

PAUL BUNYAN'S BRACHISTOCHRONE AND TAUTOCHRONE

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In this paper we concern ourselves with modified versions of the traditional brachistochrone and tautochrone problems. In the modified version of each problem the constant gravity model is replaced with an attractive inverse square law, consequently we name these the $1/r^2$ brachistochrone and $1/r^2$ tautochrone problems. With regard to the $1/r^2$ brachistochrone problem, we show that the shape of the minimizing curve is formally constructed from an infinite series of elliptic integrals, and we use a numerical optimal control technique to generate the trajectories. The $1/r^2$ tautochrone problem is solved using fractional calculus together with Lagrange's rule for tautochronous curves.

INTRODUCTION

The most classic problem in all of the calculus of variations is the brachistochrone problem. Indeed, this problem led to the development of the subject. The problem is to determine the curve joining two points in a vertical plane, along which a particle falling from rest under the influence of constant gravity travels from the higher to the lower point in the least time. It is well known that the answer is a cycloid with its cusp at the starting point. But suppose the two points are far apart and the constant gravity model is replaced with the attractive inverse square law. What is the shape of the curve now?

Another classic problem, but in the area of integral equations, is the tautochrone problem. (Whereas brachistochrone means *least time*, tautochrone means *same time*.) The problem is to determine a planar curve such that the time required for a particle to travel from rest to its lowest point, under the influence of constant gravity, is independent of its initial placement on the curve. It is wonderful that, like the brachistochrone, the solution to this problem is also the cycloid. But again, suppose the initial and final points are far apart and the constant gravity model is replaced with the attractive inverse square law. What is the solution now?

The answers to these two questions are discussed in this paper, which is organized in the following way. We first discuss brachistochrone problems: we review the traditional (constant gravity) brachistochrone problem; we then introduce, and present an optimal control solution to the $1/r^2$ brachistochrone problem. Our attention then turns towards tautochrone problems: we review the traditional tautochrone problem; we then solve the $1/r^2$ tautochrone problem, demonstrate that the solution satisfies Lagrange's rule, and discuss two interesting properties of $1/r^2$ tautochrone trajectories. After we summarize our findings, we present a brief bibliography of brachistochrone and tautochrone problems in an appendix.

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THE BRACHISTOCHRONE PROBLEMS

The Traditional Brachistochrone Problem

In this section we review the traditional brachistochrone problem. The calculus of variations is used to obtain a parametric solution that describes a cycloid.

Let the coordinate y measure down from the starting point and let the coordinate x measure along the horizontal direction. Then the position of the particle, and shape of the curve, is given by $y(x)$. The time to travel from the higher point a to the lower point b is [1]

$$t = \int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx = \int_a^b f(y, y') dx, \quad (1)$$

where $y' = dy/dx$ and g is the constant gravitational acceleration. This equation can be derived using the conservation of energy. The solution to the brachistochrone problem is the function $y(x)$ that minimizes the travel time, and it can be determined from solving the second order differential equation that comes from the associated Euler-Lagrange equation. The function $y(x)$ can also be determined from solving the first order differential equation that defines the Beltrami identity [2]—this identity, which is a first integral of the motion, exists when the integrand is not an explicit function of the independent variable. The Beltrami identity is given by

$$f(y, y') - y' \frac{\partial f}{\partial y'} = C, \quad (2)$$

where C is an unknown constant. Therefore we have

$$\sqrt{1+y'^2} - y'/\sqrt{2gy} = C\sqrt{2gy}, \quad (3)$$

or

$$y(1+y'^2) = \ell, \quad (4)$$

where $\ell = 1/(2gC^2)$. The preceding equation can be separated with the variable y on one side and x on the other to give

$$x = \int \frac{dy}{\sqrt{\ell/y - 1}}. \quad (5)$$

This equation is integrated with the substitution

$$y(\phi) = \ell \sin^2 \phi = \ell(1 - \cos 2\phi)/2 \quad (6)$$

to reveal

$$x(\phi) = \ell(2\phi - \sin 2\phi)/2. \quad (7)$$

The final conditions on x and y are used to determine the constant ℓ and the final condition on the parameter ϕ ; the initial condition on ϕ is zero. These final two equations represent a parametric solution to the brachistochrone problem and are seen to describe a cycloid with its cusp at the starting point.

The $1/r^2$ Brachistochrone Problem

In this section we introduce the $1/r^2$ brachistochrone problem. By following the method used for the traditional brachistochrone problem, we show that the formal solution of the $1/r^2$ brachistochrone problem is constructed from an infinite series of elliptic integrals.

In this development we use polar coordinates rather than Cartesian coordinates to describe the position of the particle and shape of the curve. We begin with the energy equation for a particle in an attractive inverse square law gravity field. The energy equation is [3]

$$v^2 = 2(E_0 + \mu/r), \quad (8)$$

where v is the velocity along the trajectory, E_0 is the initial energy of the particle, μ is the (positive) gravitational parameter, and r is the radial distance from the origin. But v is related to a differential element of the arc length via $v = ds/dt$. Therefore

$$ds = \sqrt{2(E_0 + \mu/r)} dt. \quad (9)$$

It is also true that ds is related to differentials in the polar coordinates r and θ through

$$ds^2 = dr^2 + r^2 d\theta^2, \quad (10)$$

which can be arranged to read

$$ds = r \sqrt{1 + r'^2/r^2} d\theta, \quad (11)$$

where $r' = dr/d\theta$. Equating the two equations for ds leads to

$$dt = \frac{r \sqrt{1 + r'^2/r^2}}{\sqrt{2(E_0 + \mu/r)}} d\theta. \quad (12)$$

The solution to this brachistochrone problem is the function $r(\theta)$ that minimizes the travel time given by the integral expression

$$t = \int_a^b \frac{r \sqrt{1 + r'^2/r^2}}{\sqrt{2(E_0 + \mu/r)}} d\theta = \int_a^b f(r, r') d\theta. \quad (13)$$

Like before, the integrand is not an explicit function of the independent variable. So, the minimizing function $r(\theta)$ can be found from solving the first order differential equation that composes Beltrami's identity. Using equation (2), Beltrami's identity for this problem can be arranged to read

$$2(1 + r'^2/r^2)(E_0 + \mu/r) = r^2/C^2. \quad (14)$$

Because the particle begins from rest, the initial energy is $E_0 = -\mu/r_0$, where r_0 is the initial radial position of the particle. Consequently we have

$$r' = r \sqrt{\frac{r^3}{\ell(1 - r/r_0)} - 1}, \quad (15)$$

where $\ell = 2\mu C^2$.

