n'+1)(n'+m)\dfrac{(\max(n,n'-1)+m)!}{(\max(n,n'-1)-m)!} \\ -n'(n'+1-m)\dfrac{(\max(n,n'+1)+m)!}{(\max(n,n'+1)-m)!} \end{array}\right); & \begin{cases} n'+n \text{ odd} \\ \text{and } m<0 \end{cases} \end{cases} \quad (2.58)$$

and

$$\mathcal{I}_3(n,n',m) = \int_{-1}^{1}\frac{1}{1-x^2}P_n^m(x)P_{n'}^m(x)\,dx$$

$$= \begin{cases} 0; \ n'+n \text{ odd} \\ \dfrac{(\min(n,n')+m)!}{m(\min(n,n')-m)!}; \ n'+n \text{ even and } m>0 \\ -\dfrac{(\max(n,n')+m)!}{m(\max(n,n')-m)!}; \ n'+n \text{ even and } m<0 \\ \text{undefined}; \ n'+n \text{ even and } m=0 \end{cases} . \quad (2.59)$$

The integrals necessary for the computation of the mutual impedance can now be written. Substituting (2.57) into (2.54)

$$I_1(n,n',m) = (-1)^m \begin{cases} 0; \ n'+n \text{ odd} \\ \dfrac{2n(n+1)}{(2n+1)}\delta_{n,n'} - m\dfrac{(n'-m)!}{(n'+m)!}\dfrac{(\min(n,n')+m)!}{(\min(n,n')-m)!}; & \begin{cases} n'+n \text{ even} \\ \text{and } m\ge 0 \end{cases} \\ \dfrac{2n(n+1)}{(2n+1)}\delta_{n,n'} + m\dfrac{(n'-m)!}{(n'+m)!}\dfrac{(\max(n,n')+m)!}{(\max(n,n')-m)!}; & \begin{cases} n'+n \text{ even} \\ \text{and } m<0 \end{cases} \end{cases} \quad (2.60)$$

Substituting (2.58) into (2.55)

$$I_2(n,n',m)=\begin{cases} 0; \; n'+n \text{ even} \\ \dfrac{(-1)^m}{(2n'+1)}\left((n'+1)\dfrac{(n'-m)!}{(n'-1+m)!}\dfrac{(\min(n,n'-1)+m)!}{(\min(n,n'-1)-m)!} - n'\dfrac{(n'+1-m)!}{(n'+m)!}\dfrac{(\min(n,n'+1)+m)!}{(\min(n,n'+1)-m)!} \right); & \begin{cases} n'+n \text{ odd} \\ \text{and } m \geq 0 \end{cases} \\ \dfrac{(-1)^{m+1}}{(2n'+1)}\left((n'+1)\dfrac{(n'-m)!}{(n'-1+m)!}\dfrac{(\max(n,n'-1)+m)!}{(\max(n,n'-1)-m)!} - n'\dfrac{(n'+1-m)!}{(n'+m)!}\dfrac{(\max(n,n'+1)+m)!}{(\max(n,n'+1)-m)!} \right); & \begin{cases} n'+n \text{ odd} \\ \text{and } m < 0 \end{cases} \end{cases} \quad (2.61)$$

Substituting (2.59) into (2.56)

$$I_3(n,n',m)=\begin{cases} 0; \; n'+n \text{ odd} \\ (-1)^{m+1} m \dfrac{(n'-m)!}{(n'+m)!}\dfrac{(\min(n,n')+m)!}{(\min(n,n')-m)!}; \; n'+n \text{ even and } m \geq 0 \\ (-1)^{m+1} m \dfrac{(n'-m)!}{(n'+m)!}\dfrac{(\max(n,n')+m)!}{(\max(n,n')-m)!}; \; n'+n \text{ even and } m < 0 \end{cases} \quad (2.62)$$

The integrals contained in (2.42) are

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = jC_{n',m}C_{n,m}\frac{h_n^{(2)}(kr')\, z_{n'}^{(i)}(kr')}{2}\left[I_2(n,n',m) - I_2(n',n,m) \right]\delta_{m,-m'} \quad (2.63)$$

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = -C_{n',m}C_{n,m}\frac{h_n^{(2)}(kr')}{2kr'}\frac{\partial}{\partial r'}\left[r' z_{n'}^{(i)}(kr') \right]\left[I_3(n,n',m) - I_1(n,n',m) \right]\delta_{m,-m'} \quad (2.64)$$

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = -C_{n',m}C_{n,m}\frac{z_{n'}^{(i)}(kr')}{2kr'}\frac{\partial}{\partial r'}\left[r' h_n^{(2)}(kr') \right]\left[I_3(n,n',m) - I_1(n',n,m) \right]\delta_{m,-m'} \quad (2.65)$$

and

$$\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \left(\bar{\mathbf{N}}^{(i)}_{n',m'}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}^{(2)}_{n,m}(\bar{\mathbf{r}}') \right) \cdot \hat{\mathbf{r}}' = jC_{n',m}C_{n,m} \frac{\frac{\partial}{\partial r'}\left[r' h^{(2)}_n(kr') \right] \frac{\partial}{\partial r'}\left[r' z^{(i)}_{n'}(kr') \right]}{2k^2 r'^2} \left[I_2(n,n',m) - I_2(n',n,m) \right] \delta_{m,-m'} \tag{2.66}$$

where, as stated previously,

$$C_{n,m} \equiv \sqrt{\frac{(2n+1)}{n(n+1)} \frac{(n-m)!}{(n+m)!}}.$$

Substituting the fields, expanded in vector spherical harmonics, into the integral for the mutual impedance leads to

$$z_{12} = z_{21} = -\frac{1}{i_1 i_2} j \frac{k}{\omega\mu} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \sum_{n=1}^{\infty} \sum_{m=-\min(n,n')}^{\min(n,n')} \left(\begin{array}{l} \left(\begin{array}{l} {}_1b^{TE}_{n',m'}\, {}_2b^{TE}_{n,m} \\ + {}_2b^{TM}_{n,m}\, {}_1b^{TM}_{n',m'} \end{array} \right) \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' \bar{\mathbf{M}}^{(i)}_{n',m'}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}^{(2)}_{n,m}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' d\theta' \\ + \left(\begin{array}{l} {}_1b^{TE}_{n',m'}\, {}_2b^{TM}_{n,m} \\ + {}_2b^{TE}_{n,m}\, {}_1b^{TM}_{n',m'} \end{array} \right) \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' \bar{\mathbf{M}}^{(i)}_{n',m'}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}^{(2)}_{n,m}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' d\theta' \\ + \left(\begin{array}{l} {}_1b^{TM}_{n',m'}\, {}_2b^{TE}_{n,m} \\ + {}_2b^{TM}_{n,m}\, {}_1b^{TE}_{n',m'} \end{array} \right) \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' \bar{\mathbf{N}}^{(i)}_{n',m'}(\bar{\mathbf{r}}') \times \bar{\mathbf{N}}^{(2)}_{n,m}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' d\theta' \\ + \left(\begin{array}{l} {}_1b^{TM}_{n',m'}\, {}_2b^{TM}_{n,m} \\ + {}_2b^{TE}_{n,m}\, {}_1b^{TE}_{n',m'} \end{array} \right) \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' \bar{\mathbf{N}}^{(i)}_{n',m'}(\bar{\mathbf{r}}') \times \bar{\mathbf{M}}^{(2)}_{n,m}(\bar{\mathbf{r}}') \cdot \hat{\mathbf{r}}' d\theta' \end{array} \right). \tag{2.67}$$

Writing (2.67) in terms of the integrals described above, the mutual impedance is

$$z_{12} = z_{21} = -\frac{1}{i_1 i_2} j \frac{k}{\omega\mu} \sum_{n'=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=-\min(n,n')}^{\min(n,n')} C_{n',m}C_{n,m} \left(\begin{array}{l} A_{n,n',m} G_{n,n'}(kr') \left[I_3(n,n',m) - I_1(n',n,m) \right] \\ + B_{n,n',m} H_{n,n'}(kr') \left[I_2(n,n',m) - I_2(n',n,m) \right] \end{array} \right) \tag{2.68}$$

where

$$\begin{aligned} A_{n,n',m} &\equiv \frac{{}_1b^{TE}_{n',-m}\, {}_2b^{TE}_{n,m} + {}_2b^{TM}_{n,m}\, {}_1b^{TM}_{n',-m}}{2} \\ B_{n,n',m} &\equiv j\left(\frac{{}_1b^{TE}_{n',-m}\, {}_2b^{TM}_{n,m} + {}_2b^{TE}_{n,m}\, {}_1b^{TM}_{n',-m}}{2} \right) \\ G_{n,n'}(kr') &\equiv \frac{z^{(i)}_{n'}(kr')}{kr'} \frac{\partial}{\partial r'}\left[r' h^{(2)}_n(kr') \right] - \frac{h^{(2)}_n(kr')}{kr'} \frac{\partial}{\partial r'}\left[r' z^{(i)}_{n'}(kr') \right] \\ H_{n,n'}(kr') &\equiv h^{(2)}_n(kr') z^{(i)}_{n'}(kr') + \frac{\frac{\partial}{\partial r'}\left[r' h^{(2)}_n(kr') \right] \frac{\partial}{\partial r'}\left[r' z^{(i)}_{n'}(kr') \right]}{k^2 r'^2} \end{aligned} \tag{2.69}$$

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Appendix I — Vector Spherical Harmonics

General vector harmonics

The vector wave equation is

$$\nabla^2\bar{\mathbf{F}} + k^2\bar{\mathbf{F}} = \nabla\nabla\cdot\bar{\mathbf{F}} - \nabla\times\nabla\times\bar{\mathbf{F}} + k^2\bar{\mathbf{F}} = 0. \qquad \text{(I-1)}$$

Three independent solutions of (I-1) are [13]

$$\bar{\mathbf{F}} = \bar{\mathbf{L}} = \nabla\psi, \qquad \text{(I-2)}$$

$$\bar{\mathbf{F}} = \bar{\mathbf{M}} = \nabla\times\hat{\mathbf{a}}\psi, \qquad \text{(I-3)}$$

and

$$\bar{\mathbf{F}} = \bar{\mathbf{N}} = \frac{1}{k}\nabla\times\bar{\mathbf{M}} = \frac{1}{k}\nabla\times\nabla\times\hat{\mathbf{a}}\psi, \qquad \text{(I-4)}$$

where ψ is a solution of the scalar wave equation

$$\nabla^2\psi + k^2\psi = 0, \qquad \text{(I-5)}$$

and $\hat{\mathbf{a}}$ is a constant unit vector. That $\bar{\mathbf{L}}$ is a solution of (I-1) is easily demonstrated. We have

$$\nabla^2\bar{\mathbf{L}} + k^2\bar{\mathbf{L}} = \nabla\nabla\cdot(\nabla\psi) - \nabla\times\nabla\times(\nabla\psi) + \nabla k^2\psi. \qquad \text{(I-6)}$$

Since $\nabla\times\nabla\psi \equiv 0$, we have

$$\nabla\left(\nabla^2\psi + k^2\psi\right) = 0. \qquad \text{(I-7)}$$

Substituting $\bar{\mathbf{M}}$ into (I-1)

$$\nabla^2\bar{\mathbf{M}} + k^2\bar{\mathbf{M}} = \nabla^2\left(\nabla\times\hat{\mathbf{a}}\psi\right) + k^2\left(\nabla\times\hat{\mathbf{a}}\psi\right). \qquad \text{(I-8)}$$

Since ∇^2 commutes with $\nabla\times$ (see Appendix II), we see that

$$\nabla\times\left(\nabla^2\hat{\mathbf{a}}\psi + k^2\hat{\mathbf{a}}\psi\right) = \nabla\times\hat{\mathbf{a}}\left(\nabla^2\psi + k^2\psi\right) = 0. \qquad \text{(I-9)}$$

Similarly, substituting $\bar{\mathbf{N}}$ into (I-1)

$$\nabla^2\bar{\mathbf{N}} + k^2\bar{\mathbf{N}} = \nabla^2\left(\frac{1}{k}\nabla\times\bar{\mathbf{M}}\right) + k^2\left(\frac{1}{k}\nabla\times\bar{\mathbf{M}}\right). \qquad \text{(I-10)}$$

Again, using the commutation property of ∇^2 and $\nabla\times$

$$\nabla^2\bar{\mathbf{N}} + k^2\bar{\mathbf{N}} = \frac{1}{k}\nabla\times\left(\nabla^2\bar{\mathbf{M}} + k^2\bar{\mathbf{M}}\right) = 0. \qquad \text{(I-11)}$$

From (I-3) and (I-4), we see that

$$\nabla\cdot\bar{\mathbf{M}} = 0, \qquad \text{(I-12)}$$

and

$$\nabla\cdot\bar{\mathbf{N}} = 0. \qquad \text{(I-13)}$$

Also, from (I-2)

$$\nabla\times\bar{\mathbf{L}} = 0, \qquad \text{(I-14)}$$

and since ψ is a solution of the wave equation,

$$\nabla\cdot\bar{\mathbf{L}} = \nabla^2\psi = -k^2\psi.$$

When the curl operator, $\nabla\times$, and $\hat{\mathbf{a}}$ are anti-commutative (which is true when $\hat{\mathbf{a}}$ = constant or $\hat{\mathbf{a}} = \hat{\mathbf{r}}$, see Appendix II), the vector harmonics are also related by

$$\bar{\mathbf{M}} = \nabla \times \hat{\mathbf{a}}\psi = -\hat{\mathbf{a}} \times \nabla\psi = -\hat{\mathbf{a}} \times \bar{\mathbf{L}} = \frac{1}{k}\nabla \times \bar{\mathbf{N}}. \tag{I-15}$$

Vector harmonics in the spherical coordinate system

In the spherical coordinate system, the requirement that $\hat{\mathbf{a}}$ be a constant can be relaxed to the extent that it can be replaced by the radial unit vector $\hat{\mathbf{r}}$ [13], because the curl operator, $\nabla \times$, and $\hat{\mathbf{r}}$ are anti-commutative.