At this point in the traditional brachistochrone problem, the counterpart to equation (15) was written in a separated form and a clever parameterization was performed that allowed a single integration in terms of elementary functions to yield the minimizing curve. The current brachistochrone problem is more challenging.

Writing equation (15) in a separated form leads to

$$\theta = \int_{r_0}^r \frac{\sqrt{1 - x/r_0}}{x \sqrt{x^3/\ell + x/r_0 - 1}} dx, \quad (16)$$

where x is a dummy variable of integration and we have taken the initial condition on θ to be zero. But within the integration interval it is true that $x/r_0 < 1$. Therefore performing a binomial series expansion on the numerator gives

$$\begin{aligned}\theta = & \int_{r_0}^r \frac{dx}{x\sqrt{x^3/\ell + x/r_0 - 1}} - \frac{1}{2r_0} \int_{r_0}^r \frac{dx}{\sqrt{x^3/\ell + x/r_0 - 1}} \\ & - \frac{1}{8r_0^2} \int_{r_0}^r \frac{x dx}{\sqrt{x^3/\ell + x/r_0 - 1}} - \frac{1}{16r_0^3} \int_{r_0}^r \frac{x^2 dx}{\sqrt{x^3/\ell + x/r_0 - 1}} \\ & - \frac{5}{128r_0^4} \int_{r_0}^r \frac{x^3 dx}{\sqrt{x^3/\ell + x/r_0 - 1}} - \dots\end{aligned}\quad (17)$$

In this form, we recognize the right-hand side as an infinite series of elliptic integrals. That is, each term has the form $\int R(x, \sqrt{P})dx$, where P is a cubic polynomial in x and R is a rational function of x and the square root of P . The terms on the right-hand side of equation (17) lead to elliptic integrals of the first, second, and third kind [4]. Elliptic integrals of the third kind become logarithmically infinite at x equal 0, consequently the path can not travel through the origin. Elliptic integrals of the second kind encounter an algebraic infinity at x equal infinity, consequently the path must begin at a finite distance from the origin.

At this point we have gained an understanding of the shape of the minimizing curve of the $1/r^2$ brachistochrone problem, but there is a problem in pursuing this solution path to completion. The problem is the computed solution is determined from integral tables that are used to evaluate the elliptic integrals of the first, second, and third kind. These evaluations rely on the roots of the cubic polynomial P , which in turn are a function of the unknown constant ℓ , which in turn depends on the boundary conditions and computed solution! This has motivated us to use a different approach and pursue a numerical solution path to our problem.

An Optimal Control Approach to the $1/r^2$ Brachistochrone Problem

In the previous section we found that the solution of the $1/r^2$ brachistochrone problem is given by an infinite series of elliptic integrals. Unfortunately, we were unable to solve the elliptic integrals and generate trajectories that are minimizing curves. Consequently in this section present an optimal control approach to generate trajectories.

The $1/r^2$ brachistochrone problem can be solved by considering the shape of the curve as a control variable and then determining the control that minimizes the maneuver time from the higher to the lower point. This approach is discussed with regard to the traditional brachistochrone problem by Bryson and Ho [5]. We begin by deriving the equations of motion for the particle.

Let the coordinate reference frame $\{e\}$ be centered at the origin and have unit vector components $(\hat{e}_r, \hat{e}_\theta)$. And let this coordinate reference frame rotate such that \hat{e}_r is always pointed directly at the particle. The position and velocity of the particle are then given by the expressions

$$\mathbf{r} = r\hat{e}_r, \quad (18)$$

$$\dot{\mathbf{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta. \quad (19)$$

But the particle velocity vector may also be written

$$\mathbf{v} = -v \sin \gamma \hat{e}_r + v \cos \gamma \hat{e}_\theta, \quad (20)$$

where v is the particle speed and γ is the angle between \hat{e}_r and a unit vector normal to the trajectory (labeled \hat{e}_n). Equating the vector components of equations (19) and (20) yields the first two differential equations of motion

$$\dot{r} = -v \sin \gamma, \quad (21)$$

$$\dot{\theta} = (v/r) \cos \gamma. \quad (22)$$

The final differential equation of motion is obtained from differentiating the energy expression given by equation (8). This leads to

$$v = (\mu/r^2) \sin \gamma. \quad (23)$$

To determine the shape of the curve that minimizes the maneuver time from the higher to the lower point, we treat $\gamma(t)$ as a control variable and solve a minimum-time optimal control problem. The task is to minimize the performance index $J = \int dt$ subject to the preceding differential equation constraints and appropriate boundary conditions. The boundary conditions are initial and final conditions on the motion:

$$\text{Initial conditions} \quad r(0) = r_0, \quad \theta(0) = 0, \quad v(0) = 0;$$

$$\text{Final conditions} \quad r(t_f) = r_f, \quad \theta(t_f) = \theta_f, \quad v(t_f) = \text{free}.$$

These conditions imply that the particle begins from rest and finishes at a defined point with an unspecified final velocity—actually, the final velocity will satisfy the energy equation. The Hamiltonian [6] for this problem is

$$H = 1 - \lambda_r v \sin \gamma + (\lambda_\theta v/r) \cos \gamma + (\lambda_v \mu/r^2) \sin \gamma, \quad (24)$$

where λ_r , λ_θ , and λ_v are the Lagrange multipliers (costates) associated with the three differential equation constraints. This Hamiltonian leads to the costate differential equations

$$\lambda_r = (\lambda_\theta v/r^2) \cos \gamma + (2\lambda_v \mu/r^3) \sin \gamma, \quad (25)$$

$$\lambda_\theta = 0, \quad (26)$$

$$\lambda_v = \lambda_r \sin \gamma - (\lambda_\theta/r) \cos \gamma, \quad (27)$$

and the optimality condition

$$0 = -(\lambda_\theta v/r) \sin \gamma + (\lambda_v \mu/r^2 - \lambda_r v) \cos \gamma. \quad (28)$$

Also, because the final time is free and the system is not an explicit function of time, the Hamiltonian must equal zero for all time. Evaluating the Hamiltonian and the optimality condition at time $t = 0$, while keeping in mind the initial condition on v , leads to $\gamma(0) = \pi/2$.

The solution to our optimal control problem—i.e., the $1/r^2$ brachistochrone problem—must simultaneously satisfy the particle differential equations of motion, the costate differential equations, the optimality condition, the constraint that the Hamiltonian equal zero for all time, and the specified initial and final conditions on the state variables r , θ , and v . The initial conditions on the costate variables and the final maneuver time are unknown, and our numerical approach is to determine these unknowns so that all the constraints are satisfied. To be clear, let us outline the procedure:

1. At time $t = 0$, we know $r(0)$, $\theta(0)$, and $v(0)$. We select $\lambda_r(0)$, $\lambda_\theta(0)$, and t_f .
2. We then integrate the differential equations forward in time while using the optimality condition to determine the control variable $\gamma(t)$.
3. Finally, we compare the final conditions resulting from the integration to the specified values r_f , θ_f and $H(t_f) = 0$, and update our selections $\lambda_r(0)$, $\lambda_\theta(0)$, and t_f based on the error.

This procedure is a numerical shooting method [7]. Within the solution procedure, the initial condition on the costate variable λ_v is obtained from evaluating the Hamiltonian at time $t = 0$. Normalizing the gravitational parameter to equal one, we find $\lambda_v(0) = -r_0^2$. When the solution procedure is complete, the shape of the minimizing curve is given in parametric form by $r = r(t)$ and $\theta = \theta(t)$.

Figures 1 and 2 show two $1/r^2$ brachistochrone trajectories. Figure 1 shows a polar plot of the minimum time trajectory beginning at $(r_0, \theta_0) = (2, 0)$ and ending at $(r_f, \theta_f) = (1, 105^\circ)$. The

missing initial conditions are $(\lambda_r, \lambda_\theta) = (2.2961, -1.087)$ and the final maneuver time is 3.7007 seconds. Figure 2 shows a trajectory from a constant (magnitude and direction) gravity model and a minimum time trajectory assuming an attractive inverse square law. (The constant gravity points left in the figure.) The minimum time trajectory dips towards the origin more than the constant gravity trajectory and takes 3.1535 seconds to complete. The constant gravity trajectory takes 3.5454 seconds to complete.

THE TAUTOCHRONE PROBLEMS

The Traditional Tautochrone Problem

As mentioned in the Introduction, the tautochrone problem is concerned with determining a planar curve such that the time required for a particle to travel from rest to its lowest point is independent of its initial placement. In this section we review the traditional tautochrone problem. We use the Laplace transform method to show that the parametric solution to the traditional tautochrone problem describes a cycloid, which is the same type of curve that solves the traditional brachistochrone problem.

Let a Cartesian coordinate reference frame be placed at the destination, and let the particle's initial coordinates be x_i and y_i in the horizontal and vertical directions. Then the travel time is given as

$$t = \int_{y_i}^0 \frac{ds}{v}, \quad (29)$$

where ds is a differential element of the arc length and v is the particle velocity. But ds and v can be expressed as functions of the vertical displacement through the equations

$$ds = -\sqrt{1 + x'^2} dy = -\phi(y) dy, \quad (30)$$

$$v = \sqrt{2g(y_i - y)}, \quad (31)$$

where $x' = dx/dy$ and g is the constant gravitational acceleration. These two equations lead to

$$t = \int_0^{y_i} \frac{\phi(y)}{\sqrt{2g(y_i - y)}} dy. \quad (32)$$

The solution to the tautochrone problem begins by determining the function $\phi(y)$ such that t is constant, and therefore independent of y_i . Equations wherein the unknown function appears under the integral sign are integral equations, and indeed, equation (32) is recognized as a special case of *Abel's integral equation* [8]. Two common approaches to solving this equation are Abel's method [8] and the Laplace transform method [9]. A solution via the Laplace transform method is highlighted below.

Let us introduce $\psi(u) = 1/\sqrt{u}$, then

$$t\sqrt{2g} = \int_0^{y_i} \phi(y) \psi(y_i - y) dy. \quad (33)$$

But the right-hand side of this equation is the convolution of ϕ and ψ , so the Laplace transform of each side produces

$$t\sqrt{2g}/s = \phi(s) \psi(s). \quad (34)$$

The Laplace transform of $\psi(u)$ can be determined and substituted into this equation. This gives

$$\phi(s) = t\sqrt{2g/(\pi s)}, \quad (35)$$

or, taking the inverse transform,

$$\phi(y) = \frac{t}{\pi} \sqrt{2g/y}. \quad (36)$$

From this point, equation (30) can be used to give

$$\frac{dx}{dy} = \sqrt{\ell/y - 1}, \quad (37)$$

where $\ell = 2gt^2/\pi^2$. This equation is integrated with the substitution

$$y(\phi) = \ell \sin^2 \phi = \ell(1 - \cos 2\phi)/2 \quad (38)$$

to reveal

$$x(\phi) = \ell(2\phi + \sin 2\phi)/2. \quad (39)$$

These two equations represent a parametric solution to the tautochrone problem. This parametric form describes a cycloid with its vertex at the origin, and therefore it has a slightly different form than the one that describes the solution to the traditional brachistochrone problem.