The set of vector spherical harmonics used here is based on the normalized vector spherical harmonics and notation as defined in Jackson [7]

$$\bar{\mathbf{X}}_{n,m}(\theta,\phi) \equiv \frac{1}{j\sqrt{n(n+1)}}\bar{\mathbf{r}} \times \nabla Y_{n,m}(\theta,\phi), \tag{I-16}$$

where $Y_{n,m}(\theta,\phi)$ are the scalar spherical harmonics given by

$$Y_{n,m}(\theta,\phi) \equiv \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)e^{jm\phi}, \tag{I-17}$$

and $P_n^m(x)$ is the associated Legendre function given by [†]

$$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x), \tag{I-18}$$

where $P_n(x)$ is the Legendre function

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{I-19}$$

For negative order, m, [7]

$$P_n^{-m}(x) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(x), \tag{I-20}$$

so that

$$Y_{n,-m}(\theta,\phi) = (-1)^m Y_{n,m}^*(\theta,\phi). \tag{I-21}$$

The scalar spherical harmonics, $Y_{n,m}(\theta,\phi)$ are orthonormal [7], so that

$$\iint_\Omega Y_{n,m}(\theta,\phi)Y_{n',m'}^*(\theta,\phi)\,d\Omega = \delta_{n,n'}\delta_{m,m'}, \tag{I-22}$$

where $\iint_\Omega d\Omega$ is $\int_0^\pi \sin\theta d\theta\int_0^{2\pi}d\phi$.

The explicit form for $\bar{\mathbf{X}}_{n,m}$ is

[†] Hanson [12], Stratton [13], Arfken [14], and Mathews and Walker [16] omit the factor $(-1)^m$, but it is included by Jackson [7], Abramowitz and Stegun [8], Chew [9], Lebedev [10], and Balanis [3].

$$\bar{\mathbf{X}}_{n,m}(\bar{\mathbf{r}}) = j\sqrt{\frac{(2n+1)}{4\pi n(n+1)}\frac{(n-m)!}{(n+m)!}}\begin{bmatrix} \frac{jm}{\sin\theta}P_n^m(\cos\theta)e^{jm\phi}\hat{\theta} \\ +\sin\theta\frac{d}{dx}P_n^m(x)\Big|_{x=\cos\theta} e^{jm\phi}\hat{\phi}\end{bmatrix}, \tag{I-23}$$

or

$$\bar{\mathbf{X}}_{n,m}(\bar{\mathbf{r}}) = \frac{j}{\sqrt{n(n+1)}}\left[\frac{jm}{\sin\theta}Y_{n,m}(\theta,\phi)\hat{\theta} + \sin\theta\frac{\partial}{\partial\cos\theta}Y_{n,m}(\theta,\phi)\hat{\phi}\right]. \tag{I-24}$$

From (I-21), we see that

$$\bar{\mathbf{X}}_{n,-m}(\bar{\mathbf{r}}) = (-1)^m\frac{j}{\sqrt{n(n+1)}}\left[\frac{-jm}{\sin\theta}Y_{n,m}^*(\theta,\phi)\hat{\theta} + \sin\theta\frac{\partial}{\partial\cos\theta}Y_{n,m}^*(\theta,\phi)\hat{\phi}\right]. \tag{I-25}$$

The usefulness of the vector spherical harmonic derives from the fact it forms a solution of the vector wave equation as

$$\nabla^2 z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi) + k^2 z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi) \equiv 0, \tag{I-26}$$

where $z_n^{(i)}(kr)$ represents any of the spherical Bessel's functions

$$z_n^{(i)}(kr) = \begin{cases} h_n^{(1)}(kr); i=1 \\ h_n^{(2)}(kr); i=2 \\ j_n(kr); i=3 \\ y_n(kr); i=4 \end{cases}. \tag{I-27}$$

$\nabla\times z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi)$ is also a solution of the vector wave equation, since the operators ∇^2 and $\nabla\times$ commute (see Appendix II),

$$\nabla^2\left(\nabla\times z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi)\right) + k^2\left(\nabla\times z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi)\right) \equiv 0. \tag{I-28}$$

Thus, $z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi)$ and $\nabla\times z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi)$ are both harmonic solutions of the wave equation. The two types of vector spherical harmonics are defined in terms of $\bar{\mathbf{X}}_{n,m}$ as

$$\bar{\mathbf{M}}_{n,m}^{(i)} \equiv z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi), \tag{I-29}$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)} \equiv \frac{1}{k}\nabla\times z_n^{(i)}(kr)\bar{\mathbf{X}}_{n,m}(\theta,\phi) = \frac{1}{k}\nabla\times\bar{\mathbf{M}}_{n,m}^{(i)}. \tag{I-30}$$

It is obvious that $\bar{\mathbf{M}}_{n,m}^{(i)}$ and $\bar{\mathbf{N}}_{n,m}^{(i)}$ correspond to the general $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ harmonics defined in (I-3) and (I-4) for the special case

$$\psi = -\frac{z_n^{(i)}(kr)}{j\sqrt{n(n+1)}}Y_{n,m}(\theta,\phi), \tag{I-31}$$

since

$$\hat{\mathbf{r}}\times\nabla z_n^{(i)}(kr) = 0. \tag{I-32}$$

It will be useful to note that

$$\begin{aligned}\nabla\times\bar{\mathbf{N}}_{l,m}^{(i)} &= \nabla\times\frac{1}{k}\nabla\times z_l^{(i)}(kr)\bar{\mathbf{X}}_{l,m}(\theta,\phi) = \nabla\times\frac{1}{k}\nabla\times\bar{\mathbf{M}}_{l,m}^{(i)} \\ &= \frac{1}{k}\nabla\nabla\cdot\bar{\mathbf{M}}_{l,m}^{(i)} - \frac{1}{k}\nabla^2\bar{\mathbf{M}}_{l,m}^{(i)}\end{aligned}, \tag{I-33}$$

and since $\nabla\cdot\bar{\mathbf{M}}_{l,m}^{(i)}=0$, and $\bar{\mathbf{M}}_{l,m}^{(i)}$ is a solution of the wave equation, we see that

$$\bar{\mathbf{M}}_{l,m}^{(i)} = \frac{1}{k}\nabla\times\bar{\mathbf{N}}_{l,m}^{(i)}. \tag{I-34}$$

This is the same result as (I-15) for the general vector-harmonic case.

The explicit forms for the normalized vector spherical harmonics are

$$\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j\sqrt{\frac{(2n+1)}{4\pi n(n+1)}\frac{(n-m)!}{(n+m)!}}\left[\begin{aligned}&\frac{jm}{\sin\theta}z_n^{(i)}(kr)P_n^m(\cos\theta)e^{jm\phi}\hat{\theta} \\ &+\sin\theta z_n^{(i)}(kr)\frac{d}{dx}P_n^m(x)\Big|_{x=\cos\theta}e^{jm\phi}\hat{\phi}\end{aligned}\right], \tag{I-35}$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j\sqrt{\frac{(2n+1)}{4\pi n(n+1)}\frac{(n-m)!}{(n+m)!}}\left\{\begin{aligned}&\frac{z_n^{(i)}(kr)}{kr}n(n+1)P_n^m(\cos\theta)e^{jm\phi}\hat{\mathbf{r}} \\ &-\frac{1}{kr}\frac{\partial}{\partial r}\left[rz_n^{(i)}(kr)\right]\sin\theta\frac{d}{dx}P_n^m(x)\Big|_{x=\cos\theta}e^{jm\phi}\hat{\theta} \\ &+\frac{1}{kr}\frac{\partial}{\partial r}\left[rz_n^{(i)}(kr)\right]\frac{jm}{\sin\theta}P_n^m(\cos\theta)e^{jm\phi}\hat{\phi}\end{aligned}\right\}. \tag{I-36}$$

Since

$$\hat{\mathbf{r}}\times\bar{\mathbf{X}}_{n,m}(\theta,\phi) = j\sqrt{\frac{2n+1}{4\pi n(n+1)}\frac{(n-m)!}{(n+m)!}}\left[-\sin\theta\frac{\partial P_n^m(x)}{\partial x}\Big|_{x=\cos\theta}\hat{\theta} + \frac{jm}{\sin\theta}P_n^m(\cos\theta)\hat{\phi}\right]e^{jm\phi} \tag{I-37}$$

we see that

$$\begin{aligned}\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) &= j\sqrt{\frac{(2n+1)}{4\pi n(n+1)}\frac{(n-m)!}{(n+m)!}}\frac{z_n^{(i)}(kr)}{kr}n(n+1)P_n^m(\cos\theta)e^{jm\phi}\hat{\mathbf{r}} \\ &+\frac{1}{kr}\frac{\partial}{\partial r}\left[rz_n^{(i)}(kr)\right]\hat{\mathbf{r}}\times\bar{\mathbf{X}}_{n,m}(\theta,\phi)\end{aligned}, \tag{I-38}$$

or

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j\sqrt{n(n+1)}\frac{z_n^{(i)}(kr)}{kr}Y_n^m(\theta,\phi)\hat{\mathbf{r}} + \frac{1}{kr}\frac{\partial}{\partial r}\left[rz_n^{(i)}(kr)\right]\hat{\mathbf{r}}\times\bar{\mathbf{X}}_{n,m}(\theta,\phi), \tag{I-39}$$

and the transverse part of $\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}})$ is simply

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) - \bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}})\cdot\hat{\mathbf{r}}\hat{\mathbf{r}} = \frac{1}{kr}\frac{\partial}{\partial r}\left[rz_n^{(i)}(kr)\right]\hat{\mathbf{r}}\times\bar{\mathbf{X}}_{n,m}(\theta,\phi). \tag{I-40}$$

These normalized vector spherical harmonics differ from those defined by Chew [9] and Stratton [13]. The harmonics used by Chew must be multiplied by a factor of

$j/\sqrt{n(n+1)}$ to produce the normalized harmonics, and from those defined by Stratton must be multiplied by a factor of $j(-1)^m \sqrt{\dfrac{(2n+1)(n-m)!}{4\pi n(n+1)(n+m)!}}$ to yield the normalized harmonics. Hanson [12] uses normalized harmonics almost identical to those used here. However, Hanson's harmonics, which are written with a different notation, must be multiplied by a factor of $j\left(m/|m|\right)^m$ to produce the ones defined here. The relation between Hanson's notation and that used here is

$$\bar{\mathbf{M}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j\left(\frac{m}{|m|}\right)^m \bar{\mathbf{F}}_{1,m,n}^{(i)}(\bar{\mathbf{r}}), \tag{I-41}$$

and

$$\bar{\mathbf{N}}_{n,m}^{(i)}(\bar{\mathbf{r}}) = j\left(\frac{m}{|m|}\right)^m \bar{\mathbf{F}}_{2,m,n}^{(i)}(\bar{\mathbf{r}}). \tag{I-42}$$

A far-field representation of the vector spherical harmonics is obtained for out-going waves, where the $z_n^i(kr)$ become $h_n^{(2)}(kr)$. Since

$$h_n^{(2)}(kr) = j^{n+1}\frac{e^{-jkr}}{kr}\sum_{p=0}^{n}\frac{\Gamma(n+p+1)}{p!\,\Gamma(n-p+1)}(2jkr)^{-p}, \tag{I-43}$$

the approximation for the far field region is,

$$h_n^{(2)}(kr) \cong j^{n+1}\frac{e^{-jkr}}{kr} \text{ as } kr \to \infty, \tag{I-44}$$

and

$$\left.\frac{\partial}{\partial z}h_n^{(2)}(z)\right|_{z=kr} \cong j^{n}\frac{e^{-jkr}}{kr} \text{ as } kr \to \infty. \tag{I-45}$$

Thus, in the far field region,

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = -\frac{j^n e^{-jkr}}{kr}\sqrt{\frac{(2n+1)}{4\pi n(n+1)}\frac{(n-m)!}{(n+m)!}}\left[\frac{jm}{\sin\theta}P_n^m(\cos\theta)\hat{\theta} + \sin\theta\frac{d}{dx}P_n^m(x)\bigg|_{x=\cos\theta}\hat{\phi}\right]e^{jm\phi}, \tag{I-46}$$

and

$$\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \frac{j^{n-1} e^{-jkr}}{kr}\sqrt{\frac{2n+1}{4\pi n(n+1)}\frac{(n-m)!}{(n+m)!}}\left[\sin\theta\frac{d}{dx}P_n^m(x)\bigg|_{x=\cos\theta}\hat{\theta} - \frac{jm}{\sin\theta}P_n^m(\cos\theta)\hat{\phi}\right]e^{jm\phi}, \tag{I-47}$$

or simply

$$\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \frac{j^{n+1}e^{-jkr}}{kr}\bar{\mathbf{X}}_{n,m}(\theta,\phi), \tag{I-48}$$

and

$$\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) = \frac{j^{n}e^{-jkr}}{kr}\hat{\mathbf{r}}\times\bar{\mathbf{X}}_{n,m}(\theta,\phi), \tag{I-49}$$

for the normalized vector spherical harmonics.