The $1/r^2$ Tautochrone Problem

In this section we introduce the $1/r^2$ tautochrone problem. Our first attempts at solving this problem involved using Abel's method and the Laplace transform method. We were not successful in obtaining a solution using either method. Consequently, we employed an uncommon approach that has been useful in the past to obtain solutions to other tautochrone problems [10, 11]. This approach uses the fractional calculus method, and its application in solving the $1/r^2$ tautochrone problem is presented below.

Beginning with the energy equation expressed in polar coordinates, the velocity of a particle beginning from rest in an attractive inverse square law gravity field can be written [3]

$$v = \sqrt{2\mu(1/r - 1/r_0)}, \quad (40)$$

where v is the velocity along the trajectory, μ is the gravitational parameter, r is the radial distance from the origin, and r_0 is the initial radial position of the particle. Also, a differential element of the arc length of the curve can be expressed as a function of the radial distance via

$$ds = \sqrt{1 + r^2\theta'^2} dr = \phi(r) dr, \quad (41)$$

where $\theta' = d\theta/dr$. From these two equations, the travel time for the particle is

$$t = \int_{r_0}^a \frac{ds}{v} = \int_{r_0}^a \frac{\phi(r)}{i\sqrt{2\mu(1/r_0 - 1/r)}} dr, \quad (42)$$

where a denotes the final radial position of the curve and $i = \sqrt{-1}$. Equation (42) is an integral equation and the solution to this tautochrone problem begins by determining the function $\phi(r)$ such that t is constant, and therefore independent of r_0 .

To use the fractional calculus methods, we first write equation (42) as a semiintegral expression. Let us introduce the functions $g(u) = 1/u$ and $\hat{\phi}(u) = u^2\phi(u)$ so that equation (42) becomes

$$it\sqrt{2\mu} = \int_a^{r_0} \frac{\hat{\phi}(r)g'(r)}{(g(r_0) - g(r))^{\frac{1}{2}}} dr, \quad (43)$$

or, dividing by the factor π on each side,

$$it\sqrt{2\mu/\pi} = \frac{1}{\Gamma(\frac{1}{2})} \int_a^{r_0} \frac{\hat{\phi}(r)g'(r)}{(g(r_0) - g(r))^{\frac{1}{2}}} dr. \quad (44)$$

$\Gamma()$ is the gamma function and the prime denotes differentiation with respect to r . The right-hand side of equation (44) is recognized as the Riemann-Liouville integral-based definition of the *semiintegral* of $\hat{\phi}(r)$ with respect to the function $g(r_0)$ [12, 13]. Using the notation introduced by Osler [13] this is written as

$$D_{g(r_0)-g(a)}^{-\frac{1}{2}} \hat{\phi}(r_0) = \frac{1}{\Gamma(\frac{1}{2})} \int_a^{r_0} \frac{\hat{\phi}(r)g'(r)}{(g(r_0) - g(r))^{\frac{1}{2}}} dr, \quad (45)$$

so equation (44) becomes

$$D_{g(r_0)-g(a)}^{-\frac{1}{2}} \hat{\phi}(r_0) = it\sqrt{2\mu/\pi}. \quad (46)$$

Taking the *semiderivative* [12] of each side of this equation with respect to $g(r_0)$ gives

$$D_{g(r_0)-g(a)}^{\frac{1}{2}} D_{g(r_0)-g(a)}^{-\frac{1}{2}} \hat{\phi}(r_0) = D_{g(r_0)-g(a)}^{\frac{1}{2}} (it\sqrt{2\mu/\pi}). \quad (47)$$

We next assume that the composition rule [14],

$$D_x^q D_x^Q f = D_x^{q+Q} f, \quad (48)$$

where f is a suitable function, is not violated for our unknown function $\hat{\phi}(r_0)$. (The requirements for obedience to this rule are that f and $D_x^Q f$ be differintegrable; we will have more to say on this in the next section.) Consequently, by way of the composition rule, equation (47) becomes

$$\begin{aligned} \hat{\phi}(r_0) &= D_{g(r_0)-g(a)}^{\frac{1}{2}} (it\sqrt{2\mu/\pi}), \\ &= it\sqrt{2\mu/\pi} D_{g(r_0)-g(a)}^{\frac{1}{2}} \{1\}, \end{aligned} \quad (49)$$

where $D_{g(r_0)-g(a)}^{\frac{1}{2}} \{1\}$ is the semiderivative of the constant value one with respect to $g(r_0)$. This expression is evaluated using the extension [15]

$$D_x^{\frac{1}{2}} f = \frac{d}{dx} (D_x^{-\frac{1}{2}} f), \quad (50)$$

so that we have

$$D_{g(r_0)-g(a)}^{\frac{1}{2}} \{1\} = \frac{d}{d(g(r_0) - g(a))} \left\{ \frac{1}{\Gamma(\frac{1}{2})} \int_a^{r_0} \frac{g'(r)}{(g(r_0) - g(r))^{\frac{1}{2}}} dr \right\}. \quad (51)$$

But the integrand is in the form of a perfect differential, therefore

$$\begin{aligned} D_{g(r_0)-g(a)}^{\frac{1}{2}} \{1\} &= \frac{d}{d(g(r_0) - g(a))} \left\{ \frac{1}{\sqrt{\pi}} \int_a^{r_0} \left[-2 \frac{d}{dr} ((g(r_0) - g(r))^{\frac{1}{2}}) \right] dr \right\} \\ &= \frac{d}{d(g(r_0) - g(a))} \left\{ -\frac{2}{\sqrt{\pi}} [(g(r_0) - g(r))^{\frac{1}{2}}] \Big|_a^{r_0} \right\} \\ &= \frac{1}{\sqrt{\pi}} (g(r_0) - g(r))^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi} (1/r_0 - 1/a)}. \end{aligned} \quad (52)$$

Folding this result into equation (49) yields

$$\hat{\phi}(r_0) = \frac{it\sqrt{2\mu}}{\pi\sqrt{1/r_0 - 1/a}}, \quad (53)$$

or, in terms of the original function ϕ ,

$$\phi(r_0) = \frac{t\sqrt{2\mu}}{\pi r_0^2 \sqrt{1/a - 1/r_0}}. \quad (54)$$

From this point, we use equation (41) to obtain the differential equation of the $1/r^2$ tautochrone problem. After some algebra, we obtain the expression

$$\frac{d\theta}{dr} = \frac{1}{r} \sqrt{\frac{a\ell}{r^3(r-a)}} - 1, \quad (55)$$

where $\ell = 2\mu t^2/\pi^2$ and we have retained the positive root.

The Satisfaction of Lagrange's Rule for the $1/r^2$ Tautochrone Problem

In the previous section we obtained equation (55) as the solution to the $1/r^2$ tautochrone problem. We must remember, however, that in arriving at this equation we assumed the composition rule was true for the unknown function $\hat{\phi}(r_0)$. Therefore equation (55) should be considered a *candidate* solution until it can somehow be confirmed that, indeed, this solution is valid: Lagrange's rule provides such confirmation.

Lagrange's rule is a sufficient condition on a force, such that the force causes a tautochronous motion of the system. Routh [16] explains this rule:

LAGRANGE'S RULE: *If the equation of motion is*

$$\ddot{x} = \dot{x}^2 \frac{f'(x)}{f(x)} + F(\dot{x}, f(x)), \quad (56)$$

where F is a homogeneous function of the first degree, and $f(x)$ is any function of x , then the time of arriving at the position determined by $f(x) = 0$ is constant, regardless of the initial position.

To demonstrate that the solution to the $1/r^2$ tautochrone problem satisfies Lagrange's rule, we first derive the equations of motion of our system. The system has two generalized coordinates and one holonomic constraint expressed in Pfaffian form, viz., equation (55) which is rewritten here

$$d\theta - \frac{1}{r} \sqrt{\frac{a\ell}{r^3(r-a)}} - 1 dr = 0. \quad (57)$$

The equations of motion for this type of system description are given by [17]

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda c_r, \quad (58)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda c_\theta, \quad (59)$$

$$c_r \dot{r} + c_\theta \dot{\theta} = 0, \quad (60)$$

where L is the system Lagrangian function, c_r and c_θ are functions that depend upon r and θ , and λ is the Lagrange multiplier associated with the constraint. The system Lagrangian function is

$$L = T - V = (\dot{r}^2 + r^2\dot{\theta}^2)/2 + \mu/r, \quad (61)$$

where T and V are the kinetic and potential energies. Consequently, the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 + \mu/r^2 = -\lambda c_r, \quad (62)$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = \lambda, \quad (63)$$

$$\theta - c_r r = 0, \quad (64)$$

where

$$c_r = \frac{1}{r} \sqrt{\frac{a\ell}{r^3(r-a)}} - 1. \quad (65)$$

Using equation (63) to eliminate λ in equation (62) gives

$$\ddot{r} - r\dot{\theta}^2 + \mu/r^2 = -c_r(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}). \quad (66)$$

And using equation (64) and its time derivative to eliminate $\dot{\theta}$ and $\ddot{\theta}$ gives

$$\ddot{r} - r\dot{r}^2 c_r^2 + \mu/r^2 = -c_r(r^2 c_r' \dot{r}^2 + \ddot{r} r^2 c_r + 2r\dot{r}^2 c_r), \quad (67)$$

where the prime indicates differentiation with respect to r . Performing some algebra leads to

$$\ddot{r} = -\dot{r}^2 \frac{(r^2 c_r c_r' + r c_r^2)}{(1 + r^2 c_r^2)} - \frac{\mu}{r^2(1 + r^2 c_r^2)}, \quad (68)$$

or, introducing $b_r = 1 + r^2 c_r^2$, we have

$$\ddot{r} = -\dot{r}^2 \frac{b_r'}{2b_r} - \frac{\mu}{r^2 b_r}. \quad (69)$$

Fully evaluating b_r and b_r' shows that

$$b_r = \frac{a\ell}{r^3(r-a)}, \quad b_r' = -b_r \frac{4r-3a}{r(r-a)}, \quad (70, 71)$$

consequently,

$$\ddot{r} = \frac{\dot{r}^2(4r-3a)}{2r(r-a)} - \frac{\mu r(r-a)}{a\ell}. \quad (72)$$

Equation (72) is the equation of motion of our system, which is a particle falling from rest under the influence of an attractive inverse square law gravity field, but subject to the holonomic constraint (expressed in Pfaffian form) that is the candidate solution to the $1/r^2$ tautochrone problem. This equation of motion is nearly in the form required by Lagrange's rule. Let us then continue to operate on equation (72) by expanding the first term on the right-hand side to produce

$$\ddot{r} = \frac{\dot{r}^2(2r-a)}{r(r-a)} - \frac{a\dot{r}^2}{2r(r-a)} - \frac{\mu r(r-a)}{a\ell}. \quad (73)$$

Defining $f(r) = r(r-a)$, we obtain

$$\ddot{r} = \dot{r}^2 \frac{f'(r)}{f(r)} - \frac{a\dot{r}^2}{2f(r)} - \frac{\mu f(r)}{a\ell}. \quad (74)$$

The last two terms of this expression form a homogenous function of $f(r)$ and \dot{r} of the first degree. Euler's homogeneous function theorem can be used to recognize this fact:

EULER'S HOMOGENEOUS FUNCTION THEOREM: *Let F be a homogeneous function of order n . Then*

$$nF(\mathbf{x}) = x_i \frac{\partial F}{\partial x_i}, \quad (75)$$

where \mathbf{x} represents the vector of variables and Einstein's summation convention has been used.