The orthogonality integrals for the normalized vector harmonics are [7]

$$\begin{aligned}\mathscr{I}_{\bar{\mathbf{M}}\cdot\bar{\mathbf{M}}}(m,n;m',n';r) &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \bar{\mathbf{M}}_{n,m}^{(i)} \cdot \left(\bar{\mathbf{M}}_{n',m'}^{(i)}\right)^* \\ &= \left|z_n^{(i)}(kr)\right|^2 \delta_{nn'}\delta_{mm'}\end{aligned} \text{, [BCB12]} \tag{I-50}$$

and

$$\begin{aligned}\mathscr{I}_{\bar{\mathbf{N}}\cdot\bar{\mathbf{N}}}(m,n;m',n';r) &\equiv \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \bar{\mathbf{N}}_{n,m}^{(i)} \cdot \left(\bar{\mathbf{N}}_{n',m'}^{(i)}\right)^* \\ &= \left[\frac{\left|z_n^{(i)}(kr)\right|^2}{k^2r^2} n(n+1) + \frac{1}{k^2r^2}\left|\frac{\partial}{\partial r}\left[rz_n^{(i)}(kr)\right]\right|^2\right]\delta_{nn'}\delta_{mm'} \\ &= \left[\left|z_n^{(i)}(kr)\right|^2 + \frac{1}{k^2r^2}\frac{\partial}{\partial r}\left[rz_n^{(i)}(kr)\frac{\partial}{\partial r}\left[rz_n^{(i)*}(kr)\right]\right]\right]\delta_{nn'}\delta_{mm'}\end{aligned} \tag{I-51}$$

or, expanding (I-51),

$$\mathscr{I}_{\bar{\mathbf{N}}\cdot\bar{\mathbf{N}}}(m,n;m',n';r) = \begin{bmatrix} \left|z_n^{(i)}(kr)\right|^2\left(1+\frac{1}{k^2r^2}\right) + \left|\frac{d}{dx}z_n^{(i)}(x)\Big|_{x=kr}\right|^2 \\ +\frac{z_n^{(i)*}(kr)}{kr}\frac{d}{dx}z_n^{(i)}(x)\Big|_{x=kr} + 3\frac{z_n^{(i)}(kr)}{kr}\frac{d}{dx}z_n^{(i)*}(x)\Big|_{x=kr} \\ +z_n^{(i)}(kr)\frac{d^2}{dx^2}z_n^{(i)*}(x)\Big|_{x=kr}\end{bmatrix}\delta_{nn'}\delta_{mm'} \tag{I-52}$$

and

$$\begin{aligned}\mathscr{I}_{\bar{\mathbf{M}}\cdot\bar{\mathbf{N}}}(m,n,m',n';r) &\equiv \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \bar{\mathbf{M}}_{n,m}^{(i)} \cdot \left(\bar{\mathbf{N}}_{n',m'}^{(i)}\right)^* \\ &= 0\end{aligned} \tag{I-53}$$

When r is small, the $\hat{\mathbf{r}}$ component of $\bar{\mathbf{N}}_{n,m}^{(i)}$ can be significant. However, it will be necessary to expand a measured field in terms of the vector spherical harmonics, and the $\hat{\mathbf{r}}$ component of the field is typically not measured. Thus, for small r, the use of (I-52) for the normalization can produce an error. Instead, we need the orthogonality integral for the transverse part of $\bar{\mathbf{N}}_{n,m}^{(i)}$

$$\begin{aligned}\mathscr{I}_{(\bar{\mathbf{N}}-\bar{\mathbf{N}}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\cdot\bar{\mathbf{N}}}(m,n;m',n';r) &\equiv \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \left(\bar{\mathbf{N}}_{n,m}^{(i)} - \bar{\mathbf{N}}_{n,m}^{(i)}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\right)\cdot\left(\bar{\mathbf{N}}_{n',m'}^{(i)}\right)^* \\ &= \frac{1}{k^2r^2}\left|\frac{\partial}{\partial r}\left[rz_n^{(i)}(kr)\right]\right|^2 \delta_{nn'}\delta_{mm'}\end{aligned} \tag{I-54}$$

Special combinations of the vector spherical harmonics

Consider the far-field representation of $\bar{\mathbf{M}}_{1,\pm1}^{(2)}(\bar{\mathbf{r}})$ obtained from (I-48)

$$\bar{\mathbf{M}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = j\frac{e^{-jkr}}{kr}\frac{1}{2}\sqrt{\frac{3}{4\pi}}\left[\hat{\theta} - j\cos\theta\hat{\phi}\right]e^{-j\phi}, \tag{I-55}$$

and

$$\bar{\mathbf{M}}_{1,1}^{(2)}(\bar{\mathbf{r}}) = j\frac{e^{-jkr}}{kr}\frac{1}{2}\sqrt{\frac{3}{4\pi}}\left[\hat{\theta} + j\cos\theta\hat{\phi}\right]e^{j\phi}. \tag{I-56}$$

For a TE field, $\bar{\mathbf{M}}_{1,-1}^{(2)}(\bar{\mathbf{r}})$ represents a right-hand circularly polarized electric field at $\theta = 0°$, while $\bar{\mathbf{M}}_{1,1}^{(2)}(\bar{\mathbf{r}})$ is the left-hand circularly polarized electric field. Now consider the far-field representation of $\bar{\mathbf{N}}_{1,\pm1}^{(2)}(\bar{\mathbf{r}})$, from (I-49)

$$\bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr}\frac{1}{2}\sqrt{\frac{3}{4\pi}}\left[-\cos\theta\hat{\theta} + j\hat{\phi}\right]e^{-j\phi}, \tag{I-57}$$

and,

$$\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr}\frac{1}{2}\sqrt{\frac{3}{4\pi}}\left[\cos\theta\hat{\theta} + j\hat{\phi}\right]e^{j\phi}. \tag{I-58}$$

Similarly, for the TM field, $\bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}})$ is the right-hand circularly polarized electric field when $\theta = 0°$, and $\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}})$ is left-hand circularly polarized. At $\theta \neq 0°$, these fields are elliptical.

For a TE field,

$$\bar{\mathbf{M}}_{1,1}^{(2)}(\bar{\mathbf{r}}) + \bar{\mathbf{M}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = j\frac{e^{-jkr}}{kr}\sqrt{\frac{3}{4\pi}}\left[\cos\phi\hat{\theta} - \cos\theta\sin\phi\hat{\phi}\right], \tag{I-59}$$

$$\bar{\mathbf{M}}_{1,1}^{(2)}(\bar{\mathbf{r}}) - \bar{\mathbf{M}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = -\frac{e^{-jkr}}{kr}\sqrt{\frac{3}{4\pi}}\left[\sin\phi\hat{\theta} + \cos\phi\cos\theta\hat{\phi}\right], \tag{I-60}$$

which represent magnetic dipoles.

For the TM field

$$\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) + \bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = j\frac{e^{-jkr}}{kr}\sqrt{\frac{3}{4\pi}}\left[\cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}\right], \tag{I-61}$$

$$\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) - \bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr}\sqrt{\frac{3}{4\pi}}\left[\cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}\right], \tag{I-62}$$

$$(1-j)\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) - (1+j)\bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr}\sqrt{\frac{3}{4\pi}}\left[\cos\theta(\sin\phi + \cos\phi)\hat{\theta} - (\sin\phi - \cos\phi)\hat{\phi}\right], \tag{I-63}$$

and

$$(1-j)\bar{\mathbf{N}}_{1,-1}^{(2)}(\bar{\mathbf{r}}) - (1+j)\bar{\mathbf{N}}_{1,1}^{(2)}(\bar{\mathbf{r}}) = \frac{e^{-jkr}}{kr}\sqrt{\frac{3}{4\pi}}\left[\cos\theta(\sin\phi - \cos\phi)\hat{\theta} + (\sin\phi + \cos\phi)\hat{\phi}\right]. \tag{I-64}$$

Equations (I-61), (I-62), (I-63), and (I-64) describe small linear dipoles oriented as E_ϕ, E_θ, $E_{45°}$, and $E_{135°}$ dipoles, respectively.

This Page Intentionally Contains No Useful Information

Explicit forms for the associated Legendre function and its derivative

Explicit Forms for $P_n^m(x)$

n \ m	0	1	2	3	4	5
0	1	0	0	0	0	0
1	x	$-\sqrt{1-x^2}$	0	0	0	0
2	$\frac{3x^2-1}{2}$	$-3x\sqrt{1-x^2}$	$3(1-x^2)$	0	0	0
3	$\frac{5x^3-3x}{2}$	$-3\frac{5x^2-1}{2}\sqrt{1-x^2}$	$15x(1-x^2)$	$-15(1-x^2)^{3/2}$	0	0
4	$\frac{35x^4-30x^2+3}{8}$	$-5\frac{7x^3-3x}{2}\sqrt{1-x^2}$	$15\frac{7x^2-1}{2}(1-x^2)$	$-105x(1-x^2)^{3/2}$	$105(1-x^2)^2$	0
5	$\frac{63x^5-70x^3+15x}{8}$	$-15\frac{21x^4-14x^2+1}{8}\sqrt{1-x^2}$	$105\frac{3x^3-x}{2}(1-x^2)$	$-105\frac{9x^2-1}{2}(1-x^2)^{3/2}$	$945x(1-x^2)^2$	$-945(1-x^2)^{5/2}$

Explicit Forms for $P_n^m(\cos\theta)$

n \ m	0	1	2	3	4	5
0	1	0	0	0	0	0
1	$\cos\theta$	$-\sin\theta$	0	0	0	0
2	$\frac{3\cos^2\theta-1}{2}$	$-3\sin\theta\cos\theta$	$3\sin^2\theta$	0	0	0
3	$\frac{5\cos^3\theta-3\cos\theta}{2}$	$-3\frac{5\cos^2\theta-1}{2}\sin\theta$	$15\cos\theta\sin^2\theta$	$-15\sin^3\theta$	0	0
4	$\frac{35\cos^4\theta-30\cos^2\theta+3}{8}$	$-5\frac{7\cos^3\theta-3\cos\theta}{2}\sin\theta$	$15\frac{7\cos^2\theta-1}{2}\sin^2\theta$	$-105\cos\theta\sin^3\theta$	$105\sin^4\theta$	0
5	$\frac{63\cos^5\theta-70\cos^3\theta+15\cos\theta}{8}$	$-15\frac{21\cos^4\theta-14\cos^2\theta+1}{8}\sin\theta$	$105\frac{3\cos^3\theta-\cos\theta}{2}\sin^2\theta$	$-105\frac{9\cos^2\theta-1}{2}\sin^3\theta$	$945\cos\theta\sin^4\theta$	$-945\sin^5\theta$

Explicit Forms for $\frac{d}{dx}P_n^m(x)$

	m					
n	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	$\frac{x}{\sqrt{1-x^2}}$	0	0	0	0
2	$3x$	$3\frac{2x^2-1}{\sqrt{1-x^2}}$	$-6x$	0	0	0
3	$3\frac{5x^2-1}{2}$	$3\frac{15x^3-11x}{2\sqrt{1-x^2}}$	$15(1-3x^2)$	$45x\sqrt{(1-x^2)}$	0	0
4	$5\frac{7x^3-3x}{2}$	$5\frac{28x^4-27x^2+3}{2\sqrt{1-x^2}}$	$30(4x-7x^3)$	$105(4x^2-1)\sqrt{(1-x^2)}$	$-420x(1-x^2)$	0
5	$15\frac{21x^4-14x^2+1}{8}$	$15\frac{105x^5-126x^3+29x}{8\sqrt{1-x^2}}$	$-105\frac{15x^4-12x^2+1}{2}$	$315\frac{15x^3-7x}{2}\sqrt{1-x^2}$	$945(1-x^2)(1-5x^2)$	$4725x(1-x^2)^{3/2}$

Explicit Forms for $\frac{d}{dx}P_n^m(\cos\theta)$

	m					
n	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	$\frac{\cos\theta}{\sin\theta}$	0	0	0	0
2	$3\cos\theta$	$3\frac{2\cos^2\theta-1}{\sin\theta}$	$-6\cos\theta$	0	0	0
3	$3\frac{5\cos^2\theta-1}{2}$	$3\frac{15\cos^3\theta-11\cos\theta}{2\sin\theta}$	$15(1-3\cos^2\theta)$	$45\cos\theta\sin\theta$	0	0
4	$5\frac{7\cos^3\theta-3\cos\theta}{2}$	$5\frac{28\cos^4\theta-27\cos^2\theta+3}{2\sin\theta}$	$30(4\cos\theta-7\cos^3\theta)$	$105(4\cos^2\theta-1)\sin\theta$	$-420\cos\theta\sin^2\theta$	0
5	$15\frac{21\cos^4\theta-14\cos^2\theta+1}{8}$	$15\frac{105\cos^5\theta-126\cos^3\theta+29\cos\theta}{8\sin\theta}$	$-105\frac{15\cos^4\theta-12\cos^2\theta+1}{2}$	$315\frac{15\cos^3\theta-7\cos\theta}{2}\sin\theta$	$945\sin^2\theta(1-5\cos^2\theta)$	$4725\cos\theta\sin^3\theta$

Explicit forms for the scalar spherical harmonic and its derivative

Explicit Forms for $Y_{n,m}(\cos\theta)$

n	$m = 0$	± 1	± 2	± 3	± 4
0	$\sqrt{\frac{1}{4\pi}}$	0	0	0	0
1	$\sqrt{\frac{3}{4\pi}}\cos\theta$	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{\pm j\phi}\sin\theta$	0	0	0
2	$\sqrt{\frac{5}{4\pi}}\frac{3\cos^2\theta-1}{2}$	$\mp\frac{3}{2}\sqrt{\frac{5}{6\pi}}e^{\pm j\phi}\sin\theta\cos\theta$	$\frac{3}{4}\sqrt{\frac{5}{6\pi}}e^{\pm j2\phi}\sin^2\theta$	0	0
3	$\sqrt{\frac{7}{4\pi}}\frac{5\cos^3\theta-3\cos\theta}{2}$	$\mp\frac{3}{4}\sqrt{\frac{7}{3\pi}}e^{\pm j\phi}\frac{5\cos^2\theta-1}{2}\sin\theta$	$\frac{15}{4}\sqrt{\frac{7}{30\pi}}e^{\pm j2\phi}\cos\theta\sin^2\theta$	$\mp\frac{5}{8}\sqrt{\frac{7}{5\pi}}e^{\pm j3\phi}\sin^3\theta$	0
4	$3\sqrt{\frac{1}{4\pi}}\frac{35\cos^4\theta-30\cos^2\theta+3}{8}$	$\mp\frac{15}{4}\sqrt{\frac{1}{5\pi}}e^{\pm j\phi}\frac{7\cos^3\theta-3\cos\theta}{2}\sin\theta$	$\frac{15}{4}\sqrt{\frac{1}{10\pi}}e^{\pm j2\phi}\frac{7\cos^2\theta-1}{2}\sin^2\theta$	$\mp\frac{105}{8}\sqrt{\frac{1}{35\pi}}e^{\pm j3\phi}\cos\theta\sin^3\theta$	$\frac{105}{16}\sqrt{\frac{1}{70\pi}}e^{\pm j4\phi}\sin^4\theta$