Let

$$F(\dot{r}, f(r)) = -\frac{a\dot{r}^2}{2f(r)} - \frac{\mu f(r)}{a\ell}, \quad (76)$$

then

$$\dot{r} \frac{\partial F}{\partial \dot{r}} + f(r) \frac{\partial F}{\partial f} = -\frac{a\dot{r}^2}{f(r)} + f(r) \left(\frac{a\dot{r}^2}{2f^2(r)} - \frac{\mu}{a\ell} \right) = F(\dot{r}, f(r)). \quad (77)$$

So

$$\ddot{r} = \dot{r}^2 \frac{f'(r)}{f(r)} + F(\dot{r}, f(r)), \quad (78)$$

where F is a homogenous function of $f(r)$ and \dot{r} of the first degree. Comparing equations (78) and (56) we see that the equation of motion is now in the form required by Lagrange's rule. Consequently the time of arriving at $f(r) = 0$, which has solution $r = a$, is constant regardless of the initial position. Thus we have fully validated the fractional calculus candidate solution to the $1/r^2$ tautochrone problem because the solution satisfies Lagrange's rule, which is a sufficient condition for tautochronous motion.

Trajectories of the $1/r^2$ Tautochrone Problem

Although we have shown that

$$\frac{d\theta}{dr} = \frac{1}{r} \sqrt{\frac{a\ell}{r^3(r-a)}} - 1 \quad (79)$$

is the solution to the $1/r^2$ tautochrone problem, we must still integrate this equation to obtain the trajectories. Unfortunately, we have not found expressions that provide the exact solution to this integral. But before we discuss a numerical integration solution, let us look more closely at this equation.

We first recall that ℓ is related to the travel time through the equation $\ell = 2\mu t^2/\pi^2$. Consequently, for a particular trajectory defined by initial and final positions, ℓ must be such that the radical remains positive. That is,

$$\ell \geq r^3(r-a)/a \quad (80)$$

throughout the motion. But r decreases during the trajectory from r_0 to a , so the right-hand side is largest at $r = r_0$. Therefore we require

$$\ell \geq r_0^3(r_0-a)/a, \quad (81)$$

and as a result we set ℓ as

$$\ell = \beta r_0^3(r_0-a)/a, \quad \text{where } \beta \geq 1. \quad (82)$$

Now notice that the right-hand side of the differential equation equals infinity at the end point $r = a$. To bypass this hazard we introduce the parameter γ such that

$$r^3(r-a) = a\ell \cos^2 \gamma, \quad (83)$$

which leads to

$$\frac{d\theta}{dr} = \tan \gamma / r. \quad (84)$$

But

$$\frac{dr}{d\gamma} = \frac{-2a\ell \cos \gamma \sin \gamma}{r^2(4r - 3a)}, \quad (85)$$

so

$$\frac{d\theta}{d\gamma} = \frac{-2a\ell \sin^2 \gamma}{r^3(4r - 3a)}, \quad (86)$$

where r must be determined from the quartic equation given by equation (83). The limits of γ can be discovered using equations (82) and (83):

$$\begin{aligned} \text{Initial condition } (r = r_0) \quad \gamma_0 &= \cos^{-1} \sqrt{1/\beta}, \\ \text{Final condition } (r = a) \quad \gamma_a &= \pi/2, \end{aligned}$$

where we have retained the positive roots. The trajectories can now be numerically determined, and to be clear, we outline the procedure:

1. The initial and final values of r are selected. These are labeled r_0 and a respectively.
2. A final value for θ is selected—the initial value is taken as 0. This specified value is labeled θ_a .
3. A value for β is selected and ℓ is computed from equation (82).
4. Equation (86) is numerically integrated over the interval $\gamma \in [\cos^{-1} \sqrt{1/\beta}, \pi/2]$, while r is determined from equation (83).
5. If the final value of θ resulting from the integration does not match the desired final value specified in step 2, then β is adjusted, ℓ is recomputed, and step 4 is repeated.

There are two interesting properties regarding $1/r^2$ tautochrone trajectories:

1. Suppose $\beta = 1$ but the final value of θ resulting from the integration is greater than θ_a , which is the desired final value. Then there is no tautochrone curve that connects $(r_0, 0)$ to (a, θ_a) . But because there exists a θ_* that is congruent to θ_a (modulo 2π), a larger β can be determined so that a tautochrone curve connects $(r_0, 0)$ to (a, θ_*) . This situation is illustrated in Fig. 3. The initial position is $(r_0, 0) = (2, 0)$, and the desired final position is $(a, \theta_a) = (1, \pi/2)$. The solid line, which corresponds to $\beta = 1$, shows that there is no tautochrone curve that connects $(2, 0)$ to $(1, \pi/2)$. The dashed line shows, however, that by increasing β to 5.6422 we are able to reach $(1, 5\pi/2)$, which also locates the desired final position.
2. Setting $\beta = 1$, a trajectory can be generated that begins at $(r_0, 0)$ and extends to (a, θ_1) , where a is a final radial position and θ_1 is the final angular position that corresponds to this choice of a and the fact that $\beta = 1$. Notice, however, that because of the congruency of θ and the decreasing nature of $dr/d\theta$, the final point can instead be reached at a later time by selecting the appropriate larger value of β . Therefore the $\beta = 1$ trajectory represents a minimum-time tautochrone curve: a *brachistochrone-tautochrone curve*, if you will. Figure 4 shows a family of brachistochrone-tautochrone curves, which begin at $r_0 = 2$ and extend to a range of a values.

CONCLUSIONS

In this paper we investigated a modified brachistochrone problem wherein the constant gravity model is replaced with an attractive inverse square law. The shape of the minimizing curve of the $1/r^2$ brachistochrone problem is formally constructed from an infinite series of elliptic integrals. Trajectories were generated by employing a common numerical approach wherein the shape of the curve is treated as a control variable and the control that minimizes the maneuver time from the higher to the lower point is computed.

We also investigated a modified tautochrone problem wherein the constant gravity model is replaced with an attractive inverse square law. Fractional calculus together with Lagrange's rule for tautochronous curves was used to solve the $1/r^2$ tautochrone problem. While there may be no tautochronous curve that strictly connects an initial position to a desired final position, the congruency of the angular variable allows all points to be reached. The congruency of the angular variable also allowed us to define minimum-time tautochronous curves, which we named brachistochrone-tautochrone curves.

APPENDIX

We use this section to briefly mention and discuss other brachistochrone and tautochrone problems.

Brachistochrone Problems

E.J. Routh in *A Treatise on Dynamics of a Particle*, Articles 590 to 606 (Dover Publications, Inc., New York, New York (1960)), has much to say regarding brachistochrone problems. In Theorem III, Article 599, Routh shows that the brachistochrone from point to point in a field $U + C$, where U is a known function of the coordinates and C is a known constant, is the same as a path of a free particle in a field $U' + C'$, provided

$$U' + C' = \frac{k^4}{4} \frac{1}{U + C}.$$

In Article 605, Exercise 1, the reader is asked to show that if the brachistochrone is a parabola when the force is parallel to the axis, then the magnitude of the force is inversely proportional to the square of the distance from the directrix. In Article 606, several central force brachistochrone problems, but with non zero initial velocity, are discussed. In Article 606, Exercise 4, the reader is asked to show that the cissoid $x(x^2 + y^2) = 2ay^2$ is a brachistochrone curve for a central repulsive force from the point $(-a, 0)$ which at the distance r from that point is proportional to $r/(r^2 + 15a^2)^2$, the particle starting from rest at the cusp.

In Exercise 5, page 44 of a text by A.R. Forsyth (*Calculus of Variations*, Cambridge University Press (1927)), one is asked to show that an ellipse is a brachistochrone between two points on its range, under a central force to one focus varying inversely as the square of the distance from the other focus.

In the *Encyclopaedic Dictionary of Mathematics for Engineers and Applied Scientists* (I.N. Sneddon editor, Pergamon Press, Inc., New York (1976)), it is mentioned that the brachistochrone problem of a particle under the action of a central force has solution

$$\theta = \int \frac{du}{\{c_1/(E - V) - u^2\}^{\frac{1}{2}}} + c_2,$$

where E is the total energy, V is the potential energy, $u = 1/r$, and c_1 and c_2 are constants of integration. Our expression for θ in the $1/r^2$ brachistochrone section of this paper satisfies this equation.

The brachistochrone with coulomb friction was studied by N. Ashby et al., in the *American Journal of Physics* (Vol. 43, No. 10, October, pp. 902-906 (1975)). This problem was also studied as a singular control problem by S.C. Lipp in *SIAM Journal of Control and Optimization* (Vol. 35, No. 2, March, pp. 562-584 (1997)).

The relativistic brachistochrone was studied by H.F. Goldstein and C.M. Bender in the *Journal of Mathematical Physics* (Vol. 27, Issue 2, pp. 507-511 (1986)).

Tautochrone Problems

E.J. Routh in *A Treatise on Dynamics of a Particle*, Article 211, has much to say regarding tautochrone problems. He mentions that if any rectifiable curve is given, then a proper force to produce a tautochronous motion can at once be assigned. A catenary is a tautochronous curve for a force acting along the ordinate equal to m^2y , where m is a constant. The equiangular spiral is a tautochronous curve for a linear central force μr tending to the pole. And, the epicycloid and hypocycloid are tautochronous curves for a linear central force tending from or to the center of the fixed circle.

Routh has more to say in *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, Articles 488 to 499. In Article 491, a theorem due to Lagrange shows that if the motion is tautochronous in a vacuum, then the motion is also tautochronous in a medium whose resistance varies as the velocity. In Article 495, tautochronous motion on any rough curve is discussed. And in Article 499, a system having one degree of freedom and described by $2T = m^2\dot{\theta}^2$, $U = f(\theta)$, has tautochronous motion if $U = C\{\int m d\theta\}^2$.

ACKNOWLEDGMENT

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REFERENCES

- [1] Goldstein, H., *Classical Mechanics*, Addison-Wesley Publishing, Reading, Massachusetts, 1980, Chapter 2, Section 3.
- [2] Weisstein, E.W., *Encyclopedia of Mathematics*, Chapman and Hall/CRC, New York, New York, 1999, pp. 107.
- [3] Battin, R.H., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Education Series, American Institute of Aeronautics and Astronautics, Inc., New York, New York, 1987, Chapter 3, Section 3, Equation (3.17).
- [4] Byrd, P.F., and Friedman, M.D., *Handbook of Elliptic Integrals for Engineers and Scientists*, Second Edition, Revised, Springer-Verlag Publishing, Berlin, Germany, 1971, Section 230.
- [5] Bryson, A.E., and Ho, Y.-C., *Applied Optimal Control*, Revised Printing, Taylor & Francis, Pennsylvania, 1975, Chapter 2, Section 7, Problem 6.
- [6] Ref. 5, Chapter 2, Section 2.3, Equation (2.3.4).
- [7] Junkins, J.L., and Turner, J.D., *Optimal Spacecraft Rotational Maneuvers*, Elsevier Science Publishers, New York, New York, 1986, pp 228.
- [8] Pogorzelski, W., *Integral Equations and their Applications*, Pergamon Press, Inc., Long Island City, New York, 1966, Vol. 1, Chapter 1, Section 1.
- [9] Andrews, L.C., and Shivamoggi, B.K., *Integral Transforms for Engineers*, SPIE Optical Engineering Press, Bellingham, Washington, 1999, Chapter 5, Section 5.
- [10] Oldham, K.B., and Spanier, J., *The Fractional Calculus*, Academic Press, Inc., New York, New York, 1974, Chapter 10, Section 2.
- [11] Kamath, S.G., "Relativistic tautochrone," *Journal of Mathematical Physics*, Vol. 33, Issue 3, 1992, pp. 934-940.
- [12] Ref. 10, Chapter 3, Section 2, Equation (3.2.3); Chapter 7, Introduction.