Explicit Forms for $\frac{dY_{n,m}(\cos\theta)}{d\cos\theta}$

n	$m = 0$	± 1	± 2	± 3	± 4
0	0	0	0	0	0
1	$\sqrt{\frac{3}{4\pi}}$	$\pm\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{\pm j\phi}\frac{\cos\theta}{\sin\theta}$	0	0	0
2	$3\sqrt{\frac{3}{4\pi}}\cos\theta$	$\pm\frac{3}{2}\sqrt{\frac{5}{6\pi}}e^{\pm j\phi}\frac{2\cos^2\theta-1}{\sin\theta}$	$-\frac{3}{2}\sqrt{\frac{5}{6\pi}}e^{\pm j2\phi}\cos\theta$	0	0
3	$3\sqrt{\frac{7}{4\pi}}\frac{5\cos^2\theta-1}{2}$	$\pm\frac{3}{4}\sqrt{\frac{7}{3\pi}}e^{\pm j\phi}\frac{15\cos^3\theta-11\cos\theta}{2\sin\theta}$	$\frac{15}{4}\sqrt{\frac{7}{30\pi}}e^{\pm j2\phi}\left(1-3\cos^2\theta\right)$	$\pm\frac{15}{8}\sqrt{\frac{7}{5\pi}}e^{\pm j3\phi}\cos\theta\sin\theta$	0
4	$3\sqrt{\frac{1}{4\pi}}\frac{35\cos^3\theta-15\cos\theta}{2}$	$\pm\frac{15}{4}\sqrt{\frac{1}{5\pi}}e^{\pm j\phi}\frac{28\cos^4\theta-27\cos^2\theta+3}{2\sin\theta}$	$\frac{15}{2}\sqrt{\frac{1}{10\pi}}e^{\pm j2\phi}\left(4\cos\theta-7\cos^3\theta\right)$	$\pm\frac{105}{8}\sqrt{\frac{1}{35\pi}}e^{\pm j3\phi}\left(4\cos^2\theta-1\right)\sin\theta$	$-\frac{105}{4}\sqrt{\frac{1}{70\pi}}e^{\pm j4\phi}\cos\theta\sin^2\theta$

Explicit expressions for the $\vec{\mathbf{X}}_{n,m}(\theta,\phi)$ vector spherical harmonic:

$\vec{\mathbf{X}}_{1,0}(\theta,\phi) = j\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\hat{\phi}$
$\vec{\mathbf{X}}_{1,-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{-j\phi}\left[\hat{\theta} - j\cos\theta\hat{\phi}\right]$
$\vec{\mathbf{X}}_{1,1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{j\phi}\left[\hat{\theta} + j\cos\theta\hat{\phi}\right]$
$\vec{\mathbf{X}}_{2,0}(\theta,\phi) = j\sqrt{\frac{5}{6\pi}}\frac{3\cos^2\theta - 1}{4}\sin\theta\hat{\phi}$
$\vec{\mathbf{X}}_{2,-1}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}e^{-j\phi}\left[\cos\theta\hat{\theta} - j\left(2\cos^2\theta - 1\right)\hat{\phi}\right]$
$\vec{\mathbf{X}}_{2,1}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}e^{j\phi}\left[\cos\theta\hat{\theta} + j\left(2\cos^2\theta - 1\right)\hat{\phi}\right]$
$\vec{\mathbf{X}}_{2,-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}e^{-j2\phi}\left[\sin\theta\hat{\theta} - j\sin\theta\cos\theta\hat{\phi}\right]$
$\vec{\mathbf{X}}_{2,2}(\theta,\phi) = -\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{j2\phi}\left[\sin\theta\hat{\theta} + j\sin\theta\cos\theta\hat{\phi}\right]$

Explicit expressions for the $\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{n,m}(\theta,\phi)$ ***vector spherical harmonic:***

$\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{1,0}(\theta,\phi)=-j\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\hat{\theta}$
$\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{1,-1}(\theta,\phi)=\frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{-j\phi}\left[j\cos\theta\hat{\theta}+\hat{\phi}\right]$
$\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{1,1}(\theta,\phi)=\frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{j\phi}\left[-j\cos\theta\hat{\theta}+\hat{\phi}\right]$
$\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{2,0}(\theta,\phi)=-j\sqrt{\frac{5}{6\pi}}\frac{3\cos^2\theta-1}{4}\sin\theta\hat{\theta}$
$\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{2,-1}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{-j\phi}\left[j(2\cos^2\theta-1)\hat{\theta}+\cos\theta\hat{\phi}\right]$
$\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{2,1}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{j\phi}\left[-j(2\cos^2\theta-1)\hat{\theta}+\cos\theta\hat{\phi}\right]$
$\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{2,-2}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{-j2\phi}\left[j\sin\theta\cos\theta\hat{\theta}+\sin\theta\hat{\phi}\right]$
$\hat{\mathbf{r}}\times\vec{\mathbf{X}}_{2,2}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{j2\phi}\left[j\sin\theta\cos\theta\hat{\theta}-\sin\theta\hat{\phi}\right]$

Explicit expressions for the $\vec{\mathbf{M}}_{n,m}^{(i)}(\vec{\mathbf{r}})$ ***vector spherical harmonics:***

$\vec{\mathbf{M}}_{1,0}^{(i)}(\vec{\mathbf{r}}) = jz_1^{(i)}(kr)\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\hat{\phi}$
$\vec{\mathbf{M}}_{1,-1}^{(i)}(\vec{\mathbf{r}}) = z_1^{(i)}(kr)\frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{-j\phi}\left[\hat{\theta} - j\cos\theta\hat{\phi}\right]$
$\vec{\mathbf{M}}_{1,1}^{(i)}(\vec{\mathbf{r}}) = z_1^{(i)}(kr)\frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{j\phi}\left[\hat{\theta} + j\cos\theta\hat{\phi}\right]$
$\vec{\mathbf{M}}_{2,0}^{(i)}(\vec{\mathbf{r}}) = jz_2^{(i)}(kr)\sqrt{\frac{5}{6\pi}}\frac{3\cos^2\theta - 1}{4}\sin\theta\hat{\phi}$
$\vec{\mathbf{M}}_{2,-1}^{(i)}(\vec{\mathbf{r}}) = z_2^{(i)}(kr)\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{-j\phi}\left[\cos\theta\hat{\theta} - j\left(2\cos^2\theta - 1\right)\hat{\phi}\right]$
$\vec{\mathbf{M}}_{2,1}^{(i)}(\vec{\mathbf{r}}) = z_2^{(i)}(kr)\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{j\phi}\left[\cos\theta\hat{\theta} + j\left(2\cos^2\theta - 1\right)\hat{\phi}\right]$
$\vec{\mathbf{M}}_{2,-2}^{(i)}(\vec{\mathbf{r}}) = z_2^{(i)}(kr)\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{-j2\phi}\left[\sin\theta\hat{\theta} - j\sin\theta\cos\theta\hat{\phi}\right]$
$\vec{\mathbf{M}}_{2,2}^{(i)}(\vec{\mathbf{r}}) = -z_2^{(i)}(kr)\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{j2\phi}\left[\sin\theta\hat{\theta} + j\sin\theta\cos\theta\hat{\phi}\right]$

Explicit expressions for the $\vec{\mathbf{N}}^{(i)}_{n,m}(\vec{\mathbf{r}})$ vector spherical harmonics:

$\vec{\mathbf{N}}^{(i)}_{1,0}(\vec{\mathbf{r}}) = j\frac{z_1^{(i)}(kr)}{kr}\sqrt{\frac{3}{2\pi}}\cos\theta\hat{\mathbf{r}} - j\frac{1}{kr}\frac{\partial}{\partial r}\left[rz_1^{(i)}(kr)\right]\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\hat{\theta}$
$\vec{\mathbf{N}}^{(i)}_{1,-1}(\vec{\mathbf{r}}) = j\frac{z_1^{(i)}(kr)}{kr}\sqrt{\frac{3}{4\pi}}e^{-j\phi}\sin\theta\hat{\mathbf{r}} + \frac{1}{kr}\frac{\partial}{\partial r}\left[rz_1^{(i)}(kr)\right]\frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{-j\phi}\left[j\cos\theta\hat{\theta} + \hat{\phi}\right]$
$\vec{\mathbf{N}}^{(i)}_{1,1}(\vec{\mathbf{r}}) = -j\frac{z_1^{(i)}(kr)}{kr}\sqrt{\frac{3}{4\pi}}e^{j\phi}\sin\theta\hat{\mathbf{r}} + \frac{1}{kr}\frac{\partial}{\partial r}\left[rz_1^{(i)}(kr)\right]\frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{j\phi}\left[-j\cos\theta\hat{\theta} + \hat{\phi}\right]$
$\vec{\mathbf{N}}^{(i)}_{2,0}(\vec{\mathbf{r}}) = j\frac{z_2^{(i)}(kr)}{kr}\frac{1}{2}\sqrt{\frac{15}{2\pi}}\left(3\cos^2\theta - 1\right)\hat{\mathbf{r}} - j\frac{1}{kr}\frac{\partial}{\partial r}\left[rz_2^{(i)}(kr)\right]\sqrt{\frac{5}{6\pi}}\frac{3\cos^2\theta - 1}{4}\sin\theta\hat{\theta}$
$\vec{\mathbf{N}}^{(i)}_{2,-1}(\vec{\mathbf{r}}) = j\frac{z_2^{(i)}(kr)}{kr}\frac{3}{2}\sqrt{\frac{5}{\pi}}e^{-j\phi}\sin\theta\cos\theta\hat{\mathbf{r}} + \frac{1}{kr}\frac{\partial}{\partial r}\left[rz_2^{(i)}(kr)\right]\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{-j\phi}\left[j\left(2\cos^2\theta - 1\right)\hat{\theta} + \cos\theta\hat{\phi}\right]$
$\vec{\mathbf{N}}^{(i)}_{2,1}(\vec{\mathbf{r}}) = -j\frac{z_2^{(i)}(kr)}{kr}\frac{3}{2}\sqrt{\frac{5}{\pi}}e^{j\phi}\sin\theta\cos\theta\hat{\mathbf{r}} + \frac{1}{kr}\frac{\partial}{\partial r}\left[rz_2^{(i)}(kr)\right]\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{j\phi}\left[-j\left(2\cos^2\theta - 1\right)\hat{\theta} + \cos\theta\hat{\phi}\right]$
$\vec{\mathbf{N}}^{(i)}_{2,-2}(\vec{\mathbf{r}}) = j\frac{z_2^{(i)}(kr)}{kr}\frac{3}{4}\sqrt{\frac{5}{\pi}}e^{-j2\phi}\sin^2\theta\hat{\mathbf{r}} + \frac{1}{kr}\frac{\partial}{\partial r}\left[rz_2^{(i)}(kr)\right]\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{-j2\phi}\left[j\sin\theta\cos\theta\hat{\theta} + \sin\theta\hat{\phi}\right]$
$\vec{\mathbf{N}}^{(i)}_{2,2}(\vec{\mathbf{r}}) = j\frac{z_2^{(i)}(kr)}{kr}\frac{3}{4}\sqrt{\frac{5}{\pi}}e^{j2\phi}\sin^2\theta\hat{\mathbf{r}} + \frac{1}{kr}\frac{\partial}{\partial r}\left[rz_2^{(i)}(kr)\right]\frac{1}{4}\sqrt{\frac{5}{\pi}}e^{j2\phi}\left[j\sin\theta\cos\theta\hat{\theta} - \sin\theta\hat{\phi}\right]$

This Page Intentionally Contains No Useful Information

Appendix II — Commutation Relations

Consider the operators ∇^2 and $\nabla\times$. The commutator is $\nabla^2(\nabla\times)-(\nabla\times)\nabla^2$. Using the representation of the operators in Cartesian coordinates, we see that

$$\begin{aligned}
\nabla^2(\nabla\times) &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\times \\
&= \left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial x}\hat{\mathbf{x}}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial y}\hat{\mathbf{y}}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial z}\hat{\mathbf{z}}\right]\times \\
&= \left[\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\hat{\mathbf{x}}+\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\hat{\mathbf{y}}+\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\hat{\mathbf{z}}\right]\times \\
&= \nabla\nabla^2\times \\
&= \left(\left[\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\right]\times\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) \\
&= (\nabla\times)\nabla^2
\end{aligned} \tag{II-1}$$

so that

$$\nabla^2(\nabla\times)-(\nabla\times)\nabla^2=0. \tag{II-2}$$

The commutator is zero, so the operators ∇^2 and $\nabla\times$ commute.

Let $\bar{\mathbf{a}}$ be a constant vector. Consider $\nabla\times\bar{\mathbf{a}}\psi$,

$$\begin{aligned}
\nabla\times &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\times\bar{\mathbf{a}}\psi \\
&= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\psi\times\bar{\mathbf{a}} \\
&= -\bar{\mathbf{a}}\times\left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\psi \\
&= -\bar{\mathbf{a}}\times\nabla\psi
\end{aligned} \tag{II-3}$$

Thus, we have the operator anti-commutative relation,

$$\nabla\times\bar{\mathbf{a}}+\bar{\mathbf{a}}\times\nabla=0. \tag{II-4}$$

Consider $\nabla\times\bar{\mathbf{r}}$ when there is no additional function to the right. We have

$$\begin{aligned}
\nabla\times\bar{\mathbf{r}} &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\times(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}) \\
&= \hat{\mathbf{x}}\times\hat{\mathbf{x}}+\hat{\mathbf{y}}\times\hat{\mathbf{y}}+\hat{\mathbf{z}}\times\hat{\mathbf{z}} \\
&= 0
\end{aligned} \tag{II-5}$$

Next consider the *operators* $\nabla\times\bar{\mathbf{r}}$ and $\bar{\mathbf{r}}\times\nabla$. Note that ∇ in $\nabla\times\bar{\mathbf{r}}$ is expected to operate on whatever function is immediately to the right of $\nabla\times\bar{\mathbf{r}}$, in addition to $\bar{\mathbf{r}}$, unlike in (II-5). Representing the operators in Cartesian coordinates, and explicitly including the right-hand function, ψ, on which the operators operate,

$$\begin{aligned}\nabla \times \bar{\mathbf{r}}\psi &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right) \times (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})\psi \\ &= \frac{\partial}{\partial x} y\psi\hat{\mathbf{z}} - \frac{\partial}{\partial x} z\psi\hat{\mathbf{y}} - \frac{\partial}{\partial y} x\psi\hat{\mathbf{z}} + \frac{\partial}{\partial y} z\psi\hat{\mathbf{x}} + \frac{\partial}{\partial z} x\psi\hat{\mathbf{y}} - \frac{\partial}{\partial z} y\psi\hat{\mathbf{x}} \\ &= -x\frac{\partial}{\partial y}\psi\hat{\mathbf{z}} + x\frac{\partial}{\partial z}\psi\hat{\mathbf{y}} + y\frac{\partial}{\partial x}\psi\hat{\mathbf{z}} - y\frac{\partial}{\partial z}\psi\hat{\mathbf{x}} - z\frac{\partial}{\partial x}\psi\hat{\mathbf{y}} + z\frac{\partial}{\partial y}\psi\hat{\mathbf{x}}, \\ &= -x\left(\frac{\partial}{\partial y}\hat{\mathbf{z}} - \frac{\partial}{\partial z}\hat{\mathbf{y}}\right)\psi - y\left(\frac{\partial}{\partial z}\hat{\mathbf{x}} - \frac{\partial}{\partial x}\hat{\mathbf{z}}\right)\psi - z\left(\frac{\partial}{\partial x}\hat{\mathbf{y}} - \frac{\partial}{\partial y}\hat{\mathbf{x}}\right)\psi \\ &= -(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \times \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\psi \\ &= -\bar{\mathbf{r}} \times \nabla\psi\end{aligned} \tag{II-6}$$

so that

$$\nabla \times \bar{\mathbf{r}}\psi + \bar{\mathbf{r}} \times \nabla\psi = 0. \tag{II-7}$$

Thus, the anti-commutator of the operators $\nabla\times$ and $\bar{\mathbf{r}}$ is zero.

Now consider $\nabla \times \hat{\mathbf{r}}\psi$, where $\hat{\mathbf{r}} \equiv \bar{\mathbf{r}}/|\bar{\mathbf{r}}|$. We have

$$\begin{aligned}\nabla \times \hat{\mathbf{r}}\psi &= \nabla \times \frac{\bar{\mathbf{r}}}{|\bar{\mathbf{r}}|}\psi \\ &= -\bar{\mathbf{r}} \times \nabla\left(\frac{\psi}{|\bar{\mathbf{r}}|}\right) + \frac{\psi}{|\bar{\mathbf{r}}|}\nabla \times \bar{\mathbf{r}} \\ &= -\hat{\mathbf{r}} \times \nabla\psi - \psi\bar{\mathbf{r}} \times \nabla\left(\frac{1}{|\bar{\mathbf{r}}|}\right). \\ &= -\hat{\mathbf{r}} \times \nabla\psi + \psi\frac{1}{|\bar{\mathbf{r}}|^2}\hat{\mathbf{r}} \times \hat{\mathbf{r}} \\ &= -\hat{\mathbf{r}} \times \nabla\psi\end{aligned} \tag{II-8}$$

Thus, we also have the operator anti-commutative relation

$$\nabla \times \hat{\mathbf{r}} + \hat{\mathbf{r}} \times \nabla = 0. \tag{II-9}$$

Appendix III Addition Theorem for Vector Spherical Harmonics

The vector-spherical-harmonic addition theorem allows a vector harmonic referenced to one coordinate system to be to be expanded in terms of vector harmonics referenced to another coordinate system, which has been translated with respect to the first. The derivation of this theorem is outlined well by Weng Cho Chew [9]. For a translation such that $\bar{\mathbf{r}} = \bar{\mathbf{r}}' + \bar{\mathbf{r}}''$, the addition theorem says [9]

$$\bar{\mathbf{M}}^{(2)}_{n,m}(\bar{\mathbf{r}}) = \sum_{n',m'} A_{n',m';n,m} \bar{\mathbf{M}}^{(i)}_{n',m'}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{N}}^{(i)}_{n',m'}(\bar{\mathbf{r}}'), \tag{III-1}$$

and

$$\bar{\mathbf{N}}^{(2)}_{n,m}(\bar{\mathbf{r}}) = \sum_{n',m'} A_{n',m';n,m} \bar{\mathbf{N}}^{(i)}_{n',m'}(\bar{\mathbf{r}}') + B_{n',m';n,m} \bar{\mathbf{M}}^{(i)}_{n',m'}(\bar{\mathbf{r}}'), \tag{III-2}$$

where

$$A_{n',m';n,m} = \frac{2\pi j^{n'-n}}{\sqrt{n(n+1)n'(n'+1)}} \sum_{n''} \begin{pmatrix} j^{n''}\left[n(n+1)+n'(n'+1)-n''(n''+1)\right] \\ \cdot A(m,n,-m',n',n''-1)\, z^{(i')}_{n''}(kr''_w)\, Y_{n'',m-m'}(\theta''_w,\phi''_w) \end{pmatrix}, \tag{III-3}$$

and

$$B_{n',m';n,m} = \frac{2\pi j^{n'-n}}{\sqrt{n(n+1)n'(n'+1)}} \sum_{n''} \begin{pmatrix} j^{n''}\left[n(n+1)+n'(n'+1)-n''(n''+1)\right] \\ \cdot B(m,n,-m',n',n'')\, z^{(i')}_{n''}(kr''_w)\, Y_{n'',m-m'}(\theta''_w,\phi''_w) \end{pmatrix}, \tag{III-4}$$

and the difference in scaling factors between the definitions of the vector spherical harmonics defined by Chew [9] and those used here has been accounted for.

In (III-1)–(III-4), the $Y_{n,m}(\theta,\phi)$ is the scalar spherical harmonic (see Appendix I),

$$Y_{n,m}(\theta,\phi) = \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}} P^m_n(\cos\theta)\, e^{jm\phi}. \tag{III-5}$$

The choice of which spherical Bessel's function, $z^{(i)}_n(kr)$, to use in (III-1)–(III-4) depends the relative sizes of r' and r'':

$$\text{in } \bar{\mathbf{M}}^{(i)}_{n',m'}(\bar{\mathbf{r}}') \text{ and } \bar{\mathbf{N}}^{(i)}_{n',m'}(\bar{\mathbf{r}}'):\ z^{(i)}_{n'}(kr') = \begin{cases} j_{n'}(kr'), & r' < r'' \\ h^{(2)}_{n'}(kr'), & r' > r'' \end{cases}, \tag{III-6}$$

and

$$\text{in } A_{n',m';n,m} \text{ and } B_{n',m';n,m}:\ z^{(i')}_{n''}(kr'') = \begin{cases} h^{(2)}_{n''}(kr''), & r' < r'' \\ j_{n''}(kr''), & r' > r'' \end{cases}. \tag{III-7}$$

Also, in (III-1)–(III-4)

$$A(m,n,-m',n',n'') = (-1)^m \sqrt{\frac{(2n+1)(2n'+1)(2n''+1)}{4\pi}} \begin{pmatrix} n & n' & n'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & n' & n'' \\ -m & m' & m-m' \end{pmatrix}, \tag{III-8}$$

$$B(m,n,-m',n',n'') = \sqrt{\frac{2n''+1}{2n''-1}} \begin{Bmatrix} -\sqrt{(n'-m')(n'+m'+1)(n''+m-m')(n''+m-m'-1)}A(m,n,-m'-1,n',n''-1) \\ +\sqrt{(n'+m')(n'-m'+1)(n''-m+m')(n''-m+m'+1)}A(m,n,-m'+1,n',n''-1) \\ +2m'\sqrt{(n''-m+m')(n''+m-m')}A(m,n,-m',n',n''-1) \end{Bmatrix}, \quad \text{(III-9)}$$

and $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3-j symbol, related to the Clebsch-Gordon coefficients as [9]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}}\left(j_1m_1j_2m_2 \middle| j_1j_2j_3,-m_3\right), \quad \text{(III-10)}$$

and the Clebsch-Gordon coefficient is [8]

$$\left(j_1j_2m_1m_2 \middle| j_1j_2j_3m_3\right) = \delta(m_3,m_1+m_2)\sqrt{\frac{(j_1+j_2-j_3)!(j_3+j_1-j_2)!(j_3+j_2-j_1)!(2j_3+1)}{(j_1+j_2+j_3+1)!}} \cdot \sum_k \frac{(-1)^k\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!}}{k!(j_1+j_2-j_3-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j_3-j_2+m_1+k)!(j_3-j_1-m_2+k)!}, \quad \text{(III-11)}$$

where

$$\delta(m,m') = \begin{cases} 1, & m=m' \\ 0, & m\neq m' \end{cases}.$$

The Wigner 3-j symbol, $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$, is nonzero only if $m_3=-m_1-m_2$ and if $j_1+j_2 \ge j_3 \ge |j_1-j_2|$. The special case $\begin{pmatrix} n & n' & n'' \\ 0 & 0 & 0 \end{pmatrix}$ is nonzero only if $n+n'+n''$ is an even integer. Notice that there is a slight difference in the notation used here for $B(m,n,-m',n',n'')$, and that used by Chew [9], who inserts an additional argument, $n''-1$, after n''. Since that argument is redundant for the usage here, it has been deleted.

From (III-8) and (III-9), we see that each term in the sum over n'', which is contained in both $A_{n',m';n,m}$ (III-3) and $B_{n',m';n,m}$ (III-4), has a factor $\begin{pmatrix} n & n' & n''-1 \\ 0 & 0 & 0 \end{pmatrix}$. This means that the sum will only contain terms where $n+n'+n''$ is odd. In addition, each term contains the factor $\begin{pmatrix} n & n' & n''-1 \\ -m & m' & m-m' \end{pmatrix}$ or $\begin{pmatrix} n & n' & n''-1 \\ -m & m'\pm1 & m-m'\mp1 \end{pmatrix}$, which will be zero unless n'' satisfies $1+|n-n'| \le n'' \le 1+n+n'$. Note that when $n''=1+|n-n'|$, then $n+n'+n''$ is odd for any n,n'. Thus, we can write

$$A_{n',m';n,m} = \frac{2\pi j^{n'-n}}{\sqrt{n(n+1)n'(n'+1)}} \sum_{\substack{n''=1+|n-n'|, \\ 3+|n-n'|,\cdots}}^{1+n+n'} \begin{matrix} j^{n''}\left[n(n+1)+n'(n'+1)-n''(n''+1)\right] \\ \cdot A(m,n,-m',n',n''-1)\, z_{n''}^{(i)}(kr''_w) Y_{n'',m-m'}(\theta''_w,\phi''_w) \end{matrix}, \quad \text{(III-12)}$$

and

$$B_{n',m':n,m} = \frac{2\pi j^{n'-n}}{\sqrt{n(n+1)n'(n'+1)}} \sum_{\substack{n''=1+|n-n'|,\\ 3+|n-n'|,\cdots}}^{1+n+n'} \begin{array}{l} j^{n''}\left[n(n+1)+n'(n'+1)-n''(n''+1)\right] \\ \cdot B(m,n,-m',n',n'')\, z_{n''}^{(i)}(kr''_w)\, Y_{n'',m-m'}(\theta''_w,\phi''_w) \end{array}. \tag{III-13}$$

Suppose the expansion of the magnetic field for an antenna is known in terms of vector spherical harmonics referenced to the origin of the unprimed coordinate system, as illustrated Figure II - 1. The magnetic field in terms of harmonics referenced to the unprimed system is

$$\bar{\mathbf{H}}(\bar{\mathbf{r}}) = j\frac{k}{\omega\mu}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[b_{n,m}^{TE}\bar{\mathbf{N}}_{n,m}^{(2)}(\bar{\mathbf{r}}) + b_{n,m}^{TM}\bar{\mathbf{M}}_{n,m}^{(2)}(\bar{\mathbf{r}})\right]. \tag{III-14}$$

In terms of harmonics referenced to the primed coordinate system, the same magnetic field is

$$\bar{\mathbf{H}}(\bar{\mathbf{r}}') = j\frac{k}{\omega\mu}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{n',m'}\begin{bmatrix}\left[b_{n,m}^{TE}B_{n',m':n,m} + b_{n,m}^{TM}A_{n',m':n,m}\right]\bar{\mathbf{M}}_{n',m'}^{(i)}(\bar{\mathbf{r}}') \\ +\left[b_{n,m}^{TE}A_{n',m':n,m} + b_{n,m}^{TM}B_{n',m':n,m}\right]\bar{\mathbf{N}}_{n',m'}^{(i)}(\bar{\mathbf{r}}')\end{bmatrix}. \tag{III-15}$$

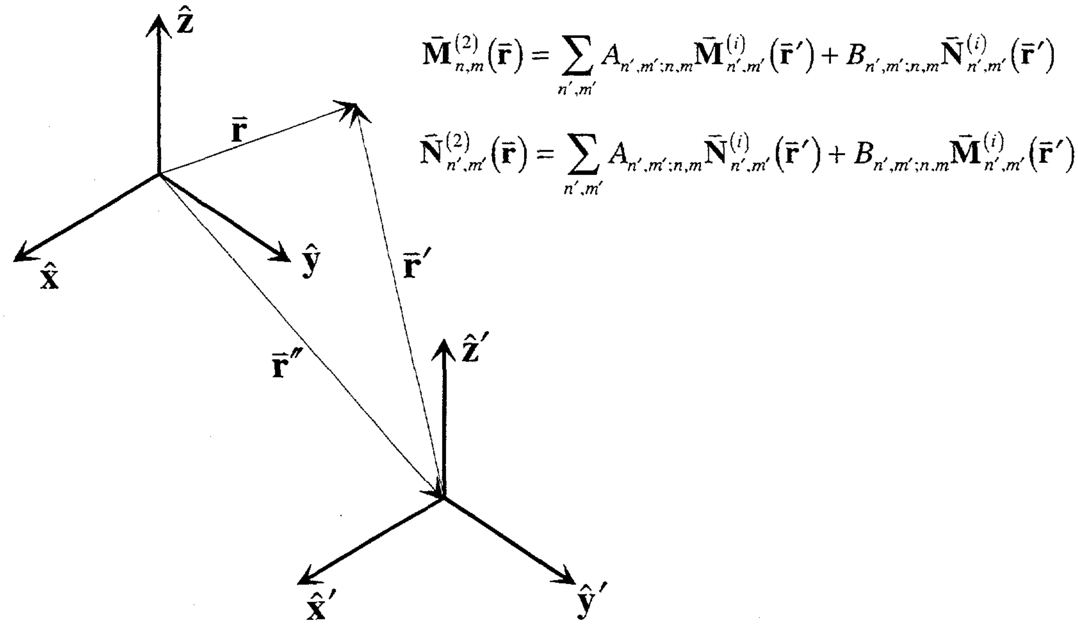

Figure II - 1 Relationship between the unprimed and primed coordinate systems for the vector-harmonic addition theorem.