- [13] Osler, T.J., "Leibniz Rule for Fractional Derivatives Generalized and an Application to Infinite Series," *SIAM Journal on Applied Mathematics*, Vol. 18, 1970, pp. 658-674, Section 3, Equation (3.1).
- [14] Ref. 10, Chapter 5, Section 7.
- [15] Ref. 10, Chapter 3, Section 2, Equation (3.2.5).
- [16] Routh, E.J., *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, Dover Publications, Inc., New York, New York, 1960, Article 490.
- [17] Ref. 1, Chapter 2, Section 4, Equations (2-29) and (2-30).

Figure 1 A $1/r^2$ Brachistochrone Example

Figure 2 Comparison of $1/r^2$ and Traditional Brachistochrone Trajectories

Figure 3 A $1/r^2$ Tautochrone Example

Figure 4 A Family of Brachistochrone-Tautochrone Curves

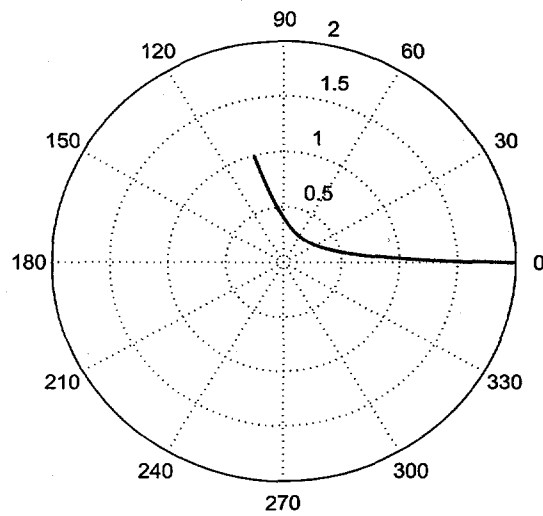


Figure 1 A $1/r^2$ Brachistochrone Example

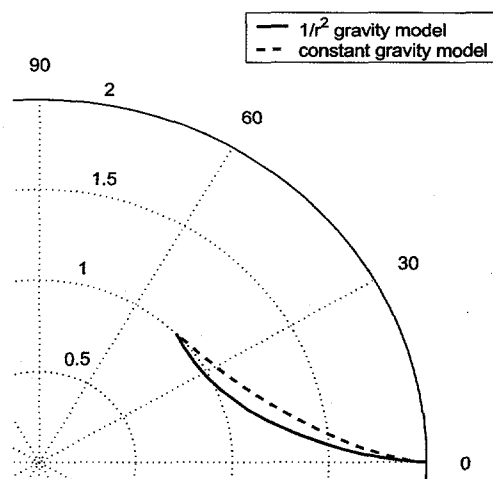


Figure 2 Comparison of $1/r^2$ and Traditional Brachistochrone Trajectories

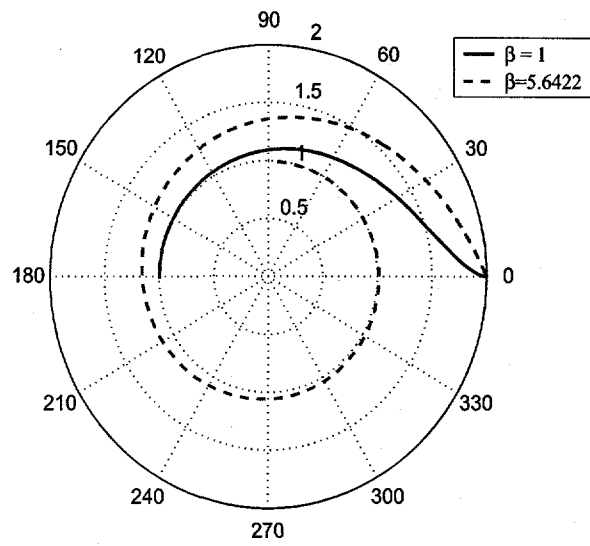


Figure 3 A $1/r^2$ Tautochrone Example

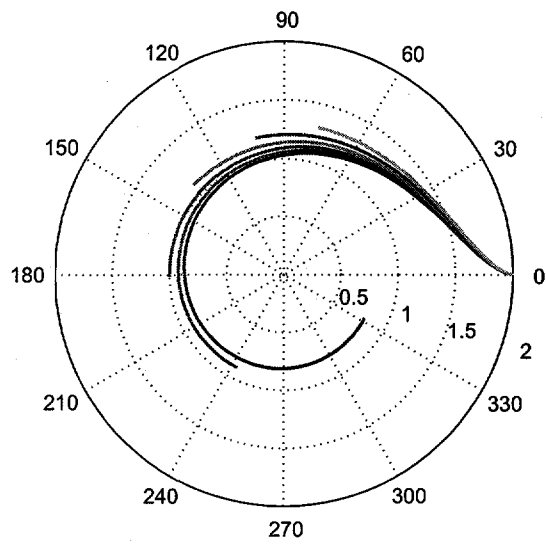


Figure 4 A Family of Brachistochrone-Tautochrone Curves