Computational considerations for the Clebsch-Gordon coefficient

The Clebsch-Gordon coefficient, (III-11), presents some potential difficulties for the numerical computation, because it contains products of factorials. While the coefficient itself is well behaved when the arguments become large, intermediate products and factors can cause numerical overflow if appropriate precautions are not taken. The logarithm of the factorial function can be computed easily for very large arguments, and should be utilized to avoid overflow. A good approach is to compute the coefficient as follows

$$\left(j_1 j_2 m_1 m_2 \middle| j_1 j_2 j_3 m_3\right) = \delta\left(m_3, m_1 + m_2\right)\sqrt{\left(2 j_3 + 1\right)}$$

$$\cdot \sum_k (-1)^k \exp\begin{bmatrix} \tfrac{1}{2}\ln\left(\left(j_1 + j_2 - j_3\right)!\right) + \tfrac{1}{2}\ln\left(\left(j_3 + j_1 - j_2\right)!\right) \\ +\tfrac{1}{2}\ln\left(\left(j_3 + j_2 - j_1\right)!\right) - \tfrac{1}{2}\ln\left(\left(j_1 + j_2 + j_3 + 1\right)!\right) \\ +\tfrac{1}{2}\ln\left(\left(j_1 + m_1\right)!\right) + \tfrac{1}{2}\ln\left(\left(j_1 - m_1\right)!\right) + \tfrac{1}{2}\ln\left(\left(j_2 + m_2\right)!\right) \\ +\tfrac{1}{2}\ln\left(\left(j_2 - m_2\right)!\right) + \tfrac{1}{2}\ln\left(\left(j_3 + m_3\right)!\right) + \tfrac{1}{2}\ln\left(\left(j_3 - m_3\right)!\right) \\ -\ln\left(k!\right) - \ln\left(\left(j_1 + j_2 - j_3 - k\right)!\right) - \ln\left(\left(j_1 - m_1 - k\right)!\right) \\ -\ln\left(\left(j_2 + m_2 - k\right)!\right) - \ln\left(\left(j_3 - j_2 + m_1 + k\right)!\right) \\ -\ln\left(\left(j_3 - j_1 - m_2 + k\right)!\right) \end{bmatrix}. \quad \text{(III-16)}$$

Appendix IV — Certain Integrals Containing Associated Legendre Functions

In the evaluation of the mutual impedance, certain integrals containing associated Legendre functions are required. These integrals are defined

$$\mathscr{I}_1(n,n',m) = \int_{-1}^{1} \left(1-x^2\right) \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_{n'}^m(x)\, dx, \tag{IV-1}$$

$$\mathscr{I}_2(n,n',m) = \int_{-1}^{1} \left[P_n^m(x) \frac{d}{dx} P_{n'}^m(x) \right] dx, \tag{IV-2}$$

and

$$\mathscr{I}_3(n,n',m) = \int_{-1}^{1} \frac{1}{1-x^2} P_n^m(x) P_{n'}^m(x)\, dx. \tag{IV-3}$$

In addition, it will prove convenient to evaluate

$$\mathscr{I}_0(n,n',m) = \int_{-1}^{1} P_n^m(x) P_{n'}^m(x)\, dx. \tag{IV-4}$$

For negative order, we use the convention [7]

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x). \tag{IV-5}$$

Establishing Orthogonality

The associated Legendre functions possess well-known orthogonality properties. The orthogonality relations are

$$\int_{-1}^{1} P_n^m(x) P_{n'}^m(x) dx = \begin{cases} 0, & n \neq n' \\ \neq 0, & n = n' \end{cases} \tag{IV-6}$$

and

$$\int_{-1}^{1} \frac{1}{1-x^2} P_n^m(x) P_n^{m'}(x) dx = \begin{cases} 0, & |m| \neq |m'| \\ \neq 0, & |m| = |m'| \end{cases} \tag{IV-7}$$

It will be useful to derive these orthogonality relations, in order to illuminate the relationship between the various integrals.

Application of the differential equation

The associated Legendre differential equation for integer degree and order is

$$\frac{d}{dx}\left[(1-x^2) \frac{d}{dx} P_n^m(x) \right] + \left[n(n+1) - \frac{m^2}{1-x^2} \right] P_n^m(x) = 0. \tag{IV-8}$$

Multiply (IV-8) by $P_{n'}^{m'}(x)$

$$P_{n'}^{m'}(x) \frac{d}{dx}\left[(1-x^2) \frac{d}{dx} P_n^m(x) \right] + \left[n(n+1) - \frac{m^2}{1-x^2} \right] P_{n'}^{m'}(x) P_n^m(x) = 0. \tag{IV-9}$$

Interchange n with n' and m with m' in (IV-9), subtract the new equation from (IV-9), and integrate over $-1 \le x \le 1$ to obtain

$$\begin{aligned}&\int_{-1}^{1} P_{n'}^{m'}(x)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n^m(x)\right]dx - \int_{-1}^{1} P_n^m(x)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_{n'}^{m'}(x)\right]dx \\ &+\left[n(n+1)-n'(n+1)\right]\int_{-1}^{1} P_{n'}^{m'}(x)P_n^m(x)dx - \left[m^2-m'^2\right]\int_{-1}^{1}\frac{1}{1-x^2}P_{n'}^{m'}(x)P_n^m(x)dx = 0\end{aligned} \quad . \qquad \text{(IV-10)}$$

Applying the integration-by-parts procedure to the first two integrals

$$\begin{aligned}&(1-x^2)P_{n'}^{m'}(x)\frac{d}{dx}P_n^m(x)\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)\frac{d}{dx}P_n^m(x)\frac{d}{dx}P_{n'}^{m'}(x)dx \\ &-(1-x^2)P_n^m(x)\frac{d}{dx}P_{n'}^{m'}(x)\Big|_{-1}^{1} + \int_{-1}^{1}(1-x^2)\frac{d}{dx}P_{n'}^{m'}(x)\frac{d}{dx}P_n^m(x)dx \\ &+\left[n(n+1)-n'(n'+1)\right]\int_{-1}^{1} P_{n'}^{m'}(x)P_n^m(x)dx - \left[m^2-m'^2\right]\int_{-1}^{1}\frac{1}{1-x^2}P_{n'}^{m'}(x)P_n^m(x)dx = 0\end{aligned} \quad . \qquad \text{(IV-11)}$$

Since $P_n^m(x)$ is finite at $x = \pm 1$ and $\frac{d}{dx}P_n^m(x)$ has no worse than a $\sqrt{1-x^2}$ singularity at $x = \pm 1$, then

$$\left[n(n+1)-n'(n'+1)\right]\int_{-1}^{1} P_{n'}^{m'}(x)P_n^m(x)dx - \left[m^2-m'^2\right]\int_{-1}^{1}\frac{1}{1-x^2}P_{n'}^{m'}(x)P_n^m(x)dx = 0 . \qquad \text{(IV-12)}$$

If $m = m'$, then

$$\int_{-1}^{1} P_n^m(x)P_{n'}^m(x)dx = \begin{cases} 0, & n \ne n' \\ \ne 0, & n = n' \end{cases} . \qquad \text{(IV-13)}$$

Thus, $P_n^m(x)$ and $P_{n'}^m(x)$ are orthogonal with weight one over the interval $-1 \le x \le 1$.
Similarly, if $n = n'$, then

$$\int_{-1}^{1}\frac{1}{1-x^2}P_n^m(x)P_n^{m'}(x)dx = \begin{cases} 0, & |m| \ne |m'| \\ \ne 0, & |m| = |m'| \end{cases} , \qquad \text{(IV-14)}$$

and $P_n^m(x)$ and $P_n^{m'}(x)$ are orthogonal with weight $1/(1-x^2)$ over the interval $-1 \le x \le 1$. Thus, the orthogonality relations have been proven. It remains to evaluate the nonzero integrals.

Relationship between the various integrals

Initially the closed-form of the integrals will be obtained for positive order, $m \ge 0$. Negative order, $m < 0$, will be handled by applying the convention (IV-5), so

$$\mathcal{I}_i(n,n',-m) = \frac{(n-m)!\,(n'-m)!}{(n+m)!\,(n'+m)!}\mathcal{I}_i(n,n',m) . \qquad \text{(IV-15)}$$

Integrating (IV-9) produces

$$\int_{-1}^{1}(1-x^2)\frac{d}{dx}P_n^m(x)\frac{d}{dx}P_{n'}^{m'}(x)dx = n(n+1)\int_{-1}^{1}P_n^m(x)P_{n'}^{m'}(x)dx$$
$$-m^2\int_{-1}^{1}\left[\frac{1}{1-x^2}\right]P_n^m(x)P_{n'}^{m'}(x)dx \quad \text{(IV-16)}$$

Thus, with $m = m'$, we obtain

$$\mathscr{I}_1(n,n',m) = n(n+1)\mathscr{I}_0(n,n',m) - m^2\mathscr{I}_3(n,n',m). \quad \text{(IV-17)}$$

Note that the integrals $\mathscr{I}_0$, $\mathscr{I}_1$, and $\mathscr{I}_3$ are symmetric with respect to interchange of n and n'. This symmetry, along with (IV-17) shows once again that

$$\int_{-1}^{1}P_n^m(x)P_{n'}^m(x)dx = 0,\ n \neq n'. \quad \text{(IV-18)}$$

Application of recursion relation

The associated Legendre functions satisfy the following recursion relations [10]

$$xP_n^m(x) = \frac{n-m+1}{2n+1}P_{n+1}^m(x) + \frac{n+m}{2n+1}P_{n-1}^m(x), \quad \text{(IV-19)}$$

and

$$(x^2-1)\frac{d}{dx}P_n^m(x) = nxP_n^m(x) - (n+m)P_{n-1}^m(x), \quad \text{(IV-20)}$$

so that

$$(x^2-1)\frac{d}{dx}P_n^m(x) = \frac{n(n-m+1)}{2n+1}P_{n+1}^m(x) - \frac{(n+m)(n+1)}{2n+1}P_{n-1}^m(x), \quad \text{(IV-21)}$$

valid for $m \geq 0$.

Substituting n' for n in (IV-21), multiplying by $P_n^m(x)/(1-x^2)$, and integrating, we obtain

$$\mathscr{I}_2(n,n',m) = \int_{-1}^{1}P_n^m(x)\frac{d}{dx}P_{n'}^m(x)dx$$
$$= -\frac{n'(n'-m'+1)}{(2n'+1)}\int_{-1}^{1}\frac{1}{(1-x^2)}P_n^m(x)P_{n'+1}^m(x)dx, \quad \text{(IV-22)}$$
$$+\frac{(n'+m)(n'+1)}{(2n'+1)}\int_{-1}^{1}\frac{1}{(1-x^2)}P_n^m(x)P_{n'-1}^m(x)dx$$

or

$$\mathscr{I}_2(n,n',m) = \frac{(n'+m)(n'+1)}{(2n'+1)}\mathscr{I}_3(n,n'-1,m) - \frac{n'(n'-m+1)}{(2n'+1)}\mathscr{I}_3(n,n'+1,m) \text{ when } m \geq 0. \quad \text{(IV-23)}$$

If expressions for the integrals $\mathscr{I}_0(n,n',m)$ and $\mathscr{I}_3(n,n',m)$ can be found, then $\mathscr{I}_1(n,n',m)$ can be obtained from (IV-17) and $\mathscr{I}_2(n,n',m)$ can be obtained from (IV-23)

Evaluation of $\mathscr{I}_0(n,n',m) = \int_{-1}^{1} P_n^m(x)\,P_{n'}^m(x)\,dx$

The explicit form of the associated Legendre function given by [†]

$$P_n^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x) \tag{IV-24}$$

where $m \geq 0$, and $P_n(x)$ is the Legendre function

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(x^2-1\right)^n . \tag{IV-25}$$

Using (IV-25) in (IV-24)

$$P_n^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{1}{2^n n!} \frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^n . \tag{IV-26}$$

Thus, the integral becomes

$$\begin{aligned} \mathscr{I}_0(n,n',m) &= \int_{-1}^{1} P_n^m(x) P_{n'}^m(x)dx \\ &= \frac{1}{2^{n+n'} n!\,n'!} \int_{-1}^{1} \left(1-x^2\right)^m \frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^n \frac{d^{n'+m}}{dx^{n'+m}}\left(x^2-1\right)^{n'} dx \end{aligned} \tag{IV-27}$$

Since we have already shown that $\mathscr{I}_0(n,n',m) = 0$ when $n \neq n'$ (IV-13), we need to evaluate

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\left(1-x^2\right)^m \frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^n \frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^n dx . \tag{IV-28}$$

Integrating by parts $(n+m)$ times

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \mathscr{I}_\Sigma(n,m) + \mathscr{I}_\int(n,m), \tag{IV-29}$$

where

$$\mathscr{I}_\Sigma(n,m) = \frac{1}{2^{2n}(n!)^2}\sum_{k=1}^{n+m}(-1)^{k-1}\frac{d^{n+m-k}}{dx^{n+m-k}}\left(x^2-1\right)^n \frac{d^{k-1}}{dx^{k-1}}\left(\left(1-x^2\right)^m \frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^n\right)\Bigg|_{-1}^{1}, \tag{IV-30}$$

and

$$\mathscr{I}_\int(n,m) = \frac{(-1)^{n+m}}{2^{2n}(n!)^2}\int_{-1}^{1}\left(x^2-1\right)^n \frac{d^{n+m}}{dx^{n+m}}\left(\left(1-x^2\right)^m \frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^n\right)dx . \tag{IV-31}$$

Applying Leibnitz' differentiation formula [‡] [14]

[†] Hanson [12], Stratton [13], Arfken [14], and Mathews and Walker [16] omit the factor $(-1)^m$, but it is included by Jackson [7], Abramowitz and Stegun [8], Chew [9], Lebedev [10], and Balanis [3].

[‡] Leibnitz' formula for the n^{th} derivative of a product:

$$\frac{d^n}{dx^n}\left[A(x)B(x)\right] = \sum_{s=0}^{n}\frac{n!}{s!(n-s)!}\frac{d^{n-s}}{dx^{n-s}}A(x)\frac{d^s}{dx^s}B(x)$$

$$\mathscr{I}_{\Sigma}(n,m)\frac{1}{2^{2n}(n!)^2}\sum_{k=1}^{n+m}\left[(-1)^{k-1}\begin{matrix}\left[\dfrac{d^{n+m-k}}{dx^{n+m-k}}(x^2-1)^n\right]\\ \cdot\displaystyle\sum_{t=0}^{k-1}\frac{(k-1)!}{t!(k-1-t)!}\left[\frac{d^{k-1-t}}{dx^{k-1-t}}(1-x^2)^m\right]\left[\frac{d^{n+m+t}}{dx^{n+m+t}}(x^2-1)^n\right]\end{matrix}\Bigg|_{-1}^{1}\right], \tag{IV-32}$$

and

$$\mathscr{I}_{\int}(n,m)=\frac{(-1)^{n+m}}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\sum_{s=0}^{n+m}\left[\begin{matrix}\dfrac{(n+m)!}{s!(n+m-s)!}\left[\dfrac{d^{n+m-s}}{dx^{n+m-s}}(1-x^2)^m\right]\\ \cdot\left[\dfrac{d^{n+m+s}}{dx^{n+m+s}}(x^2-1)^n\right]\end{matrix}\right]dx. \tag{IV-33}$$

Using the binomial expansion, we can obtain the derivatives

$$\begin{aligned}\frac{d^q}{dx^q}(1-x^2)^\alpha &= \frac{d^q}{dx^q}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)}x^{2k}\\ &=\sum_{k=q/2}^{\infty}(-1)^k\frac{\Gamma(\alpha+1)(2k)!}{k!\Gamma(\alpha+1-k)(2k-q)!}x^{2k-q}\end{aligned}, \tag{IV-34}$$

and

$$\frac{d^q}{dx^q}(x^2-1)^\alpha=(-1)^\alpha\sum_{k=q/2}^{\infty}(-1)^k\frac{\Gamma(\alpha+1)(2k)!}{k!\Gamma(\alpha+1-k)(2k-q)!}x^{2k-q}, \tag{IV-35}$$

where q is an integer and α is a real number, not necessarily an integer. For convergence, we require $|x|<1$. Note that the infinite sums in (IV-34) and (IV-35) will terminate when $k=\alpha$, for α a positive integer. However, if α is not a positive integer, an infinite number of terms will occur in the sum.

When $\alpha=n$, an integer, the derivatives also can be expanded into another useful form as follows (using equation 0.432-3 from Gradshteyn and Ryzhik [15]),

$$\frac{d^q}{dx^q}(x^2-1)^n=n!\sum_{k=0}^{\mathrm{int}(q/2)}\frac{q!}{k!(q-2k)!(n-q+k)!}2^{q-2k}x^{q-2k}(x^2-1)^{n-q+k}. \tag{IV-36}$$

or

$$\frac{d^q}{dx^q}(1-x^2)^n=(-1)^q n!\sum_{k=0}^{\mathrm{int}(q/2)}(-1)^k\frac{q!}{k!(q-2k)!(n-q+k)!}2^{q-2k}x^{q-2k}(1-x^2)^{n-q+k}. \tag{IV-37}$$

This form is useful for evaluating the derivative at the values $x=\pm1$, but *only* when the exponent is an integer. For $\alpha=n$, an integer, if $q>2n$, all the terms in (IV-36) and (IV-37) will be zero ($n-q+k$ is always negative). Additionally, if $q<n$, there will always be a factor of $(1-x^2)$ so that

$$\left.\frac{d^q}{dx^q}(1-x^2)^n\right|_{x=\pm1}=0 \text{ when } q<n, \tag{IV-38}$$

and, when $q=n$, there will be a single term

$$\frac{d^n}{dx^n}\left(1-x^2\right)^n\bigg|_{x=\pm1} = (\mp 1)^n\, n!2^n. \tag{IV-39}$$

Note that the ± 1 on the left side of (IV-39) correlates with the ∓ 1 on the right side. When $q = 2n$, the derivative has only a constant term

$$\frac{d^{2n}}{dx^{2n}}\left(1-x^2\right)^n = (-1)^n (2n)!. \tag{IV-40}$$

We see from (IV-36) that $\frac{d^q}{dx^q}\left(x^2-1\right)^n\Big|_{x=\pm1}$ can only be nonzero if $n \le q \le 2n$. Thus, in $\mathscr{I}_\Sigma(n,m)$, the factor $\left[\frac{d^{n+m-k}}{dx^{n+m-k}}\left(x^2-1\right)^n\right]_{-1}^{1}$ is zero for $k > m$, while $\left[\frac{d^{k-1-t}}{dx^{k-1-t}}\left(1-x^2\right)^m\right]_{-1}^{1}$ will be zero for $k < m+1$. Therefore, $\mathscr{I}_\Sigma(n,m)$ will be zero and

$$\begin{aligned}\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx &= \mathscr{I}_{oJ}(n,m,n,m)\\ &= \frac{(-1)^{n+m}}{2^{2n}(n!)^2}\int_{-1}^{1}\left(x^2-1\right)^n \sum_{s=0}^{n+m}\frac{(n+m)!}{s!(n+m-s)!}\left[\frac{d^{n+m-s}}{dx^{n+m-s}}\left(1-x^2\right)^m\right]\left[\frac{d^{n+m+s}}{dx^{n+m+s}}\left(x^2-1\right)^n\right]dx\end{aligned} \tag{IV-41}$$

The left derivative in (IV-41) is zero everywhere unless $s \ge n-m$, while the right derivative is zero everywhere unless $s \le n-m$. Thus only the term with $s = (n-m)$ will contribute to the integral, and

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}\frac{(n+m)!}{(n-m)!}\int_{-1}^{1}\left(x^2-1\right)^n dx. \tag{IV-42}$$

Letting $x = \cos\theta$, we obtain [15]

$$\int_{-1}^{1}\left(x^2-1\right)^n dx = (-1)^n\int_0^\pi \sin^{2n+1}\theta\, d\theta = (-1)^n\, 2\frac{(2n)!!}{(2n+1)!!} = (-1)^n\frac{n!}{\Gamma(n+3/2)}\Gamma(1/2). \tag{IV-43}$$

Since $(2n)!! = 2^n n!$ and $(2n-1)!! = 2^n\,\Gamma\left(n+\frac{1}{2}\right)/\sqrt{\pi}$ [8], it follows that

$$\frac{(2n)!}{2^n n!} = \frac{(2n)!}{(2n)!!} = (2n-1)!!,$$

and we obtain the well-known result [8]

$$\mathscr{I}_0(n,n,m) = \int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \frac{2}{(2n+1)}\frac{(n+m)!}{(n-m)!} \text{ when } m \ge 0. \tag{IV-44}$$

Using the convention for negative order (IV-5)

$$\begin{aligned}\mathscr{I}_0(n,n,-m) &= \int_{-1}^{1}\left[P_n^{-m}(x)\right]^2 dx\\ &= \left(\frac{(n-m)!}{(n+m)!}\right)^2\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \frac{2}{(2n+1)}\frac{(n-m)!}{(n+m)!} \text{ when } m \ge 0\end{aligned} \tag{IV-45}$$

Thus, combining (IV-44) and (IV-45), we have

$$\mathscr{I}_0\left(n,n',m\right)=\int_{-1}^{1}P_n^m(x)P_{n'}^m(x)dx=\begin{cases}0 \text{ when } n'\neq n\\ \dfrac{2}{(2n+1)}\dfrac{(n+m)!}{(n-m)!} \text{ when } n'=n\end{cases} \qquad \text{(IV-46)}$$

Evaluation of $\mathscr{I}_3\left(n,n',m\right)=\int\limits_{-1}^{1}\frac{1}{1-x^2}P_n^m\left(x\right)P_{n'}^m\left(x\right)dx$

Inserting (IV-26) into the integrand of $\mathscr{I}_3\left(n,n',m\right)$,

$$\begin{aligned}\mathscr{I}_3\left(n,n',m\right)&=\int_{-1}^{1}\left[\frac{1}{1-x^2}\right]P_{n'}^m(x)P_n^m(x)dx\\&=\frac{1}{2^{n+n'}n'!n!}\int_{-1}^{1}\left(1-x^2\right)^{m-1}\frac{d^{n'+m}}{dx^{n'+m}}\left(x^2-1\right)^{n'}\frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^{n}dx\end{aligned}, \qquad \text{(IV-47)}$$

where we require $m\geq 0$. We integrate by parts p times

$$\int_{-1}^{1}\left[\frac{1}{1-x^2}\right]P_{n'}^m(x)P_n^m(x)dx=$$

$$=\frac{1}{2^{n+n'}n'!n!}\left\{\begin{aligned}&\left.\sum_{k=1}^{p}(-1)^{k-1}\frac{d^{k-1}}{dx^{k-1}}\left[\left(1-x^2\right)^{m-1}\frac{d^{n'+m}}{dx^{n'+m}}\left(x^2-1\right)^{n'}\right]\frac{d^{n+m-k}}{dx^{n+m-k}}\left(x^2-1\right)^{n}\right|_{-1}^{1}\\&+(-1)^p\int_{-1}^{1}\frac{d^{n+m-p}}{dx^{n+m-p}}\left(x^2-1\right)^n\frac{d^p}{dx^p}\left[\left(1-x^2\right)^{m-1}\frac{d^{n'+m}}{dx^{n'+m}}\left(x^2-1\right)^{n'}\right]dx\end{aligned}\right\}. \qquad \text{(IV-48)}$$

Applying Leibnitz' product differentiation formula to (IV-48) gives

$$\int_{-1}^{1}\left[\frac{1}{1-x^2}\right]P_{n'}^m(x)P_n^m(x)dx=$$

$$=\frac{1}{2^{n+n'}n'!n!}\left\{\begin{aligned}&\left.\begin{aligned}&\sum_{k=1}^{p}(-1)^{k-1}\left[\frac{d^{n+m-k}}{dx^{n+m-k}}\left(x^2-1\right)^n\right]\cdot\\&\cdot\sum_{s=0}^{k-1}\frac{(k-1)!}{s!(k-1-s)!}\left[\frac{d^{k-1-s}}{dx^{k-1-s}}\left(1-x^2\right)^{m-1}\right]\left[\frac{d^{n'+m+s}}{dx^{n'+m+s}}\left(x^2-1\right)^{n'}\right]\end{aligned}\right|_{-1}^{1}\\&+(-1)^p\int_{-1}^{1}\left[\frac{d^{n+m-p}}{dx^{n+m-p}}\left(x^2-1\right)^n\right]\sum_{s=0}^{p}\frac{p!}{s!(p-s)!}\left[\frac{d^{p-s}}{dx^{p-s}}\left(1-x^2\right)^{m-1}\right]\left[\frac{d^{n'+m+s}}{dx^{n'+m+s}}\left(x^2-1\right)^{n'}\right]dx\end{aligned}\right\}. \qquad \text{(IV-49)}$$

We see that the right-hand factor in the sum outside the integral in (IV-49), $\left.\frac{d^{n+m-k}}{dx^{n+m-k}}\left(x^2-1\right)^n\right|_{x=\pm1}$, is nonzero only when $m-n\leq k\leq m$, while $\left.\frac{d^{k-1-s}}{dx^{k-1-s}}\left(1-x^2\right)^{m-1}\right|_{x=\pm1}$ is nonzero only when $m\leq k-s\leq 2m-1$. Thus, the nonzero contribution occurs for $k=m$ and $s=0$, so that for $p\geq m$

$$\int_{-1}^{1}\left[\frac{1}{1-x^2}\right]P_{n'}^m(x)P_n^m(x)dx =$$

$$=\frac{1}{2^{n+n'}n'!n!}\left\{\begin{array}{l}\left(-1\right)^{m-1}\left[\frac{d^n}{dx^n}\left(x^2-1\right)^n\right]\left[\frac{d^{m-1}}{dx^{m-1}}\left(1-x^2\right)^{m-1}\right]\left[\frac{d^{n'+m}}{dx^{n'+m}}\left(x^2-1\right)^{n'}\right]\Bigg|_{-1}^{1} \\ +\left(-1\right)^p\int_{-1}^{1}\left[\frac{d^{n+m-p}}{dx^{n+m-p}}\left(x^2-1\right)^n\right]\sum_{s=0}^{p}\frac{p!}{s!\left(p-s\right)!}\left[\frac{d^{p-s}}{dx^{p-s}}\left(1-x^2\right)^{m-1}\right]\left[\frac{d^{n'+m+s}}{dx^{n'+m+s}}\left(x^2-1\right)^{n'}\right]dx\end{array}\right\}. \quad \text{(IV-50)}$$

The integrand in (IV-50) will be nonzero for $s \geq p-2(m-1)$ and $s \leq n'-m$. If we take $p = n+m$, the integrand is nonzero only when $n-m+2 \leq s \leq n'-m$.

At this point, we have not specified the relative size of n' and n. If we choose $n \geq n'$, we see that no value of s allows a nonzero integrand. We simply obtain

$$\mathcal{I}_3\left(n,n',m\right) = \int_{-1}^{1}\left[\frac{1}{1-x^2}\right]P_{n'}^m(x)P_n^m(x)dx$$

$$=\begin{cases} 0 \text{ when } n'+n \text{ odd} \\ \dfrac{(n'+m)!}{m(n'-m)!} \text{ when } n'+n \text{ even, } n \geq n', \text{ and } m > 0 \\ \text{undefined when } m = 0 \end{cases}. \quad \text{(IV-51)}$$

Noting the symmetry of $\mathcal{I}_3(n,n',m)$ with respect to n and n', and using the convention (IV-5) for negative order ($m<0$), we write

$$\boxed{\mathcal{I}_3\left(n,n',m\right) = \int_{-1}^{1}\left[\frac{1}{1-x^2}\right]P_{n'}^m(x)P_n^m(x)dx = \begin{cases} 0 \text{ when } n'+n \text{ odd} \\ \dfrac{\left(\min\left(n,n'\right)+m\right)!}{m\left(\min\left(n,n'\right)-m\right)!} \text{ when } n'+n \text{ even and } m>0 \\ -\dfrac{\left(\max\left(n,n'\right)+m\right)!}{m\left(\max\left(n,n'\right)-m\right)!} \text{ when } n'+n \text{ even and } m<0 \\ \text{undefined when } m=0 \end{cases}} \quad \text{(IV-52)}$$

Note that $\mathcal{I}_3(n,n',m)$ is undefined when $m=0$. However, in cases where the integral is multiplied by m, letting m go to zero prior to performing the integration takes care of the problem.

Evaluation of $\mathcal{I}_1(n,n',m)=\int_{-1}^{1}\left(1-x^2\right)\frac{d}{dx}P_n^m(x)\frac{d}{dx}P_{n'}^m(x)\,dx$

The relationship between $\mathcal{I}_1(n,n',m)$, $\mathcal{I}_0(n,n',m)$, and $\mathcal{I}_3(n,n',m)$ is given in (IV-17). Substitution of (IV-46) and (IV-52) into (IV-17) gives

$$\boxed{\mathcal{I}_1(n,n',m)=\begin{cases}0 \text{ when } n'+n \text{ odd}\\ \dfrac{2n(n+1)}{(2n+1)}\dfrac{(n+m)!}{(n-m)!}\delta_{n,n'}-m\dfrac{\left(\min(n,n')+m\right)!}{\left(\min(n,n')-m\right)!} \text{ when } n'+n \text{ even and } m\ge 0\\ \dfrac{2n(n+1)}{(2n+1)}\dfrac{(n+m)!}{(n-m)!}\delta_{n,n'}+m\dfrac{\left(\max(n,n')+m\right)!}{\left(\max(n,n')-m\right)!} \text{ when } n'+n \text{ even and } m<0\end{cases}}, \qquad \text{(IV-53)}$$

where

$$\delta_{n,n'}=\begin{cases}0 \text{ when } n\ne n'\\ 1 \text{ when } n=n'\end{cases}.$$

Evaluation of $\mathcal{I}_2(n,n',m)=\int_{-1}^{1}\left[P_n^m(x)\frac{d}{dx}P_{n'}^m(x)\right]dx$

The relationship between $\mathcal{I}_2(n,n',m)$ and $\mathcal{I}_3(n,n',m)$ is given by (IV-23). Substitution of (IV-52) into (IV-23) gives

$$\mathcal{I}_2(n,n',m)=\begin{cases}0;\; n'+n \text{ even}\\ \dfrac{1}{m(2n'+1)}\left(\begin{array}{l}(n'+1)(n'+m)\dfrac{\left(\min(n,n'-1)+m\right)!}{\left(\min(n,n'-1)-m\right)!}\\ -n'(n'+1-m)\dfrac{\left(\min(n,n'+1)+m\right)!}{\left(\min(n,n'+1)-m\right)!}\end{array}\right);\; \begin{cases}n'+n \text{ odd}\\ \text{and } m>0\end{cases}\\ -\dfrac{1}{m(2n'+1)}\left(\begin{array}{l}(n'+1)(n'+m)\dfrac{\left(\max(n,n'-1)+m\right)!}{\left(\max(n,n'-1)-m\right)!}\\ -n'(n'+1-m)\dfrac{\left(\max(n,n'+1)+m\right)!}{\left(\max(n,n'+1)-m\right)!}\end{array}\right);\; \begin{cases}n'+n \text{ odd}\\ \text{and } m<0\end{cases}\end{cases} \qquad \text{(IV-54)}$$

Note that (IV-54) does not give $\mathcal{I}_2(n,n',m)$ when $m=0$ if $n'+n$ is odd. Because $\mathcal{I}_3(n,n',m=0)$ is undefined, (IV-23) cannot be used to obtain $\mathcal{I}_2(n,n',m)$ when $m=0$. Taking a direct approach, and integrating (IV-2) by parts, with $m=0$,

$$\mathscr{I}_2(n,n',0) = \int_{-1}^{1}\left[P_n^0(x)\frac{d}{dx}P_{n'}^0(x)\right]dx$$
$$= P_n^0(x)P_{n'}^0(x)\Big|_{-1}^{1} - \int_{-1}^{1}\left[P_{n'}^0(x)\frac{d}{dx}P_n^0(x)\right]dx \quad \text{(IV-55)}$$

Thus, we have

$$\mathscr{I}_2(n,n',0) + \mathscr{I}_2(n',n,0) = P_n^0(1)P_{n'}^0(1) - P_n^0(-1)P_{n'}^0(-1) = \begin{cases} 0;\ n+n' \text{ even} \\ 2;\ n+n' \text{ odd} \end{cases}. \quad \text{(IV-56)}$$

Substituting (IV-26) for $m = 0$ into (IV-2)

$$\mathscr{I}_2(n,n',0) = \int_{-1}^{1}\left[P_n^0(x)\frac{d}{dx}P_{n'}^0(x)\right]dx$$
$$= \frac{1}{2^{n+n'}n!n'!}\int_{-1}^{1}\left[\frac{d^n}{dx^n}\left(x^2-1\right)^n\frac{d^{n'+1}}{dx^{n'+1}}\left(x^2-1\right)^{n'}\right]dx \quad \text{(IV-57)}$$

Integrating (IV-57) by parts one time

$$\mathscr{I}_2(n,n',0) = \frac{1}{2^{n+n'}n!n'!}\left[\begin{array}{l}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\frac{d^{n'+1}}{dx^{n'+1}}\left(x^2-1\right)^{n'}\Big|_{-1}^{1} \\ -\int_{-1}^{-1}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\frac{d^{n'+2}}{dx^{n'+2}}\left(x^2-1\right)^{n'}dx\end{array}\right]. \quad \text{(IV-58)}$$

From (IV-38), we see that the first term in the brackets is zero. The integral can be further evaluated by the integration-by-parts procedure. Applying this procedure a total of n times,

$$\mathscr{I}_2(n,n',0) = \frac{(-1)^n}{2^{n+n'}n!n'!}\int_{-1}^{-1}\left(x^2-1\right)^n\frac{d^{n'+n+1}}{dx^{n'+n+1}}\left(x^2-1\right)^{n'}dx. \quad \text{(IV-59)}$$

From (IV-40), we see that

$$\frac{d^{n'+n+1}}{dx^{n'+n+1}}\left(x^2-1\right)^{n'} = 0 \text{ when } n \geq n'. \quad \text{(IV-60)}$$

Thus, we have

$$\mathscr{I}_2(n,n',0) = 0 \text{ when } n \geq n'. \quad \text{(IV-61)}$$

Using this information with (IV-56)

$$\mathscr{I}_2(n,n',0) = \begin{cases} 0;\ n+n' \text{ even} \\ 0;\ n+n' \text{ odd and } n > n'. \\ 2;\ n+n' \text{ odd and } n' > n \end{cases} \quad \text{(IV-62)}$$

The complete integral is

$$\mathscr{I}_2(n,n',m)=\int_{-1}^{1}\left[P_n^m(x)\frac{d}{dx}P_{n'}^m(x)\right]dx$$

$$=\begin{cases} 0; & \begin{cases} n'+n \text{ even, for all } m \\ n+n' \text{ odd, } n>n' \text{, and } m=0 \end{cases} \\ 2; \; n+n' \text{ odd, } n'>n \text{, and } m=0 & \\ \dfrac{1}{m(2n'+1)}\left(\begin{array}{l} (n'+1)(n'+m)\dfrac{(\min(n,n'-1)+m)!}{(\min(n,n'-1)-m)!} \\ -n'(n'+1-m)\dfrac{(\min(n,n'+1)+m)!}{(\min(n,n'+1)-m)!} \end{array} \right); & \begin{cases} n'+n \text{ odd} \\ \text{and } m>0 \end{cases} \\ -\dfrac{1}{m(2n'+1)}\left(\begin{array}{l} (n'+1)(n'+m)\dfrac{(\max(n,n'-1)+m)!}{(\max(n,n'-1)-m)!} \\ -n'(n'+1-m)\dfrac{(\max(n,n'+1)+m)!}{(\max(n,n'+1)-m)!} \end{array} \right); & \begin{cases} n'+n \text{ odd} \\ \text{and } m<0 \end{cases} \end{cases} \qquad \text{(IV-63)}$$

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Appendix V — Relevant Network Parameters

The relationship between the *S* parameters and the admittance matrix

Define terminal voltages and currents at the ports of a two-port network as V_1, I_1 and V_2, I_2 at ports 1 and 2, respectively. The port voltages and currents can be related through the admittance matrix

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{V-1}$$

For a microwave network, the ports are typically fed with transmission lines. In this case, it is convenient to describe the response in terms of incident and scattered port voltages, V_i^+, V_i^-, respectively. These port voltages are related through the scattering parameters

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}. \tag{V-2}$$

Associated with the incident voltage is an incident current, I_i^+, which flows into the port. Similarly, associated with the scattered voltage is a scattered current, I_i^-, flowing out of the port. The incident and scattered voltages and currents are related by the transmission-line wave admittance, $Y_{0,i}$ at the respective ports

$$\begin{bmatrix} I_1^\pm \\ I_2^\pm \end{bmatrix} = \begin{bmatrix} Y_{0,1} & 0 \\ 0 & Y_{0,2} \end{bmatrix} \begin{bmatrix} V_1^\pm \\ V_2^\pm \end{bmatrix}. \tag{V-3}$$

The total port voltage is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} + \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix}, \tag{V-4}$$

while the total port current is

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_1^+ \\ I_2^+ \end{bmatrix} - \begin{bmatrix} I_1^- \\ I_2^- \end{bmatrix} = \begin{bmatrix} Y_{0,1} & 0 \\ 0 & Y_{0,2} \end{bmatrix} \left(\begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} - \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} \right), \tag{V-5}$$

with the reference direction into the port. Substituting (V-2), (V-3), (V-4), and (V-5) into (V-1) gives the relationship between the admittance matrix and the scattering parameters

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}} Y_{0,1} & -\dfrac{2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}} Y_{0,1} \\ -\dfrac{2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}} Y_{0,2} & \dfrac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}} Y_{0,2} \end{bmatrix}. \tag{V-6}$$

Similarly

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \frac{(Y_{0,1}-Y_{11})(Y_{0,2}+Y_{22})+Y_{12}Y_{21}}{(Y_{11}+Y_{0,1})(Y_{22}+Y_{0,2})-Y_{12}Y_{21}} & \frac{-2Y_{12}Y_{0,2}}{(Y_{11}+Y_{0,1})(Y_{22}+Y_{0,2})-Y_{12}Y_{21}} \\ \frac{-2Y_{21}Y_{0,2}}{(Y_{11}+Y_{0,1})(Y_{22}+Y_{0,2})-Y_{12}Y_{21}} & \frac{(Y_{0,1}+Y_{11})(Y_{0,2}-Y_{22})+Y_{12}Y_{21}}{(Y_{11}+Y_{0,1})(Y_{22}+Y_{0,2})-Y_{12}Y_{21}} \end{bmatrix} \qquad \text{(V-7)}$$

In general, for a multiport network

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V} \qquad \text{(V-8)}$$

$$\mathbf{V}^- = \mathbf{S} \cdot \mathbf{V}^+ \qquad \text{(V-9)}$$

$$\mathbf{I}^\pm = \mathbf{Y}_0 \cdot \mathbf{V}^\pm \qquad \text{(V-10)}$$

$$\mathbf{V} = \mathbf{V}^+ + \mathbf{V}^- \qquad \text{(V-11)}$$

and

$$\mathbf{I} = \mathbf{I}^+ - \mathbf{I}^- = \mathbf{Y}_0 \cdot (\mathbf{V}^+ - \mathbf{V}^-) \qquad \text{(V-12)}$$

so that

$$\mathbf{Y} = \mathbf{Y}_0 (\mathbf{1} - \mathbf{S})(\mathbf{1} + \mathbf{S})^{-1} \qquad \text{(V-13)}$$

and

$$\mathbf{S} = (\mathbf{Y}_0 + \mathbf{Y})^{-1} (\mathbf{Y}_0 - \mathbf{Y}) \qquad \text{(V-14)}$$

The relationship between the *S* parameters and the impedance matrix

Define terminal voltages and currents at the ports of a two-port network as V_1, I_1 and V_2, I_2 at ports 1 and 2, respectively. The port voltages and currents can be related through the impedance matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \qquad \text{(V-15)}$$

For a microwave network, the ports are typically fed with transmission lines. In this case, it is convenient to describe the response in terms of incident and scattered port voltages, V_i^+, V_i^-, respectively. These port voltages are related through the scattering parameters

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}. \qquad \text{(V-16)}$$

Associated with the incident voltage is an incident current, I_i^+, which flows into the port. Similarly, associated with the scattered voltage is a scattered current, I_i^-, flowing out of the port. The incident and scattered voltages and currents are related by the transmission-line wave impedance, $Y_{0,i}$ at the respective ports

$$\begin{bmatrix} I_1^\pm \\ I_2^\pm \end{bmatrix} = \begin{bmatrix} Z_{0,1} & 0 \\ 0 & Z_{0,2} \end{bmatrix}^{-1} \begin{bmatrix} V_1^\pm \\ V_2^\pm \end{bmatrix}. \qquad \text{(V-17)}$$

The total port voltage is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} + \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix}, \qquad \text{(V-18)}$$

while the total port current is

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_1^+ \\ I_2^+ \end{bmatrix} - \begin{bmatrix} I_1^- \\ I_2^- \end{bmatrix} = \begin{bmatrix} Z_{0,1} & 0 \\ 0 & Z_{0,2} \end{bmatrix}^{-1} \left(\begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} - \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} \right), \tag{V-19}$$

with the reference direction into the port. Substituting (V-16), (V-17), (V-18), and (V-19) into (V-15) gives the relationship between the admittance matrix and the scattering parameters

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}} Z_{0,1} & \frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}} Z_{0,2} \\ \frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}} Z_{0,1} & \frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}} Z_{0,2} \end{bmatrix}. \tag{V-20}$$

Similarly

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \frac{(Z_{11}-Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}}{(Z_{11}+Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}} & \frac{2Z_{12}Z_{0,1}}{(Z_{11}+Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}} \\ \frac{2Z_{21}Z_{0,2}}{(Z_{11}+Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}} & \frac{(Z_{11}+Z_{0,1})(Z_{22}-Z_{0,2})-Z_{12}Z_{21}}{(Z_{11}+Z_{0,1})(Z_{22}+Z_{0,2})-Z_{12}Z_{21}} \end{bmatrix} \tag{V-21}$$

In general, for a multiport network

$$\mathbf{V} = \mathbf{Z} \cdot \mathbf{I}, \tag{V-22}$$

$$\mathbf{V}^- = \mathbf{S} \cdot \mathbf{V}^+, \tag{V-23}$$

$$\mathbf{V}^\pm = \mathbf{Z}_0 \cdot \mathbf{I}^\pm, \tag{V-24}$$

$$\mathbf{V} = \mathbf{V}^+ + \mathbf{V}^-, \tag{V-25}$$

and

$$\mathbf{I} = \mathbf{I}^+ - \mathbf{I}^- = \mathbf{Z}_0^{-1} \cdot \left(\mathbf{V}^+ - \mathbf{V}^- \right), \tag{V-26}$$

so that

$$\mathbf{Z} = (\mathbf{1}+\mathbf{S})(\mathbf{1}-\mathbf{S})^{-1} \mathbf{Z}_0, \tag{V-27}$$

and

$$\mathbf{S} = (\mathbf{Z}-\mathbf{Z}_0)(\mathbf{Z}+\mathbf{Z}_0)^{-1}. \tag{V-28}$$

From either (V-8) and (V-22) or (V-14) and (V-28), we see that

$$\mathbf{Y} = \mathbf{Z}^{-1}, \tag{V-29}$$

just as one would expect from the definitions of admittance and impedance in the network.

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